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Infinite dimensional sequential compactness: Sequential compactness based on barriers

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Abstract. We introduce a generalization of sequential compactness using barriers on ω extending naturally the notion introduced in [W. Kubiś and P. Szeptycki, On a topological Ramsey theorem, *Canad. Math. Bull.*, 66 (2023), 156–165]. We improve results from [C. Corral and O. Guzmán and C. López-Callejas, High dimensional sequential compactness, *Fund. Math.*] by building spaces that are $\mathcal{B}$-sequentially compact but not $\mathcal{C}$-sequentially compact when the barriers $\mathcal{B}$ and $\mathcal{C}$ satisfy certain rank assumption which turns out to be equivalent to a Katětov-order assumption. Such examples are constructed under the assumption $\mathfrak{b} = \mathfrak{c}$. We also exhibit some classes of spaces that are $\mathcal{B}$-sequentially compact for every barrier $\mathcal{B}$, including some classical classes of compact spaces from functional analysis, and as a byproduct, we obtain some results on angelic spaces. Finally, we introduce and compute some cardinal invariants naturally associated to barriers.

1 Introduction

A 2-dimensional version of sequential compactness was first considered by M. Bojańczyk, E. Kopczyński, and S. Toruńczyk in [3] where they showed that if $f : [\omega]^2 \to K$ and K is a compact metric space, then there is an infinite set $B \in [\omega]^\omega$ and $x \in K$ such that for every open $U \ni x$, there exists $m \in \omega$ such that $f''[B\backslash m]^2 \subseteq U$. In this case, x is said to be the limit of $f \restriction [B]^2$. Using this result, they show that every compact metric semigroup has an idempotent that can be defined as the limit of naturally defined $f : [\omega]^2 \to K$. It is natural to call this property 2-dimensional sequential compactness and look for higher-dimensional versions. In fact, the notion of n-sequential compactness was introduced and studied in [19] (called n-Ramsey in their paper) as a natural generalization from the case $n = 2$ to all other positive integers n. A space X is said to be *n-sequentially compact* if for every function

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$f : [\omega]^n \to X$, there is an infinite set $M \in [\omega]^\omega$ and $x \in X$ such that for every open $U \ni x$, there exists $k \in \omega$ such that $f''[M\backslash k]^n \subseteq U$. It was shown in [19] that every n-sequentially compact space is also m-sequentially compact as long as $m \leq n$, and since 1-sequential compactness coincides with the classical notion of sequential compactness, every n-sequentially compact space is also sequentially compact. However, they gave examples of n-sequentially compact spaces that are not $(n+1)$-sequentially compact (assuming CH for $n > 1$) and proved that sequentially compact spaces of character less than $\mathfrak{b}$ are n-sequentially compact for all $n \in \omega$, among other results. Many of these results were further improved in [7], where the study of n-sequentially compact spaces was carried on.

It should be mentioned that a related notion of *Ramsey convergence* was introduced and studied by H. Knaust in [15]. Given an array $\{x_{i,j} : 0 \leq i < j < \omega\}$ in a space X, it is said that it converges *Ramsey-uniformly* to a point x in X if there is an infinite set $M \in [\omega]^\omega$ such that for every open neighborhood U of x, there is $k \in \omega$ such that $x_{i,j} \in U$ for every $i, j \in M$ with $k \leq i < j$. Knaust also defines a space X to have the *Ramsey property* if given an array $\{x_{n,m} : n < m < \omega\}$ and a point x in the space X such that $\lim_{n\to\infty} \lim_{m\to\infty} x_{n,m} = x$, the array $\{x_{n,m} : n < m < \omega\}$ converges Ramsey uniformly to x. He then showed that every Rosenthal compact has the Ramsey property. In the subsequent paper [16], Knaust showed that some classes of angelic spaces have the Ramsey property, including function spaces $C_p(X)$ over quasi-Suslin spaces X.

The purpose of this paper is to extend the notion of sequential compactness to infinite dimensions. This is done using *barriers* of Nash-Williams [22], where the notion of "dimension" is captured by the rank of the barrier which can be any countable ordinal. Since the simplest examples of barriers are the families $[\omega]^n$, the notion of "Barrier-sequential compactness" is a natural one.

We start in Section 2 by studying the general theory of barriers and stating many results that are the main tools used in our proofs in later sections.

In Section 3, we study the classes of α-sequentially compact spaces and prove some basic results. For this definition to make sense, we will show that this property depends mostly on the rank of a given barrier and not on the particular recursive structure of it.

In Section 4, we show that sequentially compact spaces of small character and compact bisequential spaces are α-sequentially compact for every $\alpha < \omega_1$. As a corollary, we obtain that many classical classes of compact spaces are α-sequentially compact for every α. We close the section with some comments on angelic spaces, and we point out that a space constructed in [19] and improved in [7] is an angelic space that fails to satisfy the Ramsey property, answering a question of Knaust [15].

In Section 5, we present the constructions of spaces that are β-sequentially compact but not α-sequentially compact for $\beta < \alpha$, under the assumption of $\mathfrak{b} = \mathfrak{c}$. We also analyze further the properties mentioned in Section 3 and their relation to α-sequential compactness.

Finally, in Section 6, we define and discuss a number of cardinal invariants associated to these classes of spaces and give some topological applications.

Our terminology and notation is mostly standard. In particular, $\mathcal{N}(x)$ stands for the collection of open neighborhoods of x and for a finite set $\{b_0, \ldots, b_n\} \in [\omega]^{<\omega}$, we will always assume that it is written in increasing order.

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We will be using a number of classical cardinal invariants of the continuum including $\mathfrak{pat}$, $\mathfrak{pb}$, $\mathfrak{s}$, $\mathfrak{d}$, $\mathfrak{r}$, $\mathfrak{r}_\sigma$. We refer the reader to [2] for the background on these cardinals.

Given an almost disjoint family $\mathcal{A}$ on a countable set N, we define its *Franklin space* as then the one-point compactification of $\Psi(\mathcal{A})$ (i.e., $\mathcal{F}(\mathcal{A}) = \Psi(\mathcal{A}) \cup \{\infty\}$). Our counterexamples are almost exclusively Franklin spaces. For more on almost disjoint families and Ψ-spaces, see [14].

Any other set-theoretic notion and terminology is standard and can be found in [20], and we refer the reader to [12] for topological terminology.

2 Barriers and fronts

We now introduce the basic concepts of barriers and fronts on $[\omega]^{<\omega}$. This section is mainly based on [1] and [26], although some results are new or adapted and will be applied in subsequent sections where we will extend the theory of n-sequentially compact spaces by defining the notion of barrier sequential compactness and ultimately α-sequential compactness for all $\alpha < \omega_1$.

For $\mathcal{F} \subseteq [\omega]^{<\omega}$ and $M \in [\omega]^\omega$, let

$$\mathcal{F}|M = \{s \in \mathcal{F} : s \subseteq M\}.$$

Recall that $s \sqsubseteq t$ means that t is an end extension of s and $s \sqsubset t$ means $s \sqsubseteq t$ and $s \neq t$.

Definition 2.1 A family $\mathcal{F} \subseteq [\omega]^{<\omega}$ is

- *Ramsey*: if for every partition $\mathcal{F} = \mathcal{F}_0 \sqcup \cdots \sqcup \mathcal{F}_n$ and for every $N \in [\omega]^\omega$, there is $M \in [N]^\omega$ such that all but at most one of the restrictions $\mathcal{F}_i|M$ are empty.
- *Nash-Williams:* if $s \sqsubseteq t$ implies $s = t$ for $s, t \in \mathcal{F}$.
- *Sperner:* if $s \subseteq t$ implies $s = t$ for $s, t \in \mathcal{F}$.

Given a Ramsey family $\mathcal{F}$, applying the Ramsey property to the partition $\mathcal{F} = \mathcal{F}_0 \cup (\mathcal{F} \setminus \mathcal{F}_0)$ where $\mathcal{F}_0$ is the set of $\subseteq$-minimal elements in $\mathcal{F}$, we get the following:

Proposition 2.2 *If $\mathcal{F}$ is Ramsey, there exists $M \in [\omega]^\omega$ such that $\mathcal{F}|M$ is Sperner.*

Theorem 2.3 *[22] Every Nash-Williams family is Ramsey.*

The previous two results show that (at least if one is willing to pass to an infinite subset) the three concepts of being Ramsey, Nash-Williams, and Sperner are equivalent:

$$\text{Ramsey} \Rightarrow \text{(some restriction is) Sperner} \Rightarrow \text{Nash-Williams} \Rightarrow \text{Ramsey}.$$

Definition 2.4 A Nash-Williams family $\mathcal{F}$ such that for every $M \in [\omega]^\omega$ there is an initial segment $s \sqsubseteq M$ with $s \in \mathcal{F}$ is called a *front*. If, moreover, $\mathcal{F}$ is Sperner, we say that $\mathcal{F}$ is a *barrier* on ω.

The same argument for Proposition 2.2 can be used to show the following facts:

Fact 2.5 *If $\mathcal{F}$ is a front on a countable set M, there exists $N \in [M]^\omega$ such that $\mathcal{F}|N$ is a barrier.*

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Fact 2.6 *For every barrier $\mathcal{B}$ on M, if we partition $\mathcal{B} = \mathcal{B}_0 \sqcup \cdots \sqcup \mathcal{B}_n$, there are $N \in [M]^\omega$ and $i \leq n$ such that $\mathcal{B}_i|N$ is a barrier.*

Given a barrier $\mathcal{B}$, let

$$T(\mathcal{B}) = \{s \in [\omega]^{<\omega} : \exists t \in \mathcal{B}\ (s \sqsubseteq t)\}$$

and $\rho_{T(\mathcal{B})} : T(\mathcal{B}) \to \omega_1$ be given by

$$\rho_{T(\mathcal{B})}(s) = \sup\{\rho_{T(\mathcal{B})}(t) + 1 : t \in T(\mathcal{B}) \wedge s \sqsubset t\},$$

where $\sup \emptyset = 0$. We will omit the subindex $T(\mathcal{B})$ or replace it by $\mathcal{B}$ when no confusion arises, and we will often think of $T(\mathcal{B})$ as a subtree of $\omega^{<\omega}$.

Definition 2.7 For a barrier $\mathcal{B}$ on a countable set M, its rank is defined by $\rho(\mathcal{B}) := \rho_{T(\mathcal{B})}(\emptyset)$.

It is worth pointing out that in the definition of $T(\mathcal{B})$, we can replace the condition of being an initial segment of an element $t \in \mathcal{B}$ to being a subset of some $t' \in \mathcal{B}$ due to the following fact.

Lemma 2.8 *If $\mathcal{B}$ is a barrier on ω, then for every $s \in [\omega]^{<\omega}$, there exists $b \in \mathcal{B}$ such that $s \subseteq b$ if and only if there exists $b \in \mathcal{B}$ such that $s \sqsubseteq b$. In particular, $T(\mathcal{B}) = \{s \in [\omega]^{<\omega} : \exists b \in \mathcal{B}\ (s \subseteq b)\}$.*

Proof Let $s \in [\omega]^{<\omega}$ and $b \in \mathcal{B}$ such that $s \subseteq b$. Define $M = s \cup (\omega \setminus \max(b) + 1)$. We can find $b' \in \mathcal{B}$ such that $b' \sqsubseteq M$, and hence, $s \sqsubseteq b'$ since otherwise, $b' \sqsubset s \subseteq b$ would contradict that $\mathcal{B}$ is a $\subseteq$-antichain. ∎

Given $a = \{a_0, \ldots, a_n\}$ and $b = \{b_0, \ldots, b_k\}$ with $k < n$, denote by $a * b$ the *end replacement* of a with b, defined as follows:

$$a * b = \{a_0, \ldots, a_{n-k-1}, b_0, \ldots, b_k\}.$$

Lemma 2.9 *Let $\mathcal{B}$ be a barrier and $a \in \mathcal{B}$ enumerated as $a = \{a_0, \ldots, a_{n-1}\}$. Then, for every $b \in [\omega \setminus (a_{n-1})]^{<n}$, there exists $s \in \mathcal{B}$ such that $a * b \sqsubseteq s$. In particular, $b \notin \mathcal{B}$.*

Proof Fix $a = \{a_0, \ldots, a_{n-1}\}$ and $b = \{b_0, \ldots, b_{k-1}\}$ as in the lemma with $k < n$. We will prove the Lemma by induction on k.

If $k = 1$, then $b = \{b_0\}$ for some $b_0 \geq a_{n-1}$. Let $M \in [\omega]^\omega$ with $a * b \sqsubseteq M$. There is an $s \in \mathcal{B}$ such that $s \sqsubseteq M$. Notice that $|s| \geq n$ since otherwise, $s \sqsubset a$ would contradict that $\mathcal{B}$ is Sperner. Thus, $a * b \sqsubseteq s \in \mathcal{B}$.

Now let $k > 1$ and assume the result is true for $k - 1$. Pick any $M \in [\omega]^\omega$ such that $a * b \sqsubseteq M$ and let $s \in \mathcal{B}$ be an initial segment of M. If $|s| < n$, then $s \sqsubseteq \{a_0, \ldots, a_{n-k-1}, b_0, \ldots, b_{k-2}\} \subseteq \{a_0, \ldots, a_{n-k}, b_0, \ldots, b_{k-2}\} = a * b'$, where $b' = \{b_0, \ldots, b_{k-2}\}$. Since $|b'| = k - 1$, by the inductive hypothesis, we get that $s \subsetneq a * b' \sqsubseteq s'$ for some $s' \in \mathcal{B}$, contradicting that $\mathcal{B}$ is Sperner. Therefore, $|s| \geq n$, and hence, $a * b \sqsubseteq s \in \mathcal{B}$. ∎

Corollary 2.10 *If $\mathcal{B}$ is a barrier on ω of rank $k \in \mathbb{N}$, there exists $m \in \omega$ such that $\mathcal{B}|(\omega \setminus m) = [\omega \setminus m]^k$.*

Proof Let $\mathcal{B}$ be a barrier of rank k. If there is $a \in \mathcal{B}$ of size l, then $\rho(\mathcal{B}) \geq l$. Thus, $\mathcal{B} \subseteq [\omega]^{\leq k}$. It is also easy to see that $\mathcal{B} \cap [\omega]^k \neq \emptyset$ since otherwise, $\rho(\mathcal{B}) < k$.

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Take any $a \in \mathcal{B} \cap [\omega]^k$ and let $m = \max(a) + 1$. By Lemma 2.9, if $b \in \mathcal{B}|(\omega\backslash m)$ has size less than k, we can find $s \in \mathcal{B}$ such that $b' = a * b \sqsubseteq s$. As $|b| < k$, we have that $b \subsetneq b' \sqsubseteq s$, which contradicts that $\mathcal{B}$ is Sperner. Hence, $\mathcal{B}|(\omega\backslash m) \subseteq [\omega\backslash m]^k$.

However, for every $b \in [\omega\backslash m]^k$, we can take an infinite set M such that $b \sqsubset M$. Thus, there is an initial segment of M in $\mathcal{B}$, and from the previous inclusion, it must be the case that $b \in \mathcal{B}$. Therefore, $\mathcal{B}|(\omega\backslash m) = [\omega\backslash m]^k$. ∎

Notation 2.11 *Given a family $\mathcal{B} \subseteq [\omega]^{<\omega}$, $s \in [\omega]^{<\omega}$, and $n \in \omega$, set*

- $\mathcal{B}(s) = \{t \in \mathcal{B} : s \sqsubseteq t\}$,
- $\mathcal{B}[s] = \{t\backslash s : t \in \mathcal{B}(s)\}$,
- $s^{\frown_{\mathcal{B}}} = \{s \cup t : t \in \mathcal{B} \ \wedge \ \min(t) > \max(s)\}$,
- $\mathcal{B} + n = \{s + n : s \in \mathcal{B}\}$, *where* $s + n = \{m + n : m \in s\}$.

We write $\mathcal{B}(n)$, $\mathcal{B}[n]$, and $n^{\frown_{\mathcal{B}}}$ instead of $\mathcal{B}(\{n\})$, $\mathcal{B}[\{n\}]$, and $\{n\}^{\frown_{\mathcal{B}}}$, respectively. Notice that some confusion may arise as $n = \{0, \dots, n-1\}$, but this notation will not lead to any confusion, as $\mathcal{B}(n)$ will always be interpreted as $\mathcal{B}(\{n\})$, and the same for $\mathcal{B}[n]$ and $n^{\frown_{\mathcal{B}}}$. We see that if $\mathcal{B}$ is a barrier on M, then $\mathcal{B}[n]$ is a barrier on $M\backslash(n+1)$. Conversely, if $\mathcal{B}_n$ is a barrier on $M\backslash(n+1)$ for every $n \in M$, then $\bigcup_{n\in M} n^{\frown_{\mathcal{B}}}_n$ is a front on M, and there is an infinite set on which its restriction is a barrier. We can now describe a canonical barrier of rank ω: The *Schreier barrier* $\mathcal{S}$ is the barrier of rank ω such that $\{0\} \in \mathcal{S}$ and $\mathcal{S}[n] = [\omega\backslash(n+1)]^n$ for every $n > 0$. In other words, $s \in \mathcal{S}$ if and only if $|s| = \min(s) + 1$.

Definition 2.12 We define the notion of a uniform barrier by induction on the rank. For $\alpha = 1$, we declare the unique barrier $\mathcal{B} = [\omega]^1$ as a uniform barrier. Let $\mathcal{B}$ be a barrier on $M \in [\omega]^\omega$ with $\rho(\mathcal{B}) = \alpha$. If $\alpha > 1$, we say that $\mathcal{B}$ is a *uniform* barrier if each $\mathcal{B}[n]$ is a uniform barrier (on $M\backslash(n+1)$) and

- $\rho_{T(\mathcal{B})}(\{n\}) = \beta$ for every $n \in \omega$ if $\alpha = \beta + 1$ or
- $\{\rho_{T(\mathcal{B})}(\{n\}) : n \in \omega\}$ is an increasing sequence with limit α if α is limit.

The following result together with Corollary 2.10 shows that all barriers of rank ω are somewhere uniform and preserve its rank in such restriction.

Proposition 2.13 *If $\mathcal{B}$ is a barrier of rank ω, there exists $M \in [\omega]^\omega$ such that $\mathcal{B}|M$ is uniform and has rank ω.*

Proof By definition, we can find m_i such that $\rho(\{m_i\}) \geq i$ for every $i \in \omega$. By Corollary 2.10, we can also find k_i for every $i \in \omega$, such that $\mathcal{B}[m_i] = [\omega\backslash k_i]^{n_i}$, where $n_i = \rho(\{m_i\})$. Notice that we can pick $\{m_i : i \in \omega\}$ increasing and such that $m_{i+1} > k_i$. It is now easy to see that $M = \{m_i : i \in \omega\}$ works. ∎

A partial analogue for Proposition 2.13 for any barrier independently of its rank is also true.

Proposition 2.14 *[1] For every barrier $\mathcal{B}$ on ω, there exists an infinite set $M \in [\omega]^\omega$ such that $\mathcal{B}|M$ is uniform.*

By Corollary 2.10, the only uniform barrier of rank $k \in \omega$ is $[\omega]^k$. Thus, Lemma 2.9 states that the sequence of ranks $\{\rho(\{n\}) : n \in \omega\}$ is nondecreasing, and the sequence $\{m_i : i \in \omega\}$ can be taken to be the identity in the previous theorem. This shows that

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the only way to get a nonuniform barrier of rank ω is by embedding a nonuniform barrier $\mathcal{B}[n]$ on top of $\{n\}$; hence, essentially, every barrier of rank ω is uniform. This is as far as we can go since there is a nonuniform barrier $\mathcal{B}$ of rank $\omega+1$ such that $\mathcal{B}[n]$ is uniform for every $n \in \omega$.

Example 2.15 There is a nonuniform barrier of rank $\omega+1$.

We define a barrier $\mathcal{B}$ by describing $\mathcal{B}(n)$ for every $n \in \omega$. For every $k \in \omega$, let $\mathcal{S}_k = \{s \in [\omega]^{<\omega} : |s| = \min(s) + k\}$, (hence, the Schreier barrier is $\mathcal{S}_1$). If n is even, define $\mathcal{B}(n) = n^\frown(\mathcal{S}_n + n + 1)$. For n odd, let $\mathcal{B}(n) = n^\frown[\omega\backslash(n+1)]^n$.

Notice that for n even, $\rho_{T(\mathcal{B})}(\{n\}) = \omega$, while for n odd, $\rho_{T(\mathcal{B})}(\{n\}) = n+1$. So $\mathcal{B}$ is not uniform. Moreover, $\rho(\mathcal{B}|E) = \omega+1$, but $\rho(\mathcal{B}|O) = \omega$, where E is the set of even numbers and O is the set of odd numbers.

It is clear that every infinite subset of ω has an initial segment in $\mathcal{B}$ and that each $\mathcal{B}(n)$ is a $\subseteq$-antichain. It remains to show that if $s = \{s_0, \dots, s_i\} \in \mathcal{B}(n)$, $t = \{t_0, \dots, t_j\} \in \mathcal{B}(m)$, and $n < m$, then $t \not\subseteq s$ (the other contention is impossible as $n \in s\backslash t$).

Let us first compute the size of an element $b \in \mathcal{B}$. Let $b = \{b_0, \dots, b_k\}$. If b_0 is odd, then $b \in \mathcal{B}(b_0)$, and it has size b_0+1. Otherwise, $b \in b_0^\frown(\mathcal{S}_{b_0} + b_0 + 1)$. Let $b' = b\backslash\{b_0\} \in \mathcal{S}_{b_0} + b_0 + 1$. It is clear that $|b'| = |b' - b_0 - 1|$, and since $(b' - b_0 - 1) \in \mathcal{S}_{b_0}$ and $\min(b' - b_0 - 1) = b_1 - b_0 - 1$, we conclude that $|b'| = |b' - b_0 - 1| = (b_1 - b_0 - 1) + b_0 = b_1 - 1$. Therefore, $|b| = b_1$.

We are now ready to show that s and t are $\subseteq$-incomparable. If both n and m are odd, we have that $|t| = m+1 > n+1 = |s|$, and we are done. If both n and m are even, then $|s| = s_1$ and $|t| = t_1$, but $s_0 = n \notin t$ implies that if $t \subseteq s$, hence $|s| = s_1 \leq t_0 < t_1 = |t|$, which is a contradiction.

Similarly, if n is odd and m is even, we assume that $t \subseteq s$ yields that $|t| = t_1 > t_0 \geq s_1 \geq s_0 + 1 = |s|$, a contradiction. Finally, if n is even and m is odd, and if we assume that $t \subseteq s$, we reach a contradiction as $|t| = t_0 + 1 > t_0 \geq s_1 = |s|$.

The persistence of the rank under restriction for a uniform barrier is the key property of the notion of uniformity. We can imitate this behavior by considering a lower bound for the ranks of uniform restrictions of a given barrier $\mathcal{B}$.

Definition 2.16 Given a barrier $\mathcal{B}$, let

$$\operatorname{spec}(\mathcal{B}) = \{\alpha < \omega_1 : \exists M \in [\omega]^\omega \ (\mathcal{B}|M \text{ is uniform with rank } \alpha)\}.$$

We define the *uniform rank of* $\mathcal{B}$ as $\rho_u(\mathcal{B}) = \min(\operatorname{spec}(\mathcal{B}))$.

In [19], it is inductively proved that sequentially compact spaces of character less than $\mathfrak{b}$ are n-sequentially compact for every n. To perform the induction on $[\omega]^{n+1}$, it is enough to notice that any element of $[\omega]^{n+1}$ has a unique initial segment in $[\omega]^n$. It is also true that any element in $[\omega]^n$ is end-extended by at least one element in $[\omega]^{n+1}$. This fact is used to analyze a splitting-like cardinal invariant in [7] that helps with the construction of n-sequentially compact spaces that fail to be $(n+1)$-sequentially compact under some assumptions involving this cardinal. Other examples of results that use this fact are Ramsey's theorem itself and the proof that $\mathfrak{par}_n = \mathfrak{par}_2 = \max\{\mathfrak{b}, \mathfrak{s}\}$ in [2]. In order to obtain analogous results in our framework, we will need a generalization of this fact for barriers of any rank. The statement of

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Lemma 2.18 below and its proof appear in [1] in a slightly different way. We add here a proof for completeness.

Notation 2.17 *Given two families $\mathcal{B}, \mathcal{C} \subseteq [\omega]^{<\omega}$, we denote by $\mathcal{B} \sqsubseteq \mathcal{C}$ if*

- $\forall s \in \mathcal{B}\ \exists t \in \mathcal{C}\ (s \sqsubseteq t)$,
- $\forall t \in \mathcal{C}\ \exists s \in \mathcal{B}\ (s \sqsubseteq t)$.

Lemma 2.18 *Given two barriers $\mathcal{B}$ and $\mathcal{C}$ on a countable set $N \in [\omega]^\omega$, there is an infinite set $M \in [N]^\omega$ such that either $\mathcal{B}|M \sqsubseteq \mathcal{C}|M$ or $\mathcal{C}|M \sqsubseteq \mathcal{B}|M$. Moreover, if $\rho(\mathcal{B}) < \rho(\mathcal{C})$ and $\mathcal{C}$ is uniform, then $\mathcal{B}|M \sqsubseteq \mathcal{C}|M$ necessarily holds.*

Proof Define $\mathcal{B}_0 = \{b \in \mathcal{B} : \exists c \in \mathcal{C}(b \subseteq c)\}$. Since $\mathcal{B} = \mathcal{B}_0 \cup (\mathcal{B} \setminus \mathcal{B}_0)$, we can find, by Nash-Williams theorem, an infinite set $M \in [N]^\omega$ such that either $(\mathcal{B}\setminus\mathcal{B}_0)|M = \emptyset$ or $\mathcal{B}_0|M = \emptyset$. If $(\mathcal{B}\setminus\mathcal{B}_0)|M = \emptyset$, then we are done, as this implies that $\mathcal{B}|M \sqsubseteq \mathcal{C}|M$. To see this, notice that for every $b \in \mathcal{B}|M$ (and hence $b \in \mathcal{B}_0$), there is $c \in \mathcal{C}$ such that $b \subseteq c$, but if we define $X = b \cup (M \setminus \max(b))$, there is also $c' \in \mathcal{C}$ such that $c' \sqsubseteq X$. As $b \subseteq c \in \mathcal{C}$, it happens that $b \sqsubseteq c'$ and $c' \in \mathcal{C}|M$. For $c \in \mathcal{C}|M$, we can also find $b \in \mathcal{B}|M$ such that $b \sqsubseteq c \cup (M\setminus(\max(c)+1))$. The case where $c \sqsubset b$ is not possible since $b \in \mathcal{B}_0$, and thus, there is another $c' \in \mathcal{C}|M$ such that $b \subseteq c'$.

Assume otherwise $\mathcal{B}_0|M = \emptyset$ and define $\mathcal{C}_0 = \{c \in \mathcal{C}|M : \exists b \in \mathcal{B}|M(c \subseteq b)\}$. Again, by Nash-Williams theorem, we can find $M' \in [M]^\omega$ such that either $\mathcal{C}_0|M' = \emptyset$ or $(\mathcal{C}\setminus\mathcal{C}_0)|M' = \emptyset$. The case $\mathcal{C}_0|M' = \emptyset$ is impossible since we can find $b \in \mathcal{B}|M' \subseteq \mathcal{B}|M$ and $c \in \mathcal{C}|M'$ such that $b, c \sqsubseteq M'$ and then either $b \subseteq c$ or $c \subseteq b$ contradicting the choices of M and M'. Thus, $(\mathcal{C}\setminus\mathcal{C}_0)|M' = \emptyset$, and as in the previous case, we get that $\mathcal{C}|M' \sqsubseteq \mathcal{B}|M'$.

For the last assertion of the statement, notice that if $\mathcal{C}|M \sqsubseteq \mathcal{B}|M$, then $T(\mathcal{C}|M) \subseteq T(\mathcal{B}|M)$. In particular,

$$\rho(\mathcal{C}) = \rho(\mathcal{C}|M) = \rho_{T(\mathcal{C}|M)}(\emptyset) \leq \rho_{T(\mathcal{B}|M)}(\emptyset) = \rho(\mathcal{B}|M) \leq \rho(\mathcal{B}),$$

which finishes the proof. ∎

The previous lemma tells us that we can define a preorder on the family of all barriers on ω by defining $\mathcal{B} \leq \mathcal{C}$ if there is an infinite set M such that $\mathcal{B}|M \sqsubseteq \mathcal{C}|M$. One may be tempted to say that the partial order defined by identifying $\mathcal{B}$ and $\mathcal{C}$ if $\mathcal{B} \leq \mathcal{C}$ and $\mathcal{C} \leq \mathcal{B}$ collapses to ω_1 when restricted to the family of uniform barriers; that is, $\mathcal{B} \leq \mathcal{C}$ if and only if $\rho(\mathcal{B}) \leq \rho(\mathcal{C})$. However, the unpleasant fact about this is that for two barriers $\mathcal{B}$ and $\mathcal{C}$ of the same rank, it is not always true that $\mathcal{B} \leq \mathcal{C}$ and $\mathcal{C} \leq \mathcal{B}$. For example, if $\mathcal{B}$ is the Schreier barrier and $\mathcal{C}$ is any uniform barrier defined such that $\rho_{T(\mathcal{C})}(\{n\}) = f(n)$ for an increasing function $f \in \omega^\omega$ that is strictly bigger that the identity, then $\mathcal{C} \not\leq \mathcal{B}$. Notice that any uniform barrier $\mathcal{B}$ of rank ω is completely determined by a function $f \in \omega^\omega$ that encodes the ranks of the first level on $T(\mathcal{B})$. A more complex but naturally defined coding of the ranks of the successors of any element $s \in T(\mathcal{B})$ also determines completely the structure of any barrier $\mathcal{B}$.

We will see now that this function is the only obstruction and that, if one is willing to compress some finite intervals of ω into points, we can define a weaker relation between barriers that depends only on their ranks and extends $\leq$. This will allow us to prove, in some cases, that a property of barriers holds for all barriers of a given rank α

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(in particular, a space being "$\mathcal{B}$-sequentially compact") if and only if it holds for some single uniform barrier of the same rank (e.g., Corollary 3.5).

Definition 2.19 Given two barriers $\mathcal{B}$ and $\mathcal{C}$ on a countable set M, we write $\mathcal{C} \preceq \mathcal{B}$ if there is a finite-to-one, nondecreasing function $f \in \omega^\omega$ such that for every infinite subset $M' \in [M]^\omega$, there exists $N \in [M']^\omega$ so that $f \restriction N$ is one-to-one and

- $\forall b \in (\mathcal{B}|N)\ \exists c \in \mathcal{C}\ (c \sqsubseteq f[b])$.

Notice that if $\mathcal{B}$ is uniform and $\rho(\mathcal{C}) < \rho(\mathcal{B})$, then $\mathcal{C} \preceq \mathcal{B}$ with f being the identity map, since $\mathcal{C}|M \leq \mathcal{B}|M$ for every restriction to an infinite set M. The condition that $f \restriction N$ is one-to-one is superfluous, as we can always shrink N in order to get this property, but it will be convenient to add this to the definition to avoid saying it explicitly each time we use the preorder $\preceq$.

Having said this, it is clear that the relation $\preceq$ is transitive; that is, if $\mathcal{C} \preceq \mathcal{B}$ and $\mathcal{D} \preceq \mathcal{C}$ is witnessed by f and g, respectively, then $\mathcal{D} \preceq \mathcal{B}$, and it is witnessed by $g \circ f$.

Proposition 2.20 *If $\mathcal{B}$ and $\mathcal{C}$ are two barriers in ω, $\mathcal{B}$ is uniform, and $\rho(\mathcal{B}) \geq \rho(\mathcal{C})$, then $\mathcal{C} \preceq \mathcal{B}$.*

Proof Let us alternately define two sequences $\{a_i : i \in \omega\}$ and $\{b_i : i \in \omega\}$ such that $a_i < b_i < a_{i+1}$ for every $i \in \omega$. We will also denote by $\{k_i : i \in \omega\}$ the increasing sequence defined by these two sequences; that is, $k_{2i} = a_i$ and $k_{2i+1} = b_i$. We will also simultaneously define $f \in \omega^\omega$ which is completely determined by the sequence $\{k_i : i \in \omega\}$ as follows: $f \restriction [0, k_0) = 0$ and $f(n) = i$ if and only if $n \in [k_i, k_{i+1})$. We shall show that f satisfies the definition of $\mathcal{C} \preceq \mathcal{B}$. To save notation, let us write $\rho_{\mathcal{B}}$ and $\rho_{\mathcal{C}}$ instead of $\rho_{T(\mathcal{B})}$ and $\rho_{T(\mathcal{C})}$, respectively. Also, given $s \in [\omega]^\omega \setminus T(\mathcal{C})$, we will define $\rho_{\mathcal{C}}(s) = -1$ so that we can talk about the rank of s even if s is not in $T(\mathcal{C})$.

Step 0:

Since $\rho_{\mathcal{B}}(\emptyset) \geq \rho_{\mathcal{C}}(\emptyset)$, there are $a_0 < b_0 < \omega$ such that $\rho_{\mathcal{B}}(\{a_0\}) \geq \rho_{\mathcal{C}}(\{0\})$ and $\rho_{\mathcal{B}}(\{b_0\}) \geq \rho_{\mathcal{C}}(\{1\})$.

Now assume we have defined $\{a_i : i \leq n\}$ and $\{b_i : i \leq n\}$.

Step a:

Inductive hypothesis at n:

Let us call

$$S_a(n) = \Big\{ s \in T(\mathcal{B}) : s \subseteq \bigcup_{i<n} [a_i, b_i) \ \wedge\ \forall i < n (|s \cap [a_i, b_i)| \leq 1) \Big\}.$$

We will ensure that the following inductive hypothesis will hold throughout the construction: Given $s \in S_a(n)$,

$\phi_a(n, s) \equiv \rho_{\mathcal{B}}(s) \geq \rho_{\mathcal{C}}(f[s])$
$\psi_a(n, s) \equiv$ if $j \geq a_n$, then $\rho_{\mathcal{B}}(s \cup \{j\}) \geq \rho_{\mathcal{C}}(f[s \cup \{2n\}])$.

We will also write $\phi_a(n)$ as a shorthand for $\forall s \in S_a(n) \phi_a(n, s)$ and $\psi_a(n)$ as a shorthand for $\forall s \in S_a(n) \psi_a(n, s)$. It is worth pointing out that $S_a(n)$, $\phi_a(n, s)$ and $\psi_a(n, s)$ do not mention b_n. Note that if $j \geq a_0$, we have that $\rho_{\mathcal{B}}(\{j\}) \geq \rho_{\mathcal{B}}(\{a_0\}) \geq \rho_{\mathcal{C}}(\{0\})$ as $\mathcal{B}$ is uniform. Thus, $\psi_a(0)$ is satisfied, and $\phi_a(0)$ is vacuously satisfied too (here, $S_a(0) = \{\emptyset\}$).

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Construction of a_{n+1}:

We need to take care of all $s \in S_a(n+1)$ (note that this set is well-defined since it only depends on $\{a_i, b_i : i \leq n\}$).

First, consider $s \in S_a(n)$. By $\phi_a(n,s)$, we have that $\rho_{\mathcal{B}}(s) \geq \rho_{\mathcal{C}}(f[s])$, and thus, we can find m_s^a such that

$(\star_1^a)$ $\rho_{\mathcal{B}}(s \cup \{m\}) \geq \rho_{\mathcal{C}}(f[s] \cup \{2n+2\})$ for every $m \geq m_s^a$.

Otherwise, $s \in S_a(n+1) \backslash S_a(n)$, and we can write $s' = s \backslash \{j\}$, where $j = \max(s) \geq a_n$ and $s' \in S_a(n)$. Then, we know that $\rho_{\mathcal{B}}(s' \cup \{j\}) \geq \rho_{\mathcal{C}}(f[s'] \cup \{2n\})$ by $\psi_a(n,s)$. Since $j \in [a_n, b_n)$, we have that $f(j) = 2n$. Thus,

$(\star^a)$ $\rho_{\mathcal{B}}(s) = \rho_{\mathcal{B}}(s' \cup \{j\}) \geq \rho_{\mathcal{C}}(f[s'] \cup \{f(j)\}) = \rho_{\mathcal{C}}(f[s' \cup \{j\}]) = \rho_{\mathcal{C}}(f[s])$.

Then, there is $m_s^a \in \omega$ such that $\rho_{\mathcal{B}}(s \cup \{m_s^a\}) \geq \rho_{\mathcal{C}}(f[s] \cup \{2n+2\})$. Since $\mathcal{B}$ is uniform,

$(\star_2^a)$ $\rho_{\mathcal{B}}(s \cup \{m\}) \geq \rho_{\mathcal{C}}(f[s] \cup \{2n+2\})$ for every $m \geq m_s^a$.

Finally, define

$$a_{n+1} = \max\{m_s^a \mid s \in S_a(n+1)\}.$$

Inductive hypothesis for $n+1$:

We want to see that $\phi_a(n+1)$ and $\psi_a(n+1)$ hold. Let $s \in S_a(n+1)$. If $s \in S_a(n)$, then $\phi_a(n+1,s)$ follows from $\phi_a(n,s)$. In consequence, we can assume that $s \in S_a(n+1) \backslash S_a(n)$ and let $s' = s \backslash \{j\}$, where $j = \max(s)$. Hence, by $(\star^a)$, we have that $\rho_{\mathcal{B}}(s) \geq \rho_{\mathcal{C}}(f[s])$, which means that $\phi_n(n+1,s)$ holds and, consequently, $\phi_a(n+1)$ does.

Now let us see that $\psi_a(n+1)$ is also true. Let $s \in S_a(n+1)$ and $j \geq a_{n+1}$. If $s \in S_a(n)$, then m_a^s satisfies $(\star_1^a)$, and if $s \in S_a(n+1) \backslash S_a(n)$, we have that m_a^s satisfies $(\star_2^a)$. In either case, as $a_n + 1 \geq m_a^s$, we have that $\psi_a(n+1,s)$ holds, and so does $\psi_a(n+1)$.

Step b:

Inductive hypothesis at n:

The construction is dual to that of part a. Let us start by calling

$$S_b(n) = \left\{ s \in T(\mathcal{B}) : s \subseteq \bigcup_{i<n} [b_i, a_{i+1}) \ \wedge \ \forall i < n (|s \cap [b_i, a_{i+1})| \leq 1) \right\}.$$

The corresponding inductive formulas for $s \in S_b(n)$ are now

$\phi_b(n,s) \equiv \rho_{\mathcal{B}}(s) \geq \rho_{\mathcal{C}}(f[s])$
$\psi_b(n,s) \equiv$ if $j \geq b_n$, then $\rho_{\mathcal{B}}(s \cup \{j\}) \geq \rho_{\mathcal{C}}(f[s \cup \{2n+1\}])$.

We will write again $\phi_b(n)$ and $\psi_b(n)$ as shorthands for $\forall s \in S_b(n) \phi_a(n,s)$ and $\forall s \in S_b(n) \psi_a(n,s)$, respectively. In the particular case of ϕ_b, we have that it is the same formula as ϕ_a with different domain. Since a_{n+1} has already been defined, the interval $[b_n, a_{n+1})$ considered in $S_b(n+1)$ makes complete sense.

Construction of b_{n+1}:

Given $s \in S_b(n)$, it follows from $\phi_b(n,s)$ that $\rho_{\mathcal{B}}(s) \geq \rho_{\mathcal{C}}(f[s])$, and thus, we can find m_s^b such that

$(\star_1^b)$ $\rho_{\mathcal{B}}(s \cup \{m\}) \geq \rho_{\mathcal{C}}(f[s] \cup \{2n+3\})$ for every $m \geq m_s^b$.

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Otherwise, $s \in S_b(n+1) \setminus S_b(n)$, and we can write $s' = s \setminus \{j\}$, where $j = \max(s) \geq b_n$ and $s' \in S_b(n)$. Thus, $\rho_{\mathcal{B}}(s' \cup \{j\}) \geq \rho_{\mathcal{C}}(f[s'] \cup \{2n+1\})$ by $\psi_b(n,s)$. Since $j \in [b_n, a_{n+1})$, we have that $f(j) = 2n+1$. Then,

$(*^b)$ $\rho_{\mathcal{B}}(s) = \rho_{\mathcal{B}}(s' \cup \{j\}) \geq \rho_{\mathcal{C}}(f[s'] \cup \{f(j)\}) = \rho_{\mathcal{C}}(f[s' \cup \{j\}]) = \rho_{\mathcal{C}}(f[s])$.

We can again find $m_s^b \in \omega$ such that $\rho_{\mathcal{B}}(s \cup \{m_s^b\}) \geq \rho_{\mathcal{C}}(f[s] \cup \{2n+3\})$, and moreover,

$(*_2^b)$ $\rho_{\mathcal{B}}(s \cup \{m\}) \geq \rho_{\mathcal{C}}(f[s] \cup \{2n+3\})$ for every $m \geq m_s^b$.

We then define

$$b_{n+1} = \max\{m_s^b \mid s \in S_b(n+1)\}.$$

Inductive hypothesis for $n+1$:

If $s \in S_b(n)$, then $\phi_b(n+1,s)$ follows from $\phi_b(n,s)$, and otherwise, $s \in S_b(n+1) \setminus S_b(n)$ can be written as $s' = s \setminus \{j\}$, where $j = \max(s)$. By $(*^b)$, we have that $\rho_{\mathcal{B}}(s) \geq \rho_{\mathcal{C}}(f[s])$, and then $\phi_b(n+1)$ holds.

To see that $\psi_b(n+1)$ is true, let $s \in S_b(n+1)$ and $j \geq b_{n+1}$. Then, $b_{n+1} \geq m_s^b$, and either by $(*_1^b)$ or $(*_2^b)$, we have that $\phi_b(n+1,s)$ holds.

This finishes the construction, and it remains to prove that f works.

Clearly, f is finite-to-one and nondecreasing. Given $M \in [\omega]^\omega$, we can find an infinite subset $N \in [M]^\omega$ such that $|N \cap [k_i, k_{i+1}]| \leq 1$ for every $i \in \omega$, and either $N \cap \bigcup_{i\in\omega}[k_{2i}, k_{2i+1}) = \emptyset$ or $N \cap \bigcup_{i\in\omega}[k_{2i+1}, k_{2i+2}) = \emptyset$. Without loss of generality, assume that $N \subseteq \bigcup_{i\in\omega}[k_{2i}, k_{2i+1}) = \bigcup_{i\in\omega}[a_i, b_i)$.

Let now $b \in \mathcal{B}|N$. It follows from the choice of N that there is an $n \in \omega$ such that $b \in S_a(n)$. By $\phi_a(n,b)$, we know that $0 = \rho_{\mathcal{B}}(b) \geq \rho_{\mathcal{C}}(f[b])$. Necessarily, $\rho_{\mathcal{C}}(f[b]) \in \{0,-1\}$, but in either case, we can conclude that there is $c \in \mathcal{C}$ such that $c \sqsubseteq f[b]$. This finishes the proof. ∎

The previous lemma is new, and it is the last piece of the theory of barriers that we need in order to prove that the general notion of barrier sequential compactness extending n-sequentially compact spaces depends only on the rank of the associated uniform barrier. In the case that $\mathcal{B}$ has rank ω, we can drop the requirement that the barrier is uniform.

Corollary 2.21 *Let $\mathcal{B}$ and $\mathcal{C}$ be barriers on ω such that $\rho(\mathcal{C}) \leq \rho(\mathcal{B}) = \omega$. Then, $\mathcal{C} \preceq \mathcal{B}$.*

Proof By the fact that the preorder $\preceq$ is transitive and the previous proposition, it suffices to show that $\mathcal{S} \preceq \mathcal{B}$, where $\mathcal{S}$ is the Schreier barrier.

Since $\mathcal{B}$ has rank ω, it follows from Lemma 2.9 that for every $i \in \omega$, we can find m_i such that $\rho_{\mathcal{B}}(m) \geq i+1$ for every $m \geq m_i$. Let $f \in \omega^\omega$ be given by $f(n) = i$ if and only if $n \in [m_i, m_{i+1})$ (and we define $f \restriction m_0$ arbitrarily). We claim that f witnesses that $\mathcal{S} \preceq \mathcal{B}$.

Indeed, given $M \in [\omega]^\omega$, we can find $N_0 \in [M]^\omega$ such that $N_0 \cap [0, m_0) = \emptyset$ and $|N_0 \cap [m_i, m_{i+1})| \leq 1$ for every $i \in \omega$. Let $n_0 = \min(N_0)$, and for every $i \in \omega$, define n_{i+1} as follows: for each $j < i+1$, we can apply Corollary 2.10 and find k_j such that $\mathcal{B}[n_j] = [\omega \setminus k_j]^r$ for some $r \geq l$, where l is the unique natural number such that $n_j \in [m_l, m_{l+1})$. Pick $n_{i+1} > k_j$ for every $j < i+1$. Thus, $N = \{n_i : i \in \omega\} \in [M]^\omega$.

Fix $b \in (\mathcal{B}|N)$ and let $n_i = \min(b)$. It follows from the definitions that $b \setminus \{n_i\} \in [\omega \setminus k_j]^r$ for some $r \geq l$, where $n_i \in [m_l, m_{l+1})$. Then, $|f[b]| \geq l+1$ as f is one to one

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on N, but $f(n_i) = l$ and $\mathcal{S}(l) = \{s \in [\omega]^{<\omega} : \min(s) = l \wedge |s| = l+1\}$. Thus, we can conclude that $c \sqsubseteq f[b]$ defined as the first $l+1$ elements of $f[b]$ is an element of $\mathcal{S}$. ■

3 Infinite dimensional sequential compactness

We now define the notion of $\mathcal{B}$-sequential compactness for barriers $\mathcal{B}$ and prove necessary results to formulate the natural notion of α-sequential compactness, where $\alpha \in \omega_1$ corresponds to the rank of the barrier. All the results of this section are natural generalizations of results presented in [19] for finite n.

Definition 3.1 Let $\mathcal{B}$ be a barrier on $M \in [\omega]^\omega$, let X be a topological space, $f : \mathcal{B} \to X$, and $x \in X$. We say that f *converges* to x if for all $U \in \mathcal{N}(x)$, there is $n \in \omega$ such that $f[(\mathcal{B}|(M\backslash n))] \subseteq U$.

The word "converges" might give rise to misunderstanding, as f has countable domain; however, the meaning of this word will be understood by the context. The function f should be thought as a $\mathcal{B}$-dimensional sequence or α-dimensional sequence for $\alpha = \rho(\mathcal{B})$ rather than a classical 1-dimensional sequence indexed by the countable set $\mathcal{B}$.

Definition 3.2 Let $\mathcal{B}$ be a barrier on ω and let X be a topological space. We say that X is $\mathcal{B}$-*sequentially compact* if for all $f : \mathcal{B} \to X$, there is $M \in [\omega]^\omega$ such that $f \restriction (\mathcal{B}|M)$ converges.

The natural stratification of barriers given by their ranks will give us a natural classification of $\mathcal{B}$-sequentially compact spaces grouping them and associating them to a countable ordinal attending the following definition:

Definition 3.3 Let $\alpha < \omega_1$. We say that X is α-*sequentially compact* if X is $\mathcal{B}$-sequentially compact for all barriers $\mathcal{B}$ of rank α.

The first application of the theory developed in Section 2 and the main reason for the introduction of the order $\preceq$ is due to the next theorem:

Theorem 3.4 *If $\mathcal{C} \preceq \mathcal{B}$ and X is $\mathcal{B}$-sequentially compact, then X is also $\mathcal{C}$-sequentially compact.*

Proof Assume X is $\mathcal{B}$ sequentially compact and let $f \in \omega^\omega$ as in the definition of $\mathcal{C} \preceq \mathcal{B}$. Let $h : \mathcal{C} \to X$ and define $\hat{h} : \mathcal{B} \to X$ by $\hat{h}(b) = h(c)$ for any $c \in \mathcal{C}$ that is $\sqsubseteq$-compatible with $f[b]$. Notice that since $\mathcal{C}$ is a barrier, there is at least one of such c, and then $\hat{h}$ is well-defined.

Since X is $\mathcal{B}$-sequentially compact, there is $M \in [\omega]^\omega$ such that $\hat{h} \restriction (\mathcal{B}|M)$ converges to some $x \in X$. Take $N \in [M]^\omega$ such that

$$\forall b \in (\mathcal{B}|N)\ \exists c \in \mathcal{C}\ (c \sqsubseteq f[b])$$

given by the definition of $\mathcal{C} \preceq \mathcal{B}$, and define $M_0 = f[N]$. It turns out that $\hat{h} \restriction (\mathcal{B}|N)$ also converges to x. We shall prove that $h \restriction (\mathcal{C}|M_0)$ converges to the same $x \in X$.

To see this, fix an open neighborhood $U \in \mathcal{N}(x)$. We can find $n \in \omega$ such that $\hat{h}(b) \in U$ whenever $b \in \mathcal{B}|(N\backslash n)$. Take $c \in \mathcal{C}|(M_0\backslash f(n))$. Thus, $c = \{f(m_0), \ldots, f(m_j)\}$ for some set $a = \{m_i : i \leq j\} \subseteq N\backslash n$. Take any infinite

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set $N' \subseteq N$ such that $a \sqsubseteq N'$. We can find $b \in \mathcal{B}|N$ such that $b \sqsubseteq N'$. It follows that $a \sqsubseteq b$ since otherwise, $f[b] \sqsubset f[a] = c$ would contradict that $b \subseteq N$. Therefore, $h(c) = \hat{h}(b) \in U$, as desired. ∎

Corollary 3.5 *Let X be a topological space and $\alpha \in \omega_1$; the following are equivalent:*

(1) *X is α-sequentially compact,*
(2) *X is $\mathcal{B}$-sequentially compact for every uniform barrier of rank α,*
(3) *X is $\mathcal{B}$-sequentially compact for some uniform barrier $\mathcal{B}$ of rank α.*

Proof That (3) implies (1) follows directly from Proposition 2.20 and Theorem 3.4. The other implications are clear from the definitions. ∎

From Lemma 2.20 and Theorem 3.4, we can easily get the following result.

Corollary 3.6 *If X is α-sequentially compact and $\beta < \alpha$, then X is β-sequentially compact. In particular, if α is infinite, X is n-sequentially compact for every $n \in \omega$.*

The stronger result for barriers of rank ω given in Corollary 2.21 yields the following result:

Corollary 3.7 *If X is $\mathcal{B}$-sequentially compact for some barrier of rank ω (not necessarily uniform), then X is ω-sequentially compact.*

In general, this is not true for every countable ordinal. A trivial example is a barrier with a single node in $T(\mathcal{B})$ of rank ω (let us say $\{0\}$) and such that $\mathcal{B}|(\omega\backslash 1)$ is isomorphic to $\mathcal{S}_2$. In this case, every infinite restriction has rank ω, but the barrier itself has rank $\omega + 1$. It is not hard to modify our constructions in Section 5 to show that, consistently, there is a $\mathcal{B}$ sequentially compact space that is not $(\omega + 1)$-sequentially compact. We do not know if the requirement that $\mathcal{B}$ has an infinite restriction of rank α suffices in order to show that $\mathcal{B}$-sequentially compact spaces are α-sequentially compact. We believe that the answer is "no," but we conjecture that a weaker result is true.

Conjecture 3.8 *If X is $\mathcal{B}$-sequentially compact, then it is $\rho_u(\mathcal{B})$-sequentially compact.*

The following easy fact is a partial answer to Conjecture 3.8.

Proposition 3.9 *Let $\mathcal{B}$ be a barrier on ω and X a $\mathcal{B}$-sequentially compact space. If there is $M \in [\omega]^\omega$ such that $\mathcal{B}|M$ is uniform and either X is not $(\mathcal{B}|(\omega\backslash M))$–sequentially compact or $\omega\backslash M$ is finite, then X is $\rho(\mathcal{B}|M)$-sequentially compact. In particular, X is $\rho_u(\mathcal{B})$-sequentially compact.*

Proof By Corollary 3.5, it is enough to prove that X is $(\mathcal{B}|M)$-sequentially compact. Let $F := \omega\backslash M$. Regardless of whether F is finite or X is not $(\mathcal{B}|F)$-sequentially compact, there exists a function $g : \mathcal{B}|F \to X$ without infinite convergence subsequences (here, convergence means convergence with respect to barriers). Now let $f : \mathcal{B}|M \to X$. We want to prove that f admits an infinite convergent subsequence $f \restriction (\mathcal{B}|M')$ for some $M' \in [\omega]^\omega$. For this, let $\hat{f} : \mathcal{B} \to X$ be any common extension of both f and g. As X is $\mathcal{B}$-sequentially compact, there is $N \in [\omega]^\omega$ such that $\hat{f} \restriction N$ is convergent. Now $N \cap F$ should be finite, since otherwise, $\hat{f} \restriction (N \cap F)$ would also be an infinite convergent subsequence for g. This way, $N' := N\backslash F$ is infinite, and thus, $\hat{f} \restriction N' = f \restriction N'$ is an infinite convergent subsequence for f. ∎

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4 Some classes of ω_1-sequentially compact spaces

In [19], it is proved that compact metric spaces are n-sequentially compact for every $n \in \omega$. This result extends to any $\alpha < \omega_1$. We will start this section by giving two classes of spaces, each containing the class of compact metric spaces, that sit at the top of the hierarchy of α-sequentially compact spaces.

Definition 4.1 We say that X is *ω_1-sequentially compact* if it is α-sequentially compact for every $\alpha < \omega_1$.

The first class of spaces that we want to show to be ω_1-sequentially compact is the class of sequentially compact spaces of character less than $\mathfrak{b}$. The fact that sequentially compact spaces of character less than $\mathfrak{b}$ are n-sequentially compact for every $n \in \omega$ was proved in [19].

Theorem 4.2 *Suppose X is a sequentially compact space. If $\chi(X) < \mathfrak{b}$ then X is ω_1-sequentially compact.*

Proof We will show by induction that X is α-sequentially compact for every $\alpha < \omega_1$. If $\alpha = 1$, then X being sequentially compact is equivalent to X being 1-sequentially compact, as the only uniform barrier of rank 1 is $[\omega]^1$.

We proceed to prove the inductive step at β, regardless of it being a successor ordinal, and hence of the form $\beta = \alpha + 1$, or a limit ordinal, which would then be the $\sup \alpha_n$, where α_n form an increasing sequence. Now let $f : \mathcal{B} \to X$ be any function where $\mathcal{B}$ is uniform of rank β. For all $n \in \omega$, $\mathcal{B}[n]$ is either an α-uniform barrier on $\omega \backslash (n+1)$ or an α_n-uniform barrier if β were a limit ordinal.

For all $n \in \omega$, let $f_n : \mathcal{B}[n] \to X$ be the function mapping $s \mapsto f(\{n\} \cup s)$. By our inductive hypothesis, there is an infinite subset N_0 and a point $x_0 \in X$ such that $f_0 \restriction (\mathcal{B}[0]|N_0) \to x_0$. Let $n_0 = 0$ and $n_1 = \min N_0 \backslash \{0\}$.

Note that for any infinite subset A of ω and any $n \in \omega$, $\mathcal{B}[n]|A$ is also uniform of rank α if $\beta = \alpha + 1$ or of rank α_n if β were a limit ordinal. Now suppose that we have defined a decreasing sequence of infinite subsets $N_0 \supseteq N_1 \supseteq \cdots \supseteq N_{i-1}$ and an increasing sequence $n_0 < n_1 < \cdots < n_i$ in addition to a subset $\{x_0, \ldots, x_{i-1}\}$ of X. Consider the function $f_{n_i} : (\mathcal{B}[n_i]|N_{i-1}) \to X$. Applying our inductive hypothesis yields an infinite subset $N_i \subseteq N_{i-1}$ such that $f_{n_i} \restriction (\mathcal{B}[n_i]|N_i) \to x_i$, for some point $x_i \in X$. We also define $n_{i+1} = \min(N_i \backslash \{n_0, \ldots, n_i\})$. Thereby, we obtain a decreasing sequence of infinite subsets $N_0 \supseteq N_1 \supseteq N_2 \supseteq \ldots$, a subset $\{x_i : i \in \omega\}$ of the space X, and a strictly increasing sequence $\langle n_i : i \in \omega \rangle$ such that $f_{n_i} \restriction (\mathcal{B}[n_i]|N_i) \to x_i$, where $n_{i+1} = \min N_i \backslash \{n_0, \ldots, n_i\}$ for all $i \in \omega$.

Since $\{x_i : i \in \omega\} \subseteq X$ and X is sequentially compact, there is a convergent subsequence, $Y = \{x_{i_j} : j \in \omega\}$, with limit point x. Re-numerating the indices, we can view $Y = \{x_i : i \in \omega\}$ as the convergent subsequence. Also, let $N = \{n_i : i \in \omega\}$.

Take any open set U in X containing the point x. As $x_i \to x$, there is an integer $m_U \in \omega$ such that for all $i > m_U$, we have that $x_i \in U$. Also, for all such i, there is another integer $\phi_U(n_i)$ such that

$$(*) \qquad f_{n_i}[\mathcal{B}[n_i]|(N_i \backslash \phi_U(n_i))] \subseteq U.$$

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For any $j \leq m_U$ and any $j \notin \{n_i : i \in \omega\}$, let $\phi_U(j) = 0$. Thus, any open subset U of X containing x induces a function $\phi_U : \omega \to \omega$.

Let $\eta(x)$ be a local neighborhood base of x of minimum size. Since $\chi(X) < \mathfrak{b}$, the set of functions $\{\phi_U : U \in \eta(x)\}$ is bounded by some increasing function $\psi : \omega \to \omega$; that is, $\phi_U <^* \psi$ for every $U \in \eta(x)$.

We can find an increasing subsequence $\langle m_i : i \in \omega\rangle$ of $\langle n_i : i \in \omega\rangle$ such that $m_{i+1} > \psi(m_i)$ for every $i \in \omega$. We claim that $f \restriction (\mathcal{B}|M) \to x$, where $M = \{m_i : i \in \omega\}$.

Fix an open set $U \in \eta(x)$ and let $k \in \omega$ such that $x_j \in U$ and $\psi(j) > \phi_U(j)$ for every $j \geq k$. Let $s \in \mathcal{B}|(M\backslash\psi(k))$ and let us write $s = \{m_{k_0}, \ldots, m_{k_n}\}$ and $s' = s\backslash\{m_{k_0}\}$. Note that for every $i > 0$, we have that

$$m_{k_i} > \psi(m_{k_0}) > \phi(m_{k_0}),$$

as $m_{k_0} \geq \psi(k) \geq k$. In particular, $s' \cap \phi(m_{k_0}) = \emptyset$.

Now let $j \in \omega$ such that $m_{k_0} = n_j$. Note that if $j' > j$, then $n_{j'} \in N_j$. As any $m_{k_i} > m_{k_0}$ for $i > 0$ and this implies that $m_{k_i} = n_{j'}$ for some $j' > j$, we can conclude that $s' \subseteq N_j$.

Combining the previous arguments, we get that $s' \subseteq N_j\backslash\phi_U(n_j)$, and we already know that $s = \{n_j\} \cup s' \in \mathcal{B}$, which implies that $s' \in \mathcal{B}[n_j]$. Therefore, $f(s) = f_{n_j}(s') \in U$ by $(*)$, as desired. ∎

The second class of spaces that we shall show are ω_1-sequentially compact is the class of compact bisequential spaces. Recall that $\mathcal{U} \subseteq \mathcal{P}(X)$ clusters at x (often written as $x \in \overline{\mathcal{U}}$) if $x \in \overline{U}$ for every $U \in \mathcal{U}$. Of course, if $\mathcal{U}$ is an ultrafilter, then $\mathcal{U}$ clusters at x is equivalent to $\mathcal{U}$ converging to x.

Definition 4.3 [21] Let X be a topological space and $x \in X$. We say that X is *bisequential at* x if for every ultrafilter $\mathcal{U}$ that converges to x, there is a countable subfamily $\{A_n : n \in \omega\} \subseteq \mathcal{U}$ such that for every $W \in \mathcal{N}(x)$, there is $n \in \omega$ such that $A_n \subseteq W$. We say that X is *bisequential* if it bisequential at x for every $x \in X$.

We say that X is *(countably) bisequential* if for every ultrafilter on a (countable) subset of X that clusters at $x \in X$, there is a countable subfamily of the ultrafilter converging to x.

Given a tree $T \subseteq \omega^{<\omega}$ and $s \in T$, recall that the set of successors of s in T is $\mathrm{succ}_T(s) = \{n \in \omega : s^\frown n \in T\}$. Given $\mathcal{U} \subseteq \mathcal{P}(\omega)$, we say that T is $\mathcal{U}$-*branching* if $\mathrm{succ}_T(s) \in \mathcal{U}$ for every $s \in T$. As usual, we will omit the subindex in succ_T when there is no risk of confusion. If $\mathcal{U}$ is an ultrafilter and $\mathcal{B}$ is a barrier on ω, we define an ultrafilter $\mathcal{U}^{\mathcal{B}} \subseteq \mathcal{P}(\mathcal{B})$ by declaring $Y \in \mathcal{U}^{\mathcal{B}}$ if and only if there is an $\mathcal{U}$-branching tree $S \subseteq T(\mathcal{B})$ such that $Y = S \cap \mathcal{B}$ (here, we are identifying finite subsets with finite increasing sequences on ω).

Theorem 4.4 *Every compact countably bisequential space is ω_1-sequentially compact.*

Proof Let X be a compact countably bisequential space, let $\mathcal{B}$ be a barrier, and let $f : \mathcal{B} \to X$. Fix an ultrafilter $\mathcal{U} \subseteq \mathcal{P}(\omega)$ and define an ultrafilter $\mathcal{V} \subseteq \mathcal{P}(f[\mathcal{B}]) \subseteq \mathcal{P}(X)$ by $V \in \mathcal{V}$ if and only if $f^{-1}(V) \in \mathcal{U}^{\mathcal{B}}$.

As X is compact, we can find $x \in X$ such that $x \in \overline{\mathcal{V}}$, and by bisequentiality, there exists a decreasing family $\{V_n : n \in \omega\} \subseteq \mathcal{V}$ such that for any open neighborhood $W \in \mathcal{N}(x)$, there is $n \in \omega$ such that $V_n \subseteq W$. Define $U_n = f^{-1}(V_n)$ for every $n \in \omega$ and let $T_n = \{s \in T(\mathcal{B}) : \exists b \in U_n (s \subseteq b)\}$. Then, each T_n is a $\mathcal{U}$-branching tree.

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We now recursively define an infinite set $M = \{m_i : i \in \omega\}$. Let $m_0 = \min(\mathrm{succ}_{T_0}(\emptyset))$. If we have already defined $\{m_i : i < n\}$, define m_n such that for every $s \subseteq \{m_i : i < n\}$ with $s \in T(\mathcal{B})\backslash\mathcal{B}$, we have that $s^\frown m_n \in T_n$. The choice of m_n is possible since every relevant s is an element of T_0 and the trees T_n are $\mathcal{U}$-branching. Moreover, if $s \subseteq M$, $s \in T(\mathcal{B})\backslash\mathcal{B}$ and $\min(s) \geq m_n$, then $s \in T_k$ for every $k \geq n$.

It remains to show that $f \restriction (\mathcal{B}|M)$ converges to x. Fix an open set $W \in \mathcal{N}(x)$ and find $n \in \omega$ such that $V_n \subseteq W$. Given $s \in \mathcal{B}|(M\backslash m_n)$, let $m_k = \max(s)$ and $s' = s\backslash\{m_k\}$. By the choice of m_k, we have that $s = s'^\frown m_k \in T_n \cap \mathcal{B} = U_n$ as $k \geq n$; thus, $f(s) \in V_n \subseteq W$. Therefore, $f[\mathcal{B}|(M\backslash m_n)] \subseteq W$. ∎

The notion of n-sequential compactness in [19] was motivated by the 2-dimensional version introduced in [3], where the main application was to show that compact metric semigroups have idempotents naturally defined as the limit of a 2-dimensional sequence. From Theorem 4.4 and the results in [3], we can conclude the following:

Corollary 4.5 *Every compact countably bisequential semigroup K has an idempotent naturally representable as the Ramsey limit of a restriction of any given $f : [\omega]^2 \to K$.*

We can also now conclude that some well-known classes of spaces are ω_1-sequentially compact. It is clear that if for a class of spaces $\mathcal{G}$, hereditary under closed subsets, we have that if every separable space in $\mathcal{G}$ is ω_1-sequentially compact, then every space in the class is ω_1-sequentially compact.

Roman Pol proved that separable Rosenthal compacta are bisequential [24]; hence, Rosenthal compacta are countably bisequential. It can also be deduced from arguments of G. Debs in [9] (see [26] and Lemma 6 in [25]). Therefore, we have that every Rosenthal compact space is ω_1-sequentially compact. Since every separable Eberlein compact has weight at most $\mathfrak{c}$, and it is a result of S. Mercourakis that Eberlein compact spaces of weight at most $\mathfrak{c}$ are Rosenthal compact, we also have that Eberlein compacta are ω_1-sequentially compact. Similarly, as every separable Corson compactum is metrizable, every Corson compactum (and hence Gul'ko and Talagrand compacta) is ω_1-sequentially compact. For definitions of these classes and the results just mentioned, see [23].

A space X is *angelic* if relatively countably compact subsets of X are relatively compact, and for every relatively compact subset $A \subseteq X$ and every $x \in \overline{A}$, there is a sequence $\{x_n : n \in \omega\} \subseteq A$ that converges to x. It is a result of J. Bourgain, D. H. Fremlin, and M. Talagrand [5] that Rosenthal compacta are angelic spaces.

It was shown by Knaust in [15] that Rosenthal compacta have the Ramsey property. We recall that a space X has the Ramsey property if for every function $f : [\omega]^2 \to X$ such that $\lim_{i\to\infty} \lim_{j\to\infty} f(\{i, j\}) = x$, there is $M \in [\omega]^\omega$ such that $f \restriction [M]^2 \to x$. As any 2-sequentially compact space has the Ramsey property, the fact that Rosenthal compact spaces are ω_1-sequentially compact is a strengthening of Knaust's result.

The lists of angelic spaces with the Ramsey property was expanded in [16]; however, the question of whether every angelic space has the Ramsey property was left open. We show now that this is not the case by pointing out that the example of a sequentially compact space that is not 2-sequentially compact considered in [7] is such a counterexample.

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Theorem 4.6 *There is an angelic space without the Ramsey property.*

Proof Note that a compact space is angelic if and only if it is Fréchet. In [7], an almost disjoint family $\mathcal{A} \subseteq [\omega \times \omega]^{\omega}$, such that $\mathcal{F}(\mathcal{A})$ is Fréchet, sequentially compact, but fails to be 2-sequentially compact, is constructed. Hence, $\mathcal{F}(\mathcal{A})$ is an angelic space.

In order to prove that $\mathcal{F}(\mathcal{A})$ is not 2-sequentially compact, $\mathcal{A}$ was constructed so that $A_n := \{(n, m) : m \in \omega\} \in \mathcal{A}$ for every $n \in \omega$ and so that $G : [\omega]^2 \to \omega \times \omega$, defined by $G(\{n, m\}) = (n, m)$ for $n < m$, does not converge to any point in $\mathcal{F}(\mathcal{A})$. To avoid confusion, let's denote by $*$ the point at infinity on $\mathcal{F}(\mathcal{A})$. As $\lim_{m\to\infty}(n, m) = A_n$ and any infinite subset of $\mathcal{A}$ converges to $*$ in $\mathcal{F}(\mathcal{A})$, we get that $\lim_{n\to\infty} \lim_{m\to\infty}(n, m) = *$. Then, moreover, we have that $\mathcal{F}(\mathcal{A})$ does not have the Ramsey property. ∎

5 Some examples delineating the classes of α-sequentially compact spaces.

We will construct the counterexamples that show that the classes of α-sequentially compact spaces and related spaces do not coincide (at least consistently, in some cases), giving analogous results to those presented in [19] and [7].

We shall start this section by pointing out that neither of the Theorems 4.2 or 4.4 supersedes the other. Given $X \subseteq 2^{\omega}$, define $\mathcal{A}_X = \{\{x \restriction n : n \in \omega\} : x \in X\}$ and denote by $\mathcal{F}(\mathcal{A})$ the one-point compactifications of the Ψ-space over the almost disjoint family $\mathcal{A}$. On the one hand, if $X \subseteq 2^{\omega}$ has size $\mathfrak{c}$, then $\mathcal{F}(\mathcal{A}_X)$ is a compact bisequential space, but its character is $|\mathcal{A}_X| = \mathfrak{c} \geq \mathfrak{b}$ (so this space satisfies the assumptions of Theorem 4.2 but does not satisfy the assumptions of Theorem 4.4). On the other hand, ω_1 is a first countable, sequentially compact space that is not compact (so this space satisfies the assumptions of Theorem 4.4 but does not satisfy the assumptions of Theorem 4.2). This raises the following question:

Question 5.1 *Is there a compact and ω_1-sequentially compact space that is not countably bisequential?*

We cannot expect to conclude full bisequentiality because $\omega_1 + 1$ is a compact and ω_1-sequentially compact space that is not bisequential. There is even an Eberlein compactum (so Fréchet) with these properties (see [6]).

It is worth noting that for an almost disjoint family $\mathcal{A}$, if its Franklin space $\mathcal{F}(\mathcal{A})$ is 2-sequentially compact, then $\mathcal{A}$ is nowhere mad. Otherwise, if $\mathcal{A} \restriction X$ is mad with $X \in \mathcal{I}(\mathcal{A})^+$, we can find a function $f : [\omega]^2 \to X$ that does not converge in $\mathcal{F}(\mathcal{A})$, as mad families are never 2-sequentially compact (see [19]). Since an ad family is Fréchet if and only if it is nowhere mad, all counterexamples constructed from ad families that are at least 2-sequentially compact in [19] and [7] are also Fréchet. It was also shown in [7] that 2-sequentially compact spaces are α_3. This gives more examples of non-bisequential almost disjoint families that are Fréchet and α_3. The existence of these kind of ad families is not known in ZFC. For more consistent examples and for the definition of α_3, the reader may consult [8].

It was previously shown in [19] that the classes of n-sequentially compact spaces and $(n+1)$-sequentially compact spaces do not coincide (assuming CH for $n > 1$). This result was later improved by the authors in [7] by showing that the same holds

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under weaker assumptions (e.g., $\mathfrak{b} = \mathfrak{c}$). We will show that we can also consistently differentiate between α-sequentially compact and β-sequentially compact spaces for any $\alpha \neq \beta$ by stepping up the combinatorial analysis done in [7] to barriers.

We will say that $T \subseteq \omega^{<\omega}$ is a *Hechler tree* if for every $s \in T$, the set of successors of s is *coinitial* on ω (i.e., there exists $k_s \in \omega$ such that $\mathrm{succ}_T(s) = \{i \in \omega : s^\frown i \in T\} = \omega \backslash k_s$). We can naturally associate a function $h_T : T \to \omega$ to each Hechler tree T by defining $h_T(s) = k_s$.

For $s, t \in \omega^{<\omega}$, we say that $s \prec t$ if either $s \sqsubset t$ or $s = r^\frown l$, $t = r^\frown m$ for some $r \in \omega^{<\omega}$, and $l < m$. Here, $s \sqsubset t$ means that s is a proper initial segment of t. Enumerate $\omega^{<\omega} = \{s_i : i \in \omega\}$ such that $s_i \prec s_j$ implies $i < j$. We fix this enumeration for the rest of the paper.

Definition 5.2 Let $f \in \omega^\omega$ and let $T \subseteq \omega^{<\omega}$ be a Hechler tree. We define

- $T_f \subseteq \omega^{<\omega}$, where $\emptyset \in T_f$ and $\mathrm{succ}(s_i) = \omega \backslash f(i)$ for every $s_i \in T_f$.
- $f_T \in \omega^\omega$, where $f_T(n) = k$ if and only if $s_n \in T$ and $\mathrm{succ}_T(s_n) = \omega \backslash k$ and $f_T(n) = 0$ otherwise.

Recall that $\mathsf{FIN} \subseteq \mathcal{P}(\omega)$ is the ideal of finite sets of ω.

Definition 5.3 Given a barrier $\mathcal{B}$, we define the ideal $\mathsf{FIN}^{\mathcal{B}}$ by recursion on its rank as the set of all $X \subseteq [\omega]^{<\omega}$ such that

$$\{n \in \omega : \{s \backslash \{n\} : s \in X \cap \mathcal{B}(n)\} \notin \mathsf{FIN}^{\mathcal{B}[n]}\} \in \mathsf{FIN},$$

where $\mathsf{FIN}^{[\omega]^1} = \{\{n\} : n \in \omega\}$ works as the base case.

Notice that in the case of the barrier $\mathcal{B} = [\omega]^n$, the ideal $\mathsf{FIN}^{\mathcal{B}}$ coincides with the well-known Fubini product FIN^n. In general, the Fubini product FIN^α for a countable limit ordinal α is not uniquely determined and depends on the choice of an increasing sequence $\{\alpha_n : n \in \omega\}$ converging to α, the corresponding previous choices for each α_n, and so on. For a barrier $\mathcal{B}$ of rank α such that $\alpha_n = \rho(\mathcal{B}[n])$ forms an increasing sequence, the ideal $\mathsf{FIN}^{\mathcal{B}}$ also coincides with a Fubini product FIN^α, where the sequence $\{\alpha_n : n \in \omega\}$ is precisely the sequence converging to α used in the Fubini product. For more on these ideals and their presentations, the reader may consult [10].

It follows directly from the definition that we can characterize the ideals $\mathsf{FIN}^{\mathcal{B}}$ via Hechler trees.

Lemma 5.4 *Let $\mathcal{B}$ be a barrier and $X \in \mathsf{FIN}^{\mathcal{B}}$. Then, there exists a Hechler tree $H \subseteq \omega^{<\omega}$ such that $X \cap H \cap \mathcal{B} = \emptyset$.*

Proof We prove it by induction on $\rho(\mathcal{B})$. If $\mathcal{B} = [\omega]^1$ and $X \in \mathsf{FIN}^{\mathcal{B}} = \mathsf{FIN}$, define $H \subseteq \omega^{<\omega}$ such that $\emptyset \in H$, $\mathrm{succ}_H(\emptyset) = \omega \backslash k$, where $X \subseteq k$ and $\mathrm{succ}_H(s) = \omega$ for any $s \in H \backslash \{\emptyset\}$. This H clearly works.

Let now $\mathcal{B}$ be an arbitrary barrier with $\rho(\mathcal{B}) > 1$. We may assume that $\{k\} \notin \mathcal{B}$ for every $k \in \omega$. Fix $X \in \mathsf{FIN}^{\mathcal{B}}$. Thus, $\{n \in \omega : X \cap \mathcal{B}(n) \notin \mathsf{FIN}^{\mathcal{B}[n]}\}$ is finite, and we can find $k \in \omega$ that contains this set. Hence, for every $n \in \omega \backslash k$, there is a Hechler tree H_n with root $\{n\}$ such that $X \cap H_n \cap \mathcal{B}(n) = \emptyset$. Define $H = \{\emptyset\} \cup \bigcup_{n \geq k} H_n$. It is clear that H is a Hechler tree and $X \cap H \cap \mathcal{B} = \emptyset$. ∎

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Lemma 5.5 *For every $\mathcal{X} \subseteq \mathsf{FIN}^{\mathcal{B}}$ with $|\mathcal{X}| < \mathfrak{b}$, there is a Hechler tree $H \subseteq \omega^{<\omega}$ such that for every $X \in \mathcal{X}$, there exists $n \in \omega$ such that*

$$X \cap H \cap \mathcal{B} \subseteq \bigcup_{i \leq n} \mathcal{B}(i).$$

Proof Let $\mathcal{X} \subseteq \mathsf{FIN}^{\mathcal{B}}$ with $|\mathcal{X}| < \mathfrak{b}$. By Lemma 5.4, for every $X \in \mathcal{X}$, there is a Hechler tree H_X such that $X \cap H_X \cap \mathcal{B} = \emptyset$. Let $f_X = f_{H_X}$ for every $X \in \mathcal{X}$. Since $|\mathcal{X}| < \mathfrak{b}$, we can find $f \in \omega^\omega$ such that $f >^* f_X$ for every $X \in \mathcal{X}$. It follows from our enumeration of $\omega^{<\omega}$ that if $f >^* f_X$, then there exists $n_X \in \omega$ such that if $f_X(s) > f(s)$ for some $s \neq \emptyset$, then $s \in \mathcal{B}(i)$ for some $i \leq n_X$. Thus, letting $H = H_f$, we have that $H \backslash H_X \subseteq \bigcup_{i \leq n_X} \mathcal{B}(i)$, and this implies that

$$X \cap H \cap \mathcal{B} \subseteq \bigcup_{i \leq n_X} \mathcal{B}(i). \qquad \blacksquare$$

The following proposition appears in [19] for the particular case of barriers of the form $[\omega]^n$. The same argument shows that this is true for every barrier, so we have the following:

Proposition 5.6 *Let $\mathcal{A}$ be an almost disjoint family on a countable set N and let $\mathcal{B}$ be a barrier. If for every function $f : \mathcal{B} \to N$, there is an infinite set $X \in [\omega]^\omega$ such that*

$$|\{A \in \mathcal{A} : |f[\mathcal{B}|X] \cap A| = \omega\}| < \mathfrak{b},$$

then $\mathcal{F}(\mathcal{A})$ is $\mathcal{B}$-sequentially compact.

The following Lemma will also be helpful for proving that a space is not $\mathcal{B}$-sequentially compact for some barrier $\mathcal{B}$. We say that a barrier $\mathcal{C}$ is nontrivial if $\mathcal{C} \neq [\omega]^1$.

Lemma 5.7 *Let $\mathcal{A}$ be an almost disjoint family on a nontrivial barrier $\mathcal{C}$ such that $|A \cap \mathcal{C}(n)| \leq 1$ for every $A \in \mathcal{A}$ and every $n \in \omega$. If for every $E \in [\omega]^\omega$, there exists $A \in \mathcal{A}$ such that $A \subseteq \mathcal{C}|E$, then $\mathcal{F}(\mathcal{A})$ is not $\mathcal{C}$-sequentially compact.*

Proof As $\mathcal{C}$ is nontrivial, we may assume without loss of generality that every element of the $\mathcal{C}$ has size at least 2; in particular, $\mathcal{C}(m)$ is infinite for every $m \in \omega$.

Consider $i : \mathcal{C} \to \mathcal{C}$ to be the identity map and $E \in [\omega]^\omega$. We shall show that $i \restriction (\mathcal{C}|E)$ does not converge in $\mathcal{F}(\mathcal{A})$. Since $i[\mathcal{C}|E]$ is infinite, it does not converge to any isolated point in $\mathcal{C}$.

However, if $A \in \mathcal{A}$ and $n \in \omega$, we have that $|\mathcal{C}(m) \cap i[\mathcal{C}|E\backslash n]| = \omega$ for every $m \in E\backslash n$, but $|A \cap \mathcal{C}(m)| \leq 1$ implies that $i[\mathcal{C}|(E\backslash n)] \not\subseteq A$, and hence, it does not converge to A. It remains to show that it does not converge to ∞.

Fix the neighborhood $U = \mathcal{F}(\mathcal{A})\backslash(\{A\} \cup A)$ of ∞ for some $A \subseteq \mathcal{C}|E$ and let $n \in \omega$. As $|A \cap \mathcal{C}(i)| \leq 1$ for every $i \in \omega$, we can find $c \in A \subseteq \mathcal{C}|E$ such that $\min(c) > n$. Thus, $c \in \mathcal{C}|(E\backslash n)\backslash U$, and in consequence, $i[\mathcal{C}|(E\backslash n)] \not\subseteq U$. Since this is true for any $n \in \omega$, we get that $i \restriction (\mathcal{C}|E)$ does not converge to ∞. $\blacksquare$

The previous lemmas allow us to show that there are spaces that are $\mathcal{B}$-sequentially compact but fail to be $\mathcal{C}$-sequentially compact whenever their ideals associated satisfy a suitable relation. For this reason, it is useful to introduce the terminology of the Katětov order.

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Definition 5.8 [18] Let X and Y be countable sets, $\mathcal{I}$ and $\mathcal{J}$ ideals on X and Y, respectively, and $f : Y \to X$.

(1) f is a *Katětov* function from $(Y, \mathcal{J})$ to $(X, \mathcal{I})$ if $f^{-1}(A) \in \mathcal{J}$ for all $A \in \mathcal{I}$.
(2) $\mathcal{I} \leq_K \mathcal{J}$ ($\mathcal{I}$ *is Katětov below* $\mathcal{J}$) if there exists a Katětov function from $(Y, \mathcal{J})$ to $(X, \mathcal{I})$.

An easy consequence of the definition is the following:

Lemma 5.9 *Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on countable sets X and Y, respectively. Then, the following are equivalent:*

(1) $\mathcal{I} \not\leq_K \mathcal{J}$.
(2) *For every $f : Y \to X$, there is $B \in \mathcal{J}^+$ such that $f[B] \in \mathcal{I}$.*

Proof To see that (1) implies (2), let $f : Y \to X$. As $\mathcal{I} \not\leq_K \mathcal{J}$, f is not a Katětov function, so there is $A \in \mathcal{I}$ such that $B := f^{-1}(A) \in \mathcal{J}^+$. Now $f[B] \in \mathcal{I}$ since $f[B] \subseteq A \in \mathcal{I}$. Conversely, to see that $\mathcal{I} \not\leq_K \mathcal{J}$, let $f : Y \to X$; we want to prove that f is not a Katětov function. So let $B \in \mathcal{J}^+$ such that $A := f[B] \in \mathcal{I}$. Now $f^{-1}(A) \in \mathcal{J}^+$ since $B \subseteq f^{-1}(A)$. ∎

Besides the ideal $\mathsf{FIN}^{\mathcal{B}}$, another ideal that will be useful in the constructions of counterexamples is the following:

Definition 5.10 If $\mathcal{B}$ is a barrier on ω, then $\mathcal{G}_c(\mathcal{B})$ is the ideal on $\mathcal{B}$ such that $(\mathcal{G}_c(\mathcal{B}))^+ = \langle \{\mathcal{B}|X : X \in [\omega]^\omega\} \rangle$ (i.e., $S \subseteq \mathcal{B}$ is an element of $\mathcal{G}_c(\mathcal{B})$ if and only if there is no $X \in [\omega]^\omega$ such that $\mathcal{B}|X \subseteq S$).

Note that $(\mathcal{G}_c([\omega]^2))^+$ is the collection of all subsets of $[\omega]^2$ that contain an infinite complete subgraph; that is the reason for the notation $\mathcal{G}_c$. Also note that the fact that $\mathcal{G}_c(\mathcal{B})$ is closed under finite unions follows from Nash-Williams Theorem.

Theorem 5.11 *($\mathfrak{b} = \mathfrak{c}$) Let $\mathcal{B}$ and $\mathcal{C}$ be two barriers such that $\mathsf{FIN}^{\mathcal{C}} \not\leq_K \mathcal{G}_c(\mathcal{B})$. Then, there is an almost disjoint family $\mathcal{A}$ such that $\mathcal{F}(\mathcal{A})$ is $\mathcal{B}$-sequentially compact but not $\mathcal{C}$-sequentially compact.*

Proof Enumerate $\mathcal{C}^{\mathcal{B}} = \{f_\alpha : \alpha < \mathfrak{c}\}$ and $[\omega]^\omega = \{E_\alpha : \alpha < \mathfrak{c}\}$. For $E \in [\omega]^\omega$, let $E^\uparrow = T(\mathcal{C}|E)$. Notice that for any such E, the tree $E^\uparrow$ is everywhere ω-splitting, and hence, it has infinite intersection with any Hechler tree H inside $\mathcal{C}$; that is, $|E^\uparrow \cap H \cap \mathcal{C}| = \omega$. Moreover, for any $n \in E$, we have that

$$|E^\uparrow \cap H \cap \mathcal{C}(n)| = \omega.$$

Recursively construct $\{A_\alpha : \alpha < \mathfrak{c}\} \subseteq [\mathcal{C}]^\omega$ and $\{X_\alpha : \alpha < \mathfrak{c}\} \subseteq [\omega]^\omega$ such that for all $\beta < \alpha < \mathfrak{c}$,

(1) $f_\alpha[\mathcal{B}|X_\alpha] \in \mathsf{FIN}^{\mathcal{C}}$,
(2) $|A_\alpha \cap A_\beta| < \omega$,
(3) $|A_\alpha \cap f_\beta[\mathcal{B}|X_\beta]| < \omega$,
(4) $|A_\alpha \cap \mathcal{C}(n)| \leq 1$ for all $n \in \omega$,
(5) $A_\alpha \subseteq E_\alpha^\uparrow$.

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At step $\alpha < \mathfrak{c}$, let $X_\alpha \in [\omega]^\omega$ such that $f_\alpha[\mathcal{B}|X_\alpha] \in \mathsf{FIN}^{\mathcal{C}}$ (this set exists by Lemma 5.9). Note that also every A_β with $\beta < \alpha$ is an element of $\mathsf{FIN}^{\mathcal{C}}$. Then, we can find a Hechler tree H as in Lemma 5.5 such that

$$X \cap H \cap \mathcal{C} \subseteq \bigcup_{i \leq n_X} \mathcal{C}(i)$$

for either $X = f_\beta[\mathcal{B}|X_\beta]$ or $X = A_\beta$ for some $\beta < \alpha$. Let $A_\alpha \in [\mathcal{C}]^\omega$ be such that $A_\alpha \subseteq H \cap \mathcal{C} \cap E^\uparrow_\alpha$ and $|A_\alpha \cap \mathcal{C}(i)| \leq 1$ for every $i \in \omega$. It follows that $A_\alpha \cap X \subseteq \bigcup_{i \leq n_X} \mathcal{C}(i)$ for some $n_X \in \omega$, and as $|A_\alpha \cap \mathcal{C}(i)| \leq 1$ for every $i \leq n_X$, this intersection is finite, and (2) and (3) hold. The remaining three properties follow from the construction.

Since $\{A \in \mathcal{A} : |f_\alpha[\mathcal{B}|X_\alpha] \cap A| = \omega\} \subseteq \{A_\beta : \beta \leq \alpha\}$, and this set has size less than $\mathfrak{b}$, by Proposition 5.6, we get that $\mathcal{F}(\mathcal{A})$ is $\mathcal{B}$-sequentially compact.

However, item (5) implies that $\mathcal{F}(\mathcal{A})$ is not $\mathcal{C}$-sequentially compact by Lemma 5.7, as $\mathcal{C}$ has rank greater than 1. ∎

The natural question now is when do the barriers $\mathcal{B}$ and $\mathcal{C}$ satisfy that $\mathsf{FIN}^{\mathcal{C}} \not\leq_K \mathcal{G}_c(\mathcal{B})$? We will see below that this only depends on the rank of the barriers, assuming that at least the one with larger rank is uniform. See Theorem 5.12 and Corollary 5.14 below.

The relevance of the Katětov order in the study of subclasses of sequentially compact spaces is also demonstrated in the work of R. Filipów, K. Kowitz, and A. Kwela [13], where the authors study many subclasses of sequentially compact spaces, prove some inclusions among them, and find counterexamples by using the Katětov order relationship between some ideals naturally associated to these classes.

Recall that for $\mathcal{B}$ being a barrier, $\mathcal{B}[s]$ is a barrier on $\omega \backslash (\max(s) + 1)$. For the ease of notation, we will do some abuse of notation: for a function $g : \mathcal{B}(s) \to \mathcal{C}(t)$, we can naturally associate a function $g' : \mathcal{B}[s] \to \mathcal{C}[t]$. We use g for both functions. Also, for a Hechler tree H, we will denote by $H[n]$ its copy above $\{n\}$ – that is, a tree T with root $\{n\}$ and such that $\mathrm{succ}_T(s) = \mathrm{succ}_H(s \backslash \{n\})$. In particular, $\omega^{<\omega}[n] = \{s \in \omega^{<\omega} : n = \min(s)\}$.

Theorem 5.12 *Let $\mathcal{B}, \mathcal{C}$ be barriers with $\mathcal{C}$ uniform and $\rho(\mathcal{B}) < \rho(\mathcal{C})$. Then, $\mathsf{FIN}^{\mathcal{C}} \not\leq_K \mathcal{G}_c(\mathcal{B})$.*

Proof By Lemma 5.9, it is enough to prove that for every $f : \mathcal{B} \to \mathcal{C}$, there is $X \in [\omega]^\omega$ such that $f[\mathcal{B}|X] \in \mathsf{FIN}^{\mathcal{C}}$. The proof is by induction on $\rho(\mathcal{C})$. If $\rho(\mathcal{C}) = 1$, there is nothing to do, so we can assume that $\rho(\mathcal{C}) > 1$, and the result is true for $\mathcal{C}'$ whenever $\rho(\mathcal{C}') < \rho(\mathcal{C})$. We now prove it by induction on $\rho(\mathcal{B})$. If $\rho(\mathcal{B}) = 1$, we can find an infinite set $X \in [\omega]^\omega$ such that either $|f[\mathcal{B}|X] \cap \mathcal{C}(i)| \leq 1$ for every $i \in \omega$ or $f[\mathcal{B}|X] \subseteq \mathcal{C}(n)$ for some $n \in \omega$. Hence, assume that for any barriers $\mathcal{B}'$ and $\mathcal{C}'$ with $\rho(\mathcal{B}') < \rho(\mathcal{B}), \rho(\mathcal{C}') \leq \rho(\mathcal{C})$, and $\rho(\mathcal{B}') < \rho(\mathcal{C}')$, the result holds. We can also think of $\{\emptyset\}$ as a barrier of rank 0. In this case, the result is trivial, but it is worth mentioning since $\mathcal{B}[s]$ is a barrier of rank 0 whenever $s \in \mathcal{B}$. We will also consider these kind of barriers as part of our inductive hypothesis. We can also assume, by possibly passing to a cofinite set, that $\rho(\mathcal{C}(n)) \geq \rho(\mathcal{B})$ for every $n \in \omega$.

Define a function $\pi_n : \mathcal{B} \to 2$ for every $n \in \omega$ given by $\pi_n(s) = 0$ if and only if $f(s) \in \bigcup_{i \leq n} \mathcal{C}(i)$. By Nash-Williams, we can find an infinite set $Z_n \in [\omega]^\omega$ such that $\pi_n \restriction$

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$(\mathcal{B}|Z_n)$ is constant. If $\pi_n \restriction (\mathcal{B}|Z_n)$ is constant with value i, we will say that Z_n is a monochromatic set with color i. If for some $n \in \omega$ we can find a monochromatic set Z_n with color 0, then f maps $\mathcal{B}|Z_n$ into the union of the first $n+1$-cones of $\mathcal{C}$, which is an element of $\mathsf{FIN}(\mathcal{C})$. Hence, we can assume that there is no monochromatic set with color 0 for every $n \in \omega$.

Let us recursively define $\{X_n : n \in \omega\}$, $\{m_i : i \in \omega\}$, $\{H_n : n \in \omega\}$ and $\{g(s,n) : s \subseteq \{m_i : i < n\} \wedge n \in \omega\}$ such that

(1) $m_i = \min(X_i)$,
(2) $\pi_n \restriction (\mathcal{B}|X_n)$ is constant with value 1,
(3) $X_{n+1} \subseteq X_n \subset \omega$,
(4) $g(s,n) : \mathcal{B}[s] \to \mathcal{C}[n]$ is given by $g(s,n)(t) = f(t)\backslash\{n\}$ if and only if $f(t) \in \mathcal{C}(n)$, and otherwise, let $g(s,n)(t) \in \mathcal{C}[n]$ be arbitrary.
(5) $H_n \subseteq \omega^{<\omega}[n]$ is a Hechler tree.
(6) $g(s,n)[\mathcal{B}[s]|X_n] \cap H_n = \emptyset$ for every $s \subseteq \{m_i : i < n\}$ with $s \neq \emptyset$.

Assume X_i, m_i, H_i, and $g(s,i)$ have been defined for $i \leq n$ and s as above. Since $g(s,n+1)$ is defined from $\{m_i : i \leq n\}$ and m_{n+1} is defined from X_{n+1}, we only need to define X_{n+1} and H_{n+1}. Note that if in point (6) we have that $s \notin T(\mathcal{B})$, then $\mathcal{B}(s) = \emptyset$ and then any choice of X_n and H_n work, so we will assume in the rest of the proof that $s \in T(\mathcal{B})$.

We can find a π_{n+1} monochromatic set (with color 1) $X' \subseteq X_n$. Enumerate as $\{s_i : i \leq k\}$ for some $k \in \omega$ the set $\{s \subseteq \{m_i : i \leq n\} : s \in T(\mathcal{B})\backslash\{\emptyset\}\}$. We can recursively define $X^0_{n+1} \subseteq X'$, $X^{i+1}_{n+1} \subseteq X^i_{n+1}$ for $i < k$ and $H^i_{n+1} \subseteq \mathcal{C}[n+1]$ for $i \leq k$ such that

(*) $g(s,n+1)[\mathcal{B}[s]|X^i_{n+1}] \cap H^i_{n+1} = \emptyset$.

To see that the choices of X^i_{n+1} and H^i_{n+1} are possible, note that $g(s,n+1) : \mathcal{B}[s] \to \mathcal{C}[n+1]$ and $\rho(\mathcal{B}[s]) < \rho(\mathcal{B}) \leq \rho(\mathcal{C}[n+1])$. Then, we can apply the inductive hypothesis. Without loss of generality, we can assume that $\min(X^i_{n+1}) > m_n$ for every $i \leq k$. Also, since each H^i_{n+1} is a Hechler tree for every $i \leq k$, we can define $H_{n+1} = \bigcap_{i \leq k} H^i_{n+1}$, which is a Hechler tree in $\mathcal{C}[n+1]$. Define $X_{n+1} = X^k_{n+1}$. This finishes the construction.

Let $X = \{x_i : i \in \omega\}$ and $H = \{\emptyset\} \cup \bigcup_{n \in \omega} n^\frown H_n$. We shall show that $f[\mathcal{B}|X] \cap H = \emptyset$. Fix $s \in \mathcal{B}|X$; then, $s = (x_{i_0}, \ldots, x_{i_k})$ for some $k \in \omega$ and some sequence $i_0 < \cdots < i_k$. As $\{x_{i_0}, \ldots, x_{i_k}\} \subseteq X_{i_0}$ and X_{i_0} is π_{i_0} monochromatic with color 1, we have that $f(s) \in C(n)$ for some $n > i_0$. Let $s' = \{x_i \in s : i < n\}$ and notice that $s\backslash s' \subseteq X_n$ and $s' \neq \emptyset$. Hence, $g(s',n)$ was considered at step n in (6), and s' was indexed as s_i for some i. Since we have that $H \cap \omega^{<\omega}[n] \subseteq H^i_n$, $s\backslash s' \in \mathcal{B}[s']|X_n$, and $X_n \subseteq X^i_n$, we can conclude, by $(*)$, that $g(s',n)(s) = f(s)\backslash\{n\} \in \mathcal{C}[n]\backslash H_n$, so $f(s) = n^\frown g(s',n) \notin H$. Therefore, $f[\mathcal{B}|X] \cap H = \emptyset$, so $f[\mathcal{B}|X] \in \mathsf{FIN}^{\mathcal{C}}$. ∎

It is worth noting that the previous result gives us that for two barriers $\mathcal{B}$ and $\mathcal{C}$, the classes of $\mathcal{B}$-sequentially compact spaces and $\mathcal{C}$-sequentially compact spaces only depend on the ranks of $\mathcal{B}$ and $\mathcal{C}$ whenever the largest is uniform. If $\rho(\mathcal{B}) < \rho(\mathcal{C})$ and $\mathcal{C}$ is uniform, every $\mathcal{C}$-sequentially compact space is $\mathcal{B}$-sequentially compact by Corollaries 3.5 and 3.6. However, there are (at least consistently) examples of $\mathcal{B}$-sequentially compact spaces that are not $\mathcal{C}$-sequentially compact by Theorem 5.12.

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If otherwise, $\rho(\mathcal{B}) = \rho(\mathcal{C})$ and both of them are uniform, then the classes of $\mathcal{B}$-sequentially compact and $\mathcal{C}$-sequentially compact spaces coincide by Corollary 3.5.

A more subtle analysis can be done by considering $\rho_u(\mathcal{B})$ instead of $\rho(\mathcal{B})$, but what we have done suffices to show that (consistently) the classes of α-sequentially compact and β-sequentially compact spaces do not coincide if $\alpha \neq \beta$.

Corollary 5.13 *($\mathfrak{b} = \mathfrak{c}$) If $\{\alpha_n : n \in \omega\} \subseteq \alpha$, there exists a space that is α_n-sequentially compact for all $n \in \omega$ but fails to be α-sequentially compact. In particular, for every $\alpha < \beta < \omega_1$, there exists an α-sequentially compact space that is not β-sequentially compact.*

Proof Fix a uniform barrier $\mathcal{B}_n$ of rank α_n for every $n \in \omega$ and a barrier $\mathcal{C}$ of rank α. Repeat the proof of theorem 5.11 by enumerating

$$\bigcup_{n\in\omega} \mathcal{C}^{\mathcal{B}_n} = \{f_\alpha : \alpha < \mathfrak{c}\}.$$

The hypothesis of Theorem 5.11 for every barrier $\mathcal{B}_n$ holds by Theorem 5.12. Finally, by Corollary 3.5, we get the result. ∎

If $\mathcal{B}$ and $\mathcal{C}$ are barriers with the same rank and $\mathcal{B}$ is uniform, then $\mathsf{FIN}^{\mathcal{C}} \leq_K \mathcal{G}_c(\mathcal{B})$. To see this, note that otherwise, by Theorem 5.11, there is a space X that is $\mathcal{B}$-seq compact but not $\mathcal{C}$-seq compact under $\mathfrak{b} = \mathfrak{c}$, but this is a contradiction to Corollary 3.5 since the Katětov order is absolute among Borel ideals due to Shoenfield's absoluteness Theorem (see [17]). We then get the following corollary:

Corollary 5.14 *Let $\mathcal{B}$ and $\mathcal{C}$ be uniform barriers such that $\rho(\mathcal{B}) \leq \rho(\mathcal{C})$. Then, the following are equivalent:*

(1) $\mathsf{FIN}^{\mathcal{C}} \not\leq_K \mathcal{G}_c(\mathcal{B})$.
(2) $\rho(\mathcal{B}) < \rho(\mathcal{C})$.

It is worth pointing out that for (1) implies (2) we use that $\mathcal{B}$ is uniform while in (2) implies (1) we use the uniformity of $\mathcal{C}$.

6 Cardinal invariants associated to barriers

In this section, we define and analyze several cardinal invariants associated to barriers that play an important role in the structure of α-sequentially compact spaces. In particular, we extend a result in [19] by showing that for every $\alpha > 1$, the Cantor cube 2^κ is α-sequentially compact if and only if $\kappa < \min\{\mathfrak{b}, \mathfrak{s}\}$. When $\alpha = 1$, it is known that 2^κ is sequentially compact if and only if $\kappa < \mathfrak{s}$ (see [11]).

It is also shown in [7] that the cardinals $\mathfrak{par}_n$ are closely related to the constructions of examples of spaces that are n-sequentially compact but fail to be $(n+1)$-sequentially compact under $\mathfrak{s} = \mathfrak{b}$. This suggests that if we aim to do similar constructions for arbitrary barriers, we should start analyzing and computing the analogous cardinal invariants for $\mathfrak{par}_n$ in arbitrary barriers. If $\mathcal{B}$ is a barrier, $\pi : \mathcal{B} \to 2$, and $H \in [\omega]^\omega$, then we say that H is *monochromatic for* π if π is constant on $\mathcal{B}|H$. We say that H is *almost monochromatic* for π if there is $F \in [H]^{<\omega}$ such that $H \backslash F$ is monochromatic for π. If Π is a family of colorings on $\mathcal{B}$, then we say that H is *almost*

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monochromatic for Π if it is almost monochromatic for every $\pi \in \Pi$. We refer to $[\omega]^1$ as the trivial barrier.

Definition 6.1 If $\mathcal{B}$ is a barrier, then $\mathfrak{par}_{\mathcal{B}}$ is the minimum size of a collection of colorings on $\mathcal{B}$ without an almost monochromatic infinite set.

If $t \in [\omega]^{<\omega}\backslash\{\emptyset\}$, recall that we write t as $\{t_0, \dots, t_{n-1}\}$ in the increasing way, and if $I \subseteq n$, then $t \restriction I = \{t_i \mid i \in I\}$. Also note that $t \restriction 2 \sqsubseteq t$ for $n \geq 2$. Given a barrier $\mathcal{B}$ such that each of its elements has size at least 2, if we let $\mathcal{B} \restriction 2 := \{t \restriction 2 : t \in \mathcal{B}\}$, then $\mathcal{B} \restriction 2 = [\omega]^2$. For $n \in \omega$, the cardinal $\mathfrak{par}_n$ mentioned above is exactly $\mathfrak{par}_{[\omega]^n}$, so we will stick with the classical terminology $\mathfrak{par}_n$ when $\mathcal{B} = [\omega]^n$.

Theorem 6.2 *If* $\mathcal{B}$ *is a nontrivial barrier, then* $\mathfrak{par}_{\mathcal{B}} \leq \mathfrak{par}_2$.

Proof As $\mathcal{B}$ is a nontrivial barrier, there is $n \in \omega$ such that $\mathcal{B}|(\omega\backslash n)$ has all of its elements of size at least two. Without loss of generality, we may assume that $n = 0$. Let $\Pi = \{\pi_\alpha \mid \alpha \in \mathfrak{par}_2\}$ be a family of colorings on $[\omega]^2$ with no almost monochromatic infinite sets. Define $\widehat{\pi}_\alpha : \mathcal{B} \to 2$ for every $\alpha \in \mathfrak{par}_2$, given by $\widehat{\pi}_\alpha(t) = \pi_\alpha(t \restriction 2)$. Note that if H is a an almost monochromatic infinite set for $\widehat{\Pi} = \{\widehat{\pi}_\alpha \mid \alpha \in \mathfrak{par}_2\}$, then H is also an almost monochromatic infinite set for Π. Hence, $\widehat{\Pi}$ has no infinite almost monochromatic sets. ∎

Recall that if $\mathcal{B}$ is a barrier, the rank of $\mathcal{B}[n]$ is strictly smaller than the rank of $\mathcal{B}$ for every $n \in \omega$. This allows us to show that the same idea that proves that $\mathfrak{par}_n = \min\{\mathfrak{b}, \mathfrak{s}\}$ for every $n \in \omega$ in [2] also works for arbitrary barriers.

Theorem 6.3 *If* $\mathcal{B}$ *is a barrier such that each of its elements has size at least* 2*, then* $\mathfrak{par}_{\mathcal{B}} \geq \mathfrak{par}_2$.

Proof By induction on the rank of $\mathcal{B}$. Suppose that $\mathcal{B}$ has rank γ and the result is true for every barrier of rank strictly smaller than γ. Let $\kappa < \mathfrak{par}_2$ and let $\{\pi_\alpha : \alpha \in \kappa\}$ be a set of colorings on $\mathcal{B}$. We are going to show that there exists an almost monochromatic infinite set for this family.

For every $\alpha \in \kappa$ and every $n \in \omega$, let $\pi^n_\alpha = \mathcal{B}[n] \to 2$ given by $\pi^n_\alpha(s) = \pi_\alpha(\{n\} \cup s)$.

We recursively construct a collection $\{H_n : n \in \omega\} \subseteq [\omega]^\omega$ such that

(1) $H_{n+1} \in [H_n]^\omega$.
(2) H_{n+1} is almost monochromatic for $\{\pi^n_\alpha \restriction (\mathcal{B}[n]|H_n) : \alpha \in \kappa\}$

We start by defining $H_0 = \omega\backslash 1$, and as $B[0]$ has rank smaller than γ, by our inductive hypothesis, we know that there is $H_1 \in [\omega\backslash 1]^\omega$ such that H_1 is almost monochromatic for $\{\pi^0_\alpha \mid \alpha \in \kappa\}$. Now suppose that we have already defined H_n and consider the family $\{\pi^n_\alpha \restriction (\mathcal{B}[n]|H_n) : \alpha \in \kappa\}$. As $\mathcal{B}[n]|H_n$ has also rank smaller than γ, there is $H_{n+1} \in [H_n]^\omega$ such that H_{n+1} is almost monochromatic for this family.

Take H an infinite pseudointersection of $\{H_n : n \in \omega\}$. Note for every $n \in H$ and every $\alpha \in \kappa$, we have that H_n is almost monochromatic for π^n_α with color $j(\alpha, n) \in \{0, 1\}$. For every $\alpha \in \kappa$, define an increasing $f_\alpha \in H^\omega$ such that $H_n\backslash f_\alpha(n)$ is monochromatic for π^n_α. Also let $g \in \omega^\omega$ be increasing such that $H\backslash g(n) \subseteq H_n$. As $\mathcal{F} = \{f_\alpha : \alpha \in \kappa\} \cup \{g\} \subseteq \omega^\omega$ has size less than $\mathfrak{par}_2 \leq \mathfrak{b}$, there exists $f \in \omega^\omega$ strictly increasing that dominates $\mathcal{F}$.

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For every $i \in 2$, let $X^i_\alpha := \{n \in H : j(\alpha, n) = i\}$. Note that $\{X^0_\alpha \mid \alpha \in \kappa\}$ is a collection of less than $\mathfrak{par}_2 \leq \mathfrak{s}$ subsets of H, so there is $X \in [H]^\omega$ such that for every $\alpha \in \kappa$, there is $j(\alpha) \in \{0,1\}$ such that $X \subseteq^* X^{j(\alpha)}_\alpha$.

Let us define $J := \{x_n \mid n \in \omega\} \subseteq X \subseteq H$ such that $x_{n+1} > f(x_n)$ for every $n \in \omega$.

Claim: J is almost monochromatic for $\{\pi_\alpha \mid \alpha \in \kappa\}$.

Proof of the claim Let $\alpha \in \kappa$. We know that $X \subseteq^* X^{j(\alpha)}_\alpha$, $f_\alpha <^* f$, and $g <^* f$, so there exists $k \in \omega$ such that

(1) $X \setminus k \subseteq X^{j(\alpha)}_\alpha$,
(2) $f_\alpha(n) < f(n)$ for every $n > k$, and
(3) $g(n) < f(n)$ for every $n > k$.

Fix $s \in \mathcal{B}|(J\setminus(k+1))$ and let $n = \min(s)$ and $t = s\setminus\{n\}$. Then, we have that $n \in X^{j(\alpha)}_\alpha$ and

$$t \subseteq J\setminus f(n) \subseteq X\setminus f(n) \subseteq H\setminus f(n) \subseteq H\setminus g(n) \subseteq H_n.$$

Notice that $\min(t) > f(n) > f_\alpha(n)$; hence, $t \subseteq H_n\setminus f_\alpha(n)$. But then, $\pi_\alpha(s) = \pi^n_\alpha(t) = j(\alpha, n)$ and $j(\alpha, n) = j(\alpha)$ since $n \in X^{j(\alpha)}_\alpha$. This proves that π_α is constant on $\mathcal{B}|(J\setminus(k+1))$ with color $j(\alpha)$. ∎

The proof of the previous claim finishes the proof of the theorem.

Corollary 6.4 *$\mathfrak{par}_\mathcal{B} = \mathfrak{par}_2$ for every nontrivial barrier $\mathcal{B}$ on ω and $\mathfrak{par}_{[\omega]^1} = \mathfrak{s}$.*

Corollary 6.5 *If $\kappa < \mathfrak{par}_2$ and $\mathcal{B}$ is a barrier, then 2^κ is $\mathcal{B}$-sequentially compact. Moreover,*

$$\mathfrak{par}_2 = \min\{\kappa : 2^\kappa \text{ is not } \mathcal{B}\text{-sequentially compact}\}$$

for every nontrivial barrier $\mathcal{B}$.

Proof If $\kappa < \mathfrak{par}_2 = \min\{\mathfrak{s}, \mathfrak{b}\}$, then $\kappa < \mathfrak{s}$ and so 2^κ is sequentially compact. Similarly, it has character less than $\mathfrak{b}$ since $\kappa < \mathfrak{b}$. Therefore, 2^κ is ω_1-sequentially compact by Theorem 4.2. Conversely, $2^{\mathfrak{par}_2}$ is not even 2-sequentially compact by a result in [19]. ∎

The dual cardinal to $\mathfrak{par}$ is the cardinal $\mathfrak{hom}$. The characterization of $\mathfrak{par}$ in terms of $\mathfrak{b}$ and $\mathfrak{s}$ has also a dual version using $\mathfrak{d}$ and a variant of $\mathfrak{r}$, the dual cardinals of $\mathfrak{b}$ and $\mathfrak{s}$, respectively (see [2]).

Notation 6.6 *Let N be a countable set, $\mathcal{I}$ an ideal on N, and $X, Y \subseteq N$. Then, $X \subseteq_\mathcal{I} Y$ means that $X\setminus Y \in \mathcal{I}$.*

Definition 6.7 Given a barrier $\mathcal{B}$ let,

$$\mathfrak{hom}_\mathcal{B} = \min\{|\mathcal{H}| : \mathcal{H} \subseteq [\omega]^\omega \ \wedge\ \forall S \subseteq \mathcal{B}\ \exists M \in \mathcal{H}\ \exists i \in 2\ (\mathcal{B}|M \subseteq S^i)\}$$

and

$$\mathfrak{r}_\mathcal{B} = \min\{|\mathcal{R}| : \mathcal{R} \subseteq [\omega]^\omega\ \forall S \subseteq \mathcal{B}\ \exists R \in \mathcal{R}\ \exists i \in 2\ (\mathcal{B}|R \subseteq_{\mathsf{FIN}^\mathcal{B}} S^i)\}.$$

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A family as in the definition of $\mathfrak{hom}_{\mathcal{B}}$ is called a *homogeneous family for* $\mathcal{B}$ or a $\mathcal{B}$*-homogeneous family*. Analogously, a family as in the definition of $\mathfrak{r}_{\mathcal{B}}$ is called a *reaping family for* $\mathcal{B}$ or a $\mathcal{B}$*-reaping family*.

Recall that by the Nash-Williams theorem, for every $\pi : \mathcal{B} \to 2$, there is $M \in [\omega]^\omega$ such that $\pi \restriction (\mathcal{B}|M)$ is constant. In this way, $\mathfrak{hom}_{\mathcal{B}}$ is well-defined, and $\mathfrak{r}_{\mathcal{B}} \leq \mathfrak{hom}_{\mathcal{B}}$ since every $\mathcal{B}$-homogeneous family is a $\mathcal{B}$-reaping family. In particular, $\mathfrak{r}_{\mathcal{B}}$ is well-defined.

Note that if $\mathcal{B}$ is the trivial barrier (i.e., $\mathcal{B} = [\omega]^1$), then $\mathfrak{r}_{\mathcal{B}} = \mathfrak{hom}_{\mathcal{B}} = \mathfrak{r}$. However, if $\mathcal{B}$ is a nontrivial barrier, then $\mathfrak{hom}_2 \leq \mathfrak{hom}_{\mathcal{B}}$. To see this, we can suppose that every element of $\mathcal{B}$ has size at least 2, and thus, every coloring $\pi : [\omega]^2 \to 2$ induces a coloring of $\hat{\pi} : \mathcal{B} \to 2$ given by $\hat{\pi}(s) = \pi(s \restriction 2)$. Now an infinite homogeneous set for $\hat{\pi}$ is also homogeneous for π, which in turn gives that any homogeneous family for $\mathcal{B}$ is a homogeneous family for $[\omega]^2$, and then, $\max\{\mathfrak{d}, \mathfrak{r}_\sigma\} = \mathfrak{hom}_2 \leq \mathfrak{hom}_{\mathcal{B}}$. In particular, $\mathfrak{d} \leq \mathfrak{hom}_{\mathcal{B}}$. Ramsey ultrafilters will be helpful in our study of the cardinal $\mathfrak{hom}_{\mathcal{B}}$.

Recall that a nonprincipal ultrafilter $\mathcal{U}$ on ω is *Ramsey* if for all $\pi : [\omega]^2 \to 2$, there is $U \in \mathcal{U}$ such that $\pi \restriction [U]^2$ is constant. A family $\mathcal{F} \subseteq [\omega]^\omega$ is *selective* if for every decreasing sequence $\{Y_n \mid n \in \omega\} \subseteq \mathcal{F}$, there is $f : \omega \to \omega$ such that $f[\omega] \in \mathcal{F}$, $f(0) \in Y_0$, and $f(n+1) \in Y_{f(n)}$ for all $n \in \omega$. The following result, which according to D. Booth is mostly due to K. Kunen, relates Ramsey ultrafilters and selective families.

Theorem 6.8 *[4] If $\mathcal{U}$ is a nonprincipal ultrafilter, then $\mathcal{U}$ is Ramsey if and only if it is selective.*

We will show now that a base for a Ramsey ultrafilter is a homogeneous family for every barrier $\mathcal{B}$. Hence, if we define $\mathfrak{u}_R$ as the minimum character of a Ramsey ultrafilter,[1] we get that $\mathfrak{hom}_{\mathcal{B}} \leq \mathfrak{u}_R$.

The next theorem is a standard result of Local Ramsey theory (see [26] Chapter 7). It follows immediately from Theorem 7.42 in [26] in the same way Nash-Williams' theorem follows from Ellentuck theorem. For the convenience of reader, we sketch the argument.

Theorem 6.9 *If $\mathcal{U}$ is a Ramsey ultrafilter, $\mathcal{B}$ is a barrier on ω, and $\pi : \mathcal{B} \to 2$, then there is $U \in \mathcal{U}$ such that $\pi \restriction (\mathcal{B}|U)$ is constant.*

Proof By induction on the rank of the barrier. If $\mathcal{B}$ has rank zero (i.e, $\mathcal{B} = \{\emptyset\}$), then the result is trivial.

Suppose that $\mathcal{B}$ has rank γ and the result is true for every barrier of rank smaller than γ. As $B[n]$ is a barrier on $\omega \backslash (n+1)$ of rank less than γ, there is $V_n \in \mathcal{U}$ such that the coloring $\pi_n : B[n] \to 2$ is constant on $\mathcal{B}[n]|V_n$, where $\pi_n(s) = \pi(\{n\} \cup s)$.

For every $n \in \omega$, we know that $\pi_n \restriction (\mathcal{B}[n]|V_n)$ is constant on color $j(n) \in 2$. So take $V \in \mathcal{U}$ and $i \in 2$ such that $j(n) = i$ for all $n \in V$.

Now call $U_n = (\bigcap_{m \leq n} V_n) \cap V \cap (\omega \backslash n+1)$ and note that $\{U_n \mid n \in \omega\}$ is a decreasing sequence of elements of $\mathcal{U}$, so applying that $\mathcal{U}$ is selective, there is $f \in \omega^\omega$ such that $f[\omega] \in \mathcal{U}$, $f(0) \in U_0$, and $f(n+1) \in U_{f(n)}$ for all $n \in \omega$. Define $U := f[\omega]$.

[1] Let $\mathfrak{u}_R = \mathfrak{c}$ in case there is no such ultrafilter.

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It remains to show that $\pi \restriction (\mathcal{B}|U)$ is constant with value i. Indeed, let $s = \{a_0, \dots, a_k\} \in \mathcal{B}|U$. Then, $a_0 \in V$ and

$$\{a_1, \dots, a_k\} \subseteq U_{f(a_0)} \subseteq U_{a_0} \subseteq V_{a_0},$$

so that $\{a_1, \dots, a_k\} \in \mathcal{B}[a_0]|V_{a_0}$,. Then, $\pi(s) = \pi_{a_0}(\{a_1, \dots, a_k\}) = j(a_0) = i$. ∎

Notation 6.10 *If $R \in [\omega]^\omega$, then $\mathbb{H}(R)$ is the collection of all nonempty trees $T \subseteq R^{<\omega}$ such that $\mathrm{succ}_T(s)$ is cofinite in R for all $s \in T$.*

We will need the following folklore facts.

Proposition 6.11 *If $R \in [\omega]^\omega$ and $T \in \mathbb{H}(R)$, then there is $A \in [R]^\omega$ such that $[A]^\omega \subseteq [T]$.*

Proposition 6.12 $\mathrm{cof}(\mathsf{FIN}^{\mathcal{B}}) = \mathfrak{d}$ *for every nontrivial barrier $\mathcal{B}$.*

An explicit proof for the previous result about $\mathrm{cof}(\mathsf{FIN}^{\mathcal{B}})$, where $\mathcal{B}$ is a barrier of finite rank, appears in [7]. The same argument works for general barriers.

Theorem 6.13 $\mathfrak{hom}_{\mathcal{B}} = \max\{\mathfrak{d}, \mathfrak{r}_{\mathcal{B}}\}$ *for every nontrivial barrier $\mathcal{B}$ on ω.*

Proof We already noted that $\max\{\mathfrak{d}, \mathfrak{r}_{\mathcal{B}}\} \leq \mathfrak{hom}_{\mathcal{B}}$, so it is enough to prove the other inequality. For this, let $\mathcal{R} = \{R_\beta \mid \beta \in \mathfrak{r}_{\mathcal{B}}\}$ be a reaping family for $\mathcal{B}$, and for every $\beta \in \mathfrak{r}_{\mathcal{B}}$, let $\{X^\beta_\alpha \mid \alpha \in \mathfrak{d}\}$ cofinal in $\mathsf{FIN}^{\mathcal{B}|R_\beta}$. Now by definition of $\mathsf{FIN}^{\mathcal{B}|R_\beta}$ and Proposition 6.11, for every $\beta \in \mathfrak{r}_{\mathcal{B}}$ and every $\alpha \in \mathfrak{d}$, there are $T^\beta_\alpha \in \mathbb{H}(R_\beta)$ and $A^\beta_\alpha \in [R_\beta]^\omega$ such that

(1) $T^\beta_\alpha \cap X^\beta_\alpha = \emptyset$,
(2) $[A^\beta_\alpha]^\omega \subseteq [T^\beta_\alpha]$.

Now let $\mathcal{H} = \{A^\beta_\alpha \mid \beta \in \mathfrak{r}_\beta \wedge \alpha \in \mathfrak{d}\}$. We claim that $\mathcal{H}$ is a homogeneous family for $\mathcal{B}$. To see this, let $S \subseteq \mathcal{B}$. As $\mathcal{R}$ is a reaping family for $\mathcal{B}$, there is $\beta \in \mathfrak{r}_\beta$ and $i \in 2$ such that $\mathcal{B}|R_\beta \subseteq_{\mathsf{FIN}^{\mathcal{B}}} S^i$. Now call $X := (\mathcal{B}|R_\beta) \setminus S^i \in \mathsf{FIN}^{\mathcal{B}}$. Moreover, as $X \subseteq \mathcal{B}|R_\beta$, then $X \in \mathsf{FIN}^{\mathcal{B}|R_\beta}$. Thus, there is $\alpha \in \mathfrak{d}$ such that $X \subseteq X^\beta_\alpha$, and consequently, $X \cap T^\beta_\alpha = \emptyset$. Also we know that $[A^\beta_\alpha]^\omega \subseteq [T^\beta_\alpha]$, so $X \cap (\mathcal{B}|A^\beta_\alpha) = \emptyset$. Indeed, if $b \in X \cap (\mathcal{B}|A^\beta_\alpha)$, then there is $Z \in [A^\beta_\alpha]^\omega$ such that $b \sqsubseteq Z$, and, as $Z \in [T^\beta_\alpha]$, then $b \in T^\beta_\alpha \cap X$, which is a contradiction. This way, we have that $\mathcal{B}|A^\beta_\alpha \subseteq \mathcal{B}|R_\beta \subseteq S^i \cup X$ and $(\mathcal{B}|A^\beta_\alpha) \cap X = \emptyset$, so $\mathcal{B}|A^\beta_\alpha \subseteq S^i$, and we are done. ∎

Question 6.14 *Is $\mathfrak{hom}_{\mathcal{B}} = \mathfrak{hom}_2$ for every nontrivial barrier $\mathcal{B}$?*

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