Generalising a triangle inequality

ZOLTAN RETKES

The main goal of this paper is to give a deeper understanding of the geometrical inequality proposed by Martin Lukarevski in [1]. In order to formulate our results we shall introduce and use the following notation throughout this paper. Let $A_1A_2A_3$ be a triangle, a_1, a_2, a_3 the lengths of the sides opposite to A_1, A_2, A_3 respectively, P an arbitrary inner point of it, x_i the distance of P from the side of length a_i. Let r, R be the inradius and circumradius of the triangle, h_i the altitude belonging to side a_i, Δ the area and finally let α be a real parameter. We adopt also the use of $\sum u_i$ to refer the sum taken over the suffices $i = 1, 2, 3$. Now we are in the position to reformulate the original problem into a more general form namely: find bounds for $\sum x_i^\alpha$ in terms of r and R. The main results of our investigation are summarised in the following theorem.

Theorem 1: For any triangle $A_1A_2A_3$ and any inner point P we have the following statements:

(a) For $0 \leqslant \alpha \leqslant \frac{1}{2}$

$$\sum x_i^\alpha \leqslant 3^{1-2\alpha}\left(\sum h_i\right)^\alpha \leqslant 3^{1-2\alpha}(r+R)^{2\alpha}\left(\frac{2}{R}\right)^\alpha.$$

(b) For $\alpha < 0$

$$\sum x_i^\alpha \geqslant 3^{1-2\alpha}\left(\sum h_i\right)^\alpha \leqslant 3^{1-2\alpha}(r+R)^{2\alpha}\left(\frac{2}{R}\right)^\alpha.$$

(c) If $\alpha = 0$ then equality is valid. If $\alpha \neq 0$ then equality is valid in both (a) and (b) if, and only if, $A_1A_2A_3$ is equilateral and P is its centroid.

In order to prove Theorem 1 we need two auxiliary results organised in two lemmas below.

Lemma 1: For any inner point P of the triangle $A_1A_2A_3$,

$$\sum \sqrt{x_i} \leqslant \sqrt{\sum h_i}. \tag{1}$$

Equality is valid in (1) if, and only if, $A_1A_2A_3$ is equilateral.

Proof: Since $2\Delta = \sum a_i x_i = \sum \frac{2\Delta}{h_i} x_i$, we have $\sum \frac{x_i}{h_i} = 1$. By virtue of the Cauchy-Schwarz-Bunyakovsky inequality we infer

$$\left(\sum \sqrt{x_i}\right)^2 = \left(\sum \sqrt{\frac{x_i}{h_i}}\sqrt{h_i}\right)^2 \leqslant \left(\sum \frac{x_i}{h_i}\right)\left(\sum h_i\right) = \sum h_i$$

and the proof is complete.

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Lemma 2: Using the same notation as above we have

$$\sum h_i \leqslant (r + R)^2 \frac{2}{R}.$$

Proof: This result can be found in [2, p. 130], where the reader may explore detailed background information about related identities and inequalities as well.

Clearly Lemmas 1 and 2 imply $\sum \sqrt{x_i} \leqslant (R + r)\sqrt{\frac{2}{R}}$, the inequality originally proposed in [1].

Proof of Theorem 1

(a) Assume first $0 < \alpha \leqslant \frac{1}{2}$ which implies $0 < 2\alpha \leqslant 1$. Application of Jensen's inequality leads us to the following estimation

$$\sum x_i^{\alpha} = 3\frac{\sum \sqrt{x_i}^{2\alpha}}{3} \leqslant 3\left(\frac{\sum \sqrt{x_i}}{3}\right)^{2\alpha} = 3^{1-2\alpha}\left(\sum \sqrt{x_i}\right)^{2\alpha}.$$

By virtue of Lemma 1, the latter expression can be estimated from above by $\sum h_i$ as follows

$$3^{1-2\alpha}\left(\sum \sqrt{x_i}\right)^{2\alpha} \leqslant 3^{1-2\alpha}\left[\sqrt{\sum h_i}\right]^{2\alpha} = 3^{1-2\alpha}\left(\sum h_i\right)^{\alpha}.$$

Finally we apply Lemma 2 for h_i which gives the whole inequality chain of part (a)

$$3^{1-2\alpha}\left(\sum h_i\right)^{\alpha} \leqslant 3^{1-2\alpha}\left[(r + R)^2 \frac{2}{R}\right]^{\alpha} = 3^{1-2\alpha}(r + R)^{2\alpha}\left(\frac{2}{R}\right)^{\alpha}.$$

(b) Assume now $\alpha < 0$. Following the lines of (a), application of Jensen's inequality for the convex case gives

$$\sum x_i^{\alpha} \geqslant 3^{1-2\alpha}\left(\sum \sqrt{x_i}\right)^{2\alpha}.$$

Application of Lemmas 1 and 2 together with the fact that $f(t) = t^{\alpha}$ is strictly decreasing in t provides the inequality chain of part (b):

$$3^{1-2\alpha}\left(\sum \sqrt{x_i}\right)^{2\alpha} \geqslant 3^{1-2\alpha}\left(\sum h_i\right)^{\alpha} \geqslant 3^{1-2\alpha}(r + R)^{2\alpha}\left(\frac{2}{R}\right)^{\alpha}.$$

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(c) If $\alpha = 0$ then both sides equal 3 independently of the shape of the triangle and the location of P inside. If $\alpha \neq 0$ then the conditions of equality are the same in (a) and (b). These conditions originate from Lemma 1, Lemma 2 and Jensen's inequality. We have equality in Lemma 1 if, and only if, the triangle is equilateral; in Lemma 2 if, and only if, the triangle is equilateral and P is its centroid; in Jensen's inequality if, and only if, P is the incentre of the triangle. Combining these conditions completes the proof of part (c).

Our theorem shows that the special case of $\alpha = \frac{1}{2}$ can be extended for all exponents $\alpha \leqslant \frac{1}{2}$ and a constant (depending on α) multiple of $\left(\sum h_i\right)^{\alpha}$ can be inserted between the two sides of the original inequality. Moreover we have pretty inequalities for special choices of α.

If $\alpha = -1$ then

$$\sum \frac{1}{x_i} \geqslant \frac{27}{\sum h_i} \geqslant \frac{27}{2}\,\frac{R}{(r+R)^2}.$$

If $\alpha = -\frac{1}{2}$ then

$$\sum \frac{1}{\sqrt{x_i}} \geqslant \frac{9}{\sqrt{\sum h_i}} \geqslant \frac{9}{\sqrt{2}}\,\frac{\sqrt{R}}{r+R}.$$

We now explore other ways of obtaining bounds for $\sum x_i^{\alpha}$. A powerful and elegant way is offered by Hölder's inequality. We summarise our statements in Theorem 2:

Theorem 2:

(a) If $\alpha > 1$ or $\alpha < 0$ then

$$\sum x_i^{\alpha} \geqslant \frac{(2\Delta)^{\alpha}}{\left(\sum a_i^{\alpha/(\alpha-1)}\right)^{\alpha-1}}. \tag{2}$$

(b) If $0 < \alpha < 1$ then we have

$$\sum x_i^{\alpha} \leqslant \frac{(2\Delta)^{\alpha}}{\left(\sum a_i^{\alpha/(\alpha-1)}\right)^{\alpha-1}}. \tag{3}$$

(c) Equality is valid in both (a) and (b) if, and only if, the triangle is equilateral and P is its centroid.

In this very fortunate case we can apply the general form of Hölder's inequality for $2\Delta = \sum a_i x_i$ with conjugate indices $p = \alpha$, $\frac{1}{q} = 1 - \frac{1}{p} = \frac{\alpha-1}{\alpha}$. It simply says that

$$2\Delta = \sum x_i a_i \leqslant \left(\sum x_i^{\alpha}\right)^{1/\alpha} \left(\sum a_i^{\alpha/(\alpha-1)}\right)^{(\alpha-1)/\alpha} \qquad \text{for } \alpha > 1 \tag{4}$$

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and

$$2\Delta = \sum x_i a_i \geqslant \left(\sum x_i^{\alpha}\right)^{1/\alpha} \left(\sum a_i^{\alpha/(\alpha-1)}\right)^{(\alpha-1)/\alpha} \qquad \text{for } 0 \neq \alpha < 1. \tag{5}$$

Now (4) implies (a) for $\alpha > 1$ and from (5) we can infer (a) for $\alpha < 0$ and (b) for $0 < \alpha < 1$.

The First, Second and Third inequalities (p. 129) and the inequality for $\sum \frac{1}{a_i}$ (p. 130) of [2] provide both lower and upper bounds in terms of r and R for different sorts of symmetrical expressions of the sides and the area of the triangle. For convenience we collect these fundamental results at this point

$$2\sqrt{r(16R - 5r)} \leqslant \sum a_i \leqslant 2\sqrt{3r^2 + 4rR + 4R^2} \tag{6}$$

$$12r(2R - r) \leqslant \sum a_i^2 \leqslant 4r^2 + 8R^2 \tag{7}$$

$$r^{3/2}\sqrt{16R - 5r} \leqslant \Delta \leqslant r\sqrt{3r^2 + 4rR + 4R^2} \tag{8}$$

$$\frac{\sqrt{3}}{R} \leqslant \sum \frac{1}{a_i} \leqslant \frac{\sqrt{3}}{2r} \tag{9}$$

Equations (6), (7), (8) and (9) can be derived from the following self-contained distance formulas involving the incentre (I), centroid (G), orthocentre (H) and circumcentre (O):

$$GI^2 = \frac{2}{3}IO^2 + \frac{1}{3}IH^2 - \frac{2}{9}OH^2$$

$$IO^2 = R^2 - 2rR$$

$$IH^2 = 3r^2 + 4rR + 4R^2 - \frac{1}{4}\left(\sum a_i\right)^2$$

$$OH^2 = 9R^2 - \sum a_i^2.$$

These formulas and the identity

$$\left(\sum a_i\right)^2 = 2\sum a_i^2 + 4r(4R + r)$$

imply (6) where the right-hand side is better known as Gerretsen's inequality in the literature. (7) follows from (6) and the above identity and finally (8) is also a consequence of (6) exploiting $\Delta = rs$. It is a remarkable fact that (9) is a refinement of Euler's celebrated inequality $R \geqslant 2r$. For a detailed proof

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we refer the reader to [3]. Now we can combine these results with ours to produce new inequalities for $\alpha = \frac{1}{2}$ and $\alpha = 2$. By virtue of (3), (8) and (9) for $\alpha = \frac{1}{2}$ we have the following upper estimation

$$\sum \sqrt{x_i} \leqslant \frac{\sqrt{2\Delta}}{\left(\sum a_i^{-1}\right)^{-1/2}} = \sqrt{2\Delta \sum \frac{1}{a_i}} \leqslant \sqrt{\sqrt{3}\frac{\Delta}{r}} \leqslant \sqrt[4]{3\left(3r^2 + 4rR + 4R^2\right)}.$$

We shall encourage the reader to check that

$$\sqrt{\frac{2}{R}}\left(r + R\right) \leqslant \sqrt[4]{3\left(3r^2 + 4rR + 4R^2\right)}$$

provided that $R \geqslant 2r$, which is always hold by Euler's inequality. This shows that the upper bound above is weaker than the original inequality in [1].

For $\alpha = 2$ we have an interesting fractional lower bound by applying (2), (7) and (8)

$$\sum x_i^2 \geqslant \frac{(2\Delta)}{\sum a_i^2} \geqslant \frac{4r^3(16R - 5r)}{4r^2 + 8R^2} = \frac{r^3(16R - 5r)}{r^2 + 2R^2}.$$

Another approach to the task of finding bounds for $\sum x_i^\alpha$ is the application of Jensen's inequality. The advantage of this idea is its simplicity and it produces bounds for all exponents. If $\alpha > 1$ or $\alpha < 0$ then $f(t) = t^\alpha$ is convex and if $0 \leqslant \alpha \leqslant 1$ then concave. By virtue of Jensen's inequality we have the following

$$\sum x_i^\alpha \geqslant 3^{1-\alpha}\left(\sum x_i\right)^\alpha, \qquad \text{if } \alpha > 1 \text{ or } \alpha < 0, \tag{10}$$

$$\sum x_i^\alpha \leqslant 3^{1-\alpha}\left(\sum x_i\right)^\alpha, \qquad \text{if } 0 \leqslant \alpha \leqslant 1. \tag{11}$$

If $\alpha = 0,\ 1$ then equality holds for any inner point P, otherwise if, and only if, P is the incentre of the triangle. In order to convert these results in terms of r, R we introduce $m = \min\left\{\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}\right\}$ and $M = \max\left\{\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}\right\}$. For simplicity we assume $a_1 \leqslant a_2 \leqslant a_3$ and thus $m = \frac{1}{a_3}$ and $M = \frac{1}{a_1}$. Now by using $\sum x_i = \sum \frac{1}{a_i} a_i x_i \leqslant M\, 2\Delta$ and similarly $\sum x_i \geqslant m\, 2\Delta$ we arrive at

$$\frac{2}{a_3}\Delta \leqslant \sum x_i \leqslant \frac{2}{a_1}\Delta. \tag{12}$$

Applying (12) and (8) to (10) and (11) immediately implies

$$\sum x_i^\alpha \geqslant 3^{1-\alpha}\left(\frac{2}{a_3}\right)^\alpha r^\alpha \left(16rR - 5r^2\right)^{\alpha/2} \text{ for } \alpha > 1 \text{ or } \alpha < 0,$$

and

$$\sum x_i^\alpha \leqslant 3^{1-\alpha}\left(\frac{2}{a_1}\right)^\alpha r^\alpha \left(3r^2 + 4rR + 4R^2\right)^{\alpha/2} \text{ for } 0 \leqslant \alpha \leqslant 1.$$

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Note that if $\alpha = 1$ we have bounds for $\sum x_i$ as follows

$$\frac{2}{a_3} r\sqrt{16rR - 5r^2} \leqslant \sum x_i \leqslant \frac{2}{a_1} r\sqrt{3r^2 + 4rR + R^2}.$$

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ZOLTAN RETKES
9 North Avenue, Coventry CV2 4DH
e-mail: *tigris35711@gmail.com*

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