Canad. Math. Bull. Vol. **69** (2), 2026, pp. 388–404
http://dx.doi.org/10.4153/S0008439525101239
© The Author(s), 2025. Published by Cambridge University Press on behalf of Canadian Mathematical Society

Shadowing, sensitivity, and entropy points

Noriaki Kawaguchi

Abstract. For continuous self-maps of compact metric spaces, we explore the relationship among the shadowable points, sensitive points, and entropy points. Specifically, we show that (1) if the set of shadowable points is dense in the phase space, then any interior point of the set of sensitive points is an entropy point; and (2) if the topological entropy is zero, then the denseness of the set of shadowable points is equivalent to almost chain continuity. In addition, we present a counter-example to a question raised by Ye and Zhang regarding entropy points.

1 Introduction

Shadowing is an important concept in the theory of dynamical systems. It was initially introduced in the context of hyperbolic differentiable dynamics [3, 5] and generally refers to a situation in which coarse orbits, or *pseudo-orbits*, are approximated by true orbits. For background on the shadowing theory, we refer the reader to the monograph [4]. In [15], by localizing global shadowing into pointwise shadowings, Morales introduced the notion of *shadowable points*. In [11], a shadowable point that is also an entropy point of a certain type is characterized by the structure of pseudo-orbits. In this article, we discuss the relationship among the shadowable points, sensitive points, and entropy points. For instance, one of the main results is that if the set of shadowable points is dense in the phase space, then any point located in the interior of the set of sensitive points is an entropy point (Theorem 1.2). Another result is that if the topological entropy is zero, then the denseness of the set of shadowable points is equivalent to almost chain continuity (Theorem 1.3). In contrast to [11], a particular focus of this article is on the terminal chain components. We also present a counter-example to a question raised by Ye and Zhang [17] regarding entropy points.

We begin by defining shadowable points. Throughout, X denotes a compact metric space endowed with a metric d.

Definition 1.1 Let $f\colon X \to X$ be a continuous map and let $\xi = (x_i)_{i\geq 0}$ be a sequence of points in X. For $\delta > 0$, ξ is called a *δ-pseudo orbit* of f if $d(f(x_i), x_{i+1}) \leq \delta$ for all $i \geq 0$. For $\varepsilon > 0$, ξ is said to be *ε-shadowed* by $x \in X$ if $d(f^i(x), x_i) \leq \varepsilon$ for all $i \geq 0$. We say that $x \in X$ is a *shadowable point* for f if for any $\varepsilon > 0$, there is $\delta > 0$ such that every δ-pseudo orbit $(x_i)_{i\geq 0}$ of f with $x_0 = x$ is ε-shadowed by some $y \in X$. We denote by $Sh(f)$, the set of shadowable points for f.

Received by the editors May 17, 2025; revised September 11, 2025; accepted September 13, 2025.
Published online on Cambridge Core September 18, 2025.
AMS subject classification: **37B40**, **37B65**, **37D45**.
Keywords: Shadowing, sensitivity, entropy, chain continuity, chain component.

https://doi.org/10.4153/S0008439525101239 Published online by Cambridge University Press

For a topological space Z, a subset S of Z is called a G_δ-subset of Z if S is a countable intersection of open subsets of Z.

Remark 1.1

(1) Let $f: X \to X$ be a continuous map. For any $j, l \geq 1$, let $S_{j,l}$ denote the set of $x \in X$ such that there is an open neighborhood U of x for which every $\frac{1}{j}$-pseudo orbit $(x_i)_{i\geq 0}$ of f with $x_0 \in U$ is $\frac{1}{l}$-shadowed by some $y \in X$. We see that $S_{j,l}$ is an open subset of X for all $j, l \geq 1$ and

$$Sh(f) = \bigcap_{l\geq 1} \bigcup_{j\geq 1} S_{j,l};$$

therefore, $Sh(f)$ is a G_δ-subset of X.

(2) A continuous map $f: X \to X$ is said to have the *shadowing property* if for any $\varepsilon > 0$, there is $\delta > 0$ such that every δ-pseudo orbit of f is ε-shadowed by some point of X, which is equivalent to $X = Sh(f)$ (see [8, Lemma 2.4]).

Sensitivity is a characteristic feature of chaotic dynamical systems. It is an element of some formal definitions of chaos and intuitively means that an arbitrarily small difference in initial conditions can be amplified to be a significant difference in later states (see, e.g., [7] for an in-depth look at the concept of sensitivity). The formal definition of sensitive points is as follows.

Definition 1.2 Given a continuous map $f: X \to X$ and $r > 0$, $x \in X$ is called an *r-sensitive point* for f if for any $\varepsilon > 0$, there are $y, z \in X$ and $i \geq 0$ such that

$$\max\{d(x, y), d(x, z)\} \leq \varepsilon$$

and $d(f^i(y), f^i(z)) > r$. We denote by $\mathrm{Sen}_r(f)$ the set of r-sensitive points for f. We also define the set $\mathrm{Sen}(f)$ of *sensitive points* for f by

$$\mathrm{Sen}(f) = \bigcup_{r>0} \mathrm{Sen}_r(f).$$

For a continuous map $f: X \to X$, a subset S of X is said to be *f-invariant* if $f(S) \subset S$.

Remark 1.2

(1) For any continuous map $f: X \to X$ and $r > 0$, $\mathrm{Sen}_r(f)$ is a closed f-invariant subset of X.

(2) A continuous map $f: X \to X$ is said to be *sensitive* if $X = \mathrm{Sen}_r(f)$ for some $r > 0$.

Next, we recall the definition of entropy points. Note that the positive topological entropy is another characteristic feature of chaotic dynamical systems. The notion of entropy points is obtained by a concentration of positive topological entropy at a point [17].

Let $f: X \to X$ be a continuous map. For $n \geq 1$ and $r > 0$, a subset E of X is said to be (n, r)-*separated* if

$$\max_{0\leq i\leq n-1} d(f^i(x), f^i(y)) > r$$

https://doi.org/10.4153/S0008439525101239 Published online by Cambridge University Press

for all $x, y \in E$ with $x \neq y$. Let K be a subset of X. For $n \geq 1$ and $r > 0$, let $s_n(f, K, r)$ denote the largest cardinality of an (n, r)-separated subset of K. We define $h(f, K, r)$ and $h(f, K)$ by

$$h(f, K, r) = \limsup_{n\to\infty} \frac{1}{n} \log s_n(f, K, r)$$

and

$$h(f, K) = \lim_{r\to 0} h(f, K, r).$$

The topological entropy $h_{\text{top}}(f)$ of f is defined by $h_{\text{top}}(f) = h(f, X)$.

Definition 1.3 Let $f: X \to X$ be a continuous map. For $x \in X$, let $\mathcal{K}(x)$ denote the set of closed neighborhoods of x.

(1) $\text{Ent}(f)$ is the set of $x \in X$ such that $h(f, K) > 0$ for all $K \in \mathcal{K}(x)$.
(2) For $r > 0$, $\text{Ent}_r(f)$ is the set of $x \in X$ such that $h(f, K, r) > 0$ for all $K \in \mathcal{K}(x)$.
(3) For $r, b > 0$, $\text{Ent}_{r,b}(f)$ is the set of $x \in X$ such that $h(f, K, r) \geq b$ for all $K \in \mathcal{K}(x)$.

Remark 1.3 The following properties hold:

- $\text{Ent}(f)$, $\text{Ent}_r(f)$, $r > 0$, and $\text{Ent}_{r,b}(f)$, $r, b > 0$, are closed f-invariant subsets of X,
- $$\text{Ent}(f) \supset \text{Ent}_r(f) \supset \text{Ent}_{r,b}(f)$$
  for all $r, b > 0$;
- for any closed subset K of X and $r > 0$, if $h(f, K, r) > 0$, then $K \cap \text{Ent}_r(f) \neq \emptyset$;
- for any closed subset K of X and $r, b > 0$, if $h(f, K, r) \geq b$, then $K \cap \text{Ent}_{r,b}(f) \neq \emptyset$.

Remark 1.4 Note that $\text{Ent}_r(f) \subset \text{Sen}_r(f)$ for all $r > 0$.

Chain components, which appear in the (so-called) fundamental theorem of dynamical systems by Conley, are essential objects for a global understanding of dynamical systems [6]. Let us recall the definition of chain components.

Definition 1.4 Given a continuous map $f: X \to X$ and $\delta > 0$, a finite sequence $(x_i)_{i=0}^k$ of points in X, where $k > 0$ is a positive integer, is called a *δ-chain* of f if $d(f(x_i), x_{i+1}) \leq \delta$ for every $0 \leq i \leq k-1$. For any $x, y \in X$, the notation $x \to y$ means that for every $\delta > 0$, there is a δ-chain $(x_i)_{i=0}^k$ of f with $x_0 = x$ and $x_k = y$. The *chain recurrent set* $CR(f)$ for f is defined by

$$CR(f) = \{x \in X : x \to x\}.$$

We define a relation $\leftrightarrow$ in

$$CR(f)^2 = CR(f) \times CR(f)$$

by: for any $x, y \in CR(f)$, $x \leftrightarrow y$ if and only if $x \to y$ and $y \to x$. Note that $\leftrightarrow$ is a closed equivalence relation in $CR(f)^2$ and satisfies $x \leftrightarrow f(x)$ for all $x \in CR(f)$. An

https://doi.org/10.4153/S0008439525101239 Published online by Cambridge University Press

equivalence class C of $\leftrightarrow$ is called a *chain component* for f. We denote by $\mathcal{C}(f)$ the set of chain components for f.

Remark 1.5 The following properties hold:

- $CR(f) = \bigsqcup_{C\in\mathcal{C}(f)} C$, a disjoint union;
- every $C \in \mathcal{C}(f)$ is a closed f-invariant subset of $CR(f)$;
- for all $C \in \mathcal{C}(f)$, $f|_C : C \to C$ is chain transitive, i.e., for any $x, y \in C$ and $\delta > 0$, there is a δ-chain $(x_i)_{i=0}^k$ of $f|_C$ with $x_0 = x$ and $x_k = y$.

Following [2], we define the *terminal* chain components as follows.

Definition 1.5 Given a continuous map $f : X \to X$, we say that a closed f-invariant subset S of X is *chain stable* if for any $\varepsilon > 0$, there is $\delta > 0$ such that every δ-chain $(x_i)_{i=0}^k$ of f with $x_0 \in S$ satisfies $d(x_k, S) = \inf_{y\in S} d(x_k, y) \le \varepsilon$. We say that $C \in \mathcal{C}(f)$ is *terminal* if C is chain stable. We denote by $\mathcal{C}_{\mathrm{ter}}(f)$ the set of terminal chain components for f.

The following lemma is from [10].

Lemma 1.1 [10, Lemma 2.1] *Let $f : X \to X$ be a continuous map. For any $x \in X$, there are $C \in \mathcal{C}_{\mathrm{ter}}(f)$ and $y \in C$ such that $x \to y$.*

Given a continuous map $f : X \to X$ and $x \in X$, the *ω-limit set* $\omega(x, f)$ of x for f is defined as the set of $y \in X$ such that

$$\lim_{j\to\infty} f^{i_j}(x) = y$$

for some sequence $0 \le i_1 < i_2 < \ldots$. Note that $\omega(x, f)$ is a closed f-invariant subset of X and satisfies

$$\lim_{i\to\infty} d(f^i(x), \omega(x, f)) = 0.$$

Since, we have $y \to z$ for all $y, z \in \omega(x, f)$, there is an unique $C(x, f) \in \mathcal{C}(f)$ such that $\omega(x, f) \subset C(x, f)$ and so

$$\lim_{i\to\infty} d(f^i(x), C(x, f)) = 0.$$

The first result of this article is the following theorem. For a subset S of a topological space Z, $\overline{S}$ and $\mathrm{int}[S]$ denote the closure and the interior of S, respectively.

Theorem 1.1 *Let $f : X \to X$ be a continuous map. For any $x \in Sh(f)$ and $r > 0$,*

(1) *if $C(x, f) \in \mathcal{C}_{\mathrm{ter}}(f)$ and $x \in \mathrm{Sen}_r(f)$, then $x \in \mathrm{Ent}_s(f)$ for all $0 < s < r$;*
(2) *if $x \in \mathrm{int}[\mathrm{Sen}_r(f)]$, then $x \in \mathrm{Ent}_s(f)$ for all $0 < s < r$.*

The next lemma is a consequence of the Baire category theorem. The proof is left as an exercise for the reader.

https://doi.org/10.4153/S0008439525101239 Published online by Cambridge University Press

Lemma 1.2 *Let Z be a complete metric space. Every sequence A_j, $j \geq 1$, of closed subsets of Z satisfies* $\overline{\operatorname{int}[\bigcup_{j\geq 1} A_j]} = \overline{\bigcup_{j\geq 1} \operatorname{int}[A_j]}$.

By Theorem 1.1 and Lemma 1.2, we obtain the following theorem.

Theorem 1.2 *Given a continuous map $f: X \to X$, if $X = \overline{Sh(f)}$, then*

$$\overline{\operatorname{int}[\operatorname{Sen}(f)]} \subset \operatorname{Ent}(f).$$

Proof Letting $(r_j)_{j\geq 1}$ be a sequence of numbers with $0 < r_1 > r_2 > \dots$ and $\lim_{j\to\infty} r_j = 0$, we have

$$\operatorname{Sen}(f) = \bigcup_{j\geq 1} \operatorname{Sen}_{r_j}(f).$$

We take a sequence $(s_j)_{j\geq 1}$ of numbers such that $0 < s_j < r_j$ for all $j \geq 1$. By Theorem 1.1, we have

$$\operatorname{int}[\operatorname{Sen}_{r_j}(f)] \cap Sh(f) \subset \operatorname{Ent}_{s_j}(f)$$

for all $j \geq 1$. Note that an open subset U of X and a subset B of X satisfy

$$U \cap \overline{B} \subset \overline{U \cap B}.$$

By $X = \overline{Sh(f)}$, we obtain

$$\operatorname{int}[\operatorname{Sen}_{r_j}(f)] = \operatorname{int}[\operatorname{Sen}_{r_j}(f)] \cap \overline{Sh(f)} \subset \overline{\operatorname{int}[\operatorname{Sen}_{r_j}(f)] \cap Sh(f)} \subset \overline{\operatorname{Ent}_{s_j}(f)} = \operatorname{Ent}_{s_j}(f)$$

for all $j \geq 1$. With the aid of Lemma 1.2, we obtain

$$\overline{\operatorname{int}[\operatorname{Sen}(f)]} = \overline{\operatorname{int}[\bigcup_{j\geq 1} \operatorname{Sen}_{r_j}(f)]} = \overline{\bigcup_{j\geq 1} \operatorname{int}[\operatorname{Sen}_{r_j}(f)]} \subset \overline{\bigcup_{j\geq 1} \operatorname{Ent}_{s_j}(f)} \subset \operatorname{Ent}(f);$$

thus, the theorem has been proved. ∎

Given a continuous map $f: X \to X$, we say that $x \in X$ is

- an *equicontinuity point* for f if for any $\varepsilon > 0$, there is $\delta > 0$ such that every $y \in X$ with $d(x, y) \leq \delta$ satisfies

$$\sup_{i\geq 0} d(f^i(x), f^i(y)) \leq \varepsilon,$$

- a *chain continuity point* for f [1] if for any $\varepsilon > 0$, there is $\delta > 0$ such that every δ-pseudo orbit $(x_i)_{i\geq 0}$ of f with $x_0 = x$ is ε-shadowed by x, i.e., satisfies

$$\sup_{i\geq 0} d(f^i(x), x_i) \leq \varepsilon.$$

We denote by $EC(f)$ (resp. $CC(f)$) the set of equicontinuity (resp. chain continuity) points for f. By

$$EC(f) = X \backslash \operatorname{Sen}(f) = \bigcap_{j\geq 1} [X \backslash \operatorname{Sen}_{j^{-1}}(f)],$$

https://doi.org/10.4153/S0008439525101239 Published online by Cambridge University Press

we see that $EC(f)$ is a G_δ-subset of X. It is easy to see that

$$CC(f) = Sh(f) \cap EC(f)$$

and so $CC(f)$ is a G_δ-subset of X. We say that f is *almost chain continuous* if $CC(f)$ is a dense (G_δ-)subset of X.

Theorem 1.3 *For a continuous map $f: X \to X$, if $h_{\text{top}}(f) = 0$, then the following conditions are equivalent:*

(A) $X = \overline{Sh(f)}$;
(B) *f is almost chain continuous.*

Proof The implication (A) $\Longrightarrow$ (B): Since $Sh(f)$ is a G_δ-subset of X, if $X = \overline{Sh(f)}$, then $Sh(f)$ is a dense G_δ-subset of X. Due to Theorem 1.2, if $X = \overline{Sh(f)}$ and

$$\text{int}[\text{Sen}(f)] \neq \varnothing,$$

then $\text{Ent}(f) \neq \varnothing$ and so $h_{\text{top}}(f) > 0$. By $X = \overline{Sh(f)}$ and $h_{\text{top}}(f) = 0$, we obtain

$$\text{int}[\text{Sen}(f)] = \varnothing;$$

therefore, $EC(f) = X \backslash \text{Sen}(f)$ is a dense G_δ-subset of X. By the Baire category theorem, we conclude that

$$CC(f) = Sh(f) \cap EC(f)$$

is a dense G_δ-subset of X.

The implication (B) $\Longrightarrow$ (A) is a direct consequence of the inclusion $CC(f) \subset Sh(f)$. ∎

Remark 1.6 By Theorem 1.3, if a continuous map $f: X \to X$ satisfies $h_{\text{top}}(f) = 0$ and $X = Sh(f)$, i.e., the shadowing property, then f is almost chain continuous. An alternative proof of this can be given as follows.

Proof By Theorem A.1, for a continuous map $f: X \to X$ and $x \in X$, $x \in CC(f)$ holds exactly if $C(x, f) \in \mathcal{C}_{\text{ter}}(f)$ and $CC(f|_{C(x,f)}) = C(x, f)$. Every continuous map $f: X \to X$ satisfies $CR(f|_{CR(f)}) = CR(f)$ and

$$\mathcal{C}(f) = \mathcal{C}(f|_{CR(f)}) = \mathcal{C}_{\text{ter}}(f|_{CR(f)}).$$

Assume that $h_{\text{top}}(f) = 0$ and $X = Sh(f)$. By Theorem B.1, $X = Sh(f)$ implies $CR(f) = Sh(f|_{CR(f)})$. If $\text{Sen}(f|_{CR(f)}) \neq \varnothing$, by taking $x \in \text{Sen}(f|_{CR(f)})$ and $C \in \mathcal{C}(f)$ with $x \in C$, we obtain $C(x, f|_{CR(f)}) = C$ and $x \in \text{Sen}_r(f|_{CR(f)})$ for some $r > 0$. By $C \in \mathcal{C}_{\text{ter}}(f|_{CR(f)})$ and Theorem 1.1, we obtain $x \in \text{Ent}_s(f|_{CR(f)})$ for all $0 < s < r$ and so $h_{\text{top}}(f) > 0$, a contradiction. It follows that $\text{Sen}(f|_{CR(f)}) = \varnothing$; thus, $EC(f|_{CR(f)}) = CR(f)$. We obtain

$$CC(f|_{CR(f)}) = Sh(f|_{CR(f)}) \cap EC(f|_{CR(f)}) = CR(f)$$

https://doi.org/10.4153/S0008439525101239 Published online by Cambridge University Press

(cf. [14, Corollary 6]). This implies that $CC(f|_C) = C$ for all $C \in \mathcal{C}(f)$. As a consequence, if $h_{\text{top}}(f) = 0$ and $X = Sh(f)$, then

$$CC(f) = \{x \in X : C(x, f) \in \mathcal{C}_{\text{ter}}(f)\}.$$

On the other hand, by Theorem 1.1 of [12], $X = Sh(f)$ implies that

$$\{x \in X : C(x, f) \in \mathcal{C}_{\text{ter}}(f)\}$$

is a dense G_δ-subset X. By these conditions, we conclude that $CC(f)$ is a dense G_δ-subset X, i.e., f is almost chain continuous. ■

Let $f : X \to X$ be a continuous map. For $\delta, r > 0$ and $n \geq 1$, we say that two δ-chains $(x_i)_{i=0}^n$, $(y_i)_{i=0}^n$ of f is (n, r)-*separated* if $d(x_i, y_i) > r$ for some $0 \leq i \leq n$. Let

$$s_n(f, X, r, \delta)$$

denote the largest cardinality of a set of pairwise (n, r)-separated δ-chains of f. The next lemma is from [13].

Lemma 1.3 (Misiurewicz)

$$h_{\text{top}}(f) = \lim_{r \to 0} \lim_{\delta \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s_n(f, X, r, \delta).$$

Let $f : X \to X$ be a continuous map. Following [17], we define the set $\text{Ent}_{up}(f)$ of *uniform entropy points* for f by

$$\text{Ent}_{up}(f) = \bigcup_{r,b>0} \text{Ent}_{r,b}(f).$$

Note that

$$\text{Ent}(f) = \overline{\text{Ent}_{up}(f)}.$$

The following lemma can be proved by Lemma 1.3 and a similar argument as in the proof of Theorem 1.2 in [11]. Note that the "only if" part is rather trivial.

Lemma 1.4 *For a continuous map $f : X \to X$ and $C \in \mathcal{C}_{\text{ter}}(f)$, $h_{\text{top}}(f|_C) > 0$ holds if and only if $C \cap \text{Ent}_{up}(f) \neq \emptyset$.*

The following lemma is proved in [11] (see [11, Lemmas 1.2 and 3.1]).

Lemma 1.5 *Let $f : X \to X$ be a continuous map. Let $x, y \in X$ and $r, b > 0$. If $x \to y$, then:*

(1) *$x \in Sh(f)$ implies $y \in Sh(f)$;*
(2) *$x \in Sh(f)$ and $y \in \text{Ent}_r(f)$ imply $x \in \text{Ent}_s(f)$ for all $0 < s < r$;*
(3) *$x \in Sh(f)$ and $y \in \text{Ent}_{r,b}(f)$ imply $x \in \text{Ent}_{s,b}(f)$ for all $0 < s < r$.*

By using Lemmas 1.1, 1.4, and 1.5, we obtain the following theorem.

https://doi.org/10.4153/S0008439525101239 Published online by Cambridge University Press

Theorem 1.4 *For a continuous map $f: X \to X$, if $X = Sh(f)$, then the following conditions are equivalent:*

(A) $X = \mathrm{Ent}_{up}(f)$;
(B) $h_{\mathrm{top}}(f|_C) > 0$ *for all* $C \in \mathcal{C}_{\mathrm{ter}}(f)$.

Proof The implication (A) $\Longrightarrow$ (B) is a direct consequence of Lemma 1.4.

The implication (B) $\Longrightarrow$ (A): Due to Lemma 1.1, for any $x \in X$, there are $C \in \mathcal{C}_{\mathrm{ter}}(f)$ and $y \in C$ such that $x \to y$. By $h_{\mathrm{top}}(f|_C) > 0$, we obtain $z \in \mathrm{Ent}_{r,b}(f)$ for some $z \in C$ and $r, b > 0$. Since $y, z \in C$, we have $y \to z$. This with $x \to y$ implies $x \to z$. By $x \in Sh(f)$ and Lemma 1.5, we obtain $x \in \mathrm{Ent}_{s,b}(f)$ for all $0 < s < r$; thus, $x \in \mathrm{Ent}_{up}(f)$. Since $x \in X$ is arbitrary, we conclude that $X = \mathrm{Ent}_{up}(f)$. ∎

We recall the definition of C-entropy points from [17]. For a continuous map $f: X \to X$ and an open cover $\mathcal{U}$ of X, let $h(f, \mathcal{U})$ denote the entropy of f relative to $\mathcal{U}$ (see [16] for details). We say that $(x, y) \in X^2$ with $x \neq y$ is an *entropy pair* for f if for any closed neighborhoods A of x and B of y,

$$h(f, \{X \backslash A, X \backslash B\}) > 0$$

whenever $A \cap B = \varnothing$. We denote by $E_2(f)$ the set of entropy pairs for f. We say that $x \in X$ is a *C-entropy point* for f if $(x, y) \in E_2(f)$ for some $y \in X$ with $x \neq y$. Let $E_1(f)$ denote the set of C-entropy points for f. By Theorem 3.4 of [17], we know that $E_1(f) \subset \mathrm{Ent}(f)$. Ye and Zhang [17] raised the following question.

Question [17, Question 6.11] *Does $E_1(f) \subset \mathrm{Ent}_{up}(f)$ hold for any homeomorphism $f: X \to X$?*

In Section 3, we present an example of a homeomorphism $f: X \to X$ with the following properties:

(1) $X = Sh(f)$;
(2) there exists $C \in \mathcal{C}_{\mathrm{ter}}(f)$ such that C is an infinite set and satisfies $C \subset E_1(f) \backslash \mathrm{Ent}_{up}(f)$.

This article consists of three sections and three appendices. In the next section, we prove Theorem 1.1. In Section 3, we present the example just mentioned above. In Appendix A, we prove Theorem A.1. In Appendices B and C, Theorems B.1 and C.1 are proved.

2 Proof of Theorem 1.1

In this section, we prove Theorem 1.1. For the proof, we need a few lemmas.

Let $f: X \to X$ be a continuous map. For a closed f-invariant subset S of X and $r > 0$, we denote by $\mathrm{Sen}_r^*(f|_S)$ the set of $x \in S$ such that for any $\delta > 0$, there are δ-chains $(x_i)_{i=0}^k$, $(y_i)_{i=0}^k$ of $f|_S$ with $x_0 = y_0 = x$ and $d(x_k, y_k) > r$. For $x \in X$ and $\varepsilon > 0$, let $B_\varepsilon(x)$ denote the closed ε-ball centered at x: $B_\varepsilon(x) = \{y \in X : d(x, y) \leq \varepsilon\}$.

https://doi.org/10.4153/S0008439525101239 Published online by Cambridge University Press

Lemma 2.1 *Let $f: X \to X$ be a continuous map and let $C \in \mathcal{C}(f)$. For any $x \in C$ and $r > 0$, if*

$$x \in Sh(f) \cap \mathrm{Sen}_r^*(f|_C),$$

then $x \in \mathrm{Ent}_s(f)$ for all $0 < s < r$.

Proof We fix $s, \varepsilon > 0$ such that $s + 2\varepsilon < r$. By $x \in Sh(f)$, we obtain $\delta > 0$ such that every δ-pseudo orbit $(x_i)_{i\geq 0}$ of f with $x_0 = x$ is ε-shadowed by some $y \in X$. Since $x \in \mathrm{Sen}_r^*(f|_C)$, there is a pair

$$(\alpha_0, \alpha_1) = ((x_i^{(0)})_{i=0}^k, (x_i^{(1)})_{i=0}^k)$$

of δ-chains of $f|_C$ such that $x_0^{(0)} = x_0^{(1)} = x$ and $d(x_k^{(0)}, x_k^{(1)}) > r$. Since $x, x_k^{(0)}, x_k^{(1)} \in C$, we have $x_k^{(0)} \to x$ and $x_k^{(1)} \to x$; therefore, there is a pair

$$(\beta_0, \beta_1) = ((y_i^{(0)})_{i=0}^l, (y_i^{(1)})_{i=0}^m)$$

of δ-chains of f such that $y_0^{(0)} = x_k^{(0)}$, $y_0^{(1)} = x_k^{(1)}$, and $y_l^{(0)} = y_m^{(1)} = x$. Let $n = 2k + l + m$ and let

$$(\gamma_0, \gamma_1) = (\alpha_0\beta_0\alpha_1\beta_1, \alpha_1\beta_1\alpha_0\beta_0) = ((z_i^{(0)})_{i=0}^n, (z_i^{(1)})_{i=0}^n),$$

a pair of δ-chains of f. Note that $z_0^{(0)} = z_0^{(1)} = z_n^{(0)} = z_n^{(1)} = x$ and

$$d(z_k^{(0)}, z_k^{(1)}) = d(x_k^{(0)}, x_k^{(1)}) > r.$$

For any $N \geq 1$ and $u = (u_j)_{j=1}^N \in \{0,1\}^N$, let

$$\gamma_u = (w_i^{(u)})_{i=0}^{nN} = \gamma_{u_1}\gamma_{u_2}\cdots\gamma_{u_N},$$

a δ-chain of f. Since $w_0^{(u)} = x$, by the choice of δ, we have $x_u \in X$ such that

$$d(f^i(x_u), w_i^{(u)}) \leq \varepsilon$$

for all $0 \leq i \leq nN$. Note that

$$d(x, x_u) = d(w_0^{(u)}, x_u) \leq \varepsilon$$

and so $x_u \in B_\varepsilon(x)$ for all $u \in \{0,1\}^N$. For any $u, v \in \{0,1\}^N$, $u \neq v$ implies $u_j \neq v_j$ and so

$$\begin{aligned} d(f^{k+(j-1)n}(x_u), f^{k+(j-1)n}(x_v)) &\geq d(w_{k+(j-1)n}^{(u)}, w_{k+(j-1)n}^{(v)}) - 2\varepsilon \\ &= d(z_k^{(0)}, z_k^{(1)}) - 2\varepsilon > r - 2\varepsilon > s \end{aligned}$$

for some $1 \leq j \leq N$. It follows that $\{x_u : u \in \{0,1\}^N\}$ is an (nN, s)-separated subset of $B_\varepsilon(x)$ with

$$|\{x_u : u \in \{0,1\}^N\}| = 2^N.$$

https://doi.org/10.4153/S0008439525101239 Published online by Cambridge University Press

Since $N \geq 1$ is arbitrary, we obtain

$$h(f, B_\varepsilon(x), s) \geq \limsup_{N\to\infty} \frac{1}{nN} \log s_{nN}(f, B_\varepsilon(x), s)$$
$$\geq \limsup_{N\to\infty} \frac{1}{nN} \log 2^N = \frac{1}{n} \log 2 > 0.$$

Since $\varepsilon > 0$ with $s + 2\varepsilon < r$ is arbitrary, we conclude that $x \in \mathrm{Ent}_s(f)$, completing the proof of the lemma. ∎

Lemma 2.2 *Let $f: X \to X$ be a continuous map and let $C \in \mathcal{C}_{\mathrm{ter}}(f)$. For any $x \in C$ and $r > 0$, $x \in \mathrm{Sen}_r(f)$ implies $x \in \mathrm{Sen}_s^*(f|_C)$ for all $0 < s < r$.*

Proof Since $x \in C$ and $C \in \mathcal{C}_{\mathrm{ter}}(f)$, for any $\varepsilon > 0$, there is $\gamma > 0$ such that every γ-chain $(x_i)_{i=0}^k$ of f with $x_0 = x$ satisfies

$$d(x_i, C) \leq \varepsilon$$

for all $0 \leq i \leq k$. By $x \in \mathrm{Sen}_r(f)$, we obtain a pair

$$((x_i^{(0)})_{i=0}^k, (x_i^{(1)})_{i=0}^k)$$

of γ-chains of f with $x_0^{(0)} = x_0^{(1)} = x$ and $d(x_k^{(0)}, x_k^{(1)}) > r$. For each $j \in \{0, 1\}$, we take a sequence $y_i^{(j)} \in C$, $0 \leq i \leq k$, with $y_0^{(j)} = x_0^{(j)} = x$ and

$$d(x_i^{(j)}, y_i^{(j)}) = d(x_i^{(0)}, C) \leq \varepsilon$$

for all $0 < i \leq k$. For every $\delta > 0$, if $\varepsilon, \gamma > 0$ are sufficiently small, then

$$((y_i^{(0)})_{i=0}^k, (y_i^{(1)})_{i=0}^k)$$

is a pair of δ-chains of $f|_C$ such that $y_0^{(0)} = y_0^{(1)} = x$ and $d(y_k^{(0)}, y_k^{(1)}) > s$. This implies $x \in \mathrm{Sen}_s^*(f|_C)$; therefore, the lemma has been proved. ∎

By using these lemmas, we prove Theorem 1.1.

Proof of Theorem 1.1 (1): We take $\varepsilon > 0$ such that $s + 2\varepsilon < r$. Let $y \in \omega(x, f)$. Since $\mathrm{Sen}_r(f)$ is a closed f-invariant subset of X, $x \in \mathrm{Sen}_r(f)$ implies $\omega(x, f) \subset \mathrm{Sen}_r(f)$. Note that

$$\omega(x, f) \subset C(x, f).$$

By $C(x, f) \in \mathcal{C}_{\mathrm{ter}}(f)$ and Lemma 2.2, we obtain

$$y \in \mathrm{Sen}_r(f) \cap C(x, f) \subset \mathrm{Sen}_{s+2\varepsilon}^*(f|_{C(x,f)}).$$

Note that $y \in \omega(x, f)$ implies $x \to y$. By $x \in Sh(f)$ and Lemma 1.5, we obtain $y \in Sh(f)$ and so

$$y \in Sh(f) \cap \mathrm{Sen}_{s+2\varepsilon}^*(f|_{C(x,f)}).$$

This with Lemma 2.1 implies $y \in \mathrm{Ent}_{s+\varepsilon}(f)$. By $x \to y$, $x \in Sh(f)$, and Lemma 1.5, we conclude that $x \in \mathrm{Ent}_s(f)$.

https://doi.org/10.4153/S0008439525101239 Published online by Cambridge University Press

(2): Due to Lemma 1.1, there are $C \in \mathcal{C}_{\text{ter}}(f)$ and $y \in C$ such that $x \to y$. Let $(\varepsilon_j)_{j\geq 1}$ be a sequence of numbers with $0 < \varepsilon_1 > \varepsilon_2 > \dots$ and $\lim_{j\to\infty} \varepsilon_j = 0$. By $x \in Sh(f)$, for each $j \geq 1$, we have $\delta_j > 0$ such that every δ_j-pseudo orbit $(x_i)_{i\geq 0}$ of f with $x_0 = x$ is ε_j-shadowed by some $z \in X$. Since $x \to y$, for each $j \geq 1$, there is a δ_j-chain $(x_i^{(j)})_{i=0}^{k_j}$ of f with $x_0^{(j)} = x$ and $x_{k_j}^{(j)} = y$. We obtain a sequence $x_j \in X$, $j \geq 1$, such that

$$d(f^i(x_j), x_i^{(j)}) \leq \varepsilon_j$$

for all $j \geq 1$ and $0 \leq i \leq k_j$. Note that

$$d(x_j, x) = d(x_j, x_0^{(j)}) \leq \varepsilon_j$$

and

$$d(f^{k_j}(x_j), y) = d(f^{k_j}(x_j), x_{k_j}^{(j)}) \leq \varepsilon_j$$

for all $j \geq 1$. Since $x \in \text{int}[\text{Sen}_r(f)]$, we have $x_j \in \text{Sen}_r(f)$ for all j large enough. Since $\text{Sen}_r(f)$ is a closed f-invariant subset of X, it follows that $f^{k_j}(x_j) \in \text{Sen}_r(f)$ for all j large enough; thus, by $\lim_{j\to\infty} f^{k_j}(x_j) = y$, we obtain

$$y \in \text{Sen}_r(f) \cap C.$$

Note that $C \in \mathcal{C}_{\text{ter}}(f)$, $x \to y$, and $x \in Sh(f)$. The rest of the proof is similar to that of (1). ∎

3 Example

In this section, we present an example of a homeomorphism $f: X \to X$ with the following properties:

(1) $X = Sh(f)$;
(2) there exists $C \in \mathcal{C}_{\text{ter}}(f)$ such that C is an infinite set and satisfies $C \subset E_1(f) \backslash \text{Ent}_{up}(f)$.

The following example is taken from [10]. We modify Example 4.3 of [10] to obtain the homeomorphism.

Example 3.1 Let $\sigma: [-1,1]^{\mathbb{Z}} \to [-1,1]^{\mathbb{Z}}$ be the shift map, i.e.,

$$\sigma(x)_n = x_{n+1}$$

for all $x = (x_n)_{n\in\mathbb{Z}} \in [-1,1]^{\mathbb{Z}}$ and $n \in \mathbb{Z}$. Let d be the metric on $[-1,1]^{\mathbb{Z}}$ defined by

$$d(x,y) = \sup_{n\in\mathbb{Z}} 2^{-|n|}|x_n - y_n|$$

for all $x = (x_n)_{n\in\mathbb{Z}}$, $y = (y_n)_{n\in\mathbb{Z}} \in [-1,1]^{\mathbb{Z}}$. We fix a sequence $s = (s_k)_{k\geq 1}$ of numbers with $0 < s_1 < s_2 < \dots$ and $\lim_{k\to\infty} s_k = 1$. Put

$$S = \{-1, 1\} \cup \{-s_k : k \geq 1\} \cup \{s_k : k \geq 1\},$$

a closed subset of $[-1,1]$.

https://doi.org/10.4153/S0008439525101239 Published online by Cambridge University Press

We define a closed σ-invariant subset X of $S^{\mathbb{Z}}$ by for any $x = (x_n)_{n\in\mathbb{Z}} \in S^{\mathbb{Z}}$, $x \in X$ if and only if the following conditions are satisfied:

- $|x_n| \le |x_{n+1}|$ for all $n \in \mathbb{Z}$;
- for any $n \in \mathbb{Z}$ and $k \ge 1$, if $x_n = s_k$, then $x_{n+j} = -s_k$ for all $1 \le j \le k$;
- for every $n \in \mathbb{Z}$, if $x_n = 1$, then $x_{n+j} = -1$ for all $j \ge 1$.

We define $z = (z_n)_{n\in\mathbb{Z}}$ and $z^{(m)} = (z_n^{(m)})_{n\in\mathbb{Z}}$, $m \in \mathbb{Z}$, as follows:

- $z_n = -1$ for all $n \in \mathbb{Z}$;
- $z_m^{(m)} = 1$ for all $m \in \mathbb{Z}$;
- $z_n^{(m)} = -1$ for all $m, n \in \mathbb{Z}$ with $m \ne n$.

Let $f = \sigma|_X : X \to X$, $X_k = X \cap \{-s_k, s_k\}^{\mathbb{Z}}$, $k \ge 1$, and let

$$X_\infty = X \cap \{-1, 1\}^{\mathbb{Z}} = \{z\} \cup \{z^{(m)} : m \in \mathbb{Z}\}.$$

A similar argument as in Example 4.3 of [10] shows that

$$CR(f) = \{x = (x_n)_{n\in\mathbb{Z}} \in X : |x_n| = |x_{n+1}| \text{ for all } n \in \mathbb{Z}\} = X_\infty \cup \bigcup_{k\ge1} X_k$$

and $\mathcal{C}(f) = \{X_\infty\} \cup \{X_k : k \ge 1\}$. Note that $\mathcal{C}_{\text{ter}}(f) = \{X_\infty\}$.

As in Example 4.3 of [10], by taking a sequence $s = (s_k)_{k\ge1}$ that rapidly converges to 1, we can show that $X = Sh(f)$, i.e., f has the shadowing property. Since $X_\infty \in \mathcal{C}_{\text{ter}}(f)$ and $h_{\text{top}}(f|_{X_\infty}) = 0$, by Lemma 1.4, we obtain

$$X_\infty \cap \text{Ent}_{up}(f) = \varnothing.$$

Note that the sequence X_k, $k \ge 1$, satisfies

$$\lim_{k\to\infty} X_k = X_\infty$$

with respect to the Hausdorff distance. For every $k \ge 1$, since X_k is a mixing subshift of finite type, $f|_{X_k}$ is mixing and satisfies the shadowing property. This implies that for every $k \ge 1$, $f|_{X_k}$ has uniformly positive entropy (upe) and so satisfies

$$(x, y) \in E_2(f)$$

for all $x, y \in X_k$ with $x \ne y$ (see, e.g., [9] for details). Given any $u, v \in X_\infty$ with $u \ne v$, by taking two sequences $(x_k)_{k\ge1}$, $(y_k)_{k\ge1}$ such that

- $x_k, y_k \in X_k$ and $x_k \ne y_k$ for all $k \ge 1$;
- $\lim_{k\to\infty} x_k = u$ and $\lim_{k\to\infty} y_k = v$,

we obtain $(u, v) \in E_2(f)$. It follows that $X_\infty \subset E_1(f)$; thus, this example satisfies the desired properties.

Appendix A: A theorem on chain continuity points

In this appendix, we prove the following theorem.

Theorem A.1 *Let $f : X \to X$ be a continuous map. Let $x \in X$ and $C = C(x, f)$. The following conditions are equivalent:*

https://doi.org/10.4153/S0008439525101239 Published online by Cambridge University Press

(A) $x \in CC(f)$;
(B) $C \in \mathcal{C}_{\mathrm{ter}}(f)$ *and* $C \subset CC(f)$;
(C) $C \in \mathcal{C}_{\mathrm{ter}}(f)$ *and* $CC(f|_C) = C$;
(D) $C \in \mathcal{C}_{\mathrm{ter}}(f)$ *and* $CC(f|_C) \neq \emptyset$.

For the proof, we need several lemmas.

Lemma A.1 *For a continuous map $f: X \to X$ and $x, y \in X$, if $x \in CC(f)$ and $x \to y$, then $y \in CC(f)$.*

Proof Let $\varepsilon > 0$. By $x \in CC(f)$, we have $\delta > 0$ such that every δ-pseudo orbit $(x_i)_{i\geq 0}$ of f with $x_0 = x$ is ε-shadowed by x. Let $\xi = (y_i)_{i\geq 0}$ be a δ-pseudo orbit of f with $y_0 = y$. We fix a sequence $(\delta_j)_{j\geq 1}$ of numbers with $\delta \geq \delta_1 > \delta_2 > \dots$ and $\lim_{j\to\infty} \delta_j = 0$. Since $x \to y$, for every $j \geq 1$, there is a δ_j-chain $\alpha_j = (x_i^{(j)})_{i=0}^{k_j}$ of f with $x_0^{(j)} = x$ and $x_{k_j}^{(j)} = y$. By $x \in CC(f)$ and $\lim_{j\to\infty} \delta_j = 0$, we obtain $\lim_{j\to\infty} f^{k_j}(x) = y$. For each $j \geq 1$, let

$$\xi_j = (y_i^{(j)})_{i\geq 0} = \alpha_j \xi,$$

a δ-pseudo orbit of f. Given any $j \geq 1$, since $y_0^{(j)} = x_0^{(j)} = x$, we have

$$\sup_{i\geq 0} d(f^i(x), y_i^{(j)}) \leq \varepsilon$$

which implies

$$\sup_{i\geq 0} d(f^i(f^{k_j}(x)), y_i) = \sup_{i\geq 0} d(f^{i+k_j}(x), y_{i+k_j}^{(j)}) \leq \varepsilon.$$

Letting $j \to \infty$, we obtain

$$\sup_{i\geq 0} d(f^i(y), y_i) \leq \varepsilon.$$

Since ξ is arbitrary, we conclude that $y \in CC(f)$, proving the lemma. ∎

Lemma A.2 *For a continuous map $f: X \to X$ and $C \in \mathcal{C}(f)$, if $C \cap CC(f) \neq \emptyset$, then $C \in \mathcal{C}_{\mathrm{ter}}(f)$.*

Proof Let $x \in C \cap CC(f)$. Let $y \in C$, $z \in X$ and $y \to z$. By $x, y \in C$, we have $x \to y$. By $x \to y$ and $y \to z$, we obtain $x \to z$. Since $x \in CC(f)$, this implies $z \in C$; thus, we conclude that $C \in \mathcal{C}_{\mathrm{ter}}(f)$. ∎

Lemma A.3 *Let $f: X \to X$ be a continuous map and let $C \in \mathcal{C}(f)$. If $C \in \mathcal{C}_{\mathrm{ter}}(f)$, then $CC(f|_C) \subset CC(f)$.*

Proof Let $x \in CC(f|_C)$ and $\varepsilon > 0$. By $x \in CC(f|_C)$, we have $\delta > 0$ such that every δ-pseudo orbit $(x_i)_{i\geq 0}$ of $f|_C$ with $x_0 = x$ is ε-shadowed by x. Let $0 < \beta \leq \varepsilon$. Since

https://doi.org/10.4153/S0008439525101239 Published online by Cambridge University Press

$C \in \mathcal{C}_{\text{ter}}(f)$ and $x \in C$, there is $\gamma > 0$ such that every γ-pseudo orbit $\xi = (y_i)_{i\geq 0}$ of f with $y_0 = x$ satisfies

$$\sup_{i\geq 0} d(y_i, C) \leq \beta.$$

If $\beta, \gamma > 0$ are sufficiently small, then by letting $x_0 = x$ and taking $x_i \in C$, $i > 0$, such that

$$d(y_i, x_i) = d(y_i, C) \leq \beta$$

for all $i > 0$, we obtain a δ-pseudo orbit $\xi' = (x_i)_{i\geq 0}$ of $f|_C$ with $x_0 = x$. It follows that

$$\sup_{i\geq 0} d(f^i(x), x_i) \leq \varepsilon$$

and so

$$\sup_{i\geq 0} d(f^i(x), y_i) \leq \varepsilon + \beta \leq 2\varepsilon.$$

Since ξ is arbitrary, we obtain $x \in CC(f)$ and thus $CC(f|_C) \subset CC(f)$, proving the lemma. ∎

Lemma A.4 *Let $f: X \to X$ be a continuous map. Let $x \in X$ and let $C = C(x, f)$. If $C \in \mathcal{C}_{\text{ter}}(f)$ and $CC(f|_C) \neq \emptyset$, then $x \in CC(f)$.*

Proof Let $y \in CC(f|_C)$ and $z \in \omega(x, f)$. By $C \in \mathcal{C}_{\text{ter}}(f)$ and Lemma A.3, we obtain $y \in CC(f)$. Since $\omega(x, f) \subset C$, we have $y, z \in C$ and so $y \to z$. By Lemma A.1, we obtain $z \in CC(f)$. Then, for any $\varepsilon > 0$, there is $\delta > 0$ such that every δ-pseudo orbit $(x_i)_{i\geq 0}$ of f with $x_0 = z$ is ε-shadowed by z. For $0 < \beta \leq 2\varepsilon$, we fix $k > 0$ with $d(z, f^k(x)) \leq \beta$. By continuity of f, we have $\gamma > 0$ such that every γ-pseudo orbit $\xi = (y_i)_{i\geq 0}$ of f with $y_0 = x$ satisfies

$$\sup_{0\leq i\leq k} d(f^i(x), y_i) \leq \beta$$

and so

$$d(z, y_k) \leq d(z, f^k(x)) + d(f^k(x), y_k) \leq 2\beta.$$

If $\beta, \gamma > 0$ are sufficiently small, then

$$(z, f^{k+1}(x), f^{k+2}(x), \dots)$$

and

$$(z, y_{k+1}, y_{k+2}, \dots)$$

are δ-pseudo orbits of f and so ε-shadowed by z, which implies

$$\sup_{i\geq k+1} d(f^i(x), y_i) \leq 2\varepsilon.$$

https://doi.org/10.4153/S0008439525101239 Published online by Cambridge University Press

Note that

$$\sup_{0\le i\le k} d(f^i(x), y_i) \le \beta \le 2\varepsilon.$$

Since ξ is arbitrary, we conclude that $x \in CC(f)$, completing the proof. ∎

By these lemmas, we prove Theorem A.1.

Proof of Theorem A.1 The implication (A) $\Longrightarrow$ (B) is a consequence of Lemmas A.1 and A.2. (B) $\Longrightarrow$ (C) and (C) $\Longrightarrow$ (D) are obvious by definition. (D) $\Longrightarrow$ (A) is a consequence of Lemma A.4. This completes the proof of the theorem. ∎

Appendix B: A theorem on chain recurrent shadowable points

The aim of this appendix is to prove the following theorem.

Theorem B.1 *Every continuous map $f: X \to X$ satisfies $Sh(f) \cap CR(f) \subset Sh(f|_{CR(f)})$.*

Let $f: X \to X$ be a continuous map. For $x, y \in X$ and $\delta > 0$, the notation $x \to_\delta y$ means that there is a δ-chain $(x_i)_{i=0}^k$ of f with $x_0 = x$ and $x_k = y$. We define an equivalence relation $\leftrightarrow_\delta$ in

$$CR(f)^2 = CR(f) \times CR(f)$$

by: for any $x, y \in CR(f)$, $x \leftrightarrow_\delta y$ if and only if $x \to_\delta y$ and $y \to_\delta x$. It is easy to see that $x \leftrightarrow_\delta f(x)$ for all $x \in CR(f)$. It is also not difficult to see that $x \leftrightarrow_\delta y$ for all $x, y \in CR(f)$ with $d(x, y) \le \delta$.

Proof of Theorem B.1 Let $x \in Sh(f) \cap CR(f)$. Then, for any $\varepsilon > 0$, we have $\delta > 0$ such that every δ-pseudo orbit $(x_i)_{i\ge 0}$ of f with $x_0 = x$ is ε-shadowed by some $y \in X$. Let $\xi = (x_i)_{i\ge 0}$ be a δ-pseudo orbit of $f|_{CR(f)}$ with $x_0 = x$. For every $i \ge 0$, since $d(f(x_i), x_{i+1}) \le \delta$, we have $x_i \leftrightarrow_\delta f(x_i)$ and $f(x_i) \leftrightarrow_\delta x_{i+1}$; therefore, $x_i \leftrightarrow_\delta x_{i+1}$. This implies that $x_0 \leftrightarrow_\delta x_k$ for all $k > 0$. Then, for any $k > 0$, there is a δ-chain $(y_i)_{i=0}^l$ of f with $y_0 = x_k$ and $y_l = x_0$. Since

$$\xi_k = (x_0, x_1, \dots, x_{k-1}, y_0, y_1, \dots, y_{l-1}, x_0, x_1, \dots, x_{k-1}, y_0, y_1, \dots, y_{l-1}, \dots)$$

is a δ-pseudo orbit of f with $x_0 = x$, letting

$$X_k = \{x \in X : \xi_k \text{ is } \varepsilon\text{-shadowed by } x\},$$

we have $X_k \ne \varnothing$. Note that X_k is a closed f^{k+l}-invariant subset of X. By taking $u \in X_k$ and $v \in \omega(u, f^{k+l})$, we obtain $v \in X_k \cap CR(f^{k+l}) \subset X_k \cap CR(f)$ and thus

$$CR(f) \cap \bigcap_{i=0}^{k-1} f^{-i}(B_\varepsilon(x_i)) \ne \varnothing.$$

It follows that:

$$CR(f) \cap \bigcap_{i=0}^{\infty} f^{-i}(B_\varepsilon(x_i)) = \bigcap_{k>0} \left(CR(f) \cap \bigcap_{i=0}^{k-1} f^{-i}(B_\varepsilon(x_i)) \right) \ne \varnothing,$$

https://doi.org/10.4153/S0008439525101239 Published online by Cambridge University Press

i.e., ξ is ε-shadowed by some $y \in CR(f)$. Since ξ is arbitrary, we obtain $x \in Sh(f|_{CR(f)})$, proving the theorem. ∎

Appendix C: A theorem on shadowable points

In this appendix, we prove the following theorem (Theorem C.1) that is not used elsewhere in this article but illustrates how "shadowable points" localize the concept of shadowing. Given a continuous map $f: X \to X$ and a subset S of X, we say that f has the *shadowing on* S if for any $\varepsilon > 0$, there is $\delta > 0$ such that every δ-pseudo orbit $(x_i)_{i\geq 0}$ of f with $x_i \in S$ for all $i \geq 0$ is ε-shadowed by some $x \in X$.

Theorem C.1 *Let $f: X \to X$ be a continuous map. A closed subset A of X satisfies $A \subset Sh(f)$ if and only if there is a closed f-invariant subset S of X such that*

(1) $A \subset S$;
(2) *S is chain stable;*
(3) *f has the shadowing on S.*

Proof First, we prove the "if" part. Let $x \in S$ and $0 < \beta \leq \varepsilon$. By (3), we have $\delta > 0$ such that every δ-pseudo orbit $(x_i)_{i\geq 0}$ of f with $x_i \in S$ for all $i \geq 0$ is ε-shadowed by some $y \in X$. By (2), there is $\gamma > 0$ such that every γ-pseudo orbit $\xi = (y_i)_{i\geq 0}$ of f with $y_0 = x$ satisfies

$$\sup_{i\geq 0} d(y_i, S) \leq \beta.$$

If $\beta, \gamma > 0$ are sufficiently small, then by taking $x_i \in S$, $i \geq 0$, such that

$$d(y_i, x_i) = d(y_i, S) \leq \beta$$

for every $i \geq 0$, we obtain a δ-pseudo orbit $\xi' = (x_i)_{i\geq 0}$ of f with $x_i \in S$ for all $i \geq 0$. It follows that:

$$\sup_{i\geq 0} d(f^i(y), x_i) \leq \varepsilon$$

and so

$$\sup_{i\geq 0} d(f^i(y), y_i) \leq \varepsilon + \beta \leq 2\varepsilon$$

for some $y \in X$. Since ξ is arbitrary, it follows that $x \in Sh(f)$ and thus $S \subset Sh(f)$. By (1), we obtain $A \subset Sh(f)$.

Next we prove the "only if" part. Let

$$S = A \cup \{y \in X : x \to y \text{ for some } x \in A\}.$$

Since A is closed in X and the relation $\to$ is closed in X^2, S is a closed subset of X. Note that $z \to f(z)$ for all $z \in X$. For any $x \in A$, we have $x \to f(x)$ and so $f(x) \in S$. For any $x \in A$, $y \in X$ with $x \to y$, we have $y \to f(y)$ and so $x \to f(y)$; therefore, $f(y) \in S$. It follows that S is f-invariant. Since $y \in S$ for all $x \in S$, $y \in X$ with $x \to y$, it also follows that S is chain stable. By $A \subset Sh(f)$ and Lemma 1.5, we have $S \subset Sh(f)$. By

https://doi.org/10.4153/S0008439525101239 Published online by Cambridge University Press

using Lemma 2.4 in [8], we conclude that f has the shadowing on S, completing the proof. ■

Acknowledgements The author would like to thank the reviewer for a careful reading of the manuscript and helpful comments.

References

[1] E. Akin, *On chain continuity*. Discrete Contin. Dyn. Syst. **2**(1996), 111–120.
[2] E. Akin, M. Hurley, and J. Kennedy, *Dynamics of topologically generic homeomorphisms*. Mem. Amer. Math. Soc. **164**(2003), 130 pp.
[3] D. V. Anosov, *Geodesic flows on closed Riemann manifolds with negative curvature*. Proc. Steklov Inst. Math. **90**(1967), 235.
[4] N. Aoki and K. Hiraide, *Topological theory of dynamical systems. Recent advances*, North–Holland Mathematical Library, 52, North–Holland Publishing Co., North-Holland, Amsterdam, 1994.
[5] R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Lecture Notes in Mathematics, 470, Springer–Verlag, Springer Berlin, Heidelberg, 1975.
[6] C. Conley, *Isolated invariant sets and the Morse index*, CBMS Regional Conference Series in Mathematics, 38, American Mathematical Society, Providence, RI, 1978.
[7] E. Glasner and B. Weiss, *Sensitive dependence on initial conditions*. Nonlinearity **6**(1993), 1067–1075.
[8] N. Kawaguchi, *Quantitative shadowable points*. Dyn. Syst. **32**(2017), 504–518.
[9] N. Kawaguchi, *Distributionally chaotic maps are C^0-dense*. Proc. Am. Math. Soc. **147**(2019), 5339–5348.
[10] N. Kawaguchi, *Generic and dense distributional chaos with shadowing*. J. Differ. Equ. Appl. **27**(2021), 1456–1481.
[11] N. Kawaguchi, *Some results on shadowing and local entropy properties of dynamical systems*. Bull. Braz. Math. Soc. (N.S.) **55**(2024), 18, 29 pp.
[12] N. Kawaguchi, *Shadowing and the basins of terminal chain components*. Can. Math. Bull. **68**(2025), 187–197.
[13] M. Misiurewicz, *Remark on the definition of topological entropy*. In: Dynamical systems and partial differential equations (Caracas, 1984), Universidad Simón Bolívar, Caracas, 1986, pp. 65–67.
[14] T. K. S. Moothathu, *Implications of pseudo-orbit tracing property for continuous maps on compacta*. Topol. Appl. **158**(2011), 2232–2239.
[15] C.A. Morales, *Shadowable points*. Dyn. Syst. **31**(2016), 347–356.
[16] P. Walters, *An introduction to ergodic theory*, Graduate Texts in Mathematics, 79, Springer, Springer New York, NY, 1982.
[17] X. Ye and G. Zhang, *Entropy points and applications*. Trans. Am. Math. Soc. **359**(2007), 6167–6186.

Department of Mathematical and Computing Science, School of Computing, Institute of Science Tokyo, 2-12-1 Ookayama, Meguro-ku, Tokyo 152-8552, Japan
e-mail: gknoriaki@gmail.com

https://doi.org/10.4153/S0008439525101239 Published online by Cambridge University Press