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DISCRETE TIME REPRESENTATIONS OF COINTEGRATED CONTINUOUS TIME MODELS WITH MIXED SAMPLE DATA

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This paper derives an exact discrete time representation corresponding to a triangular cointegrated continuous time system with mixed stock and flow variables and observable stochastic trends. The discrete time model inherits the triangular structure of the underlying continuous time system and does not suffer from the apparent excess differencing that has been found in some related work. It can therefore serve as a basis for the study of the asymptotic sampling properties of estimators of the model's parameters. Some further analytical and computational results that enable Gaussian estimation to be implemented are also provided.

DEDICATION

I dedicate this paper to the memory of Rex Bergstrom, my teacher and colleague, who inspired and guided my own work on continuous time econometrics.

1. INTRODUCTION

To his students and colleagues, Rex Bergstrom's passion for economics and econometrics was clearly evident. Nowhere was this more apparent than when he was discussing continuous time methods, an area in which he worked extensively for much of his career and for which he is perhaps best known. Indeed, he continued

I would like to thank participants at the Workshop in memory of Rex Bergstrom held at the University of Westminster Business School, 2nd December, 2005 and at the A.R. Bergstrom Memorial Conference held at the University of Essex, 24th–25th May, 2006, for comments on some related results that are contained in Chambers (2006). The results in the present paper were derived in response to the comments, in particular, of Peter Phillips, to whom I am grateful. I also thank Rob Taylor and four anonymous referees for helpful comments that have improved the exposition and the presentation of the results. The solution methods employed in this paper make extensive use of the techniques developed by Rex Bergstrom in his pioneering work on continuous time models, and the proposed Gaussian estimation method is very much in the spirit of his work. Address correspondence to Professor Marcus J. Chambers, Department of Economics, University of Essex, Wivenhoe Park, Colchester, Essex CO4 3SQ, England. Tel: +44 1206 872756; fax: +44 1206 872724; e-mail: mchamb@essex.ac.uk.

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working on his continuous time macroeconomic model of the U.K. and the related econometric techniques beyond his retirement in 1992, and it is an ensuing article, Bergstrom (1997), that provides the motivation for the present paper.

The model considered in Bergstrom (1997) is a system of mixed first- and second-order stochastic differential equations with unobservable stochastic trends and mixed stock and flow variables, and it corresponds to the form of the macroeconometric model of the U.K. developed in Bergstrom and Nowman (2007). The model allows for cointegration among the variables in the system, and the unobservable stochastic trends are eliminated from the discrete time representation by differencing *all* of the variables, the resulting system being notable for the fact that it is written entirely in terms of the first differences of the observations. At first sight, the discrete time model therefore appears to correspond to a system in which all variables are integrated of order one and not cointegrated, but Bergstrom (2009) provides an example that demonstrates that, in a model with unobservable stochastic trends, the value of the Gaussian likelihood function is unaffected whether the entire system is differenced or whether it is differenced only in directions orthogonal to the cointegration vectors. Since Gaussian estimation was the motivation for the derivation of the exact discrete time model, such excess differencing was therefore shown to be unimportant, but Bergstrom (1997, p.485) did acknowledge that a study of the asymptotic sampling properties of the Gaussian estimator would be aided by the derivation of a discrete time model that more closely resembles cointegrated systems formulated directly in discrete time.

The model considered in this paper is a triangular system of first-order stochastic differential equations with mixed stock and flow variables and observable stochastic trends. Although the stochastic trends are observable, it is necessary, in deriving the discrete time representation, to deal with unobservable integrals of the stock components and unobservable levels of the flow components of these trends. Instead of differencing, the stochastic trends are eliminated by a method that replaces the unobservable component with its observable counterpart and assigns the difference (whose dynamic stochastic properties are known) to the disturbance term. The resulting discrete time model inherits the triangular format of the underlying continuous time system and would therefore be suitable as a basis for studying the asymptotic properties of estimators of its parameters. In addition to a discussion of some issues pertinent to estimation and inference in this system, the autocovariance matrices of the discrete time disturbance vector that are utilized in specifiying the time domain Gaussian likelihood function are also derived, and some computational issues are briefly discussed as an aid to the implementation of these formulas.

The method employed in this paper for the treatment of the stochastic trends relies on their increments being defined (identified) in terms of a random measure (in addition to a deterministic trend component). A related paper, Chambers (2006), considered a model in which there was feedback from the cointegrating errors onto the stochastic trends, and although two solution methods were

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examined, both resulted in the apparent excess differencing inherent in Bergstrom (1997) and also required stronger assumptions than utilized here with a mixed sample of stocks and flows (the excess differencing was not present when the sample consisted entirely of stocks *or* flows). The triangular specification appears to offer tangible advantages in deriving a convenient discrete time representation that are not necessarily present in other specifications.

It is perhaps also worth mentioning that alternative approaches exist for handling cointegrated continuous time models that do not rely on the exact discrete time model of the type derived in Bergstrom (1997) and that do not display the excess differencing. Harvey and Stock (1985) proposed the Kalman filter as a method of handling mixed stock and flow data generated by a continuous time autoregressive model with a view to computing the Gaussian likelihood function. The algorithm was subsequently extended by Zadrozny (1988) to incorporate, inter alia, mixed-frequency data, continuous time moving average disturbances, and exogenous variables, and by Harvey and Stock (1988) to handle systems with common stochastic trends. Exact discrete time models in the form of first-order error correction models have been derived by Phillips (1991b), Chambers (2003) and Chambers and McCrorie (2007), where in each of these cases the dynamics, potentially generated by higher-order systems, are assigned to the disturbance vector. Phillips suggests frequency domain regression techniques for the estimation of the cointegrating vectors; Chambers uses the representation in a theoretical analysis of the asymptotic efficiency of optimal estimators; and Chambers and McCrorie use frequency domain techniques to approximate the likelihood function and also derive the asymptotic properties of the resulting estimators for the parameters of the cointegrating vectors as well as those governing the dynamics. These methods should be viewed as alternatives to those outlined here.

2. THE MODEL AND ITS SOLUTION

To examine the issues raised above, we shall consider a continuous time random n-vector $y(t)$ that is partitioned into two subvectors $y_1(t)$ and $y_2(t)$ of dimensions $n_1 \times 1$ and $n_2 \times 1$, respectively, where $n_1 + n_2 = n$. We shall assume that the elements of $y(t)$ are nonstationary I(1) processes but that there exist n_1 stationary linear cointegrating relationships of the form $y_1(t) - \Pi y_2(t)$, where Π is an $n_1 \times n_2$ matrix of cointegrating parameters. Identification is achieved by normalizing these relationships on the elements of $y_1(t)$. It is further assumed that $y(t)$ satisfies the following first-order stochastic differential equation system:

$$dy(t) = \left[a + bt + AB'y(t)\right] dt + \zeta(dt), \quad t > 0, \tag{1}$$

where $y(t)$ is an $n \times 1$ vector of continuous time random variables, $\zeta(dt)$ is an $n \times 1$ vector of random measures satisfying $\mathrm{E}[\zeta(dt)] = 0$, $\mathrm{E}[\zeta(dt)\zeta(dt)'] = \Sigma dt$ and $\mathrm{E}[\zeta(\Delta_1)\zeta(\Delta_2)'] = 0$ for disjoint intervals Δ_1 and Δ_2, a and b are $n \times 1$ vectors of constants, $B' = (I_{n_1}, -\Pi)$ and $A = (C', 0_{n_1 \times n_2})'$. In circumstances in which the cointegrating relationships themselves contain intercepts and deterministic trend

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terms, the vectors a and b can be restricted appropriately; details of how this is achieved in discrete time cointegrated systems can be found in Pesaran, Shin, and Smith (2000) and carry over straightforwardly to the continuous time model. In terms of the vectors $y_1(t)$ and $y_2(t)$, the system (1) implies that

$$dy_1(t) = [a_1 + b_1 t + C(y_1(t) - \Pi y_2(t))]\,dt + \zeta_1(dt), \quad t > 0, \tag{2}$$

$$dy_2(t) = [a_2 + b_2 t]\,dt + \zeta_2(dt), \quad t > 0, \tag{3}$$

in which the vectors a, b, and $\zeta(dt)$ have been partitioned conformably with y_1 and y_2. The n_2 stochastic trends in the system are depicted by (3), while the stationary increments in $y_1(t)$ depend on the equilibrium errors $y_1(t) - \Pi y_2(t)$. Hence the system is triangular of the type proposed by Phillips (1991a) for discrete time models and Phillips (1991b) for continuous time systems.

The vectors $y_1(t)$ and $y_2(t)$ shall each be assumed to comprise both stock and flow variables and, without loss of generality, we shall write

$$y_1(t) = \begin{pmatrix} y_1^s(t) \\ y_1^f(t) \end{pmatrix}, \quad y_2(t) = \begin{pmatrix} y_2^s(t) \\ y_2^f(t) \end{pmatrix},$$

where, for $i = 1, 2$, $y_i^s(t)$ is an $n_i^s \times 1$ vector of stock variables, $y_i^f(t)$ is an $n_i^f \times 1$ vector of flow variables, and $n_i^s + n_i^f = n_i$. The observations on these vectors take the form

$$y_{1t} = \begin{pmatrix} y_{1t}^s \\ y_{1t}^f \end{pmatrix} = \begin{pmatrix} y_1^s(t) \\ \int_{t-1}^t y_1^f(r)\,dr \end{pmatrix}, \quad y_{2t} = \begin{pmatrix} y_{2t}^s \\ y_{2t}^f \end{pmatrix} = \begin{pmatrix} y_2^s(t) \\ \int_{t-1}^t y_2^f(r)\,dr \end{pmatrix}, \quad t = 1, \ldots, T,$$

the flow components being observed as integrals of the underlying rate of flow over the observation interval and where T denotes sample size. The first n_i^s elements of the initial state vectors $y_i(0)$ $(i = 1, 2)$ are also observed and equal to $y_i^s(0)$, while $y_i^f(0)$ is unknown. The objective is to relate the unknown parameters of the model (1) to the discrete time observations, which shall be arranged in the $n \times 1$ vector $y_t = [y_{1t}', y_{2t}']'$. It is also convenient to arrange the unobservable components $w_{it}^f = y_i^f(t)$ and $w_{it}^s = \int_{t-1}^t y_i^s(r)\,dr$ into the $n_i \times 1$ vectors

$$w_{1t} = \begin{pmatrix} w_{1t}^s \\ w_{1t}^f \end{pmatrix} = \begin{pmatrix} \int_{t-1}^t y_1^s(r)\,dr \\ y_1^f(t) \end{pmatrix}, \quad w_{2t} = \begin{pmatrix} w_{2t}^s \\ w_{2t}^f \end{pmatrix} = \begin{pmatrix} \int_{t-1}^t y_2^s(r)\,dr \\ y_2^f(t) \end{pmatrix}, \quad t = 1, \ldots, T.$$

As shown below, the key to deriving discrete time representations that can be used for estimation is the ability to solve out the elements of the unobservable vectors w_{it} from the difference equations that are implied by the unique mean square solution to (1), which (initialized at $t = 0$) is given by

$$y(t) = e^{tAB'} y(0) + \int_0^t e^{(t-s)AB'}(a + bs)\,ds + \int_0^t e^{(t-s)AB'}\zeta(ds), \quad t > 0, \tag{4}$$

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where $e^{tAB'} = \sum_{j=0}^{\infty}(tAB')^j/j!$ denotes the matrix exponential. It can be shown (see Appendix A) that the $n \times n$ matrix exponential $e^{tAB'}$ can be expressed more simply in terms of the $n_1 \times n_1$ matrix e^{tC} as follows:

$$e^{tAB'} = I_n + G(e^{tC} - I_{n_1})B', \tag{5}$$

where $G = (I_{n_1}, 0_{n_1 \times n_2})'$. It is clear from this representation that the matrix $e^{tAB'} - I_n$ retains the reduced rank of AB'. Indeed, standard manipulations applied to (4), of the type utilized in the proof of Theorem 2(c) of Bergstrom (1984), allied with (5), yield

$$\Delta y(t) = GJB'y(t-1) + c_t + \eta_t, \quad t = 1, \ldots, T, \tag{6}$$

where

$$J = \begin{pmatrix} J_s \\ J_f \end{pmatrix} = \begin{pmatrix} J_{ss} & J_{sf} \\ J_{fs} & J_{ff} \end{pmatrix} = e^C - I_{n_1}, \tag{7}$$

$$c_t = \int_{t-1}^{t} e^{(t-s)AB'}(a + bs)\,ds, \tag{8}$$

$$\eta_t = \int_{t-1}^{t} e^{(t-s)AB'}\zeta(ds). \tag{9}$$

If the sample was comprised entirely of stock variables, then (6) would provide the exact discrete model. However, with a mixed sample, it is also necessary to deal with flow variables, and so defining $\bar{y}(t) = \int_{t-1}^{t} y(r)\,dr$ and integrating (6) over $(t-1, t)$ yields

$$\Delta \bar{y}(t) = GJB'\bar{y}(t-1) + \psi_t + \upsilon_t, \quad t = 2, \ldots, T, \tag{10}$$

$$\psi_t = \int_{t-1}^{t}\int_{r-1}^{r} e^{(r-s)AB'}(a + bs)\,ds\,dr, \tag{11}$$

$$\upsilon_t = \int_{t-1}^{t}\int_{r-1}^{r} e^{(r-s)AB'}\zeta(ds)\,dr. \tag{12}$$

In order to relate the model's parameters to $y(0)$ with flow variables, it is necessary to also integrate (4) over the interval $(0, 1)$:

$$\bar{y}_1 - y(0) = GKB'y(0) + \widetilde{\psi}_1 + \widetilde{\upsilon}_1, \tag{13}$$

$$K = \begin{pmatrix} K_s \\ K_f \end{pmatrix} = \begin{pmatrix} K_{ss} & K_{sf} \\ K_{fs} & K_{ff} \end{pmatrix} = \int_0^1 e^{rC}dr - I_{n_1}, \tag{14}$$

$$\widetilde{\psi}_1 = \int_0^1\int_0^r e^{(r-s)AB'}(a + bs)\,ds\,dr, \tag{15}$$

$$\widetilde{\upsilon}_1 = \int_0^1\int_0^r e^{(r-s)AB'}\zeta(ds)\,dr. \tag{16}$$

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The representations in (10) and (13) would constitute the exact discrete model if all variables were flows. The challenge with a mixed sample of stocks and flows is to utilize the information encapsulated in (6), (10), and (13) in order to derive a discrete time model in terms of observable variables.

3. AN EXACT DISCRETE TIME REPRESENTATION

One of the key aspects in the derivation of a discrete time representation is the ability to replace the unobservable components ω_{2t} with the observable y_{2t} and then to incorporate the difference $\omega_{2t} - y_{2t}$ in the model's disturbance vector. Lemma 1 generalizes Lemma A2 of Chambers (2003) to provide the necessary result.

LEMMA 1. *Let the $n_2 \times 1$ vector $y_2(t)$ be generated by (3). Then, for $t = 1, \ldots, T$,*

$$y_2(t) - \int_{t-1}^{t} y_2(r)\,dr = \phi_{2t} + \epsilon_{2t}, \tag{17}$$

where

$$\phi_{2t} = \left[\left(\tfrac{1}{2}a_2 - \tfrac{1}{6}b_2\right) + \tfrac{1}{2}b_2 t\right],$$

$$\epsilon_{2t} = \int_{t-1}^{t} [1 - (t-s)]\zeta_2(ds).$$

Lemma 1 shows that the required differences are stationary around a deterministic trend component. The exact discrete time model of interest is presented in Theorem 1, in which vectors and matrices are partitioned conformably with the stock and flow components of y_1 and y_2 (so that, for example, the vector τ_{1t}^s is $n_1^s \times 1$ and the matrix C_{fs} is $n_1^f \times n_1^s$). The theorem relies on the following assumption.

Assumption 1. The $n_1^s \times n_1^s$ matrix C_{ss} and the $n_1^f \times n_1^f$ matrix J_{ff} are nonsingular.

The relevance of Assumption 1 is described in Remarks 2 and 3 following the theorem.

THEOREM 1. *Let $y(t)$ be generated by (1). Then, under Assumption 1, the observable vector y_t satisfies*

$$y_1 - y(0) = \Phi_{10} B' y(0) + \tau_1 + \xi_1, \tag{18}$$

$$\Delta y_2 = \Phi_{21} B' y_1 + \Phi_{20} B' y(0) + \tau_2 + \xi_2, \tag{19}$$

$$\Delta y_t = \Phi_0 B' y_{t-1} + \Phi_1 \Delta y_{t-1} + \tau_t + \xi_t, \quad t = 3, \ldots, T, \tag{20}$$

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where, in addition to terms that have been defined previously,

$$\Phi_{10} = \begin{pmatrix} J_s \\ K_f \\ 0 \\ 0 \end{pmatrix}, \quad \tau_1 = \begin{pmatrix} c_{11}^s \\ \widetilde{\psi}_{11}^f \\ c_{21}^s \\ \widetilde{\psi}_{21}^f \end{pmatrix}, \quad \xi_1 = \begin{pmatrix} \eta_{11}^s \\ \widetilde{\upsilon}_{11}^f \\ \eta_{21}^s \\ \widetilde{\upsilon}_{21}^f \end{pmatrix},$$

$$\Phi_{21} = \begin{pmatrix} J_s \\ J_f \\ 0 \\ 0 \end{pmatrix}, \quad \Phi_{20} = \begin{pmatrix} J_{sf}(J_f - K_f) \\ J_{fs}(K_s - J_s) \\ 0 \\ 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} \tau_{12}^s \\ \tau_{12}^f \\ c_{22}^s \\ \psi_{22}^f \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} \xi_{12}^s \\ \xi_{12}^f \\ \eta_{22}^s \\ \upsilon_{22}^f \end{pmatrix},$$

$$\Phi_0 = \begin{pmatrix} P_0 \\ 0 \end{pmatrix}, \quad P_0 = \begin{pmatrix} P_0^{ss} & P_0^{sf} \\ 0 & P_0^{ff} \end{pmatrix}, \quad \Phi_1 = \begin{pmatrix} P_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_1 = \begin{pmatrix} P_1^{ss} & 0 \\ P_1^{fs} & 0 \end{pmatrix},$$

$$\tau_t = \begin{pmatrix} \tau_{1t}^s \\ \tau_{1t}^f \\ c_{2t}^s \\ \psi_{2t}^f \end{pmatrix}, \quad \xi_t = \begin{pmatrix} \xi_{1t}^s \\ \xi_{1t}^f \\ \eta_{2t}^s \\ \upsilon_{2t}^f \end{pmatrix},$$

and all remaining terms are defined in Table 1 in Appendix B.

Remark 1. The exact discrete model (20) retains the triangular structure of the continuous time system with the stationary terms Δy_{t-1} and $B' y_{t-1}$ on the right-hand side. It does not suffer from the apparent excess differencing that has been found in other work, such as Bergstrom (1997) and Chambers (2006), and could therefore be used as a basis for studying the asymptotic sampling properties of the Gaussian estimator outlined in the next section.

Remark 2. One of the key features in deriving the exact discrete model in (20) is the ability to eliminate unobservable components and replace them with observables. Lemma 1 enables this to be done for the stock and flow components of w_{2t}, while Assumption 1 enables w_{1t}^s and w_{1t}^f to be eliminated via the nonsingularity of C_{ss} and J_{ff}, respectively; for details, see (A.32) and the following equations in Appendix A. The resulting adjustment terms, such as $w_{2t}^s - y_{2t}^s$, are assigned to the random disturbances (such as ξ_{2t}^s), and their serial correlation properties can be handled appropriately (as presented in Theorem 2 below). This is not dissimilar to the approach taken by Robinson and Hualde (2003) in a discrete time cointegrated system displaying long memory.

Remark 3. The nonsingularity of the matrices C_{ss} and J_{ff} in Assumption 1 has precedents in the literature on the derivation of exact discrete models corresponding to continuous time systems. Agbeyegbe (1987), for example, assumes

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nonsingularity of the equivalent of C_{ss} in a first-order system with mixed sample but requires, in addition, the nonsingularity of the equivalent of the entire matrix C, assumptions that are also used by Simos (1996) in a first-order system with observable common stochastic trends. Nonsingularity assumptions concerning submatrices of the matrix exponential (corresponding to e^C) are also common and can be found, for example, in Bergstrom (1997) and Chambers (1999), both of which deal with nonstationary and cointegrated systems. In Assumption 1, J_{ff} is a submatrix of $e^C - I_{n_1}$, and although e^C will always be nonsingular even if C is singular (see Abadir and Magnus, 2005, p. 262), this property does not in itself ensure that J_{ff} is nonsingular; hence the need for Assumption 1.

Remark 4. Another key component in deriving (20) is the specification (and identification) of the stochastic trends in the continuous time system and, hence, in the resulting discrete time system. The variables y_2 are observable stochastic trends whose increments evolve as stationary random components around a deterministic trend. When there is feedback from the equilibrium deviations $y_1 - \Pi y_2$ onto dy_2 the substitutions that are possible with Lemma 1 are no longer valid. Such a system is analyzed in Chambers (2006) and the two solution methods presented there (which both exhibit the apparent excess differencing) are only applicable when $n_1^s + n_2^s \geq n_1$ or when the additional condition $n_1^f + n_2^f \geq n_1$ also holds. Assumption 1, used here, appears to be less restrictive.

Remark 5. The cointegrating relationships $y_1 - \Pi y_2$ are normalized on the vector y_1, but in some circumstances interest may concern feedback from the long run deviations in a structural system of the form $\Pi_1 y_1 - \Pi_2 y_2$, where Π_1 is an $n_1 \times n_1$ matrix that contains, in conjunction with Π_2, sufficient normalizations, exclusions, and other restrictions to ensure identification. Such a structural model may take the form

$$dy_1(t) = [a_1 + b_1 t + C_0(\Pi_1 y_1(t) - \Pi_2 y_2(t))]\,dt + \zeta_1(dt), \quad t > 0,$$

$$dy_2(t) = [a_2 + b_2 t]\,dt + \zeta_2(dt), \quad t > 0,$$

which is incorporated in the model here by defining $\Pi = \Pi_1^{-1}\Pi_2$ and $C = C_0\Pi_1$.

4. ESTIMATION, INFERENCE, AND COMPUTATION

In order to discuss appropriate methods of estimating the unknown parameters of the model (1), it is necessary to derive the precise properties satisfied by the sequence of disturbance vectors $\xi_1, \ldots, \xi_T$. The derivation of the required autocovariance structure is facilitated by first reducing the expressions involving double integrals of the random measure $\zeta(dt)$, such as υ_t and $\widetilde{\upsilon}_1$, to single integrals using the approach first employed by Phillips (1978) and subsequently used and extended by Bergstrom (1997) and McCrorie (2000); details can be found in (A.40) and (A.41) in Appendix A. Theorem 2 provides the autocovariances of the discrete time disturbance vector ξ_t.

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THEOREM 2. *Let ξ_t be defined as in Theorem 1. Then ξ_t is a vector MA(1) process satisfying*

$$\mathrm{E}(\xi_1\xi_1') = \Omega_{11} = \int_0^1 J_2(s)\Sigma J_2(s)'ds,$$

$$\mathrm{E}(\xi_2\xi_2') = 7\Omega_{22} = \int_0^1 J_2(s)\Sigma J_2(s)'ds + \int_0^1 J_3(s)\Sigma J_3(s)'ds,$$

$$\mathrm{E}(\xi_2\xi_1') = \Omega_{21} = \int_0^1 J_3(s)\Sigma J_2(s)'ds,$$

$$\mathrm{E}(\xi_t\xi_t') = \Omega_0 = \int_0^1 J_0(s)\Sigma J_0(s)'ds + \int_0^1 J_1(s)\Sigma J_1(s)'ds, \quad t = 3, \ldots, T,$$

$$\mathrm{E}(\xi_3\xi_2') = \Omega_{32} = \int_0^1 J_1(s)\Sigma J_2(s)'ds,$$

$$\mathrm{E}(\xi_t\xi_{t-1}') = \Omega_1 = \int_0^1 J_1(s)\Sigma J_0(s)'ds, \quad t = 4, \ldots, T,$$

$$\mathrm{E}(\xi_t\xi_{t-j}') = 0, \quad j > 1,$$

where the matrix functions $J_i(s)$ $(i = 0, 1, 2, 3)$ are defined in Table 2 in Appendix B.

The MA(1) disturbance vector ξ_t displays some initial heteroskedasticity in the sense that $\mathrm{E}(\xi_1\xi_1')$ and $\mathrm{E}(\xi_2\xi_2')$ are different from $\mathrm{E}(\xi_t\xi_t')$ $(t = 3, \ldots, T)$; there are also differences in the initial first-order autocovariance matrices.

Theorems 1 and 2 show that the exact discrete model is a reduced rank VARMA(2,1) system whose parameter matrices and autocovariances are, in general, complicated functions of the parameters of the underlying continuous time system. Methods for estimating discrete time reduced rank VARMA models, although without such restrictions, have been proposed by Yap and Reinsel (1995), Lütkepohl and Claessen (1997), and Poskitt (2006), and could potentially be adapted to incorporate the restrictions arising here. It should be clear, however, that reduced rank VAR methods, such as those advocated by Johansen (1991), would not be appropriate for estimating all the parameters of (1) due to the inherent dynamic misspecification and subsequent neglect of the restrictions imposed by temporal aggregation, although such methods can be utilized for testing the cointegrating rank and estimating the cointegrating vectors, as discussed below.

Rather than attempting to modify these methods for discrete time systems, we focus, instead, on an exact Gaussian procedure that is very much in the Bergstrom tradition. In the context of a cointegrated model, such an approach is really only worthwhile if interest also centers on the continuous time dynamics as well as the cointegration vectors. This is because, as pointed out by Phillips (1991b), the cointegration parameters (the elements of Π) are not subject to aliasing and can

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be identified directly from discrete time data, which suggests that the usual discrete time methods of testing for cointegration rank remain valid for the discrete time observations generated by a continuous time system. Phillips (1991b) recommends frequency domain regression methods to estimate the cointegrating parameters with the dynamics treated nonparametrically, although other approaches are possible. Hence there is no gain in efficiency in the estimates of the long-run parameters by carrying out the full Gaussian estimation procedure described below. But in cases where the continuous time dynamics are taken seriously, such an approach is necessary. It should be noted that the short-run (dynamic) parameters are subject to the usual aliasing identification problem even though the long-run (cointegrating) parameters are not. Results on identification in continuous time systems can be found in Phillips (1973), Hansen and Sargent (1983), and McCrorie (2003), and we assume that sufficient restrictions are present to ensure identification of the short-run parameters.

In order to derive the Gaussian likelihood function, it is convenient to define the $nT \times 1$ vector $\xi = (\xi_1', \ldots, \xi_T')'$ and its $nT \times nT$ block-diagonal Toeplitz covariance matrix $\Omega = \mathrm{E}(\xi\xi')$, the nonzero blocks of which are defined in Theorem 2. Assuming that ξ is multivariate Gaussian (which is equivalent to assuming that $\zeta(dt)$ is Gaussian or, equivalently, the increment of a Brownian motion process), minus twice the negative of the logarithm of the Gaussian likelihood function can be written (ignoring the constant term)

$$\ln L = \ln|\Omega| + \xi'\Omega^{-1}\xi.$$

Computation of $\ln L$ requires the determinant and inverse of the $nT \times nT$ sparse covariance matrix Ω. Bergstrom (1985, 1990) provides details of a computationally efficient recursive algorithm for computing these quantities that exploits the sparse nature of Ω and that has been used in subsequent empirical applications, including Bergstrom and Nowman (2007). The Gaussian estimator of the short-run parameters can be expected to converge at the rate $\sqrt{T}$ to a limiting normal distribution, while the Gaussian estimator of the cointegrating parameters can be expected to converge at the rate T to a limiting mixed normal distribution. These properties have been established for a frequency domain Gaussian estimator of the parameters of a more general continuous time system than that considered here by Chambers and McCrorie (2007) and can be expected to hold in the current context as well.

Further inspection of the formulae in Theorems 1 and 2 reveals that a number of terms, such as c_t, ψ_t, $\widetilde{\psi}_1$, and K, as well as the autocovariances, involve integrals of the exponential of the matrix AB'. Theorem 3 contains computable representations for the relevant expressions and utilizes the following assumption.

Assumption 2. The $n_1 \times n_1$ matrix C is nonsingular.

Assumption 2 extends part of Assumption 1 to ensure that it is the entire matrix C, and not just the submatrix C_{ss}, that is nonsingular. This aids in deriving the expressions below, because integrals involving C then have exact closed form

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solutions, but it is not strictly necessary, because numerical methods based on the series representation of the matrix exponential could otherwise be used in practical circumstances; see, for example, Lemma 1 of Chambers (1999).

THEOREM 3. *Under Assumption 2, the vectors c_t, ψ_t, and $\widetilde{\psi}_1$, and the matrix K have the following representations:*

$$c_t = \int_{t-1}^{t} e^{(t-s)AB'}(a+bs)\,ds = H_0 a - \left(\tfrac{1}{2}I_n + H_1 B'\right) b + H_0 bt,$$

$$\psi_t = \int_{t-1}^{t}\int_{r-1}^{r} e^{(r-s)AB'}(a+bs)\,ds\,dr = H_0 a - \left(\tfrac{1}{2}(I_n + H_0) + H_1 B'\right) b + H_0 bt,$$

$$\widetilde{\psi}_1 = \int_{0}^{1}\int_{0}^{r} e^{(r-s)AB'}(a+bs)\,ds\,dr = \left(\tfrac{1}{2}I_n + H_2 B'\right) a + \left(\tfrac{1}{6}I_n + (H_3 - H_4)B'\right) b,$$

$$K = \int_0^1 e^{rC}dr - I_{n_1} = C^{-1}J - I_{n_1},$$

where $H_0 = I_n + GKB'$,

$$H_1 = G\left(K_1 - \tfrac{1}{2}I_{n_1}\right), \quad H_2 = G\left(K_2 - \tfrac{1}{2}I_{n_1}\right),$$

$$H_3 = G\left(C^{-1}K_1 - \tfrac{1}{2}C^{-1} - \tfrac{1}{3}I_{n_1}\right), \quad H_4 = G\left(C^{-1}K_1 - C^{-2}K - \tfrac{1}{6}I_{n_1}\right),$$

$K_1 = C^{-1}e^C - C^{-2}J$, *and* $K_2 = C^{-2}J - C^{-1}$. *Furthermore, the functions* $J_0(s)$, $J_1(s)$, *and* $J_2(s)$ *have the representation*

$$J_i(s) = J_{i0} + J_{i1}s + J_{i2}e^{sC}B', \quad i = 0,1,2,3,$$

where the matrices J_{ik} $(i = 0,1,2,3; k = 0,1,2)$ *are defined in Table 3 in Appendix B. The integrals determining the autocovariance matrices then have the representation*

$$\begin{aligned}\int_0^1 J_i(s)\Sigma J_k(s)'ds &= \int_0^1 \left[J_{i0} + J_{i1}s + J_{i2}e^{sC}B'\right]\Sigma\left[J_{k0} + J_{k1}s + J_{k2}e^{sC}B'\right]' ds \\ &= J_{i0}\Sigma J_{k0}' + \tfrac{1}{2}J_{i0}\Sigma J_{k1}' + J_{i0}\Sigma BK' J_{k2}' + \tfrac{1}{2}J_{i1}\Sigma J_{k0}' \\ &\quad + \tfrac{1}{3}J_{i1}\Sigma J_{k1}' + J_{i1}\Sigma BK_1' J_{k2}' + J_{i2}KB'\Sigma J_{k0}' \\ &\quad + J_{i2}K_1 B'\Sigma J_{k1}' + J_{i2}K_3 J_{k2}', \end{aligned} \tag{21}$$

where $K_3 = \int_0^1 e^{sC}B'\Sigma B e^{sC'}ds$.

The only remaining integral and matrix exponential that require computation are therefore K_3 and e^C. Phillips (1973) shows that the elements of K_3 satisfy

$$\operatorname{vec}(K_3) = [(B\otimes I_n) + (I_n \otimes B)]^{-1}\left[\left(e^C \otimes e^C\right) - I_{n^2}\right]\operatorname{vec}(\Sigma),$$

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where vec(A) stacks the columns of an $n \times m$ matrix A to form an $nm \times 1$ vector and $\otimes$ denotes the Kronecker product operator. Alternatively, both K_3 and e^C can be obtained from the computation of a single matrix exponential, this being

$$M^* = \exp\begin{pmatrix} -C & B'\Sigma B \\ 0 & C' \end{pmatrix} = \begin{pmatrix} M_{11}^* & M_{12}^* \\ 0 & M_{22}^* \end{pmatrix}.$$

Jewitt and McCrorie (2005), using results of van Loan (1978), show that $K_3 = M_{22}^{*\prime} M_{12}^*$ and $e^C = M_{22}^{*\prime}$. Hence the problem is reduced to the computation of either e^C or e^{M^*}, and a number of methods exist for this purpose. The most straightforward of these, though not necessarily the most efficient, is truncation of the infinite series representation at some sufficiently large integer N, resulting in the approximations

$$e^C \approx (e^C)_N = I_{n_1} + \sum_{j=1}^{N} \frac{C^j}{j!} \quad \text{and} \quad e^{M^*} \approx (e^{M^*})_N = I_{2n_1} + \sum_{j=1}^{N} \frac{(M^*)^j}{j!}.$$

The integer N can be chosen such that the elements of the differences $(e^C)_N - (e^C)_{N-1}$ and $(e^{M^*})_N - (e^{M^*})_{N-1}$ are sufficiently small in absolute value. A potentially more efficient method exploits the highly accurate rational approximations to scalar exponentials due, inter alia, to Hastings (1955). In this procedure, the matrix C (or M^*) is diagonalized and the exponentials of the scalar eigenvalues are computed using the rational approximations, from which the matrix exponential can then be formed. An investigation of alternative methods of computing matrix exponentials can be found in Jewitt and McCrorie (2005). Although they recommend a method based on a trigonometric representation, they conclude that the computational problem does not appear to be ill-conditioned for the models typically considered in the continuous time macroeconometrics literature, so that the truncation method should be adequate, provided standard IEEE double precision arithmetic is used.

5. CONCLUDING COMMENTS

This paper has derived an exact discrete model corresponding to a triangular cointegrated continuous time model with mixed stock and flow variables and observable stochastic trends. In the context of this model, it is shown that it is possible to derive a discrete time model that retains the triangular structure of the continuous time system and that doesn't suffer from the apparent excess differencing observed in some recent work on cointegrated continuous time models, although the underlying models are specified differently. Although the continuous time model considered is of first order, the methods can be straightforwardly extended to higher-order as well as to mixed-order models, though with an associated

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increase in the complexity of the derivations. The results in this paper also provide a framework within which the asymptotic properties of the time domain Gaussian estimator of the parameters of a cointegrated continuous time system can be derived.

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APPENDIX A: Proofs and Supplementary Results

Proof of (5). Consider $e^{tAB'} = I_n + \sum_{j=1}^{\infty} t^j (AB')^j / j!$. Because $A = GC$ and $B'G = I_{n_1}$ it is easy to show that $(AB')^j = GC^j B'$, hence

$$e^{tAB'} = I_n + G\left(\sum_{j=1}^{\infty} \frac{(tC)^j}{j!}\right) B' = I_n + G(e^{tC} - I_{n_1})B'$$

as required. ■

Proof of Lemma 1. The solution to (3) is given by

$$y_2(t) = y_2(0) + \int_0^t (a_2 + b_2 s)\, ds + \int_0^t \zeta_2(ds) = y_2(0) + a_2 t + \tfrac{1}{2} b_2 t^2 + \int_0^t \zeta_2(ds). \quad \textbf{(A.1)}$$

Integrating the solution over $(t-1, t]$ gives

$$\int_{t-1}^t y_2(r)dr = y_2(0) + \int_{t-1}^t \left(a_2 r + \tfrac{1}{2} b_2 r^2\right) dr + \int_{t-1}^t \int_0^r \zeta_2(ds)\, dr.$$

Evaluating the integral of the deterministic component and noting that

$$\int_{t-1}^t \int_0^r \zeta_2(ds)\, dr = \int_0^{t-1} \left(\int_{t-1}^t dr\right) \zeta_2(ds) + \int_{t-1}^t \left(\int_s^t dr\right) \zeta_2(ds)$$

$$= \int_0^{t-1} \zeta_2(ds) + \int_{t-1}^t (t-s)\zeta_2(ds)$$

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yields

$$\int_{t-1}^{t} y_2(r)\,dr = y_2(0) + a_2\left(t-\tfrac{1}{2}\right) + \tfrac{1}{6}b_2(3t^2-3t+1) + \int_0^{t-1}\zeta_2(ds) + \int_{t-1}^{t}(t-s)\zeta_2(ds). \tag{A.2}$$

The expression in Lemma 1 is obtained by taking the difference of (A.1) and (A.2). ■

Proof of Theorem 1. To derive (18), setting $t=1$ in (6) gives $y_1 - y(0) = GJB'y(0) + c_1 + \eta_1$, implying that

$$y_{11}^s - y_1^s(0) = J_s B' y(0) + c_{11}^s + \eta_{11}^s, \tag{A.3}$$

$$y_{21}^s - y_2^s(0) = c_{21}^s + \eta_{21}^s. \tag{A.4}$$

For the flow components, we integrate the solution (4) over $(0,1)$:

$$\begin{aligned}\int_0^1 y(r)\,dr &= \int_0^1 e^{rAB'}\,dr\,y(0) + \int_0^1\int_0^r e^{(r-s)AB'}(a+bs)\,ds\,dr \\ &\quad + \int_0^1\int_0^r e^{(r-s)AB'}\zeta(ds)\,dr \\ &= \left(I_n + GKB'\right)y(0) + \widetilde{\psi}_1 + \widetilde{\upsilon}_1,\end{aligned} \tag{A.5}$$

from which it follows that

$$y_{11}^f - y_1^f(0) = K_f B' y(0) + \widetilde{\psi}_{11}^f + \widetilde{\upsilon}_{11}^f, \tag{A.6}$$

$$y_{21}^f - y_2^f(0) = \widetilde{\psi}_{21}^f + \widetilde{\upsilon}_{21}^f. \tag{A.7}$$

Combining (A.3), (A.4), (A.6), and (A.7) yields (18).

For the remaining equations, it is convenient to first split (6) and (10) conformably with y_1 and y_2, which gives, respectively:

$$\Delta y_1(t) = JB'y(t-1) + c_{1t} + \eta_{1t}, \tag{A.8}$$

$$\Delta y_2(t) = c_{2t} + \eta_{2t}, \tag{A.9}$$

$$\Delta \bar{y}_1(t) = JB'\bar{y}(t-1) + \psi_{1t} + \upsilon_{1t}, \tag{A.10}$$

$$\Delta \bar{y}_2(t) = \psi_{2t} + \upsilon_{2t}. \tag{A.11}$$

Furthermore, defining $u_t^s = y_{1t}^s - \Pi_{ss}y_{2t}^s - \Pi_{sf}w_{2t}^f$, $u_t^f = w_{1t}^f - \Pi_{fs}y_{2t}^s - \Pi_{ff}w_{2t}^f$,

$$\bar{u}_t^s = w_{1t}^s - \Pi_{ss}w_{2t}^s - \Pi_{sf}y_{2t}^f, \tag{A.12}$$

and $\bar{u}_t^f = y_{1t}^f - \Pi_{fs}w_{2t}^s - \Pi_{ff}y_{2t}^f$, and partitioning into the stock and flow subcomponents:

$$\Delta y_{1t}^s = J_{ss}u_{t-1}^s + J_{sf}u_{t-1}^f + c_{1t}^s + \eta_{1t}^s, \tag{A.13}$$

$$\Delta w_{1t}^f = J_{fs}u_{t-1}^s + J_{ff}u_{t-1}^f + c_{1t}^f + \eta_{1t}^f, \tag{A.14}$$

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$$\Delta y_{2t}^s = c_{2t}^s + \eta_{2t}^s, \tag{A.15}$$

$$\Delta w_{2t}^f = c_{2t}^f + \eta_{2t}^f, \tag{A.16}$$

$$\Delta w_{1t}^s = J_{ss}\bar{u}_{t-1}^s + J_{sf}\bar{u}_{t-1}^f + \psi_{1t}^s + \upsilon_{1t}^s, \tag{A.17}$$

$$\Delta y_{1t}^f = J_{fs}\bar{u}_{t-1}^s + J_{ff}\bar{u}_{t-1}^f + \psi_{1t}^f + \upsilon_{1t}^f, \tag{A.18}$$

$$\Delta w_{2t}^s = \psi_{2t}^s + \upsilon_{2t}^s, \tag{A.19}$$

$$\Delta y_{2t}^f = \psi_{2t}^f + \upsilon_{2t}^f. \tag{A.20}$$

To establish (19), we first set $t = 2$ in (A.13) to give

$$\Delta y_{12}^s = J_{ss}(y_{11}^s - \Pi_{ss}y_{21}^s - \Pi_{sf}w_{21}^f) + J_{sf}(w_{11}^f - \Pi_{fs}y_{21}^s - \Pi_{ff}w_{21}^f) + c_{12}^s + \eta_{12}^s. \tag{A.21}$$

The objective is to eliminate w_{11}^f and w_{21}^f and to replace them with the observable y_{11}^f and y_{21}^f, respectively. Setting $t = 1$ in (A.14) gives

$$w_{11}^f - y_1^f(0) = J_f B' y(0) + c_{11}^f + \eta_{11}^f, \tag{A.22}$$

and taking the difference of (A.22) and (A.6) yields

$$w_{11}^f - y_{11}^f = (J_f - K_f)B' y(0) + (c_{11}^f - \widetilde{\psi}_{11}^f) + (\eta_{11}^f - \widetilde{\upsilon}_{11}^f). \tag{A.23}$$

Furthermore, from (A.16) we have $w_{21}^f = y_2^f(0) + c_{21}^f + \eta_{21}^f$, while (A.5) yields $y_{21}^f = y_2^f(0) + \widetilde{\psi}_{21}^f + \widetilde{\upsilon}_{21}^f$ so that

$$w_{21}^f - y_{21}^f = (c_{21}^f - \widetilde{\psi}_{21}^f) + (\eta_{21}^f - \widetilde{\upsilon}_{21}^f). \tag{A.24}$$

Using (A.23) and (A.24) in (A.21) results in

$$\Delta y_{12}^s = J_s B' y_1 + J_{sf}(J_f - K_f)B' y(0) + \tau_{12}^s + \xi_{12}^s. \tag{A.25}$$

For the flow components, setting $t = 2$ in (A.19) gives

$$\Delta y_{12}^f = J_{fs}(w_{11}^s - \Pi_{ss}w_{21}^s - \Pi_{sf}y_{21}^f) + J_{ff}(y_{11}^f - \Pi_{fs}w_{21}^s - \Pi_{ff}y_{21}^f) + \psi_{12}^f + \upsilon_{12}^f. \tag{A.26}$$

The aim is to replace w_{11}^s and w_{21}^s with the observable y_{11}^s and y_{21}^s respectively. From (A.5), $w_{11}^s = y_1^s(0) + K_s B' y(0) + \widetilde{\psi}_{11}^s + \widetilde{\upsilon}_{11}^s$ while from (A.3), $y_{11}^s = y_1^s(0) + J_s B' y(0) + c_{11}^s + \eta_{11}^s$, implying that

$$w_{11}^s - y_{11}^s = (K_s - J_s)B' y(0) + (\widetilde{\psi}_{11}^s - c_{11}^s) + (\widetilde{\upsilon}_{11}^s - \eta_{11}^s). \tag{A.27}$$

Also from (A.5), $w_{21}^s = y_2^s(0) + \widetilde{\psi}_{21}^s + \widetilde{\upsilon}_{21}^s$, while (A.15) gives $y_{21}^s = y_2^s(0) + c_{21}^s + \eta_{21}^s$ and hence

$$w_{21}^s - y_{21}^s = (\widetilde{\psi}_{21}^s - c_{21}^s) + (\widetilde{\upsilon}_{21}^s - \eta_{21}^s). \tag{A.28}$$

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Using (A.27) and (A.28) in (A.26) yields

$$\Delta y_{12}^f = J_f B' y_1 + J_{fs}(K_s - J_s)B' y(0) + \tau_{12}^f + \xi_{12}^f. \tag{A.29}$$

Combining (A.25) and (A.29), along with the equations obtained when setting $t = 2$ in (A.15) and (A.20), yields (19) as required.

To demonstrate the validity of (20), we first note that (A.15) and (A.20) characterize the evolution of Δy_{2t}, while in (A.13) and (A.18) it is necessary to eliminate the unobservables contained in u_{t-1}^j and $\bar{u}_{t-1}^j$ $(j = s, f)$. Lemma 1 enables this to be done straightforwardly for w_{2t}^s and w_{2t}^f:

$$u_t^s = y_{1t}^s - \Pi_{ss} y_{2t}^s - \Pi_{sf} y_{2t}^f - \Pi_{sf}(w_{2t}^f - y_{2t}^f) = B_s' y_t - \Pi_{sf}(\phi_{2t}^f + \epsilon_{2t}^f), \tag{A.30}$$

$$\bar{u}_t^f = y_{1t}^f - \Pi_{fs} y_{2t}^s - \Pi_{ff} y_{2t}^f - \Pi_{fs}(w_{2t}^s - y_{2t}^s) = B_f' y_t + \Pi_{fs}(\phi_{2t}^s + \epsilon_{2t}^s). \tag{A.31}$$

The treatment of w_{1t}^f (in u_t^f) and w_{1t}^s (in $\bar{u}_t^s$) is trickier. To facilitate their treatment, integrating (2) yields

$$\Delta y_1(t) = h_{1t} + C[\bar{y}_1(t) - \Pi \bar{y}_2(t)] + e_{1t}, \tag{A.32}$$

and partitioning into the stock and flow components, we obtain

$$\Delta y_{1t}^s = h_{1t}^s + C_{ss}\bar{u}_t^s + C_{sf}\bar{u}_t^f + e_{1t}^s, \tag{A.33}$$

$$\Delta w_{1t}^f = h_{1t}^f + C_{fs}\bar{u}_t^s + C_{ff}\bar{u}_t^f + e_{1t}^f. \tag{A.34}$$

Under Assumption 1 we can solve (A.33) for $\bar{u}_t^s$:

$$\bar{u}_t^s = C_{ss}^{-1}(\Delta y_{1t}^s - h_{1t}^s - C_{sf}\bar{u}_t^f - e_{1t}^s), \tag{A.35}$$

and, equating the right-hand sides of (A.34) and (A.14), we can solve for u_{t-1}^f:

$$u_{t-1}^f = J_{ff}^{-1}\left(h_{1t}^f - c_{1t}^f - J_{fs}u_{t-1}^s + C_{fs}\bar{u}_t^s + C_{ff}\bar{u}_t^f + e_{1t}^f - \eta_{1t}^f\right). \tag{A.36}$$

Using (A.17), (A.19), and (A.20) in (A.12), we obtain the following dynamic equation for $\bar{u}_t^s$:

$$\bar{u}_t^s = (J_{ss} + I_{n_1^s})\bar{u}_{t-1}^s + J_{sf}\bar{u}_{t-1}^f + B_s'(\psi_t + \upsilon_t). \tag{A.37}$$

In a similar way, we also find that

$$\bar{u}_t^f = J_{fs}\bar{u}_{t-1}^s + (J_{ff} + I_{n_1^f})\bar{u}_{t-1}^f + B_f'(\psi_t + \upsilon_t). \tag{A.38}$$

Substituting (A.37) and (A.38) into (A.36) and eliminating $\bar{u}_{t-1}^s$ using (A.35) yields

$$u_{t-1}^f = g_t^f + \Gamma_1 \Delta y_{1,t-1}^s + \Gamma_2 u_{t-1}^s + \Gamma_3 \bar{u}_{t-1}^f + z_t^f. \tag{A.39}$$

The stock component of (20) is obtained by using (A.39) to eliminate u_{t-1}^f in (A.13) and then using (A.30) and (A.31) to eliminate the remaining terms in u_t^s and $\bar{u}_t^f$. Finally, the

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derivation of the flow component of (20) is obtained by the straightforward substitution of (A.35) in (A.18) and the elimination of the remaining $\bar{u}^f_{t-1}$ term by (A.31). ■

Proof of Theorem 2. From their definitions in Theorem 1, the disturbance vectors ξ_t have the representations

$$\xi_1 = S_6\eta_1 + S_7\widetilde{v}_1,$$

$$\xi_2 = S_6\eta_2 + S_8\eta_1 + S_7v_2 - S_8\widetilde{v}_1,$$

$$\xi_t = S_1\eta_t + S_2v_t + S_3e_t + S_4e_{t-1} + S_5\epsilon_{t-1}, \quad t = 3,\ldots,T,$$

where $e_t = \int_{t-1}^t \zeta(dr)$, $\epsilon_t = \int_{t-1}^t [1-(t-s)]\zeta(ds)$, and the selection matrices S_i $(i = 1,\ldots,8)$ are defined in Table 2 in Appendix B. Now $\eta_t = \int_{t-1}^t F_0(t-s)\zeta(ds)$, and we can write

$$\begin{aligned} v_t &= \int_{t-1}^t \int_{r-1}^r e^{(r-s)A}\zeta(ds)\,dr \\ &= \int_s^t \int_{t-1}^t e^{(r-s)A}\zeta(ds)\,dr + \int_{t-1}^{s+1} \int_{t-2}^{t-1} e^{(r-s)A}\zeta(ds)\,dr \\ &= \int_{t-1}^t F_1(t-s)\zeta(ds) + \int_{t-2}^{t-1} F_2(t-1-s)\zeta(ds), \end{aligned} \tag{A.40}$$

where $F_1(t-s) = \int_s^t e^{(r-s)AB'}dr$ and $F_2(t-1-s) = \int_{t-1}^{s+1} e^{(r-s)AB'}dr$. The expressions for $F_1(s)$ and $F_2(s)$ are obtained by making the appropriate substitutions. Furthermore,

$$\widetilde{v}_1 = \int_0^1 \int_0^r e^{(r-s)AB'}\zeta(ds)\,dr = \int_0^1 \int_s^1 e^{(r-s)AB'}dr\zeta(ds) = \int_0^1 F_1(1-s)\zeta(ds), \tag{A.41}$$

while $\epsilon_t = \int_{t-1}^t F_3(t-s)\zeta(ds)$ and $e_t = \int_{t-1}^t \zeta(ds)$. Substituting these expressions into those for the ξ_t, we obtain the representations

$$\xi_1 = \int_0^1 J_2(1-s)\zeta(ds),$$

$$\xi_2 = \int_1^2 J_2(2-s)\zeta(ds) + \int_0^1 J_3(1-s)\zeta(ds),$$

$$\xi_t = \int_{t-1}^t J_0(t-s)\zeta(ds) + \int_{t-2}^{t-1} J_1(t-1-s)\zeta(ds), \quad t = 3,\ldots,T,$$

from which the autocovariances follow immediately. ■

Proof of Theorem 3. First, note that c_t can be written $c_t = H_0a + H_5(t)b$, where $H_0 = \int_0^1 e^{rAB'}dr$ and $H_5(t) = \int_0^1 e^{rAB'}(t-r)\,dr = H_0t - \int_0^1 e^{rAB'}r\,dr$. The expression for H_0 in the Theorem is obtained by integrating the expression for $e^{rAB'}$ in (5), while

$$\int_0^1 e^{rAB'}r\,dr = \tfrac{1}{2}I_n + G\int_0^1 r(e^{rC} - I_{n_1})\,dr B',$$

which can be shown to be equal to H_1. The expression for ψ_t is obtained by integrating c_t again, i.e., $\psi_t = \int_{t-1}^t c_r\,dr$, and a similar procedure is applied to obtain $\widetilde{\psi}_1$, although here

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the integrals are over $(0,1]$ and $(0,r]$. The matrix K is straightforward under Assumption 2. The expressions for the covariance kernels $J_i(s)$ are obtained by using the definitions in Theorem 2 in terms of the selection matrices S_j and the functions $F_j(s)$, the latter of which can be evaluated by undertaking the appropriate integration where necessary. The integrals involving the covariance kernels are then obtained by integrating the products term by term and evaluating. ■

APPENDIX B: Tables

TABLE 1. Definition of additional matrices and vectors in Theorem 1

	Submatrices			
	$[\cdot]^{ss}$	$[\cdot]^{sf}$	$[\cdot]^{fs}$	$[\cdot]^{ff}$
P_0	$J_{ss}+J_{sf}\Gamma_2$	$J_{sf}\Gamma_3$	0	$J_{ff}-J_{fs}C_{ss}^{-1}C_{sf}$
P_1	$J_{sf}\Gamma_1$	0	$J_{fs}C_{ss}^{-1}$	0
M	—	$J_{ss}\Pi_{sf}+J_{sf}\Pi_{ff}$	$J_{fs}\Pi_{sf}+J_{ff}\Pi_{ff}$	—
Γ_1	$J_{ff}^{-1}(C_{fs}(J_{ss}+I_{n_1^s})+C_{ff}J_{fs})C_{ss}^{-1}$			
Γ_2	$-J_{ff}^{-1}J_{fs}$			
Γ_3	$J_{ff}^{-1}(C_{fs}J_{sf}+C_{ff}(J_{ff}+I_{n_1^f}))-\Gamma_1 C_{sf}$			
Γ_4	$C_{fs}B_s'+C_{ff}B_f'$			
	Subvectors			
	$[\cdot]^{s}$		$[\cdot]^{f}$	
τ_{12}	$c_{12}^s-M_{sf}(c_{21}^f-\psi_{21}^f)+J_{sf}(c_{11}^f-\widetilde{\psi}_{11}^f)$		$\psi_{12}^f-M_{fs}(\widetilde{\psi}_{21}^s-c_{21}^s)+J_{fs}(\widetilde{\psi}_{11}^s-c_{11}^s)$	
τ_{1t}	$c_{1t}^s+J_{sf}g_t^f-P_0^{ss}\Pi_{sf}\phi_{2,t-1}^f+P_0^{sf}\Pi_{fs}\phi_{2,t-1}^s$		$\psi_{1t}^f-P_1^{fs}h_{1,t-1}^s+P_0^{ff}\Pi_{fs}\phi_{2,t-1}^s$	
ξ_{12}	$\eta_{12}^s-M_{sf}(\eta_{21}^f-\widetilde{v}_{21}^f)+J_{sf}(\eta_{11}^f-\widetilde{v}_{11}^f)$		$v_{1t}^f-P_1^{fs}e_{1,t-1}^s+P_0^{ff}\Pi_{fs}\epsilon_{2,t-1}^s$	
ξ_{1t}	$\eta_{1t}^s+J_{sf}z_t^f-P_0^{ss}\Pi_{sf}\epsilon_{2,t-1}^f+P_0^{sf}\Pi_{fs}\epsilon_{2,t-1}^s$		$\xi_{1t}^f=v_{1t}^f-P_1^{fs}e_{1,t-1}^s+P_0^{ff}\Pi_{fs}\epsilon_{2,t-1}^s$	
g_t	—		$J_{ff}^{-1}(h_{1t}^f-c_{1t}^f+\Gamma_4\psi_t)-\Gamma_1 h_{1,t-1}^s$	
z_t	—		$J_{ff}^{-1}(e_{1t}^f-\eta_{1t}^f+\Gamma_4 v_t)-\Gamma_1 e_{1,t-1}^s$	
h_{1t}	$a_1+b_1\left(t-\frac{1}{2}\right)$			
e_{1t}	$\int_{t-1}^{t}\zeta_1(dr)$			

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TABLE 2. Definition of matrix functions and selection matrices in Theorem 2

Matrix functions

$$J_0(s) = S_1 F_0(s) + S_2 F_1(s) + S_3, \qquad J_1(s) = S_2 F_2(s) + S_5 F_3(s) + S_4,$$
$$J_2(s) = S_6 F_0(s) + S_7 F_1(s), \qquad J_3(s) = S_8 F_0(s) + S_7 F_2(s) - S_8 F_1(s),$$
$$F_0(s) = e^{sAB'}, \qquad F_1(s) = \int_0^s e^{rAB'} dr,$$
$$F_2(s) = \int_s^1 e^{rAB'} dr, \qquad F_3(s) = (1-s) I_n,$$

Selection matrices

$$S_1 = \begin{pmatrix} I_{n_1^s} & -J_{sf} J_{ff}^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_{n_2^s} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad S_2 = \begin{pmatrix} J_{sf} J_{ff}^{-1} \Gamma_4 & & & \\ 0 & I_{n_1^f} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{n_2^f} \end{pmatrix},$$

$$S_3 = \begin{pmatrix} 0 & J_{sf} J_{ff}^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad S_4 = \begin{pmatrix} -J_{sf} J_{ff}^{-1} \Gamma_1 & 0 & 0 & 0 \\ -P_1^{fs} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$S_5 = \begin{pmatrix} 0 & 0 & P_0^{sf} \Pi_{fs} & -P_0^{ss} \Pi_{sf} \\ 0 & 0 & P_0^{ff} \Pi_{fs} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad S_6 = \begin{pmatrix} I_{n_1^s} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_{n_2^s} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$S_7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & I_{n_1^f} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{n_2^f} \end{pmatrix}, \qquad S_8 = \begin{pmatrix} 0 & J_{sf} & 0 & -M_{sf} \\ -J_{fs} & 0 & M_{fs} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note: The column and row partitions contain n_1^s, n_1^f, n_2^s, and n_2^f elements, respectively.

TABLE 3. Definition of the J_{ik} matrices in Theorem 3

	k		
i	0	1	2
0	$S_1(I_n - GB') - S_2 GC^{-1} B' + S_3$	$S_2(I_n - GB')$	$S_1 G + S_2 GC^{-1}$
1	$S_2(I_n - GB') + S_2 GC^{-1} e^C B' + S_4$	$-S_2(I_n - GB')$	$-S_2 GC^{-1}$
2	$S_6(I_n - GB') - S_7 GC^{-1} B'$	$S_7(I_n - GB')$	$S_6 G + S_7 GC^{-1}$
3	$-J_{31} + S_7 GC^{-1} e^C B' + S_8 GC^{-1} B'$	$-(S_7 + S_8)(I_n - GB')$	$S_8 G - (S_7 + S_8) GC^{-1}$

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