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Constructions of arcbodies

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1. *Introduction*

The concept of *arcbody* was introduced in [4] as a generalization of prime tangles used by Lickorish [3] and Bleiler [1]. The pair (A, l), where A is a compact 3-manifold with boundary ∂A, and l is a 1-submanifold properly embedded in A, is an arcbody if:

(i) the inclusion $\partial A - l \hookrightarrow A - l$ is monic, i.e. $A - l$ has incompressible boundary;
(ii) no component of (A, l) is homeomorphic to $(D^2, 0) \times I$ or $(D^2, 0) \times S^1$; and
(iii) every sphere of ∂A intersects l.

In this paper we give some technical results which serve to recognize when a pair (A, l) is an arcbody. In particular, in § 2 we define a disc-sum of arcbodies and we obtain families of arcbodies which generalize the ones obtained in [5] from fractures of 3-braid links. In § 3, we give a method for checking if a pair (A, l) is an arcbody, when A is a handlebody and l is a set of curves parallel to the boundary.

Given a fracturable link L, the possible closed incompressible surfaces in the exterior of L are obtained by 'tubing' the surfaces that fracture L. In [4] it is proved that, after tubing, one of the two pairs bounded by the tubed surface is still an arcbody ([4], lemma 1·1). The other part, which has been 'drilled', may or may not be an arcbody. In § 4, we analyse some cases of 'drilling', and we give also applications and examples.

Throughout the paper, we work in the PL-category. The notation $U(-)$ will denote a small enough closed regular neighbourhood of $(-)$. As usual, $\bar{A}$ is the closure of A.

Let (A, l) be an arcbody. A properly embedded disc D in A, intersecting l transversally in exactly one interior point, is a cutting-disc if D does not separate A into two parts such that one of them is a 3-ball intersecting l in a spanning arc.

2. *Disc-sum of arcbodies*

Definition 2·1. Let (A, l_A) be an arcbody. An arc α in A, joining a point a_1 in l_A with a point a_2 in ∂A and such that $\operatorname{int}\alpha \subset \operatorname{int}(A - l_A)$, is called an *adding-arc* if α is not contained in any properly embedded disc D in A intersecting l_A transversally in just one point.

Denote by $U(\alpha)$ a regular neighbourhood of the adding-arc α in A such that $(U(\alpha), l_A \cap U(\alpha))$ is homeomorphic to $(D^2 \times I, \{0\} \times I\}$. Then $\Delta_\alpha = \partial U(\alpha) - \partial A$ is a disc intersecting l_A transversally in two points.

Definition 2·2. Let α, β be adding-arcs in the arcbodies (A, l_A) and (B, l_B) respectively. The *disc-sum* $\Delta(\alpha, \beta, f)$ of (A, l_A) and (B, l_B) is the pair $(A \Delta_f B, l_A \Delta_f l_B)$, where $A \Delta_f B$

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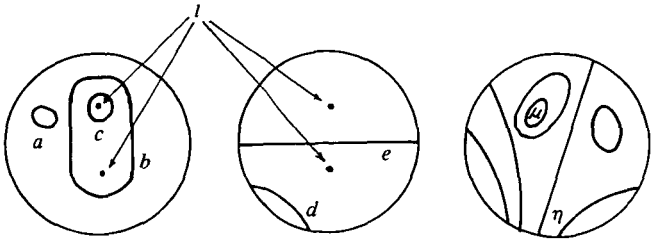


Fig. 2.1. $D \cap \Delta$.

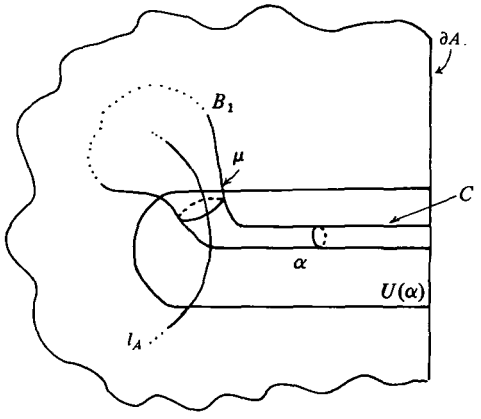


Fig. 2.2

is the result of pasting $\overline{A - U(\alpha)}$ and $\overline{B - U(\beta)}$ along Δ_α and Δ_β by a homeomorphism $f\colon \Delta_\alpha \to \Delta_\beta$ which sends $\Delta_\alpha \cap l_A$ to $\Delta_\beta \cap l_B$, and $l_A \Delta_f l_B$ is the result of pasting

$$l_A \cap \overline{(A - U(\alpha))} \quad \text{and} \quad l_B \cap \overline{(B - U(\beta))}$$

by $f|_{\Delta_\alpha \cap l_A}$.

Note that, given α and β, there exist two different homeomorphisms f (up to isotopy).

Definition 2·3. An arcbody (A, l) is *irreducible* if the manifold $A - l$ is irreducible, i.e. every embedded 2-sphere in $A - l$ bounds a 3-ball in $A - l$.

THEOREM 2·1. *The disc-sum of two irreducible arcbodies is an irreducible arcbody.*

Proof. Let (M, l) be the disc-sum $\Delta(\alpha, \beta, f)$ of the irreducible arcbodies (A, l_A) and (B, l_B). We shall prove that the inclusion $\partial M - l \hookrightarrow M - l$ is a monic map. Let γ be a simple closed curve in $\partial M - l$ bounding a properly embedded disc D in $M - l$. The intersection of D with the pasting disc $\Delta = \Delta_\alpha \equiv_f \Delta_\beta$ consists of closed simple curves and arcs. According to their relative position with respect to $\Delta \cap l$, they can be classified into the five classes shown in Fig. 2·1 (i) and (ii), and denoted by a, b, c, d, e.

Let μ be a closed simple curve of $D \cap \Delta$ innermost in D. Then μ bounds a disc B_1 in D, contained in $A - l_A$ or $B - l_B$ (say $A - l_A$).

Next we prove that if μ belongs to the class a, it can be eliminated by isotopy of D. A curve of class a bounds a disc B_2 in $\Delta - l$. Then the 2-sphere $S = B_1 \cup_\mu B_2$ is contained in the irreducible manifold $A - l_A$. Thus, S bounds a 3-ball in $A - l_A$, hence in $M - l$, which can be used to isotope B_1 over B_2 eliminating μ. If μ is not innermost in Δ, this isotopy reduces more than one intersection curve.

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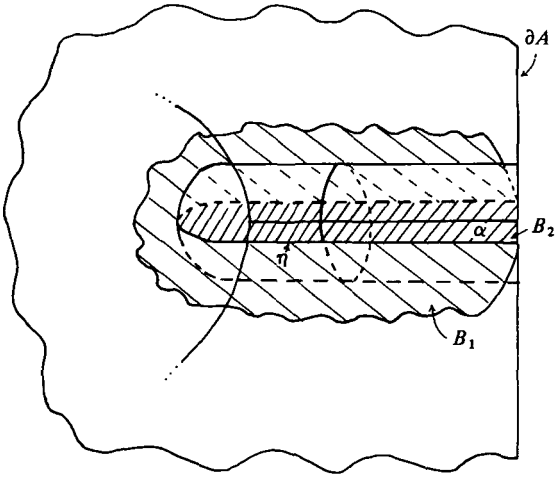


Fig. 2·3

Suppose μ is in the class b. Let B_2 be a properly embedded disc in $U(\alpha)$, cutting α transversally in exactly one point and bounded by μ. Then $S = B_1 \cup_\mu B_2$ is a 2-sphere in $A - l_A$, which has to be the boundary of a 3-ball in $A - l_A$, because $A - l_A$ is irreducible. But a_1 and a_2 are in different components of $A - S$. Hence neither of these components can intersect $A - l_A$ in a 3-ball. This contradiction shows that μ cannot be in the class b.

Suppose μ belongs to the class c. Let C be an annulus properly embedded in $U(\alpha)$ such that C contains α, C cuts l_A transversally in only one point (the origin of α) and μ is one of the two components of ∂C. (See Fig. 2·2.)

Then $B_1 \cup C$ is a properly embedded disc in A, cutting l_A transversally in a_1, and containing α. This contradicts the condition of adding-arc for α. Therefore μ cannot be in the class c.

Let η be an arc of $D \cap \Delta$ outermost in D. Then η together with an arc λ_1 in ∂D bounds a disc B_1 in D whose interior does not intersect Δ. Assume, for instance, that B_1 is contained in $A - l_A$.

If η is in the class d it can be eliminated by isotopy. To prove it, consider the arc λ_2 in $\partial\Delta$ such that $\eta \cup \lambda_2$ bounds a disc B_2 in $\Delta - l$. Then $\lambda_1 \cup \lambda_2$ is a closed curve in $A - l_A$ bounding the disc $B_1 \cup B_2$ in $A - l_A$. As (A, l_A) is arcbody, $\lambda_1 \cup \lambda_2$ is the boundary of a disc B_3 in ∂A. Then the 2-sphere $B_1 \cup B_2 \cup B_3$ contained in $A - l_A$ bounds a 3-ball in $A - l_A$, which can be used to isotope $B_1 \cup B_3$ over B_2 eliminating the intersection η. If η is not outermost in Δ, this isotopy reduces more than one intersection.

Let us prove that η cannot belong to the class e. If so, consider a properly embedded disc B_2 in $U(\alpha)$ containing α, cutting l_A transversally in only one point a_1 (an end point of α), and such that $\partial B_2 = \eta \cup \lambda_2$, where λ_2 is an arc contained in ∂A (Fig. 2·3). Then $B_1 \cup_\eta B_2$ is a disc properly embedded in A, containing α and intersecting l_A transversally in a_1. This contradicts the fact that α is an adding-arc.

We have proved that we can eliminate all intersection curves and arcs in $D \cap \Delta$ by isotopy, and we can assume that D is contained in $\overline{A - U(\alpha)} - l_A$ or $\overline{B - U(\beta)} - l_B$ (say in $\overline{A - U(\alpha)} - l_A$). As (A, l_A) is an arcbody, γ bounds a disc E in $\partial A - l_A$. In order to demonstrate that E is contained in $\partial M - l$, we have to show that a_2 (end-point of α)

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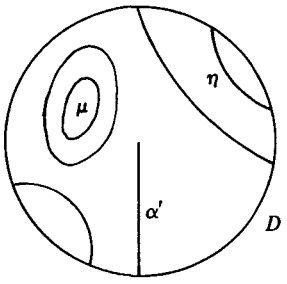


Fig. 2·4. $D \cap \Delta$.

is not in E. For, if it is, α is contained in the 3-ball B^3 bounded by $E \cup D$ (this 3-ball B^3 exists and is contained in $A - l_A$, because $A - l_A$ is irreducible). Since $a_1 = l_A \cap \alpha$ is in B^3, and B^3 is contained in $A - l_A$, this is a contradiction.

Showing that (M, l) satisfies conditions (ii) and (iii) of the definition of an arcbody is an easy exercise.

Next, we prove that (M, l) is irreducible. Let S be a 2-sphere embedded in $M - l$. The intersection of S with Δ consists of closed curves, in the classes a, b, c according to their relative position with respect to $\Delta \cap l$. (See Fig. 2·1 (i).)

As has been shown above, intersection curves in the class a can be eliminated by isotopy. Intersection curves in the class b contradict the fact that (A, l_A) and (B, l_B) are irreducible. And intersection curves of class c are impossible because α and β are adding-arcs. Then, after isotopy, we can assume that S is contained in $A - U(\alpha) - l_A$ or $B - U(\beta) - l_B$. By the irreducibility of (A, l_A) and (B, l_B), S bounds a 3-ball in $M - l$. |

LEMMA 2·1. *Let (M, l) be the disc-sum $\Delta(\alpha, \beta, f)$ of the irreducible arcbodies (A, l_A) and (B, l_B). If α' is an adding-arc in (A, l_A), disjoint from α, then the image of α' by the inclusion map $A - U(\alpha) \hookrightarrow M$ is an adding-arc in (M, l).*

Proof. Suppose α' is not an adding-arc in (M, l). Then, there exists a properly embedded disc D in M containing α' and intersecting l transversally in one point (necessarily an end point a_1' of α'). All the intersection curves and arcs of D with Δ can be eliminated in the same way as in the proof of Theorem 2·1, because all of them are disjoint from α'. (See Fig. 2·4.)

Then, after isotopy, D is contained in $A - U(\alpha)$. Thus α' is not an adding-arc in (A, l_A). |

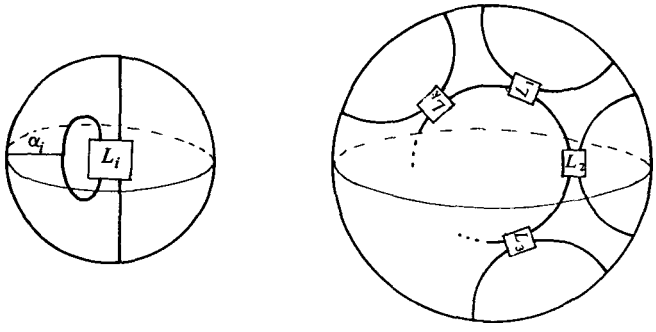
Next, we present a family of arcbodies, coming from non-split non-trivial prime links in S^3, which we shall use to construct more complicated arcbodies through disc-sum.

LEMMA 2·2. *Let x_0 be a point on a non-split non-trivial prime link $\hat{L}$ in S^3. Then the pair (A_L, L), where A_L is the exterior in S^3 of a regular neighbourhood $U(x_0)$ of x_0, and L is $\hat{L} \cap A_L$, is an irreducible arcbody with no cutting-discs.*

Proof. (A_L, L) is an arcbody because $\hat{L}$ is not the trivial knot. It is irreducible because A_L is a 3-ball and $\hat{L}$ is non-split. Since $\hat{L}$ is prime, this arcbody has no cutting-discs.

In Fig. 2·5 (*a*) we represent a general arcbody (A_{L_i}, L_i) in the class of the arcbodies defined in Lemma 2·2, where an explicit arc α_i has been chosen.

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(*a*) (*b*)

Fig. 2·5

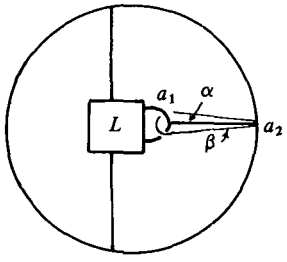


Fig. 2·6

COROLLARY 2·1. *The pair* $(A_{L_1,\ldots,L_E}, L)$ *of* Fig. 2·5(*b*), *where* α_i *is an adding-arc in* (A_{L_i}, L_i) *of* Fig. 2·5(*a*), *is an irreducible arcbody.*

These arcbodies $(A_{L_1,\ldots,L_k}, L)$ generalize the ones obtained from 3-braids [5], where $\hat{L}_i$ were toroidal links $(2,n)$ with fixed α_i.

There exists a useful criterion to determine whether an arc α_i in A_{L_i} of Fig. 2·5(*a*) is an adding-arc: *Let* (A,l) *be an arcbody with no cutting-discs, and let* α *be an arc joining* $a_1 \in l$ *with* $a_2 \in \partial A$ *such that* $\operatorname{int}\alpha \subset \operatorname{int}(A-l)$. *Then* α *is an adding-arc if and only if* α *is not parallel to* $l \cup \partial A$, *i.e. there does not exist a disc* Δ *embedded in* A *such that*

$$\operatorname{int}\Delta \subset \operatorname{int}(A-l) \quad \textit{and} \quad \partial\Delta = \alpha \cup \beta, \quad \textit{where} \quad \beta \subset l \cup \partial A.$$

For if such a disc exists, it is clear that α is not an adding-arc. Conversely, if α is not an adding-arc, there exists a properly embedded disc D in A containing α and cutting l transversally in just one point (a_1, necessarily). As D cannot be a cutting-disc, D separates A in two parts, one of which is a 3-ball B^3 intersecting l in a spanning arc. Then there exists a disc Δ in B^3 which makes α parallel to $l \cup \partial A$.

Now, to prove that α in (A_L, L) (see Fig. 2·6) is not parallel to $L \cup \partial A_L$, it is enough to prove that the homotopy class β of the loop $\bar{\alpha} * m_{a_1} * \alpha$ is not a peripheral element of $\pi_1(\overline{A_L - U(L)}, a_2)$. Here $\bar{\alpha}$ is the arc α starting in a_2 and ending in a_1, and m_{a_1} is a meridian of L close to a_1. Note that the base point a_2 for π_1 is important.

Example 2·1. Let $\hat{L}$ be the trefoil knot. We are going to prove that the arc α of Fig. 2·7 is an adding-arc.

The group $G = \pi_1(\overline{A_L - U(L)}, a_2)$ has the following presentation:

$$G = \langle x, y | xyx = yxy \rangle,$$

and y is the homotopy class of $\bar{\alpha} * m_{a_1} * \alpha$.

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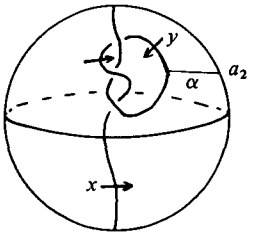


Fig. 2·7

Let ω be the representation of G in the permutation group of three symbols Σ_3 given by

$$\omega: G \to \Sigma_3$$
$$x \mapsto (1,2)$$
$$y \mapsto (2,3).$$

The peripheral subgroup H based in a_2 is generated by x and $xy^{-1}x^{-1}x^{-1}y^{-1}$. Then $\omega(H) = ((1,2), \text{identity})$. As $\omega(y) = (2,3)$ does not belong to $\omega(H)$, we conclude that y is not a peripheral element, therefore that α is an adding-arc.

3. *Arcbodies whose curves are parallel to the boundary*

Definition 3·1. Let M be a 3-manifold with boundary ∂M, and let L be a compact 1-submanifold properly embedded in M. (L is a union of closed curves L_i and arcs L'_j.) Then L *is parallel to the boundary* if there is a copy F of $L \times I$ embedded in M with $L \times 0 \equiv L$ and the remainder $\partial(L \times I)$, which will be denoted by $\tilde{L}$, lies in the boundary ∂M.

Note that if (M, L) is an arcbody and L is parallel to the boundary, L only has closed curves.

LEMMA 3·1. *Let (A, L) be an irreducible arcbody with L parallel to the boundary. Then $\partial A - \tilde{L}$ is incompressible in A.*

Proof. Let $U(F)$ be a regular neighbourhood of F in A. As the pair $(A, \partial A - \tilde{L})$ is homeomorphic to $(\overline{A - U(F)}, \partial A - U(F))$, $\partial A - \tilde{L}$ is incompressible in A if and only if $\partial A - U(F)$ is incompressible in $\overline{A - U(F)}$.

To prove that $\partial A - U(F)$ is incompressible in $\overline{A - U(F)}$, consider a disc D properly embedded in $\overline{A - U(F)}$, whose boundary $\alpha = \partial D$ lies in $\partial A - U(F)$. As (A, L) is arcbody, and α bounds a disc in $A - L$, α bounds a disc Δ in ∂A. Two cases are possible:

(1) The disc Δ does not intersect $U(F)$. Then α is inessential in $\partial A - U(F)$. Thus D is not a compression disc.

(2) The disc Δ intersects $U(F)$. Then Δ contains at least one component $\tilde{L}_i$ of $\tilde{L}$. The 2-sphere $D \cup \Delta$ is contained in $A - L$, and bounds a 3-ball B in $A - L$, because (A, L) is irreducible. Then every arc γ in F joining L_i and $\tilde{L}_i$ must interest $D \subset \partial B$. Thus $U(F) \cap D$ is non-empty. This contradicts the choice of D. Hence case (2) is not possible. ∎

The following Lemma establishes, under certain conditions, a converse to Lemma 3·1.

LEMMA 3·2. *Let M be an irreducible compact connected 3-manifold whose boundary ∂M is a connected surface of positive genus. Let L be a set of pairwise disjoint simple closed*

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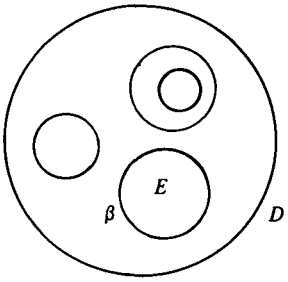


Fig. 3·1

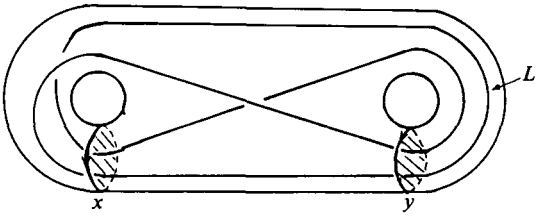


Fig. 3·2

essential curves in ∂M. *We assume also that if* M *is* $D^2 \times S^1$, L *is not* $\{x\} \times S^1$. *If* $\partial M - L$ *is incompressible in* M, *then the pair* (M', L) *is an arcbody, where* M' *is* M *collared* (M' *is the result of pasting* M *to* $\partial M \times [0,1]$ *along* ∂M *and* $\partial M \times \{0\}$ *respectively*).

Proof. Let us prove that $i\colon \partial M' \hookrightarrow M' - L$ is a monic map. Suppose γ is a simple closed curve in $\partial M'$ bounding a disc D properly embedded in $M' - L$. This disc D intersects $\partial M \equiv \partial M \times \{0\}$ in simple closed curves (see Fig. 3·1). Let β be one of them innermost in D. Then β bounds a disc E contained in D and lying in M or in $\partial M \times [0,1]$. In both cases there exists a disc Δ contained in $\partial M \times \{0\} \equiv \partial M$, which does not intersect L and is bounded by β. This is because $\partial M - L$ is incompressible in M, and $\partial M \times [0,1]$ is a 3-manifold with incompressible boundary and L consists of essential curves. The union $E \cup \Delta$ is a 2-sphere bounding a 3-ball B in M or in $\partial M \times [0,1]$, because these 3-manifolds are irreducible. Then B permits us to isotope E over Δ eliminating the intersection curve β.

Therefore, after all the intersection curves of $D \cap (\partial M \times \{0\})$ have been eliminated by isotopy, D is contained in $\partial M \times [0,1]$, which has incompressible boundary. Hence γ bounds a disc in $\partial M \times \{1\} = \partial M'$ proving that i is a monic map.

Conditions (ii) and (iii) of the definition of an arcbody are easy consequences of the hypothesis of the Lemma. ∎

These two Lemmas and Lyon's algorithm [6] allow us to check easily whether a pair (H_n, L), where H_n is a handlebody ($n \geqslant 1$) and L is a family of closed curves in H_n parallel to the boundary, is or is not an irreducible arcbody. The result is given in the following Corollary.

COROLLARY 3·1. *The pair* (H_n, L), *where* L *is a set of closed curves parallel to the boundary in the handlebody* H_n ($n \geqslant 1$), *is an irreducible arcbody if and only if the following conditions hold*:

(1) (H_n, L) *is not homeomorphic to* $(D^2 \times S^1, \{x\} \times S^1)$.

(2) *No component of* L *bounds a disc in* H_n.

(3) $H_n - \tilde{L}$ *is incompressible in* M. (*Use Lyon's algorithm.*)

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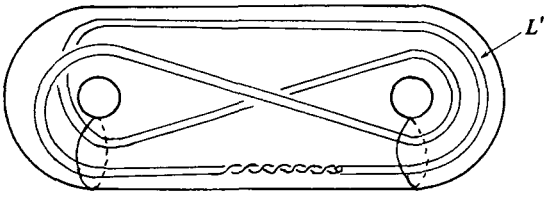


Fig. 4·1

Example 3·1. Consider (H_2, L) of Fig. 3·2 which obviously satisfies (1) and (2). Let x, y be the generators of the first homotopy group of H_2, whose duals are the discs represented in Fig. 3·2 by the same notation x, y. Then $\tilde{L}$ belongs to the homotopy class $xyxy^{-1}$, which binds (it uses all the letters and it is length irreducible; see [6]). Therefore (H_2, L) is an irreducible arcbody.

4. *Drilling arcbodies*

The following Lemma assures that the result of drilling closed components of L in an arcbody (A, L) with certain properties, is an arcbody. This fact can be used to modify L without changing A (see Corollary 4·1).

LEMMA 4·1. *Let (A, L) be an irreducible arcbody such that A does not contain non-separating 2-spheres and let l be one closed component of L $(L = l \cup L_1)$. Then*

$$(A - \operatorname{int} U(l), L_1)$$

is an irreducible arcbody such that $A - \operatorname{int} U(l)$ does not contain non-separating 2-spheres.

Proof. Let α be an essential simple closed curve contained in $\partial U(l)$ and assume α bounds an embedded disc D in $A - \operatorname{int} U(l) - L_1$, then $\partial(U(D) \cup U(l))$ does not bound a 3-ball. ∎

As $(A - \operatorname{int} U(l), L_1)$ still satisfies the hypothesis of Lemma 4·1 we can again drill any closed component of L_1, if such exists, and obtain also an irreducible arcbody with no non-separating 2-spheres.

COROLLARY 4·1. *Let (V, L_V), (A, L) be irreducible arcbodies such that V is a solid torus and there are no non-separating 2-spheres in A. Let $h\colon V \to U(l)$ be a homeomorphism, where l is a closed component of L $(L = l \cup L_1)$. Then the pair $(A, L_1 \cup h(L_V))$ is an irreducible arcbody.*

Proof. By Lemma 4·1, the pair $(A - \operatorname{int} U(l), L_1)$ is an irreducible arcbody. The pair $(A, L_1 \cup h(L_V))$ is the result of pasting (V_1, L_V) to $(A - \operatorname{int} U(l), L_1)$ along $h|_{\partial V}\colon \partial V \to \partial U(l)$. Then $(A, L_1 \cup h(L_V))$ is an arcbody ([4], Lemma 1·3) and it is irreducible. ∎

Note that l is a companion of $h(L_V)$. (See [7], p. 110.)

Example 4·1. The pair (H_2, L') of Fig. 4·1 is an irreducible arcbody, because L of the irreducible arcbody (H_2, L) of Example 3·1 (Fig. 3·2), is a companion of L'.

LEMMA 4·2. *Let (M, L) be a pair where M is an irreducible connected compact 3-manifold with incompressible boundary, and L is an arc properly embedded in M. Then (M, L) is an arcbody if $\partial M - L$ does not contain a 2-sphere, and L, followed by an arc in ∂M connecting the two points of ∂L, represents a non-peripheral element of $\pi_1(M, x_0)$, where x_0 is some point of ∂L.*

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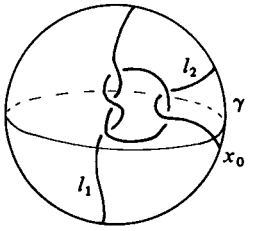


Fig. 4·2

Proof. To show that (M, L) satisfies property (i) of the definition of an arcbody, i.e. $\partial M - L \hookrightarrow M - L$ is a monic map, assume α is an essential loop in $\partial M - L$. If α is essential in ∂M, α does not bound a disc in M, because M has incompressible boundary. Then α is not contractible in M, hence is not contractible in $M - L$. If α is inessential in ∂M, there exists a disc D in ∂M, bounded by α, and such that D contains at least one point of L, because α is essential in $\partial M - L$. Two cases are possible:

(1) D intersects L in one point. Then α is not contractible in $M - L$. For if Δ is a disc properly embedded in $M - L$, bounded by α, the 2-sphere $D \cup_\alpha \Delta$ intersects L in one point, and is the boundary of a 3-ball in M (recall M is irreducible), which is impossible.

(2) D contains the two points of ∂L. Suppose α is contractible in $M - L$. Then there exists a disc Δ in $M - L$, bounded by α. The 2-sphere $D \cup \Delta$ bounds a 3-ball B^3 which contains L. An isotopy sending Δ to D along B^3 projects L to a (singular) arc β in D. Let γ be an arc in ∂M connecting the two points of ∂L. Then the loop $L * \gamma * \gamma^{-1} * \beta^{-1}$ is null-homotopic, hence $L * \gamma$ is homotopic to $\beta * \gamma$, which is a peripheral element of $\pi_1(M, x_0)$. This contradicts the hypothesis of the Lemma. Therefore α is not contractible in $M - L$.

The pair (M, L) satisfies conditions (ii) and (iii) of the definition of an arcbody, because M has incompressible boundary and $\partial M - L$ does not contain a 2-sphere. ∎

Example 4·2. Consider the arcbody $(A, l_1 \cup l_2)$ of Fig. 4.2. We are going to use the preceding Lemma to prove that $(\overline{A - U(l_1)}, l_2)$, the result of drilling l_1, is an arcbody. The manifold $\overline{A - U(l_1)}$ has incompressible boundary, because it is the exterior of the trefoil knot.

The loop $l_2 * \gamma$ represents the element y of Example 2·1, and it was proved there that y is not a peripheral element in $\pi_1(\overline{A - U(l_1)}, x_0)$. However, the pair $(\overline{A - U(l_2)}, l_1)$, the result of drilling l_2, is not an arcbody.

The following theorem gives another sufficient condition for obtaining an arcbody after drilling an arc in an arcbody.

THEOREM 4·1. *Let (A, l) be an arcbody, where A is either a 3-ball or a 3-manifold with compressible boundary, and l is an arc. Then $(A - U(l), \varnothing)$ is an arcbody.*

Proof. If A is a 3-ball, $\overline{A - U(l)}$ is the exterior of a non-trivial knot, then it has incompressible boundary, hence $(\overline{A - U(l)}, \varnothing)$ is an arcbody (see Fig. 4·3).

If A is a 3-manifold with compressible boundary, the result is a consequence of the following theorem of Jaco ([2], theorem 2).

THEOREM (Jaco). *Let M be a 3-manifold and let J be a simple closed curve in ∂M such that $\partial M - J$ is incompressible. If M has compressible boundary, then the 3-manifold obtained by adding a 2-handle to M along J has incompressible boundary.*

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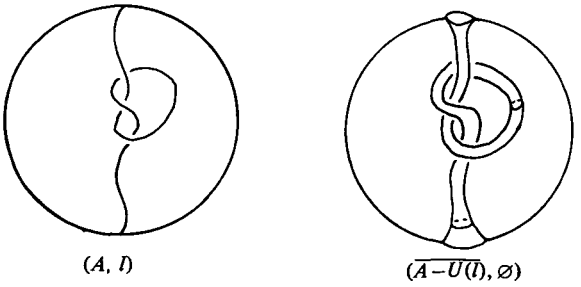


Fig. 4·3

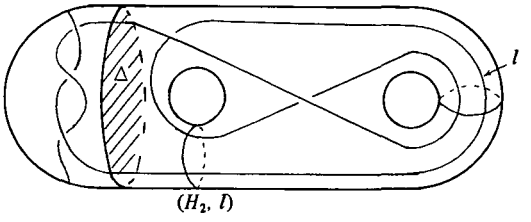


Fig. 4·4

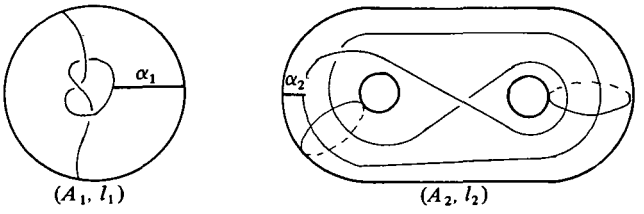


Fig. 4·5

Let M be $\overline{A - U(l)}$, and J the core of the annulus $\partial U(l) - \partial A$. Then, condition (i) for (A, l), i.e. the fact that $\partial A - l \hookrightarrow A - l$ is a monic map, implies that $\partial M - J$ is incompressible in M. As the 3-manifold obtained by adding a 2-handle to M along J is A, it has compressible boundary. Then the above theorem implies that M (i.e. $\overline{A - U(l)}$) has incompressible boundary. Hence $(\overline{A - U(l)}, \varnothing)$ is an arcbody. |

Example 4·3. The pair $(A, \varnothing) = (\overline{H_2 - U(l)}, \varnothing)$, where (H_2, l) is the pair of Fig. 4·4, is an irreducible arcbody. This is because (H_2, l) is an irreducible arcbody. It is the disc sum of the irreducible arcbodies (A_1, l_1) and (A_2, l_2) of Fig. 4·5.

Recall that (A_1, l_1) is an arcbody by Lemma 2·2, and it was shown in Example 2·1 that α_1 is an adding-disc. The pair (A_2, l_2) is the arcbody (H_2, L) of Example 3·1, and α_2 is an adding-arc because all properly embedded discs in H_2, intersecting L transversally, contain an even number of points of L.

It remains as an open problem when drilling arcs in a general arcbody produces an arcbody. The complete solution for arcbodies $(A_{L_1, \ldots, L_k}, L)$ coming from 3-braids may be deduced from [5]: the only case of drilling which does not give an arcbody is when the drilling arc is the arc in (A_{L_i}, L_i), where $\hat{L}_i$ is the Hopf link of two components.

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REFERENCES

[1] S. A. Bleiler. Knots prime on many strings. *Trans. Amer. Math. Soc.* **282** (1984), 385–402.
[2] W. Jaco. Adding a 2-handle to a 3-manifold: an application to property *R*. *Proc. Amer. Math. Soc.* **92** (1984), 288–292.
[3] W. B. R. Lickorish. Prime knots and tangles. *Trans. Amer. Math. Soc.* **267** (1981), 321-332.
[4] M. T. Lozano. Arcbodies. *Math. Proc. Cambridge Philos. Soc.* **94** (1983), 253–260.
[5] M. T. Lozano and J. Przytycki. Incompressible surfaces in the exterior of a closed 3-braid. I. *Math. Proc. Cambridge Philos. Soc.* **98** (1985), 275–299.
[6] C. H. Lyon. Incompressible surfaces in the boundary of a handlebody – an algorithm. *Canad. J. Math.* **32** (1980), 590–595.
[7] D. Rolfsen. *Knots and Links* (Publish or Perish, Inc., 1976).

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