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The set of uniquely ergodic interval exchange transformations is path-connected

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Abstract. Let π be a non-degenerate permutation on at least four symbols. We show that the set of uniquely ergodic interval exchange transformations with permutation π is path-connected.

1. *Introduction*

An interval exchange transformation (from now on abbreviated to *IET*) is a piecewise isometric map of an interval to itself that rearranges sub-intervals according to a permutation π (see §2 for formal definitions). While simple to define, interval exchange transformations have deep and interesting dynamical properties as well as connections to foliations on Riemann surfaces, billiards in polygons and smooth flows on surfaces and therefore have been the subject of intense study over the last years. See [**Z06**], [**Y10**] or [**V06**] for good surveys.

In this article we are concerned not with the dynamics of a single IET, but the set of all IETs with a given permutation π of n symbols. This set carries a natural topology and is homeomorphic to an Euclidean simplex Δ_π of dimension $n-1$. It is known that unless the permutation has obvious combinatorial obstructions most IETs are uniquely ergodic.

THEOREM. (Masur [**M82**], Veech [**V82**]) *Given an irreducible permutation, almost every IET with that permutation is uniquely ergodic with respect to Lebesgue measure.*

On the other hand, it is known that the set of non-uniquely ergodic IETs is also fairly large in a geometric sense.

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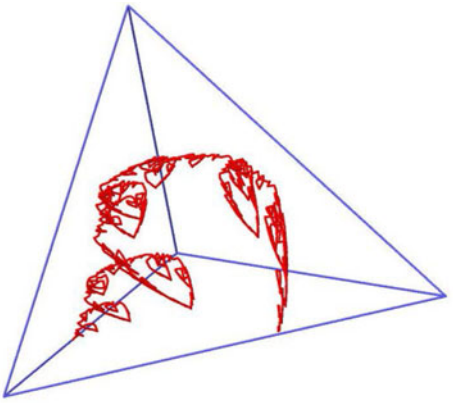


FIGURE 1. A path of uniquely ergodic 4-IETs.

THEOREM. (Masur and Smillie [**MS91**]) *The set of minimal and not uniquely ergodic IETs with a non-degenerate permutation on n intervals has Hausdorff dimension greater than* $n - 2$.

The Hausdorff dimension of minimal and not uniquely ergodic 4-IETs has been computed by the Athreya and the first named author.

THEOREM. (Athreya and Chaika [**AC**]) *The set of minimal and not uniquely ergodic IETs with a non-degenerate permutation on* 4 *intervals has Hausdorff dimension* $5/2$.

In this article, we investigate the set of uniquely ergodic IETs from a topological point of view. More precisely, the main result of this paper is the following theorem.

THEOREM 1.1. *Let* $n \geq 4$ *and let* π *be any non-degenerate permutation on n symbols. Then the set of uniquely ergodic unit length IETs with permutation* π *is path-connected.*

See Figure 1 for a path of uniquely ergodic 4-IETs.

For the formal definition of non-degenerate permutations, see §2. Intuitively, non-degenerate means that no induced map on a sub-interval has less than n singularities. In the case of $n = 2$, the space of 2-IETs is equal to the space of rotations; where such an IET is uniquely ergodic if and only if the rotation angle is irrational, which is clearly a disconnected subset.

3-IETs can be thought of as the induced map (see Definition 2.3) of a rotation by $(1 - L_1)/(1 + L_2)$ on $[0, 1/1 + L_2)$ (compare [**KS67**, **K75**]) and they are uniquely ergodic if and only if $(1 - L_1)/(1 + L_2)$ is irrational. So a set $(1 - L_1)/(1 + L_2) = y \in \mathbb{Q}$ gives a curve (or plane if one does not normalize the lengths of an IET) of non-minimal (and so not uniquely ergodic) IETs that disconnect the space of 3-IETs.

Thus the bound on n in Theorem 1.1 is optimal.

Outline of proof. The proof of Theorem 1.1 begins with a reduction to the case of $n = 4$. Namely, the space of n-IETs contains many copies of the space of 4-IETs by setting the length of enough intervals to be zero—and in these subspaces we know that

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the uniquely ergodic IETs are path-connected. Studying the combinatorics of a non-degenerate permutation, we can show that the union U of all these subspaces is path-connected. Using a limiting argument, we can then show that each uniquely ergodic IET can be connected by a path to a point in U.

In the case of $n = 4$, we are able to give an explicit inductive procedure that constructs paths in the space of uniquely ergodic IETs. Unique ergodicity of an IET T can be detected by the Rauzy induction of T. Thus, ideally, one would want to construct paths c where Rauzy induction has desirable properties for each $c(t)$.

A key problem, however, is that Rauzy induction is undefined on a countable set of codimension one planes in the simplex. Continuous paths necessarily cross these 'fail planes' and, in general, most points on such a plane do not correspond to uniquely ergodic IETs.

This suggests the following naive strategy. Start by joining two IETs S, T by a straight line segment $S \to T$. If this line intersects the plane where Rauzy induction fails, select a uniquely ergodic point R on that plane, and replace the line segment by a concatenation $S \to R \to T$ of two segments. Now both $S \to R$ and $R \to T$ share the same first Rauzy induction step. We can continue this process iteratively, always replacing straight segments by concatenations after some number of Rauzy steps.

There are two main issues with this approach: why does the sequence of approximate paths converge, and why are all points on it uniquely ergodic?

We deal with both of these issues simultaneously by taking care how we choose the intermediate IETs on the fail planes. More precisely, we want that limit points which do not eventually follow the Rauzy expansion of one of the points on a fail plane to have infinitely many matrices which define contracting maps on the simplex in their Rauzy expansion (see §6). This will yield both unique ergodicity and continuity at these limit points. For the points on the fail planes, unique ergodicity is clear from the construction, but continuity of the limit path at these points needs to be proved by a different argument (see §5).

Section 2 of the article describes some necessary background on IETs and Rauzy induction, while §3 contains some basic results and notation on matrices that is used throughout.

Section 4 describes the basic mechanism used to construct the paths in the case $n = 4$ (depending on an explicit construction done in §8). The necessary results to show continuity at points on 'fail planes' are developed in §5, and the results for other points are developed in §6. Finally, §7 contains the proof of Theorem 1.1 for $n = 4$. In §9, we extend the results to n-IETs.

Questions. The work in this article suggests several possible directions of further research. One possibility is to ask further topological questions about sets of specific IETs. In particular, we ask the following.

Question 1. (Mladen Bestvina) In the case of n-IETs for $n \geq 5$, does the set of uniquely ergodic n-IETs satisfy higher connectivity properties?

Question 2. Is the set of minimal n-IETs path-connected?

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Another example is a question that inspired this work. The space of measured foliations on a Riemann surface of genus g is homeomorphic to a sphere of dimension $6g-7$. The set of uniquely ergodic laminations has full measure [**M82**], and one can ask the following question.

Question 3. For a surface of genus $g \geq 2$, is the set of uniquely ergodic laminations path-connected?

There is a closely related space $\mathcal{EL}$ (the space of ending laminations) which has been studied extensively. Ending lamination space is known to be connected and locally path-connected [**G09**, **LS09**], and has even been identified explicitly in some low-complexity cases [**HP11**, **G11**]. However, the set of uniquely ergodic foliations is only a subset of ending lamination space, and so these results do not yield any direct information about the topology of the set of uniquely ergodic laminations. Somewhat in the line of our results is a result by Leininger and Schleimer which ensures the existence of spheres in the set of uniquely ergodic laminations [**LS11**]. To the authors' knowledge, Question 3 is still open.

Once could also try to adapt the methods of the current paper to the setting of surface foliations by building flat surfaces from IETs. The direct analogue of the case $n = 4$ (which all our work is inductively based on) would concern foliations in the stratum $\mathcal{H}(2)$ in genus two. However, in this stratum, every saddle connection defines a simple closed curve, and thus arational foliations are not path-connected. Indeed in every Rauzy class there are permutations where the IETs corresponding to arational foliations are not path-connected. However, one can still ask the following question.

Question 4. For a suitable permutation π, is the set of all IETs corresponding to arational uniquely ergodic foliations path-connected?

2. *Interval exchange transformations and Rauzy induction*

In this section, we fix the notation for IETs that we will use, and also collect some well-known results that we will use throughout. We refer the reader to [**V06**] for a detailed treatment. The only non-standard part of this is §2.1.

We denote a permutation π on the symbols $1, \ldots, n$ by the list containing the image of the symbols under the inverse permutation π^{-1}. That is, the identity permutation on four symbols is denoted by (1234), the transposition of the first two symbols is (2134). The permutation which maps every symbol to the next one (cyclically) would be (4123). The reason for this convention will be clear later.

Definition 2.1. Given $\hat{L} = (l_1, l_2, \ldots, l_d)$, where $l_i \geq 0$, we obtain d sub-intervals of the interval $[0, \sum_{i=1}^{d} l_i)$

$$I_1 = [0, l_1),\ I_2 = [l_1, l_1 + l_2),\ \ldots,\ I_d = [l_1 + \cdots + l_{d-1},\ l_1 + \cdots + l_{d-1} + l_d).$$

Given a permutation π on the set $\{1, 2, \ldots, d\}$, we obtain a d-IET

$$T : \left[0, \sum_{i=1}^{d} l_i\right) \to \left[0, \sum_{i=1}^{d} l_i\right)$$

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which exchanges the intervals I_i according to π. That is, if $x \in I_j$, then

$$T(x) = x - \sum_{k<j} l_k + \sum_{\pi(k')<\pi(j)} l_{k'}.$$

If such a T is given, we denote its (unnormalized) length vector by $\hat{L}(T)$ and its permutation by $\pi(T)$.

With our convention, $\pi(T)$ is then described by the order of intervals from left to right after applying the IET. If T is an IET, we denote by

$$L(T) = \hat{L}(T)/|\hat{L}(T)|_1$$

the (normalized) length vector of T. We say that an IET is *normalized* if it is defined on the unit interval. Every IET can be rescaled to a normalized one, and we will usually identify IETs which just differ by such a rescaling without explicit mention. When we speak about a length vector we mean the normalized one, unless specified explicitly.

Intuitively, a permutation on n symbols is degenerate if any IET with that permutation has either fewer than $n-1$ discontinuities or an induced map of it (see below for the definition) has fewer than $n-1$ discontinuities. The technical conditions are as follows [**V78**, §3].

Definition 2.2. A permutation π on n symbols is *degenerate* if there is some $j < n$ so that one of the following holds:

(1) $\pi(j+1) = \pi(j) + 1$;
(2) $\pi(j) = n$, $\pi(j+1) = 1$ and $\pi(1) = \pi(n) + 1$;
(3) $\pi(j+1) = 1$ and $\pi(1) = \pi(j) + 1$; or
(4) $\pi(j+1) = \pi(n) + 1$ and $\pi(j) = n$.

Definition 2.3. Let $T : [0, 1) \to [0, 1)$ be measurable and Lebesgue measure preserving and $A \subset [0, 1)$. The *induced map of T on A* is $T|_A : A \to A$ given by

$$T|_A(x) = T^{n(x)}(x), \quad n(x) = \min\{n > 0 : T^n(x) \in A\}.$$

Recall that the Poincaré recurrence theorem guarantees that $T|_A$ is defined almost everywhere in A.

Next, we briefly describe the most important points of *Rauzy induction*: the renormalization method that we will use throughout to study IETs. Our treatment of Rauzy induction will be the same as in [**V82**, §7].

Let T be a n-IET with permutation π. Let δ_+ be the rightmost discontinuity of T and let δ_- be the rightmost discontinuity of T^{-1}. Let $\delta_{\max} = \max\{\delta_+, \delta_-\}$. Consider the induced map of T on $[0, \delta_{\max})$ denoted by $T|_{[0,\delta_{\max})}$. The result is again an IET, perhaps with a different permutation. We can renormalize it so that it is once again an n-IET on $[0, 1)$. That is, let $R(T)(x) = 1/\delta_{\max} T|_{[0,\delta_{\max})}(x\delta_{\max})$. This is the Rauzy induction of T.

If $\delta_+ \neq \delta_-$, then the permutation and length vector of the IET $R(T)$ can be combinatorially determined. Otherwise, we will say (slightly abusing terminology) that *Rauzy induction fails*.

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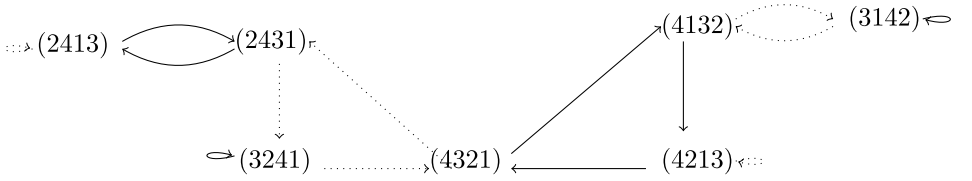


FIGURE 2. The Rauzy diagram for $n = 4$.

Supposing that Rauzy induction does not fail, if $\delta_{\max} = \delta_+$, we say that the first step in Rauzy induction is a. In this case, the permutation of $R(T)$ is given by

$$\pi'(j) = \begin{cases} \pi(j), & j \le \pi^{-1}(n), \\ pi(n), & j = \pi^{-1}(n) + 1, \\ \pi(j-1), & \text{otherwise.} \end{cases}$$

We keep track of what has happened to the interval lengths under Rauzy induction by a matrix $M(T, 1)$ (see Lemma 2.4 below), where

$$M(T, 1)[ij] = \begin{cases} \delta_{i,j}, & j \le \pi^{-1}(n), \\ \delta_{i,j-1}, & j > \pi^{-1}(n) \text{ and } i \ne n, \\ \delta_{\pi^{-1}(n)+1,j}, & i = n. \end{cases}$$

If $\delta_{\max} = \delta_-$, we say that the first step in Rauzy induction is b. In this case, the permutation of $R(T)$ is given by

$$\pi'(j) = \begin{cases} \pi(j), & \pi(j) \le \pi(n), \\ \pi(j) + 1, & \pi(n) < \pi(j) < n, \\ \pi(n) + 1, & \pi(j) = n. \end{cases}$$

Again, we keep track of what has happened to the interval lengths under Rauzy induction by a matrix

$$M(T, 1)[ij] = \begin{cases} 1, & i = n \text{ and } j = \pi^{-1}(n), \\ \delta_{i,j}, & \text{otherwise.} \end{cases}$$

In the case where Rauzy induction fails (i.e., $\delta_+ = \delta_-$), it is not well defined if we are in case a or case b. However, the formulas for $R(T)$ from case a and case b still make sense and we write $R_a(T)$ and $R_b(T)$ for these. Both of these describe the IET T; in these descriptions there are intervals of length zero. See §2.1 for details on this.

The change in permutation under Rauzy induction can be depicted in a *Rauzy diagram*. Compare Figure 2 for the diagram in the case $n = 4$.

The matrices described above depend on whether the step is a or b and the permutation $\pi(T)$. The following well-known lemmas, which are immediate calculations, help motivate the definition of $M(T, 1)$.

LEMMA 2.4. *If $R(T) = S$, then the length vector $L(T)$ is a scalar multiple of $M(T, 1)$ $L(S)$.*

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We will denote the positive orthant by

$$\mathbb{R}^d_+ = \{(x_1, \ldots, x_d) \in \mathbb{R}^d | x_i > 0 \text{ for all } i\}$$

and the open Euclidean simplex by

$$\mathring{\Delta}_d = \left\{(x_1, \ldots, x_d) \in \mathbb{R}^d_+, \sum_{i=1}^{d} x_i = 1\right\}.$$

We let $M\Delta = M\mathbb{R}^+_d \cap \mathring{\Delta}_d$.

LEMMA 2.5. *Let T be some IET. An IET S with permutation $\pi(S) = \pi(T)$ and whose length vector $L(S)$ is contained in $M(T, 1)\Delta$ has the same first step of Rauzy induction as T.*

We define the nth matrix of Rauzy induction by

$$M(T, n) = M(T, n-1)M(R^{n-1}(T), 1).$$

It follows from Lemma 2.5 that, for an IET with length vector in $M(T, n)\Delta$ and permutation π, the first n steps of Rauzy induction agree with T.

The set of all normalized n-IETs with a given permutation π can be naturally identified with a simplex which we denote by Δ_π. The subset of Δ_π corresponding to IETs on which Rauzy induction fails is a codimension-1 simplex embedded in Δ_π. Iterated Rauzy induction defines a partition of Δ_π into smaller simplices in the following way. In each of the full-measure simplices in the complement of the plane where Rauzy induction fails, the set where Rauzy induction is defined once but not twice again is a union of two codimension-1 simplices. We denote by $\mathcal{P}_k$ the full-measure partition of Δ_π into the (open) simplices on which Rauzy induction is defined for the first k steps. A simplex in $\mathcal{P}_k$ consists of all the IETs which follow the same first k steps in the Rauzy diagram under Rauzy induction.

The projective linear map defined by the matrix $M(T, k)$ maps the standard simplex $\Delta_{\pi'}$ (where π' is the permutation corresponding to the IET $R^k(T)$) to the simplex in $\mathcal{P}_k$ which contains T.

We will also need a criterion that ensures unique ergodicity of IETs. A *Rauzy path* is a finite or infinite path in a Rauzy diagram $\mathcal{R}$. Associated to a Rauzy path is the product of matrices describing the change on lengths of the intervals which we will call the *Rauzy matrix* of the path. The following theorem is well known.

THEOREM 2.6. (Veech [**V78**, p. 225]) *Suppose that T is an IET where $R^n(T)$ is defined for all $n \geq 1$ and is such that $\bigcap_{n\geq 1} M(T, n)\Delta = \{T\}$. Then T is uniquely ergodic.*

To check that the prerequisite of Theorem 2.6 is satisfied, it suffices to check that the angle between the columns of $M(T, n)$ converges to zero, since $M(T, n)\Delta$ consists of convex combinations of the columns of $M(T, n)$.

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2.1. *Shadows of Rauzy paths.* We need to make the following consideration for our proof to hold for all uniquely ergodic 4-IETs, as opposed to just uniquely ergodic 4-IETs that have all powers of Rauzy induction defined.

To set up some notation, let T be a minimal d-IET with the length of some interval equal to zero. We define $\hat{T}$ to be the minimal $(d-1)$-IET given by 'forgetting the interval of T with length 0'.

For example if $L(T) = (\alpha, 0, 1-\alpha)$ and $\pi(T) = (321)$, then $L(\hat{T}) = (\alpha, 1-\alpha)$ and $\pi(\hat{T}) = (21)$.

Assume that $R^n(\hat{T})$ is defined for all n. We define T_ϵ to be the d-IET with $\pi(T_\epsilon) = \pi(T)$ and

$$L(T_\epsilon) = \begin{cases} L(T) & \text{if } L(T) \neq 0, \\ \epsilon & \text{otherwise.} \end{cases}$$

PROPOSITION 2.7. *With notation as above, for all $\ell > 0$, there exist (possibly two different) numbers n and $\sigma_n \in \{1, \ldots, d\}$ so that:*

- $\lim_{\epsilon \to 0} L(R^n T_\epsilon)_i = L(R^\ell \hat{T})_{i - c_{i,\sigma_n}}$ *for all i;*
- *for any i, $j \neq \sigma_n$, $\pi(R^n(T_\epsilon))(i) > \pi(R^n(T_\epsilon))(j)$ if and only if*

$$\pi(R^\ell(\hat{T}))(i - c_{i,\sigma_n}) > \pi(R^\ell(\hat{T}))(j - c_{j,\sigma_n})$$

for all small enough ϵ, where $c_{k,\ell} = \begin{cases} 0 & \text{if } k < \ell, \\ 1 & \text{if } k > \ell, \\ \text{undefined} & \text{if } k = \ell. \end{cases}$

Furthermore, given an n as above, for all ϵ small enough, $M(T_\epsilon, n)$ and $M(\hat{T}, \ell)$ are related in the following way: after deleting the σ_nth column of $M(T_\epsilon, n)$, one is left with a $d \times (d-1)$ matrix with a row of all zeros, so that when it is deleted, one has $M(T, \ell)$.

Proof. We prove the two claims above by induction on ℓ. We assume that this is true for ℓ with corresponding number n. We say that j is in *critical position* if $j \in \{d, \pi^{-1}d\}$. The proof breaks into two parts: when σ_n is not in critical position, in which case the corresponding number for $\ell + 1$ is $n + 1$, and when σ_n is in critical position, in which case the corresponding number for $\ell + 1$ is $n + 2$.

Case 1. σ_n is not in critical position. In this case, it is straightforward for $\ell + 1$ and the corresponding number is $n + 1$ with

$$\sigma_{n+1} = \begin{cases} \sigma_n & \text{if } R(R^n T_\epsilon) \text{ is 'b' or } R(R^n T_\epsilon) \text{ is 'a' and } \sigma_n < \pi^{-1}(d), \\ \sigma_n + 1 & \text{otherwise.} \end{cases}$$

Case 2. σ_n is in a critical position. Let $j = d$ if $\sigma_n = \pi^{-1}(d)$ and $\pi^{-1}(d)$ if $\sigma_n = d$. By conclusion 1 (which we are inductively assuming), for all small enough ϵ, $L(T_\epsilon)_j > L(T_\epsilon)_{\sigma_n} = \epsilon$. Set

$$\sigma_{n+1} = \begin{cases} \pi^{-1}(d) + 1 & \text{if } \sigma_n = d, \\ \sigma_n & \text{otherwise.} \end{cases}$$

Conclusion 1 follows inductively. Indeed, except for one entry, $L(R^{n+1} T_\epsilon)$ is a permutation of $L(R^n T_\epsilon)$ and the exceptional entry is $L(R^n T_\epsilon)_j - \epsilon$. Conclusion 2 is

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straightforward to check, as is the claim on the matrices (the deleted column should be σ_{n+1}). Moreover, σ_{n+1} is not in the critical position so we may follow Case 1 for $\ell + 1$, which will correspond to $n + 2$. To see that σ_{n+1} is not in critical position, if $\sigma_n = d$, then $\pi^{-1}(d) \neq d - 1$ by the minimality of $R^\ell \hat{T}$ (which follows from our assumptions on $\hat{T}$). The case of $\sigma_n = \pi^{-1}(d)$ is similar. □

Remark 2.8. This proposition is useful for understanding minimal IETs with non-zero interval lengths that will have a failure of Rauzy induction. Indeed, assume that T is a minimal d-IET that has R^k defined on it, but R^{k+1} is not defined on it. This means that the two intervals of $R^k T$ in critical position have the same length. One can formally continue Rauzy induction (by either R_a or R_b) one step and have an IET, S, with the length of one entry zero. One also gets a matrix $M = M(T, k)M'$, where M' is the matrix given by the formal step or Rauzy induction. We may now forget the entry of S that has zero length and obtain a $(d - 1)$-IET, $\hat{S}$. We may relate its path under Rauzy induction to S_ϵ, as in the proposition. To recover an approximation to T, we just multiply the matrices and lengths we obtain for S by M. This procedure can be successively iterated if there is a subsequent failure of Rauzy induction for $\hat{S}$.

3. *Matrix terminology and combining matrices*

In this section, we will collect some notation and terminology on 4×4 matrices with non-negative integral entries.

If A is such a matrix, then we will denote by $C_i(A)$ the ith column of A. We will often call i the *index* of the column in such a context. If P is some property that a column of a matrix (or a vector in general) may have, we will say that the *index i or column i has P* if $C_i(A)$ has P, when the matrix A is clear from the context.

We use the 1-norm for vectors throughout, so $|C_i(A)|$ will denote the sum of the entries in the ith column. We will often need to consider the column of A with the largest sum of entries. We denote this column by $C_{max}(A)$. Should there be several columns which all have the largest entry sum, we adopt the convention that $C_{max}(A)$ is the column with smallest index realizing this maximum.

Recall that in a matrix product AB, the column $C_j(AB)$ is a sum of the columns of A, with coefficients from $C_j(B)$: that is,

$$C_j(AB) = \sum_{i=1}^{4} (C_j(B))_i C_i(A).$$

Thus, we say that *B adds column i to column j (or column i is added to column j)* if the ith entry of $C_j(B)$ is positive (usually, it will be the case that the jth entry of $C_j(B)$ is also positive, to make the terminology completely justified, but we do not insist on this). Note that it is possible for a column to add to itself. Equivalently, we may say that *column j has column i added to it*. If there is any j so that column i is added to column j, then we simply say that *column i is added to another column*. Similarly, we simply say that *column i has a column added to it* if there is a corresponding j.

This terminology will be used, in particular, when considering powers of some matrix B.

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Next, we introduce the central new notion of this section. Combining matrices will be generalizations of positive matrices, in the sense shown in Lemma 3.4 below.

Definition 3.1. A matrix M is called *combining* if there are two groups of columns, *active* and *passive* ones. We require that:

(1) there are at least two active columns;
(2) at most one column of M is neither active nor passive; this column is called *idle* if it exists;
(3) every column that is added to a (distinct) other column is active; and
(4) every active column is added to all active columns (including itself) and each passive column has at least one active column added to it.

A finite Rauzy path P is said to be *combining* if the corresponding matrix is combining.

Example 3.2. (1) Every positive matrix is combining and every column is active.

(2) The matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 2 & 2 & 2 \end{pmatrix}$$

is combining. Namely, columns one and four are active, while two and three are passive.

(3) The matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 2 & 2 \end{pmatrix}$$

is *not* combining. Column two has columns one, three and four added to it. Thus, all three of those would need to be active. However, column four does not have column three added to it, which violates the definition of active.

(4) The matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

is combining, where columns three and four are active, one is passive and column two is idle.

THEOREM. (Perron–Frobenius) *If M is an $n \times n$ matrix with all entries positive, then there exists a unique largest (in absolute value) eigenvalue, called the Perron–Frobenius eigenvector. The corresponding eigenvector can be chosen to be positive (i.e., has all entries positive) and is called the Perron–Frobenius eigenvector. It is the only eigenvector with that property.*

One can observe that matrices as in the previous theorem act as a contraction in the so-called Hilbert projective metric and we obtain the following result (see, e.g., [**V78**, p. 240]).

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PROPOSITION 3.3. *If M is an $n \times n$ matrix with all entries positive, then any non-negative (non-zero) vector v has that $M^n v$ converges exponentially quickly to the direction of the Perron–Frobenius eigenvector.*

We conclude with a lemma generalizing the Perron–Frobenius theorem to the case of combining matrices. It will be used frequently in the subsequent work.

In its formulation, we will call the jth entry $(C_i(A))_j$ of a column *active* (or *passive*, or *idle*), if the corresponding index j has this property (i.e., the column $C_j(A)$ is active, passive, or idle).

LEMMA 3.4. *Let A be a combining matrix. Then there are numbers $E, \gamma' > 1$ with the following properties. Let $n > 3$ be any integer, and let i, j be indices corresponding to active or passive columns.*

(i) $|C_i(A^n)|/|C_j(A^n)| \le E$.

(ii) *Any two of the active entries in columns i or j of A^n differ by a factor of at most E.*

(iii)
$$\sin \angle(C_i(A^n), C_j(A^n)) \le \frac{E}{\gamma'^n}.$$

The proof requires the following standard lemma (the second part follows from the fact that all norms on $\mathbb{R}^n$ are equivalent). In it, and below, $\|v\|$ denotes the usual Euclidean norm.

LEMMA 3.5. (Law of sines) *Let $v, w \in \mathbb{R}^n$. Then*
$$|\sin \angle(v + w, w)| = \frac{\|v\|}{\|v + w\|} |\sin \angle(v, w)|$$
and, for some universal constant S,
$$|\sin \angle(v + w, w)| \le S \frac{|v|}{|v + w|} |\sin \angle(v, w)|.$$

Proof of Lemma 3.4. Suppose that $\tau \le 4$ columns are active, and let $Y \subset \mathbb{R}^4$ be the τ-dimensional subspace which is preserved by A and so that the matrix $\hat{A}$ describing the action of A on Y is positive. Thus, by the Perron–Frobenius theorem, there is a unique positive eigenvector $\hat{w} \in Y$ of $\hat{A}$, which has positive eigenvalue μ that is the largest eigenvalue in absolute value. If $\tau < 4$, consider the action of $\hat{A}$ on the invariant subspace of dimension $\tau - 1$ which does not contain $\hat{w}$; call the corresponding matrix $\tilde{A}$. There exists $\gamma < \mu$ and C so that $\|\tilde{A}^k\|_{\text{op}} \le C\gamma^k$.

(i) By Proposition 3.3, it follows that the projection of the active columns to the τ-dimensional subspace in the previous paragraph does not lie in the $\hat{A}$ invariant $\tau - 1$ dimensional subspace complemented to the Perron–Frobenius eigenvector. So the sizes of these columns grow proportionally to the Perron–Frobenius eigenvalue of $\hat{A}$. The passive columns get some multiples of active columns added to them. These active columns are growing exponentially (according to the Perron–Frobenius eigenvalue) and so the column is proportional to the last summand once n is large enough. For small n, proportionality also holds simply by finiteness. If we consider a passive column v of A^n, it has the form

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$\sum_{j=1}^{n-1} w_j$, where the w_j are active columns of A^j. Since the active columns are growing exponentially, w_{n-1} is proportional to the largest active column of A^n.

(ii) This is true for powers of the τ-by-τ matrix $\hat{A}$. Indeed, by Proposition 3.3, the ratio of the entries converges to the ratio of the entries of the Perron–Frobenius eigenvector. Arguing as in (i), the passive column(s) are sums of vectors with this property, so they inherit it.

(iii) Arguing as before, by Proposition 3.3 the active columns of A converge exponentially fast to $\hat{w}$ under taking powers, and the angle between the active columns decreases as required.

Consider next a passive column of A^n. This column has the form $\sum_{i=0}^{n} v_i$, where v_0 is the initial passive column of A, and each v_i is a linear combination of active columns of A^i. In particular, each v_i has norm between $D_1\mu^i$ and $D_2\mu^i$, where μ is the Perron–Frobenius eigenvalue and D_1, D_2 depend on the matrix A.

Next, note that

$$\sin\angle\Big(\sum_{i=0}^{n} v_i,\, \hat{w}\Big) \le \sin\angle\Big(\sum_{i=0}^{n} v_i,\, \sum_{i=n/2}^{n} v_i\Big) + \sin\angle\Big(\sum_{i=n/2}^{n} v_i,\, \hat{w}\Big)$$

since (absolute values of) angles satisfy the triangle inequality and $\sin(\theta+\phi) \le \sin(\theta) + \sin(\phi)$ for all $0 \le \theta, \phi$ with $\theta, \phi < \pi/2$.

Since $(\sum_{i=1}^{n/2} D_2\mu^i)/(\sum_{i=n/2}^{n} D_1\mu^i)$ decays exponentially in n, Lemma 3.5 implies that the first summand decays exponentially as well.

On the other hand, as $\sum_{i=n/2}^{n} v_i$ is a positive linear combination of the active columns of $A^{n/2}$, the second summand decays exponentially as well, by the previous comments. □

4. *Building blocks*

This section sets up most of the novel terminology used to construct the paths necessary to prove the following theorem.

THEOREM 4.1. *The set of uniquely ergodic 4-IETs with permutation in the Rauzy class of* (4321) *is path-connected.*

In the introduction, we gave a first intuitive description of how these paths will be constructed: we begin with a line segment connecting the two IETs we want to join. Such a segment will cross a codimension-1-subsimplex in the simplex of 4-IETs where Rauzy induction fails and we therefore lose control over the behaviour of the IETs (compare §2 for the terminology of Rauzy induction failing). A *building block* is an instruction on how to cross this simplex where Rauzy induction fails, and intuitively corresponds to replacing the straight segment by a concatenation of two segments. One could then apply Rauzy induction and recursively repeat this procedure, trying to take a limit of the resulting paths (compare Figure 3).

In practice, we will proceed slightly differently. Instead of taking limits of actual paths, we will directly encode points on the limit path by a sequence of building blocks and show that they define a path. Such building block sequences are a way to encode Rauzy induction for as long as it is defined, and we then choose a uniquely ergodic IET when it fails to be defined.

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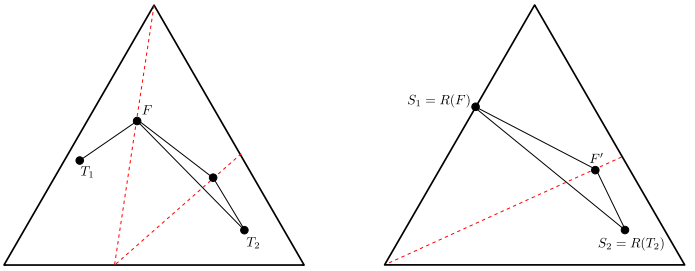


FIGURE 3. The idea behind building blocks (dimension reduced to two). Any path joining T_1 to T_2 necessarily crosses the (dotted) segment containing F, where Rauzy induction fails. The building block (T_1, F, T_2) gives an instruction on how to cross this segment. Then, one can apply Rauzy induction on both sides; here we have pictured the situation on the right. S_1, S_2 are the Rauzy inductions of F, T_2. Again, paths connecting those IETs need to pass a (dotted) segment where Rauzy induction fails. This appears in the original simplex as a segment where Rauzy induction is defined once, but not twice. The building block (S_1, F', S_2) instructs how to cross these segments.

Definition 4.2. A *building block* b is a triple of IETs $b = (T_1, F, T_2)$ such that:

(1) T_1, F, T_2 are uniquely ergodic with the same underlying permutation π; and
(2) F lies on the plane given by the failure of the first step of Rauzy induction.

We call T_1, T_2 the *endpoints* of the building block and F the *midpoint*.

The next definition explains how to apply Rauzy induction on a building block. Intuitively, we would like to say that a building block $b' = (S_1, G, S_2)$ is left compatible with $b = (T_1, F, T_2)$ if the IETs T_1 and F share a number of common Rauzy steps and the outcome is the pair S_1, S_2 of IETs. We formally need to phrase this slightly differently to avoid the problem that Rauzy induction is not well defined for F.

Definition 4.3. A building block $b' = (S_1, G, S_2)$ is *left compatible* with a building block $b = (T_1, F, T_2)$ if the following hold.

(1) $S_1 = R^{k_1}(T_1)$.
(2) Every point on the straight line between T_1 and F has Rauzy induction defined for k_1 steps (and it agrees with the one of S_1).
(3) $L(F) = M(T_1, k_1)L(S_2)$.

We then write $b \xrightarrow{L} b'$.

The *Rauzy matrix associated to the compatibility* $b \xrightarrow{L} b'$ is then defined to be the Rauzy matrix $M(T_1, k_1)$ (describing Rauzy expansion for all points on the segment between T_1 and F).

Similarly, a building block $b' = (S_1, G, S_2)$ is *right compatible* with a building block $b = (T_1, F, T_2)$ if the following hold.

(1) $S_2 = R^{k_2}(T_2)$.
(2) Every point on the straight line between F and T_2 has Rauzy induction defined for k_2 steps (and it agrees with the one of S_2).
(3) $L(F) = M(T_2, k_2)L(S_1)$.

We then write $b \xrightarrow{R} b'$.

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The *Rauzy matrix associated to the compatibility* $b \xrightarrow{R} b'$ is then defined to be the Rauzy matrix $M(T_2, k_2)$ (describing Rauzy expansion for all points on the segment between F and T_2).

We take *compatible* and $\to$ to mean left compatible or right compatible.

With the definition of compatibility, we can now define building block sequences, which can be thought of as the analogue of Rauzy expansions in our formalism.

Definition 4.4. A *building block sequence* is a (finite or infinite) sequence $b_i = (T_1^i, F^i, T_2^i)$ of building blocks so that b_{i+1} is compatible with b_i for all i. We often write such a sequence as $b_1 \to b_2 \to \cdots$.

For a finite building block sequence $b_1 \to \cdots \to b_n$, we define the *associated Rauzy matrix* $M(b_1 \to \cdots \to b_n)$ as the product $M = M_1 \dots M_{n-1}$, where M_i is the Rauzy matrix associated to the compatibility $b_i \to b_{i+1}$.

Next, we need to define the *depth* of a building block sequence. The definition is somewhat technical, but the core idea is as follows. Consider a building block sequence $b_i = (T_1^i, F^i, T_2)$, where the right endpoint T_2 is, in fact, always the same IET. Hence, even though this is an infinite sequence, it captures only the dynamics of the one IET T_2, which may lie on a fail plane of Rauzy induction. Hence, for such sequences, we should not expect good contraction properties for the associated sequence of Rauzy matrices. At the other extreme, one can consider a sequence which always alternates between choosing a left and right compatible block. Such a sequence will encode an IET which has Rauzy induction defined for all times, and will turn out to have good contracting properties. This will be crucial both in showing continuity of the paths and for verifying Veech's criterion for unique ergodicity (Theorem 2.6).

The notion of depth will allow us to distinguish these behaviours (and many in between). To state it formally, note that a building block sequence $b_1 \to \cdots \to b_n$ defines a 'direction sequence' $d_1, \dots, d_{n-1}$, where $d_i \in \{L, R\}$ and $d_i = L$ if and only if b_{i+1} is left compatible with b_i. We say that d_i *swaps direction at most s times* if there are indices $1 = t_0 < t_1 < \cdots < t_{s+1} = n$ so that d_i is constant for $t_j \le i < t_{j+1}$.

Definition 4.5. Let $b_1 \to \cdots \to b_n$ be a building block sequence. The *depth* of the building block sequence is the smallest s so that the associated direction sequence d_i swaps direction at most s times.

As an important example, a building block sequence $b_1 \to \cdots \to b_n$ has depth zero if either, for all i, b_i is left compatible to b_{i-1} or, for all i, b_i is right compatible to b_{i-1}.

Intuitively, the sequence $b_1 \to \cdots \to b_n$ has depth k if the compatibilities $b_i \to b_{i+1}$ swap k times between choosing L or R.

The main phenomenon responsible for long building block sequences which do not increase depth is given by the following, which formalises the idea that a short sequence which starts and ends with the same building block can be repeated without increasing depth (corresponding to inserting a large power of a non-positive matrix in the Rauzy expansion of an IET).

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Definition 4.6.

(i) A *building block loop* is a building block sequence $b_1 \to \cdots \to b_n$ such that the last element b_n is compatible with the first element b_1.

(ii) Such a loop is *minimal* if $b_i \neq b_j$ for all $1 \leq i < j \leq n$.

(iii) The *depth* of a building block loop $b_1 \to \cdots \to b_n$ is defined to be the depth of the building block sequence $b_1 \to \cdots \to b_n \to b_1$.

(iv) If $b_1 \to \cdots \to b_n$ is a depth zero building block loop, then one of the endpoints of b_1 is contained in $M(b_1 \to \cdots \to b_n \to b_1)\Delta$. We call that IET the *endpoint of the building block loop*.

We will frequently use product notation for building block sequences. That is, let P be a building block sequence $b_1 \to \cdots \to b_n$ and let Q be a building block sequence $b_{n+1} \to \cdots \to b_m$. We say that P is compatible with Q if b_n is compatible with b_{n+1}. In that case, we denote by $P \to Q$ the sequence

$$b_1 \to \cdots \to b_n \to b_{n+1} \to \cdots \to b_m$$

obtained by concatenating P and Q. Similarly, if P is a building block loop $b_1, \ldots, b_n$, the P^n is the sequence $P \to \cdots \to P$ obtained by concatenating P with itself n times.

Definition 4.7. If P is a building block loop $b_1 \to \cdots \to b_n$, then we say that a building block sequence $c_1 \to c_2 \to c_3$ compatible with P *leaves the loop* P if the sequence $c_1 \to c_2 \to c_3$ does not contain P as an initial segment and is not contained in P as an initial segment.

In §8, we will show the following proposition by an explicit construction. It is very technical, and some of its properties deserve a brief explanation. *Transitivity* and *completeness* are straightforward. *Allowed breakpoints* is purely of technical convenience, and will make certain checks in §8 easier. We now explain the other properties.

In the sketch of the construction given above, we indicated that the matrix product corresponding to an infinite-depth building block sequence has good contraction properties. This would be particularly easy if it were a product of positive matrices of bounded size. Unfortunately, this cannot be arranged. The best we can do is *almost positivity*: whenever the depth increases enough, the matrix (while not quite positive) will contract angle between columns when multiplying†. A remaining issue is that there can be loops (in the sense of the definition above) of building blocks which do not increase depth, and whose corresponding matrices are not positive, but have very large entries. *Combining loops* and *isolated idle* will deal with this issue: they ensure that after repeating such a loop for a long time, the idle column is soon added to another column, decreasing the maximum angle betweens columns to verify Veech's criterion for unique ergodicity. The formal statement is as follows.

† This is a little bit subtle: the fact that each column of a matrix B has two non-zero entries implies that the corresponding column of AB is the sum of at least two columns of A. To ensure contraction, we need that all but at most one of the columns of A have similar size and the other column (if it exists) to be smaller than the other columns.

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PROPOSITION 4.8. *There is a finite set $\mathcal{B}$ of building blocks with the following properties for some number $c > 0$.*

Transitivity. *For every permutation π in the Rauzy class of* (4321)*, there is a building block in the set $\mathcal{B}$ whose endpoints lie on different sides of the fail plane.*

Completeness. *Every building block in $\mathcal{B}$ has a left and right compatible building block in the set $\mathcal{B}$.*

Combining loops. *If $b_1 \to \cdots \to b_k$ is a minimal depth zero loop formed from building blocks in the set $\mathcal{B}$, then the matrix $M(b_1 \to \cdots \to b_k \to b_1)$ is combining.*

Isolated idle. *Suppose $P = b_1 \to \cdots$ is a building block loop of depth zero formed by building blocks in the set $\mathcal{B}$, so that $M(P \to b_1)$ has an idle column. Then, for any building block sequence $b_1 \to b_2 \to b_3$ which leaves P formed by building blocks in the set $\mathcal{B}$, each column of $M(P \to b_1 \to b_2 \to b_3)$ has at least two non-zero entries.*

Allowed breakpoints. *There is a subset $\mathcal{B}' \subset \mathcal{B}$ of* allowed breakpoints *such that in every infinite building block sequence $b_1 \to b_2 \to \cdots$, up to discarding a finite initial segment, at least one block in $b_i \to \cdots \to b_{i+c}$ is an allowed breakpoint for each i. In particular, in each minimal building block loop of depth zero, at least one block (the 'initial one') is allowed.*

Almost positivity. *There is a number $c \geq 0$ so that every building block sequence formed by building blocks in the set $\mathcal{B}$, starting at an allowed breakpoint and of depth at least c, has an initial segment $b_1 \to \cdots \to b_k$ with $k \leq c$ so that $M(b_1 \to \cdots \to b_k)$ is* almost positive*: i.e., it has $\tau > 1$ rows with all entries positive and the other rows are rows of the identity matrix. By this, we mean that they contain exactly one entry* 1*, and all other entries are* 0*. Furthermore, b_k is an allowed breakpoint.*

In §§5–7, we will show how to use Proposition 4.8 to show Theorem 4.1. Hence, we fix once and for all a finite set $\mathcal{B}$ as given by Proposition 4.8. Any mention of building blocks will refer to elements of this set. We will also refer to the properties guaranteed by Proposition 4.8 by their name, without explicit mention of Proposition 4.8.

Following the sketch outlined in the introduction, there is a general dichotomy of points on the paths: finite-depth points eventually follow the Rauzy induction of a fixed uniquely ergodic IET (which is the left or right endpoint of a building block). Continuity at these points will be shown using the tools in §5. Infinite-depth points, on the other hand, are those that do not eventually lie on a fail plane, and whose Rauzy induction does not eventually follow one of the endpoints. For these points, convergence, unique ergodicity and continuity all need to be checked, using techniques developed in §6. Finally, in §7, we collect the pieces and prove Theorem 4.1.

Depending on the taste of the reader, it may be beneficial to read §7 first in order to get an overview of how the proof uses the tools and only read §§5 and 6 later in order to understand the more technical parts of the argument.

5. *Finite depth*

LEMMA 5.1. *Let $P = b_1 \to \cdots \to b_k$ be a minimal depth zero loop. Denote by b the index of the column which is idle if it exists. We consider the building block sequence P^n.*

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Then there exists a vector V such that

$$\frac{C_i(M(P^n))}{|C_i(M(P^n))|} \to V$$

for all $i \neq b$.

The vector V is the length vector of the endpoint of the building block loop P.

Furthermore, the sizes of the columns $i \neq b$ grow exponentially, and the size of column b is constant.

Here, and beyond, we adopt the convention that the *size* of a column is its 1–norm.

Proof. The vector $(C_i(M(P^n)))/(|C_i(M(P^n))|)$ is contained in $M(P^n)\Delta$, just as the endpoint of the building block loop P (compare Definition 4.6iv). By *combining loops*, the matrix $M(P)$ is combining, and thus all claims follow from Lemma 3.4. □

LEMMA 5.2. *Let P be a minimal depth zero loop, and let P_0 be a building block sequence compatible with P. We consider building block sequences of the form $P_k = P_0 \to P^{n_k} \to Q_k$, where Q_k is a building block sequence of length three which leaves the loop P and $\lim_{k\to\infty} n_k = \infty$.*

Let T_∞ be the IET corresponding to the endpoint of P defined by the loop. Let $\{T_i\}_{i\in\mathbb{N}}$ be IETs so that $T_k \in M(P_k)\Delta$ (i.e., the Rauzy induction of T_k agrees with the one defined by P_k).

Then the T_k converge (as a sequence of points in the simplex) to $M(P_0)T_\infty$.

Proof. To begin, note that we may assume that P_0 is empty. Namely, the Rauzy induction corresponding to P_0 defines a continuous map $M(P_0)$ of the simplex which does not change the result.

We fix $m_k > 0$ so that the Rauzy expansion of the building block sequence P^{n_k-1} has m_k Rauzy steps. Further, let m_k' be the number of Rauzy steps in $P \to Q_k$. Thus, the initial $N_k = m_k + m_k'$ Rauzy steps of the interval exchange T_k agree with the Rauzy expansion defined by P_k.

Next note that the (unnormalized) length vector of T_k satisfies

$$\hat{L}(T_k) = M(P^{n_k}Q_k)\hat{L}(R^{N_k}(T_k)).$$

The length vector for T_k is $L(T_k) = \hat{L}(T_k)/|\hat{L}(T_k)|_1$.

Write $L(R^{m_k}(T_k)) = \sum_{i=1}^4 a_i e_i$. We have $\sum_{i=1}^4 a_i = 1$. We let v_i denote the ith column of $M(P^{n_k-1})$. Note that, in both of these short cuts, we are suppressing the dependence of the a_i and the v_i on k.

$$L(T_k) = \frac{\sum_{i=1}^4 a_i v_i}{\sum_{i=1}^4 a_i |v_i|_1}.$$

Next, we claim that there is a number $\epsilon_0 > 0$ (depending only on the set of building blocks and not k) such that at least two of the a_i are at least ϵ_0. Namely,

$$(a_1, \ldots, a_4) = M(P \to Q_k)v/|M(P \to Q_k)v|_1$$

for some vector v with non-negative entries and $|v|_1 = 1$. Hence, at least one entry v_i of v has size at least $1/4$. By the isolated idle property, each column of $M(P \to Q_k)$ has at least two positive entries, and hence $M(P \to Q_k)e_i$ has at least two non-zero entries for

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each i. Thus, at least two entries of the (unnormalized) vector $M(P \to Q_k)v$ have size at least $1/4$. On the other hand, note that $M(P \to Q_k)$ is a matrix whose entries can be bounded by some constant which does not depend on k (since the set of building blocks is finite). Thus, $|M(P \to Q_k)v|_1$ can likewise be bounded by some constant independent of k. This implies the existence of ϵ_0 as claimed.

Now suppose that we are given some $\epsilon > 0$. We choose K so that for all $n > K$ the following hold (where V is the vector given by Lemma 5.1).

(i) For all but at most one i,
$$|v_i - |v_i|_1 V|_1 < \epsilon |v_i|_1.$$
We call the i where this fails the *bad* i and v_i the *bad column*. We call other indices $j \neq i$ and the corresponding columns v_j *good*. This property can be ensured by Lemma 5.1.

(ii) If there is a bad column v_i, then
$$\frac{a_i v_i}{\sum_{j\text{good}} a_j |v_j|} < \epsilon.$$
This is possible since at least one coefficient a_j of a good column v_j is at least ϵ_0, and the sizes of the good columns grow exponentially, while the size of the bad column is uniformly bounded by Lemma 5.1.

Now, let $n > K$ be given.

We first consider the case in which there is no bad i for $M(P^N)$. In that case, we compute
$$|L(T_k) - V|_1 = \left| \frac{\sum_{i=1}^4 a_i v_i - \sum_{i=1}^4 a_i |v_i|_1 V}{\sum_{i=1}^4 a_i |v_i|_1} \right|_1 \leq \frac{\sum_{i=1}^4 a_i |v_i - |v_i|_1 V|_1}{\sum_{i=1}^4 a_i |v_i|_1}.$$
Thus by (i),
$$|L(T_k) - V|_1 < \frac{\sum_{i=1}^4 a_i |v_i|_1 \epsilon}{\sum_{i=1}^4 a_i |v_i|_1} = \epsilon.$$
Now suppose that there is a bad i, which, without loss of generality, we may assume to be one.

In this case, we compute
$$L(T_k) = \frac{\sum_{i=1}^4 a_i v_i}{\sum_{i=1}^4 a_i |v_i|_1} = \frac{a_1 v_1}{\sum_{i=1}^4 a_i |v_i|_1} + \frac{\sum_{i=2}^4 a_i v_i}{\sum_{i=1}^4 a_i |v_i|_1}.$$
By property (ii), the first summand has size at most ϵ. Now note that for any numbers $K < D$ and c,
$$\frac{K}{D} - \frac{K}{c+D} = \frac{cK}{D(c+D)} < c\frac{K}{D}\frac{1}{D} < c\frac{1}{D}.$$
We apply this estimate for K being each of the four entries of the vector $\sum_{i=2}^4 a_i v_i$ with $D = \sum_{i\text{good}} a_i |v_i|_1$ and $c = a_1 |v_1|_1$ to conclude that the second summand in the expression for $L(T_k)$ is within 4ϵ of
$$\frac{\sum_{i=2}^4 a_i v_i}{\sum_{i=2}^4 a_i |v_i|_1}.$$
Arguing as in the first case, this is within ϵ of V. Thus, in conclusion, $|L(T_k) - V|_1 < 6\epsilon$.

Thus, the normalized length vectors $L(T_k)$ converge to V. □

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COROLLARY 5.3. *Let P be an infinite Rauzy path so that at least three columns increase in size by an unbounded amount† and all columns which do increase by an unbounded amount converge projectively to some fixed vector V. Let $Q_1, \ldots, Q_k$ be Rauzy paths with positive associated Rauzy matrices.*

Then, for every $\epsilon > 0$, there is a $N > 0$ so that if $n > N$ and T is any IET whose initial n-steps of Rauzy induction agree with P and then are followed by some Q_i, then $L(T)$ is within ϵ of V.

Proof. By assumption of the Corollary, for any ϵ, there is an N so that for all $n > N$ conditions (i) and (ii) stated in the proof of Lemma 5.2 hold for the column v_i of the Rauzy matrix $M(P_n)$ describing the first n steps of P (where V is now the vector given by the assumption).

Furthermore, as Q_i is assumed to be positive, $L(R^N(P_n Q_i))$ has at least two entries of size at least ϵ_0 (arguing as in the previous proof, with positivity of Q_i and the fact that $\|Q_i\|$ are uniformly bounded since there are only k such matrices, replacing the part of the argument involving the isolated idle property).

From here, one can finish the proof exactly as in the proof of Lemma 5.2 above. □

6. *Infinite depth*

In this section, we are concerned with analysing infinite-depth points. Here, the situation is much more involved compared with §5. Convergence, continuity and unique ergodicity will all follow from the same mechanism, which we first sketch.

We will fix an infinite building block sequence of infinite depth and we will study the associated infinite matrix product given by the Rauzy matrices. The steps in the proof are as follows. First, we show that the ratio of the top to second smallest column in this matrix product is uniformly bounded. This means that adding any but the smallest column to another column moves it closer to the others by a definite amount. Note that because of the possible presence of an idle column, we cannot assume that the ratio of the largest to smallest is bounded as well. However, this will only occur in the presence of long loops. In Lemma 6.9, we establish contraction if we have long enough loops. With these results, we can then prove the angle contraction result we need by applying Lemma 6.9 if we loop enough and, otherwise, we obtain almost positive matrices of a bounded size, which provides a contraction (because of Theorem 6.3).

Throughout, we consider an infinite building block sequence (b_i) of infinite depth. This, in particular, means that the sequence b_i does not end in an infinite power of a depth zero loop.

We will rewrite the sequence $b_1 \to b_2 \to \cdots$ as a sequence of building block paths in various ways to control the sizes and directions of the columns of the corresponding matrices.

As a first step, we will consider the regrouping $B_1 \to B_2 \to \cdots$, where each B_i is either a power of a minimal depth zero loop starting at an allowed breakpoint or a single building block, and the regrouping is maximal with that property (that is, if B_i is a power

† Note that all matrices of Rauzy induction have only non-negative entries and thus such an increase is automatically monotone.

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of a minimal depth zero loop starting at an allowed breakpoint, then the initial length-3-segment of building blocks defined by the sequence following B_i leaves the loop). By our assumption, each B_i is a finite building block sequence, and the sequence (B_i) is infinite.

As we are interested only in convergence, we may also discard a finite initial segment, and we do so without further mention (this is a minor technical point that makes the construction in §8 a bit easier).

Later, we will further group the B_i which do not correspond to powers of minimal depth zero loops. In this regrouping, we will also take care that these new groups begin at allowed breakpoints. We will not emphasize this point in the subsequent work; it only serves to reduce the number of cases needed to check in §8. During a first read, it is advisable to assume that every block is an allowed breakpoint.

Throughout this section, we denote by $A_i = M(B_i)$ the matrix corresponding to B_i.

By Proposition 4.8, these matrices satisfy the following corollary.

COROLLARY 6.1. *There are numbers $c, N > 0$ such that, for all i corresponding to an allowed breakpoint (i.e., the first building block in the sequence defining A_i is an allowed breakpoint):*

(P) the product $A_i \cdot \cdots \cdot A_{i+c}$ is positive and the initial block of the sequence corresponding to A_{i+c+1} is an allowed breakpoint; or

(R) there exists $c' \leq c$ so that $A_i \cdot \cdots \cdot A_{i+c'}$ is almost positive and the initial block of the sequence corresponding to $A_{i+c'+1}$ is an allowed breakpoint.

Furthermore, each matrix A_i satisfies either:

(S) $\|A_i\| < N$; or

(C) A_i is a power of a combining matrix B with $\|B\| < N$.

The following lemma follows from the finiteness of the set of building blocks and property (C).

LEMMA 6.2. *For all k, there exists n so that if $\|A_i\| > n$, then A_i is at least the kth power of a combining matrix.*

6.1. *Column size bound.* We now begin to study the infinite matrix product $\prod A_i$. The first intermediate goal will be the following theorem.

THEOREM 6.3. *There is a number $K > 0$ so that, for all n, the ratio of the L^1 norm of the largest to the second smallest column of $\prod_{i=1}^{n} A_i$ is at most K.*

The proof requires several preliminary lemmas, beginning with the following elementary estimate.

LEMMA 6.4. *Let $a, b, c, d > 0$ be four positive numbers so that $b \leq Kd$. Then*

$$\frac{a+b}{c+d} \leq \frac{a}{c} \quad \text{or} \quad \frac{a+b}{c+d} \leq K.$$

Proof. Suppose that the first estimate is not satisfied. Hence

$$\frac{a+b}{c+d} > \frac{a}{c} \quad \Leftrightarrow \quad ca + cb > ca + da \quad \Leftrightarrow \quad cb > da.$$

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Thus,

$$\frac{a+b}{c+d} = \frac{ad+bd}{cd+d^2} < \frac{bc+bd}{cd+d^2} = \frac{b}{d} \leq K. \qquad \square$$

LEMMA 6.5. *Suppose B is a 4×4 matrix with non-negative entries. Let A be a matrix whose entries are bounded by $K \geq 1$.*

(i) *Suppose that A has all positive entries. Then*

$$\frac{|C_i(BA)|}{|C_j(BA)|} \leq K$$

for all i, j.

(ii) *Suppose that A has only positive entries in at least two rows, and that the other rows are equal to rows of the identity matrix. Then*

$$\max_{i,j} \frac{|C_i(BA)|}{|C_j(BA)|} \leq \max_{i,j} \frac{|C_i(B)|}{|C_j(B)|} \quad \text{or} \quad \max_{i,j} \frac{|C_i(BA)|}{|C_j(BA)|} \leq 2K,$$

where all maxima are taken over i, $j \in \{1, 2, 3, 4\}$.

Proof. (i) The ith column of BA is obtained by summing the columns of B according to the entries of the ith column of A. In such a linear combination, each column of B appears at least once, and at most K times by the assumption on A. Thus, the l_1-norm of any column is at least equal to $|C_1(B)| + \cdots + |C_4(B)|$ and at most $K(|C_1(B)| + \cdots + |C_4(B)|)$. This shows (i).

(ii) Suppose, for ease of notation, that rows three and four of A are the ones which are not positive (if only one non-positive row exists, then the argument works analogously). By the assumption, every column of BA is the sum of some multiples of columns one and two of B and possibly one copy of column three or four. We thus have for the two indices i corresponding to columns with two non-zero entries (call such indices *nice*)

$$|C_i(BA)| \geq (|C_1(B)| + |C_2(B)|), \quad |C_i(BA)| \leq K(|C_1(B)| + |C_2(B)|)$$

and for the other two indices

$$|C_i(BA)| \geq |C_{\pi(i)}(B)| + (|C_1(B)| + |C_2(B)|),$$
$$|C_i(BA)| \leq |C_{\pi(i)}(B)| + K(|C_1(B)| + |C_2(B)|)$$

for some $\pi(i) \in \{1, 2, 3, 4\}$.

Now, let i, j be such that

$$\max_{a,b} \frac{|C_a(BA)|}{|C_b(BA)|} = \frac{|C_i(BA)|}{|C_j(BA)|}.$$

If i is nice, we can estimate this as

$$\frac{|C_i(BA)|}{|C_j(BA)|} \leq \frac{K(|C_1(B)| + |C_2(B)|)}{(|C_1(B)| + |C_2(B)|)} = K.$$

Otherwise, we assume, without loss of generality, that $|C_1(B)| \leq |C_2(B)|$ and estimate

$$\frac{|C_i(BA)|}{|C_j(BA)|} \leq \frac{|C_{\pi(i)}(B)| + K(|C_1(B)| + |C_2(B)|)}{(|C_1(B)| + |C_2(B)|)} \leq \frac{|C_{\pi(i)}(B)| + 2K|C_2(B)|}{|C_1(B)| + |C_2(B)|}.$$

Applying Lemma 6.4, we obtain that this last term is either $\leq 2K$ or less than $(|C_{\pi(i)}(B)|)/(|C_1(B)|)$. In both cases, the lemma follows. $\square$

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Recall that an entry in a combining matrix A is *active* if it is in row i and column i of A is active.

Definition 6.6. If M is a matrix, we denote by $C_u(M)$ the column of M whose norm is second smallest.

LEMMA 6.7. *Given $M > 0$, there are constants N, $K > 0$ with the following properties. Suppose that D is any non-negative matrix with $\|D\|_{\text{op}} \leq M$, B is any non-negative matrix and A is a matrix corresponding to a nth power of a minimal depth zero loop with $n \geq N$. Then*

$$\frac{|C_{\max}(BDA)|}{|C_u(BDA)|} \leq \max\left\{K, \frac{|C_{\max}(B)|}{|C_u(B)|}\right\}.$$

Proof. Since the set $\mathcal{B}$ of building blocks is finite, it suffices to show the statement of the lemma assuming that A is a power of a fixed combining matrix A_0 (and then take the largest of the resulting constants N, K).

Next, observe that $(|C_{\max}(BD)|)/(|C_u(BD)|) \leq \|D\|_{\text{op}}(|C_{\max}(B)|)/(|C_u(B)|)$ (where operator norms are taken with respect to the l_1–norm; noting that $|C_u(BD)| \geq |C_u(B)|$, as D is invertible with integer entries). Thus, as $\|D\|_{\text{op}} \leq M$, the lemma is a consequence of the following claim.

Claim. Let A_0 be combining and let $\epsilon > 0$. Then there are constants K, $N > 0$ such that, for any $n > N$,

$$\frac{|C_{\max}(BA_0^n)|}{|C_u(BA_0^n)|} \leq \max\left\{K, \epsilon\frac{|C_{\max}(B)|}{|C_u(B)|}\right\}.$$

The rest of the proof is concerned with establishing this claim. There are several cases, depending on which columns of BA_0^n are the largest and second smallest. To ease notation, we assume that columns one and two of A_0 are active, three is passive and four is idle. If there is no idle column, some of the cases below are simply vacuous. Additionally, we will say that a column of B or BA_0^i is active, passive or idle if the corresponding column of A_0 has that property.

As a first reduction, note that if any active column of B is larger than the idle column (should it exist), we may replace B by BA_0^2 in the claim (at the cost of replacing the constant N we find by $N + 2$, and replacing ϵ by $\epsilon/\|A_0\|^2$); which has the effect of adding both active columns to all of the non-idle columns of B and making the idle column the smallest. Hence, we may assume that either the idle column of B is the smallest (Case 1 below), or that both active columns are smaller than the idle column (Cases 2A and 2B below).

Additionally, possibly multiplying B by another factor A_0^2 (modifying constants as above), we may assume that the sizes of the active columns of B differ by at most $L > 0$ (which depends only on A_0), as they are added to each other at least once and at most $\|A_0^2\|$ times.

Case 1. The idle column of B is smallest. In this case, note that any columns involved in computing $|C_{\max}(BA_0^i)|$, $|C_u(BA_0^i)|$ (for any $i > 0$) are active or passive—this is, in fact, the only statement needed for the arguments in this case.

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Let E be the constant from Lemma 3.4 applied to A_0. Let $m_i(n)+1$ be the smallest and let $M_i(n)$ be the largest of the active entries in the column i of A_0^n. By Lemma 3.4, $M_i(n) \le E(m_i(n)+1) \le (E+1)m_j(n)$ for any active or passive i, j (assuming that n is large enough so that $m_j(n) \ge E$).

Now fix active or passive columns i, j. Thus

$$|C_j(BA_0^n)| \ge |C_j(B)| + m_j(n)(|C_1(B)| + |C_2(B)|)$$

and

$$|C_i(BA_0^n)| \le |C_i(B)| + (E+1)m_j(n)(|C_1(B)| + |C_2(B)|).$$

First, consider the case when $C_{\max}(BA_0^n)$ is realized by an active column (and $C_u(BA_0^n)$ by an active or passive column). In that case, we can estimate

$$\frac{|C_{\max}(BA_0^n)|}{|C_u(BA_0^n)|} \le \frac{3(E+1)m_j(n)\max\{|C_1(B)|, |C_2(B)|\}}{2m_j(n)\min\{|C_1(B)|, |C_2(B)|\}} \le 3(E+1)L$$

and the proof is complete, having bounded the desired ratio by a constant (here, L is the constant introduced before the case distinctions). Next, consider the case when $C_{\max}(BA_0^n)$ is realized by the passive column (and thus $C_u(BA_0^n)$ is realized by an active column). If the size of the passive column of B is comparable to the active ones, the same argument as above yields a constant bound. We may thus assume that

$$\frac{|C_{\max}(B)|}{|C_u(B)|} = \frac{|C_3(B)|}{\min\{|C_1(B)|, |C_2(B)|\}} > (E+1)L\frac{3}{\epsilon}.$$

In this case, we estimate

$$\begin{aligned}\frac{|C_{\max}(BA_0^k)|}{|C_u(BA_0^k)|} &\le \frac{|C_3(B)| + 2(E+1)m_j(n)\max\{|C_1(B)|, |C_2(B)|\}}{2m_j(n)\min\{|C_1(B)|, |C_2(B)|\}} \\ &\le (E+1)L + \frac{|C_3(B)|}{2m_j(n)\min\{|C_1(B)|, |C_2(B)|\}} \\ &\le (E+1)L + \frac{|C_{\max}(B)|}{|C_u(B)|}\frac{1}{2m_j(n)} \le \left(\frac{\epsilon}{3} + \frac{1}{2m_j(n)}\right)\frac{|C_{\max}(B)|}{|C_u(B)|}.\end{aligned}$$

As $m_j(n)$ grows exponentially in n, there is an N so that $1/2m_j(n) < \epsilon/3$ for all $n > N$, which finishes this case.

Case 2A. The passive column of B is the largest, and the active columns are the smallest. To set up, let N_0 be the constant from Case 1 that is required for a threshold of $\epsilon = 1$. Let $J > 0$ be the largest number so that in BA_0^k for $k < J$ the passive column is largest, and the two active columns are the smallest.

For any $k < J$, both columns involved in

$$\frac{|C_{\max}(BA_0^k)|}{|C_u(BA_0^k)|}$$

are therefore active or passive, and hence the arguments of Case 1 apply verbatim; let N_1 be the bound obtained from Case 1 for the threshold $\epsilon/\|A_0\|^{N_0}$.

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Should $J < N_1$, then we restart the argument with BA_0^J in place of B and $\epsilon/\|A_0^J\|$ in place of ϵ. Since we cannot enter Case 2A or B again, the sum of N_1 and the resulting bound N_2 is the desired bound.

If n is so that $N_1 < n < J$, then the claim of the lemma follows directly by Case 1, as by the definition of J the prerequisites of Case 1 hold for BA_0^n.

Otherwise, if $N_1 < J \leq n < J + N_0$, then

$$\frac{|C_{\max}(BA_0^n)|}{|C_u(BA_0^n)|} < \|A_0^{N_0}\| \frac{|C_{\max}(BA_0^J)|}{|C_u(BA_0^J)|},$$

which yields the claim as BA_0^J satisfies the desired bounds for $\epsilon/\|A_0\|^{N_0}$.

Finally, consider the case when $J + N_0 \leq n$. In that case, we write $BA_0^n = (BA_0^J)A_0^{n-J}$, and note that $B' = BA_0^J$ satisfies the desired column size bound and does not fall into Case 2A or B by definition of J. Further, $n - J > N_0$, and thus the claim of the lemma follows.

Case 2B. The idle column of B is the largest. The structure of this case is very similar to Case 2A. Let N_0 be the bound obtained from Cases 1 and 2A for the threshold $\epsilon = 1$.

Let $J > 0$ be maximal such that, for all $k < J$, the idle column of BA_0^k is the largest. Note that the size of the active and passive columns in BA_0^k grow exponentially in k. Thus, for all $k < J$,

$$\frac{|C_{\max}(BA_0^k)|}{|C_u(BA_0^k)|} < \xi^k \frac{|C_{\max}(B)|}{|C_u(B)|}$$

for some $0 < \xi < 1$. Let N_1 be such that $\xi^{N_1} < \epsilon/\|A_0^{N_0}\|$.

If $N_1 > J$, then we restart the argument with BA_0^J in place of B and $\epsilon/\|A_0^J\|$ in place of ϵ. Since we cannot enter Case 2B again, the sum of N_1 and the resulting bound N_2 is the desired bound.

Otherwise, if $N_1 \leq n < J$, we have the desired column size bound for BA_0^n from the formula above. If $J \leq n < J + N_0$, then

$$\frac{|C_{\max}(BA_0^n)|}{|C_u(BA_0^n)|} < \|A_0^{N_0}\| \frac{|C_{\max}(BA_0^J)|}{|C_u(BA_0^J)|},$$

which yields the claim.

Finally, consider the case that $J + N_0 \leq n$. In that case, we write $BA_0^n = (BA_0^J)A_0^{n-J}$, and note that $B' = BA_0^J$ satisfies the desired column size bound and does not fall into Case 2B by definition of J. Further, $n - J > N_0$, and thus the claim of the lemma follows. □

Proof of Theorem 6.3. The strategy to show the theorem is to first define a regrouping $D_k = \prod_{i=i(k)}^{i(k+1)-1} A_i$, where $i(k)$ are chosen so that $i(k+1) - i(k) \leq c$ and so that the products $\prod_{i=1}^n D_i$ satisfy

$$\frac{|C_u(\prod_{j=1}^m D_j)|}{|C_{\max}(\prod_{j=1}^m D_j)|} \leq H \tag{1}$$

for some K independent of m; furthermore, the difference between any product $\prod A_i$ and a suitably chosen $\prod D_j$ will have small operator norm, and thus implies the theorem.

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We now begin with the formal proof. First, we will inductively define some constants. Note that the set of all paths $B_1 \to \cdots \to B_k$ so that no B_i is at least a third power of a minimal depth zero building block loop and so that $k \leq c$ is finite. Thus, the set of operator norms $\|M(B_1 \to \cdots \to B_k)\|$ of the corresponding matrices is bounded; let M_1 be such a bound.

Next, let N_1 be the number guaranteed by Lemma 6.7 applied to the bound $M = M_1$. In particular, this implies that if A is any matrix and $B_1 \to \cdots \to B_k \to P^n$, where $B_1 \to \cdots \to B_k$ is as above, P is a minimal depth zero loop and $n \geq N_1$, then

$$\frac{|C_{\max}(AM(b_1 \to \cdots \to b_k \to P^n))|}{|C_u(AM(b_1 \to \cdots \to b_k \to P^n))|} \leq \max\left\{K, \frac{|C_{\max}(A)|}{|C_u(A)|}\right\}.$$

Suppose now that N_i, M_i are defined. We define M_{i+1} to be a bound for the operator norm of matrices $M(B_1 \to \cdots \to B_k)$ defined by sequences with $k \leq c$ and so that no B_i is a minimal depth zero loop with power at least N_i. Then, define N_{i+1} to be the output of Lemma 6.7 applied to $M = M_{i+1}$.

Finally, let K_2 be the output of Lemma 6.7 applied to the bound $M = M_c$.

We can now describe the grouping D_i which will satisfy equation (1) for $H = \max(K, K_2)$.

Namely, suppose that D_i, $i \leq n$ is defined so that equation (1) is satisfied for $m \leq n$ and $H = \max(K, K_2)$. Let A_N be the final matrix appearing in the product defining D_n. By almost positivity, for some number $k \leq c$, the matrix $A_{N+1} \dots A_{N+k}$ is almost positive (i.e., satisfies the property of almost positivity). There are now two cases.

(1) $\|A_{N+1} \dots A_{N+k}\| \leq M_c$. Then we put $D_{n+1} = A_{N+1} \dots A_{N+k}$ and note that, by Lemma 6.5,

$$\frac{|C_{\max}(D_1 \dots D_n D_{n+1})|}{|C_u(D_1 \dots D_n D_{n+1})|} \leq \max\left\{K_2, \frac{|C_{\max}(D_1 \dots D_n)|}{|C_u(D_1 \dots D_n)|}\right\}.$$

(2) $\|A_{N+1} \dots A_{N+k}\| > M_c$. Then at least one of the A_j, $N+1 \leq j \leq N+k$ corresponds to at least a N_cth power of a minimal depth zero loop. Let r be the minimal such j. We put $D_{n+1} = A_{N+1} \dots A_{r-1} A_r$. Note that $\|A_{N+1} \dots A_{r-1}\| \leq M_c$ by minimality of r. Thus, by our choice of N_c,

$$\frac{|C_{\max}(D_1 \dots D_n D_{n+1})|}{|C_u(D_1 \dots D_n D_{n+1})|} \leq \max\left\{K, \frac{|C_{\max}(D_1 \dots D_n)|}{|C_u(D_1 \dots D_n)|}\right\}.$$

By construction, the endpoints of minimal depth zero building block loops are admissible breakpoints, and therefore D_{n+1} is of the desired form to apply Corollary 6.1.

Thus, in any case,

$$\frac{|C_{\max}(D_1 \dots D_n D_{n+1})|}{|C_u(D_1 \dots D_n D_{n+1})|} \leq \max\left\{\max\{K, K_2\}, \frac{|C_{\max}(D_1 \dots D_n)|}{|C_u(D_1 \dots D_n)|}\right\}.$$

This shows equation (1) for $m = n+1$. By induction, equation (1) therefore follows for all m.

The theorem (i.e., equation (1) for products of A_i instead of D_i) now follows simply because, by construction, the difference between $\prod A_i$ and a suitably chosen $\prod D_j$ has operator norm at most M_c, and hence $M_c H$ works as a column size bound for the A_i. □

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COROLLARY 6.8. *For any $\zeta > 0$, there is an L with the following property. Let $B = \prod_{i=1}^{N} A_i$, and suppose that A is at least the Lth power of a minimal depth zero loop. Then the idle column of BA^L has size at most ζ times the size of any active or passive column of BA^L.*

Proof. By Theorem 6.3, the quotient of the largest and second smallest sizes of columns of B is at most K. If the idle column of B is not already the smallest, then one of the active columns has size at least $1/K$ times the size of the idle column. After K applications of A, this active column was added to all non-idle columns, so the idle column has become the smallest. Since every further application of A increases the size of every non-idle column, the desired L exists. □

6.2. *Angle contraction.*

LEMMA 6.9. *Given $D \geq 0$, there are constants $K' > 0$, $\xi > 1$ with the following property. Let B be a matrix so that $\max_{1\leq i,j\leq 4} \angle(C_i(B), C_j(B)) \leq \alpha$, $|C_{\max}(B)|/|C_u(B)| \leq D$ and let A_0 be the matrix defined by a minimal depth zero loop. Then*

$$\max_{i,j \text{ active or passive}} \sin \angle(C_i(BA_0^k), C_j(BA_0^k)) < K' \frac{\sin(\alpha)}{\xi^k}.$$

Proof. Because there are only finitely many minimal depth zero loops in our building blocks, it suffices to show the lemma for a fixed combining matrix A_0.

The proof is similar to Lemma 3.4. We consider the action of A_0 on the subspace spanned by the active columns, described by a matrix $\hat{A}_0$. It has a Perron–Frobenius eigenvector $[a_1, \ldots, a_\tau]$ (where $2 \leq \tau \leq 4$), with all $a_i > 0$ and $\sum a_i = 1$. Denote the corresponding eigenvalue by μ. We also consider the action of $\hat{A}_0$ on the invariant $(\tau - 1)$-dimensional subspace not containing the Perron–Frobenius eigenvector $[a_1, \ldots, a_\tau]$. Call the linear transformations describing this action $\tilde{A}_0$. There exists C, ν, where $\nu < \mu$ and $\|\tilde{A}_0^k\|_{\text{op}} \leq C\nu^k$ (all operator norms are with respect to the L^1 norm). Thus there exists constants N, $\gamma > 1$ so that $(\mu^k)/(C\nu^k) \geq \gamma^k$ for all $k \geq N$. Write $C_i(\hat{A}_0^k) = \lambda[a_1, \ldots, a_\tau] + w$, where λ is chosen to be the largest possible so that $C_i(\hat{A}_0^k) - \lambda[a_1, \ldots, a_\tau] \in \mathbb{R}_+^\tau$. There exists C, M so that, for all $k > M$,

$$\frac{\lambda}{|w|} > C\gamma^k. \tag{2}$$

Indeed, decompose $C_i(\hat{A}_0^k) = \lambda'[a_1, \ldots, a_\tau] + w'$, where w' is in the subspace $\hat{A}_0$ acts on. By the above, $(\lambda'/|w'|) > \gamma^k$ for all $k > N$. Now, letting ζ be the minimal number so that $\zeta[a_1, \ldots, a_\tau] + w' \in \mathbb{R}_+^\tau$ for all large enough k gives equation (2)†.

Now let $A = A_0^k$, where $k \geq N$. Consider the active columns $C_{i_1}(B), \ldots, C_{i_\tau}(B)$ and the vector $\tilde{a} \in \mathbb{R}_+^4$, where $\tilde{a}_i$ is zero if C_i is not active and is a_b where b is the corresponding column of $\hat{A}_0$ if C_i is active. Because $\hat{A}_0$ captures the action of A_0 on the active columns, $C_{i_b}(A) = \lambda\tilde{a} + w_b$, where λ is chosen to be the largest possible with $w_b = C_{i_b} - \lambda\tilde{a} \in \mathbb{R}_+^4$; then $(|w_b|/\lambda) \leq C'(1/\gamma^k)$, as shown above (where C' is some constant depending only on A_0).

† In particular, $\zeta \leq (|w'|/\min a_i)$.

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We first show that there is a constant D' with

$$\sin(\angle(B\tilde{a}, C_{i_b}(BA))) \le D' \frac{|w_b|}{\lambda} \sin(\angle(Bw_b, B\tilde{a})). \tag{3}$$

Namely, using Lemma 3.5,

$$\sin(\angle(B\tilde{a}, C_{i_b}(BA))) = \sin(\angle(B\lambda\tilde{a}, B\lambda\tilde{a} + Bw_b)) \le S \frac{|Bw_b|}{|B\lambda\tilde{a} + Bw_b|} \sin(\angle(B\tilde{a}, Bw_b)).$$

Let j be an active entry so that $|C_j(B)| \ge |C_u(B)|$. Then

$$|B\lambda\tilde{a} + Bw_b| \ge |B\lambda\tilde{a}| \ge \lambda |C_j(B)|\tilde{a}_j \ge \lambda |C_u(B)|\tilde{a}_j$$

and thus

$$\frac{|Bw_b|}{|B\lambda\tilde{a} + Bw_b|} \le \frac{|C_{\max}(B)| \cdot |w_b|}{\lambda |C_u(B)|\tilde{a}_j} \le \frac{D}{\tilde{a}_j} \frac{|w_b|}{\lambda}.$$

This implies equation (3).

Now by our assumption on B, $\angle(Bv, Bw) \le \alpha$ for any $v, w \in \mathbb{R}^d_+$. So applying the conclusion of the previous paragraph to $(|w_b|/\lambda)$, $\sin(\angle(Bw_b, B\tilde{a})) \le C''(1/\gamma^k) \sin(\alpha)$.

Hence the claim of the lemma follows for active columns and for $k > M$. The extension to passive column analogous to the corresponding step in the proof of Lemma 3.4 part (iii).

Finally, we may absorb the requirement that $k \ge M$, N by choosing K' large enough. □

We next introduce some notation. If v, w are line segments in $\mathbb{R}^3$, let $|w|$ denote the length of w and let Φ_v denote the orthogonal projection onto the direction of v. We also need a fact from Euclidean geometry.

LEMMA 6.10. *For all $c_2 < 1$, there exists $c_1 < 1$ with the following property. Let $\Delta' \subset \mathbb{R}^3$ be a (not necessarily regular) Euclidean simplex with diameter τ, and let w be a line segment in Δ' with length $|w| \ge c_1\tau$.*

Then there is an edge e of Δ' with length at least $|w|$ so that $|\Phi_e(w)| \ge c_2|w|$.

Proof. To begin, note that the claim of the lemma holds for a fixed (possibly degenerate) simplex by convexity of the distance function. Next, note that the claim is invariant under scaling the simplex. Hence, it suffices to show it for simplices of diameter one. The space of possibly degenerate simplices of diameter one is compact, and so the lemma follows. □

LEMMA 6.11. *For every N, there exists $F_1 < 1$ so that if B is a product of matrices from Corollary 6.1, then $A = A_i \cdots A_{i+k}$ is a sub-product satisfying (P) or (R) with $k \le c$ and $\|A\|_{\mathrm{op}} \le N$. Then*

$$\operatorname{diam}(BA\Delta) \le F_1 \operatorname{diam}(B\Delta).$$

Proof. The strategy of the proof is to consider $BA\Delta \subset B\Delta$ and apply Lemma 6.10 to show that any line segment $w \subset BA\Delta$ is less than $F_1\operatorname{diam}(B\Delta)$.

Consider the (normalized) simplices $B\Delta$ and $BA\Delta$ with vertices $p_1, \ldots, p_4$ and $q_1, \ldots, q_4$, respectively. These correspond to the directions of the columns $C_j(B)$ and $C_j(BA)$. Let τ be the length of the longest edge of $B\Delta$ (which is also its diameter).

Let E be the set of edges with length at least $c_3\operatorname{diam}(B\Delta)$. To show the lemma, it suffices to show that there is a constant $0 < c_3 < 1$ so that

$$|\Phi_e(\overrightarrow{q_i q_j})| < c_3|e| \quad \text{for all } i, j \text{ and } e \in E. \tag{4}$$

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Indeed, assuming (4), there exists q_i, q_j so that $|\overrightarrow{q_i q_j}| = \operatorname{diam}(BA\Delta)$. So applying Lemma 6.10 with $c_2 = c_3$ establishes Lemma 6.11 with F_1 the corresponding c_1.

To prove (4), let $e \in E$ be arbitrary. For concreteness, assume that $e = \overrightarrow{p_1 p_2}$. Let j be such that every entry in row j of A is positive and $|C_j(B)| \geq |C_u(B)|$ (such a j exists as at least two rows of A are positive). Up to exchanging p_1 and p_2, we may also assume that $|\Phi_e(\overrightarrow{p_1 p_j})| \geq (1/2)|e|$.

Note that

$$C_i(BA) = \sum a_{k,i} C_k(B),$$

where $a_{k,i} \leq N$ for all k, i and $a_{j,i} \geq 1$ for all i. Thus,

$$q_i = \sum a_{k,i} \frac{|C_k(B)|}{|C_i(BA)|} p_k.$$

Since $\|A\| < N$ and $|C_j(B)| \geq |C_u(B)|$, there exists some $\epsilon > 0$ (dependent only on N and the constant K from Theorem 6.3) so that $(|C_j(B)|)/(|C_i(BA)|) > \epsilon$.

By the above,

$$\Phi_e(q_i) = \sum a_{k,i} \frac{|C_k(B)|}{|C_i(BA)|} \Phi_e(p_k).$$

The coefficient $a_{j,i}(|C_j(B)|)/(|C_i(BA)|)$ of $\Phi_e(p_j)$ in this sum is at least ϵ. Since $|\Phi_e(\overrightarrow{p_1 p_j})| \geq (1/2)|e|$, there is some $c_4 < 1$ so that, for all i, $\Phi_e(q_i)$ is within $c_4 \operatorname{diam}(B\Delta)$ of p_2†. If $c_3 = \sqrt{c_4}$, we obtain $|\Phi_e(\overrightarrow{q_i q_j})| < c_3|e|$ for all i, j and e with $|e| > c_3 \operatorname{diam}(B\Delta)$, which establishes (4) and thus the lemma. □

LEMMA 6.12. *There exists $N > 0$ and $F < 1$ so that if B is a product of matrices from Corollary 6.1, $A = A_i \cdots A_{i+k}$ is a sub-product satisfying (P) or (R) with $k \leq c$ and $\|A\|_{\text{op}} > N$, then*

$$\sin(\max \angle(C_i(BAA_{i+k+1} \cdot \cdots \cdot A_{r+k+3}), C_j(BAA_{i+k+1} \cdot \cdots \cdot A_{r+k+3})))$$
$$\leq F \sin(\max \angle(C_i(B), C_j(B))).$$

Proof. The number N will be given by Lemma 6.2, so that any A with $\|A\| > N$ contains a matrix corresponding to at least ℓ powers of a minimal depth zero loop—where ℓ will be chosen below.

To begin, note that

$$\sin(\max \angle(C_i(XY), C_j(XY))) \leq \sin(\max \angle(C_i(X), C_j(X)))$$

for any matrices X, Y with non-negative entries (as $XY\Delta \subset X\Delta$). By the choice of N, there is an s, $i \leq s \leq i + k$ so that A_s is the matrix defined by a minimal depth zero loop taken at least ℓ times. By the preceding remark it suffices to show that

$$\sin(\max \angle(C_i(BAA'), C_j(BAA'))) \leq F \sin(\max \angle(C_i(B), C_j(B))),$$

where $A = A_s$ is a matrix corresponding to the ℓth power of a combining matrix, and A' is a matrix corresponding to three building block steps. We emphasize that this is

† Indeed, it is a convex combination of points on the line in direction e and there is a point within $(1/2)|e|$ of p_2 with coefficient at least ϵ and the furthest any point is from p_2 is $\operatorname{diam}(B\Delta)$.

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not necessarily the product of three A_i, but may be a smaller product (if some of the A_i correspond to loops). In particular, we may assume that $\|A'\|_{\mathrm{op}} \leq L$ for some universal L.

For simplicity of notation, we abbreviate $\beta = \sin(\max \angle(C_i(B), C_j(B)))$. By Lemma 6.9 and our choice of ℓ,

$$\max_{i,j \text{active or passive}} \sin \angle(C_i(BA), C_j(BA)) < K' \frac{\beta}{\xi^k},$$

where active and passive refers to the columns of A. Note that we have proved the lemma if A has no idle column; we may thus suppose that column b of A is idle.

Next, consider any column $C_k(BAA')$ and write it as a sum of columns of BA, giving

$$C_k(BAA') = a'_{kb} C_b(BA) + \sum_{i \neq b} a'_{ki} C_i(BA).$$

By the isolated idle property, every column of A' has at least two non-zero entries, and thus $\sum_{i \neq b} a'_{ki} \geq 1$.

Furthermore, the sum $\sum_i a'_{ki}$ is at most $\|A'\|_{\mathrm{op}} \leq L$. Thus,

$$\frac{|a'_{kb} C_b(BA)|}{|\sum_{i \neq b} a'_{ki} C_i(BA)|} \leq L \frac{|C_b(BA)|}{|C_i(BA)|}$$

for some index i corresponding to an active or passive column. Hence, by Lemma 3.5,

$$\begin{aligned} &\sin \angle\left(C_k(BAA'), \sum_{i \neq b} a'_{ki} C_i(BA)\right) \\ &\quad \leq LS \frac{|C_b(BA)|}{|C_i(BA)|} \sin \angle\left(\sum_{i \neq b} a'_{ki} C_i(BA), a'_{bi} C_b(BA)\right) \leq LS \frac{|C_b(BA)|}{|C_i(BA)|} \beta. \end{aligned}$$

In other words, for each j, there is an active or passive index i so that $\sin \angle(C_j(BAA'), \overline{v}) \leq LS(|C_b(BA)|)/(|C_i(BA)|)\beta$, where $\overline{v}$ is a convex combination of active or passive columns of BA.

On the other hand,

$$\begin{aligned} &\sin \angle\left(\sum_{i \neq b} a'_{ki} C_i(BA), \sum_{i \neq b} a'_{mi} C_i(BA)\right) \\ &\quad \leq \sin \max_{r,s \text{active,passive}} \angle C_r(BA), C_s(BA) \leq K' \frac{\beta}{\xi^k}. \end{aligned}$$

Since (absolute values of) angles satisfy the triangle inequality and $\sin(\theta + \phi) \leq \sin(\theta) + \sin(\phi)$ for all $0 \leq \theta, \phi$ with $\theta, \phi < \pi/2$, we then know that for any k, m there are active or passive i, j with

$$\begin{aligned} &\sin(\angle(C_k(BAA'), C_m(BAA'))) \\ &\quad < LS \frac{|C_b(BA)|}{|C_i(BA)|} \beta + K' \frac{\beta}{\xi^k} + LS \frac{|C_b(BA)|}{|C_j(BA)|} \beta \\ &\quad \leq \left(LS \frac{|C_b(BA)|}{|C_i(BA)|} + \frac{K'}{\xi^k} + LS \frac{|C_b(BA)|}{|C_j(BA)|}\right) \beta. \end{aligned}$$

By Corollary 6.8, ℓ may be chosen large enough so that $LS(|C_b(BA)|)/(|C_j(BA)|) < 1/4$ for any active/passive index j and, by enlarging ℓ further, we may assume that

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$K'/\xi^k < 1/4$ as well. Then

$$\sin(\angle(C_k(BAA'), C_m(BAA'))) < \tfrac{3}{4}\sin(\max \angle(C_i(B), C_j(B))),$$

which shows the desired outcome. □

The following proposition will be used to imply convergence, continuity and unique ergodicity at an infinite-depth point.

PROPOSITION 6.13. *Let A_i be as above. Then $(\prod A_i)\mathbb{R}^4_+$ converges to a single line.*

Proof. To prove the proposition, we inductively group the product $(\prod A_i)$ into sub-products of length at most c so that the initial blocks corresponding to the first matrix in each sub-product is an allowed breakpoint, and so that each sub-product satisfies (P) or (R). Depending on the norm of the next block we apply Lemma 6.11 or 6.12 to conclude that either the diameter of the image of Δ, or $\max \sin \angle$ decreases by a definite factor. Since one of these happens an infinite number of times, the proposition follows. □

7. *Connecting building block endpoints*

THEOREM 7.1. *Let $\mathcal{B}$ be a complete finite set of building blocks, and suppose that S_1, S_2 are, respectively, left and right endpoints for a building block. Then there exists a path of uniquely ergodic IETs connecting S_1 and S_2.*

Proof. Let $\pi = \pi(S_1) = \pi(S_2)$ be the permutation of the two points we want to connect. We will obtain the path c connecting S_1 to S_2 in Δ_π as the limit of approximating paths c_n. However, before we define the actual paths c_n, we first construct a combinatorial object that will serve as a guideline on how to build the paths.

We define a rooted oriented bivalent tree $\mathcal{T}$ in the following way. Each vertex will correspond to a building block $b \in \mathcal{B}$ and has two outgoing edges. The two vertices joined to b are the left and right compatible building blocks to b in $\mathcal{B}$. The root corresponds to the building block $(S_1, F, S_2) \in \mathcal{B}$, which exists by our hypothesis.

Suppose v is a vertex of $\mathcal{T}$. Then there is a unique path ρ which joins the root of $\mathcal{T}$ to v. This path ρ corresponds to a building block sequence, which, in turn, defines a Rauzy matrix $M(\rho)$. If v corresponds to the building block (T_1, F, T_2), then we say that the triple of IETs $(M(\rho)T_1, M(\rho)F, M(\rho)T_2) \in \Delta_\pi$ is defined by v.

Consider the set $v_0, \ldots, v_{2^n}$ of all vertices of distance n to the root of $\mathcal{T}$, and let $(T_1^{(i)}, F^{(i)}T_2^{(i)})$ be the triple of IETs defined by v_i. Note that (by construction) $T_2^{(i)} = T_1^{(i+1)}$. We define c_n as the concatenation of straight line segments in Δ_π connecting $T_1^{(i)}$ to $T_2^{(i)}$ for all i, parametrized so that each straight segment is traversed with constant speed on an interval of length 2^{-n}.

Explicitly, the first path c_0 is simply the straight line segment connecting S_1 to S_2. The path c_1 is the concatenation of the straight line connecting S_1 to F and F to S_2, where (S_1, F, S_2) is a building block in $\mathcal{B}$. Similarly, c_{i+i} is constructed by replacing each straight line segment in c_i by a concatenation of two line segments according to the building block which (after the suitable number of Rauzy steps) has the same endpoints.

Note that by definition c_i and c_{i+1} agree at all t which correspond to IETs defined by vertices of distance at most i to the root of $\mathcal{T}$.

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We call any point $c_i(t)$ which corresponds to an IET defined by some vertex v of $\mathcal{T}$ a *problematic point* of c_i. The depth of $c_i(t)$ is defined to be the depth of the building block sequence corresponding to the unique path joining the root of $\mathcal{T}$ to v.

Explicitly, this means that the boundary points $c_1(0)$, $c_1(1)$ are of depth zero. Suppose we have assigned a depth to all the problematic points of c_i. Then, each problematic point p of c_{i+1} is adjacent to two problematic points p_l, p_r of c_i. If d_l, d_r are the depths of these points, then the depth of p is $\min(d_l, d_r) + 1$.

By the definition of building blocks, every problematic point corresponds to a uniquely ergodic IET.

For any $t \in [0, 1]$ which is not a multiple of 2^{-m} for some m, we now claim that the sequence $c_n(t)$ converges in Δ_n. Namely, suppose that $t \in [k2^{-M}, (k+1)2^{-M}]$. Then for each $i > M$, the initial part of the Rauzy expansion of $c_i(t)$ agrees with the Rauzy expansion of the building block sequence corresponding to a suitable vertex v_M of distance M to the root of the tree $\mathcal{T}$. Increasing M to an $N > M$, it follows from the construction that v_M is contained in the (unique) oriented path joining the root of the tree $\mathcal{T}$ to v_N. Let ρ be the infinite oriented path in $\mathcal{T}$ which contains all v_i. Since t is not a dyadic fraction, the path ρ cannot eventually only make left or only right turns (as that would, e.g. for left turns, mean that $k2^{-N} \leq t \leq k2^{-N} + 2^{-L}$ for all L).

Thus, the path ρ defines a building block sequence as in §6. Therefore Proposition 6.13 implies that for any ϵ there is an $N > 0$ such that, for all $i > N$, the values $c_i(t)$ differ by at most a distance of ϵ (since they share the same initial segment of building blocks). This shows that $c_i(t)$ converges.

In fact, the same argument shows that $\lim_{n\to\infty} c_n(t)$ is a uniquely ergodic IET by Theorem 2.6 and that the limiting function is continuous at such t.

It remains to show continuity at the problematic points. We show continuity using the limit formulation. By construction, points on c_n close to a depth L point follow the expansion P of the endpoint for longer and longer times. But, in this situation, Lemma 5.2 shows that the normalized length vectors converge, and thus implies the desired continuity. □

COROLLARY 7.2. *Let T_1, T_2 be two 4-IETs which each become the left or right endpoint of a building block after a finite number of Rauzy steps. Then T_1 and T_2 can be joined by a path of uniquely ergodic IETs.*

Proof. Let K be large enough so that T_1, T_2 become endpoints after K Rauzy steps. Now consider $\mathcal{P}_K$, the set of fail planes of Rauzy induction in Δ_π within the first K steps, and choose endpoints of building blocks on all of the fail planes in $\mathcal{P}_K$ separating T_1 from T_2. Two such points which are not separated by a fail plane can be joined by a path as in the previous theorem, by applying Rauzy induction first. The Corollary follows by concatenating these finitely many paths. □

To finish the proof of Theorem 4.1, we need to be able to join IETs which are not themselves endpoints of building blocks. This will be done with a limiting argument via the following lemmas.

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LEMMA 7.3. *Let (X, d) be a metric space, let (p_i) be a sequence in X and let $p_\infty \in X$ be so that:*
(1) *(p_i) converges to p_∞;*
(2) *p_i, p_{i+1} are path-connected by a path $P_i \subset X$; and*
(3) *$\lim_{i\to\infty} diam(P_i) = 0$.*
Then p_1 is path-connected to p_∞.

Proof. Let $\psi(t) = P_j(2^j t)$, where $t \in [1 - 2^{-j+1}, 1 - 2^{-j}]$ and $\psi(1) = p_\infty$. Because the P_i are continuous and $P_i(1) = P_{i+1}(0)$, ψ is continuous everywhere except possibly one. Notice by conditions (1) and (3) $\lim_{t\to 1^-} \psi(t) = p_\infty$. So ψ extends to a path connecting p_1 to p_∞. □

LEMMA 7.4. *Let $\mathcal{B}$ be a finite complete set of building blocks. Let T be an IET so that $R^k(T)$ is defined for all k. Then there exists a sequence of IETs (S_i) so that $\pi(R^k(S_j)) = \pi(R^k(T))$ for all $k < j$ and either $R^{i-1}(S_i) = R^{i-1}(S_{i+1})$ or they are left and right endpoints from a building block triple.*

Proof. Because $\mathcal{B}$ is complete, for each vertex of the Rauzy diagram there is a corresponding building block where the right-hand side takes one (forward pointing) edge and the left-hand side takes the other. Thus, we can follow the expansion of T and choose the desired building blocks. □

THEOREM 4.1. *The set of uniquely ergodic unit length 4-IETs with permutation in the Rauzy class of* (4321) *is path-connected.*

Proof. Let E by a uniquely ergodic 4-IET with permutation (4321) so that $R^k(E)$ is defined for all k. As paths may be concatenated, it suffices to show that, for each uniquely ergodic S with permutation (4321), there is a path joining it to E.

The first case we consider is when S is a uniquely ergodic IET that has $R^k(S)$ defined for all k. Let S_i be a sequence converging to S given by Lemma 7.4. By Corollary 7.2, for each i, there is a path, l_i, of uniquely ergodic IETs that connect S_i to S_{i+1} and that stays in $M(S, i)\Delta$. These satisfy condition (2) of Lemma 7.3. Because S is uniquely ergodic and so $M(S, k)\Delta$ converges to a point, the l_i satisfy condition (3) of Lemma 7.3. For the same reason, the endpoints of the l_i are converging to S, satisfying condition (1) of Lemma 7.3. Repeat the same for E in place of S. Thus, Lemma 7.3 yields the desired path.

We next consider the case of a uniquely ergodic IET S that does not have $R^k(S)$ defined for all k. There exists an interval J so that $S|_J$ has the minimal number of intervals of any induced map of S. As an IET on fewer intervals, $S|_J$ has infinite Rauzy induction. By Proposition 2.7 and Remark 2.8, there exists a path in the Rauzy diagram $\mathcal{R}(4321)$ whose actions on the relevant columns reflect the action on $S|_J$ on the appropriate Rauzy class. Choose once and for all finite Rauzy paths P_i with positive associated matrix for each permutation in the Rauzy diagram.

Now we choose a sequence $\hat{T}_i$ of building block endpoints converging to T that share longer and longer segments of this path and then are followed by one of the P_i. We can connect the $\hat{T}_i$ and satisfy condition (2) of Lemma 7.3. Following T in Rauzy induction leads to at least three columns pointing increasingly in the direction of the length vector

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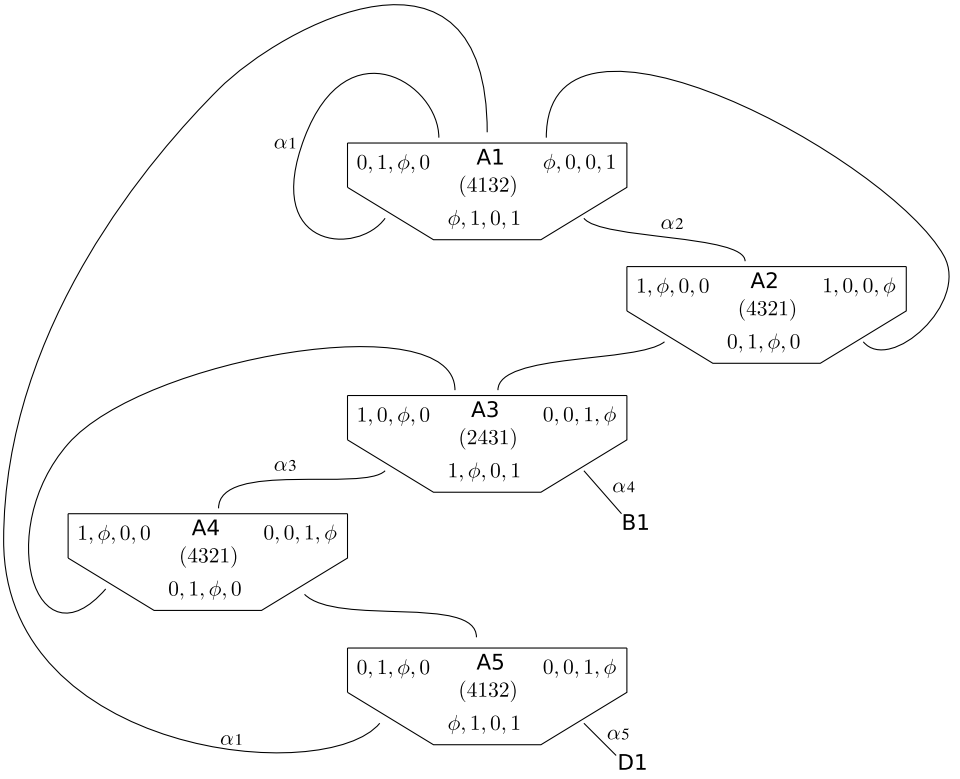


FIGURE 4. Diagram A.

of T and having increasing column norm by unique ergodicity. By Corollary 5.3, after the application of one of the P, all four columns point in roughly this direction. Also by Corollary 5.3, the paths connecting $\hat{T}_i$ to $\hat{T}_{i+1}$ have diameter going to zero as i goes to infinity†. By Corollary 5.3, condition (1) of Lemma 7.3 is satisfied. Because any IET in $M(T_{n+k}, n+k)\Delta$ also satisfies the assumptions of Corollary 5.3, condition (3) of Lemma 7.3 is satisfied. □

8. *Explicitly producing building blocks*

In this section, we prove Proposition 4.8 by explicit construction. The building blocks and their relations are depicted in Figures 4–6. Unlabelled edges correspond to single Rauzy steps (left and right corresponds to type a and b Rauzy induction). Labelled edges of the graphs correspond to several common Rauzy steps (the Rauzy steps corresponding to each path can be found in Appendix A). Edges ending in capital letters lead to the corresponding blocks in a different diagram.

By simply observing the Rauzy diagram and Figures 4–7 it is immediate that the set of building blocks satisfies the *transitivity* and *completeness* properties claimed in Proposition 4.8. Note that all blocks in diagram 7 are sources (building block sequences

† This is because, at some point, they have (at least) three large columns that almost point in T's direction and all steps after that they follow building block matrices.

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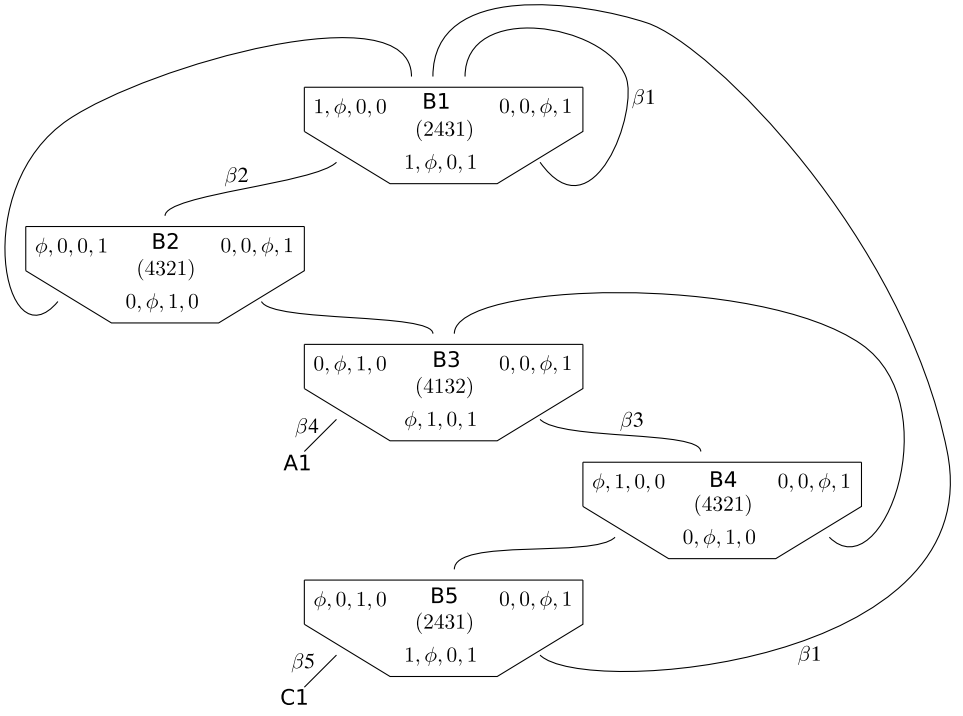


FIGURE 5. Diagram B.

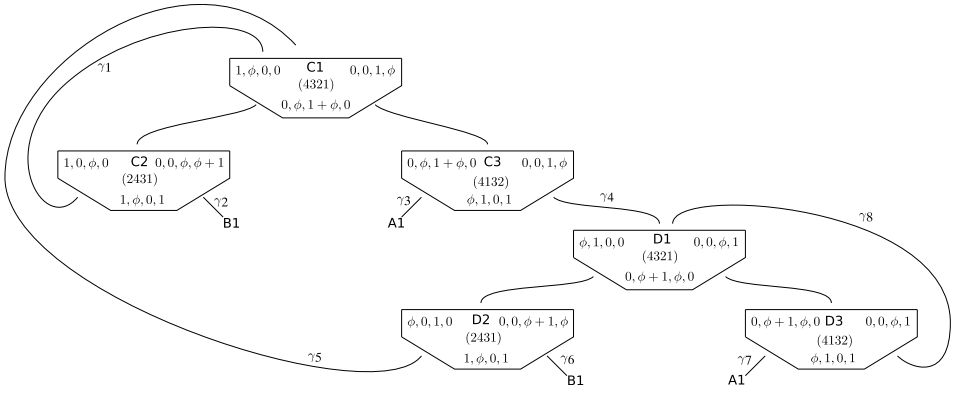


FIGURE 6. Diagram CD.

can only involve these in a uniformly short initial segment), and hence are not relevant for the checks we have to do here. To check the *Combining loops* and *Isolated idle* properties, one checks in the diagrams that the following paths are the only periodic paths of depth zero, and that their matrices have the desired properties. We declare all of their initial points to be allowed breakpoints.

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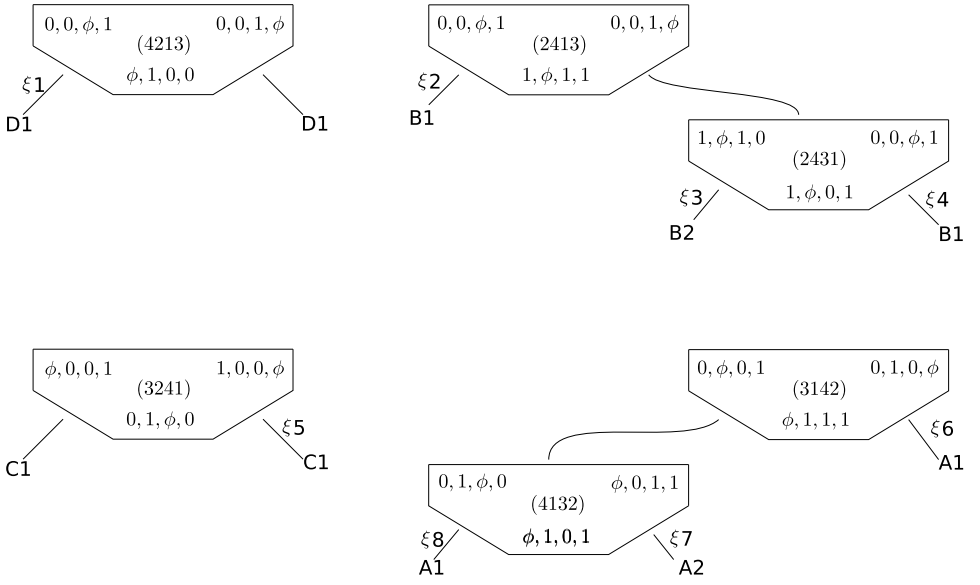


FIGURE 7. Diagram EFGH.

(1) $A1 \to A1$ corresponds to $M_{\alpha 1}$.

$$M_{\alpha 1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Columns two and three are active, four is passive. Column one is idle. The matrix products which can obtained by adding a length–3 building block sequence which leaves the loop $A1 \to A1$ are

$$M_{\alpha 1} M_{\alpha 2} M_a^{(4321)} M_{\alpha 3},\ M_{\alpha 1} M_{\alpha 2} M_a^{(4321)} M_{\alpha 4},\ M_{\alpha 1} M_{\alpha 2} M_b^{(4321)} M_{\alpha 1},$$
$$M_{\alpha 1} M_{\alpha 2} M_b^{(4321)} M_{\alpha 2},$$

all of which have at least two non-zero entries in each column. Thus, the isolated idle property holds for this loop.

(2) $A2 \to A1 \to A2$ corresponds to $M_b^{(4321)} M_{\alpha 2}$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 2 & 2 & 2 \end{pmatrix}.$$

Columns one and four are active, two and three are passive.

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(3) $A3 \to A4 \to A3$ corresponds to $M_{\alpha 3} M_a^{(4321)}$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Columns one and three are active, two and four are passive.

(4) $B1 \to B1$ corresponds to $M_{\beta 1}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}.$$

Columns three and four are active, and column one is passive. Column two is idle. The matrix products which can obtained by adding a length–3 building block sequence which leaves the loop $B1 \to B1$ are

$$M_{\beta 1} M_{\beta 2} M_a^{(4321)} M_{\beta 1},\ M_{\beta 1} M_{\beta 2} M_a^{(4321)} M_{\beta 2},\ M_{\beta 1} M_{\beta 2} M_b^{(4321)} M_{\beta 3},$$
$$M_{\beta 1} M_{\beta 2} M_b^{(4321)} M_{\beta 4},$$

all of which have at least two non-zero entries in each column. Thus, the isolated idle property holds.

(5) $B1 \to B2 \to B1$ corresponds to $M_{\beta 2} M_a^{(4321)}$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Columns one and two are active, columns three and four are passive.

(6) $B3 \to B4 \to B3$ corresponds to $M_{\beta 3} M_b^{(4321)}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Columns three and four are active, columns one and two are passive.

(7) $C1 \to C2 \to C1$ corresponds to $M_a^{(4321)} M_{\gamma 1}$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Columns one and two are active, columns three and four are passive.

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(8) $D1 \to D3 \to D1$ corresponds to $M_b^{(4321)} M_{\gamma 8}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Columns three and four are active, columns one and two are passive.

It remains to check for *almost positivity* (and define our allowed breakpoints), i.e., we need to show that paths that increase depth enough yield almost positive matrices.

We begin by noting two reductions that describe why this is a finite task.

- The content of Proposition 4.8 is to find sub-products which are almost positive. We may arrange that each such subblock starts at $A1$, $A3$, $B1$, $B3$, $D1$, $C1$ (these will be our allowed breakpoints), as all loops and all positive or almost positive sequences produced in this section for Proposition 4.8 will end at a such a point. The first, initial segment leading to one of these points has uniformly small length (as can be checked in the diagrams) and does not affect the convergence argument in §6.
- All minimal depth zero loops have matrices with positive entries on the diagonal. Since multiplying by such a matrix only increases the size of entries, any path that yields a positive matrix will still have this property if any number of loops is inserted.
- To check whether a matrix product is (almost) positive, the exact entries of the factor matrices do not matter; only which entries are zero and which are positive. The matrices A corresponding to loops have the property that whichever entries of A^n are positive does not depend on n for $n \geq 2$.

  Hence, if a path without loops yields an almost positive, yet not positive matrix, there are a finite number of modifications (add loops of length ≤ 2) that need to be considered. If all such modifications yield positive matrices, the proof is complete.

The desired property now follows by a lengthy case-by-case check. We only list the cases for paths starting at $A1$ in detail. All other cases are checked in a completely analogous manner†.

The following observation helps to reduce the number of cases.

Observation 8.1. Say that a 4×4 matrix A is *weakly positive* if every entry is non-negative and there is at most one zero entry, which is not on the diagonal.

If A is a weakly positive matrix and B is almost positive or weakly positive, then AB is positive.

Similarly, if A is a weakly positive matrix and BB' is positive, then BAB' is positive.

Thus, it suffices to show a slightly weaker version of almost positivity: *any path that increases depth enough yields a matrix that is almost positive, or is weakly positive.* Here, 'increases depth enough' is taken to mean that there is a uniform constant K, so any path which increases depth by at least K.

Paths that begin with the loop α_1 at $A1$. There are four possibilities that can follow the α_1-loop.

† Which can also be automated and done with computer algebra systems—see the second author's webpage for a SAGE script doing this task.

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(1) The path $A1 \to A2 \to A3 \to B1$, corresponding to the matrix

$$\alpha_1\alpha_2 M_a^{(4321)}\alpha_4 = \begin{pmatrix} 2 & 2 & 1 & 2 \\ 2 & 1 & 2 & 4 \\ 2 & 0 & 1 & 3 \\ 1 & 1 & 1 & 2 \end{pmatrix},$$

which is weakly positive.

(2) The path $A1 \to A2 \to A3 \to A4 \to A5 \to D1$, corresponding to the matrix

$$\alpha_1\alpha_2 M_a^{(4321)}\alpha_3 M_b^{(4321)}\alpha_5 = \begin{pmatrix} 4 & 5 & 4 & 2 \\ 3 & 5 & 3 & 1 \\ 1 & 2 & 2 & 0 \\ 2 & 3 & 2 & 1 \end{pmatrix},$$

which is weakly positive.

(3) The loop $A1 \to A2 \to A3 \to A4 \to A5 \to A1$, which immediately leads to a positive matrix.

(4) The loop $A1 \to A2 \to A1$, leading to

$$\alpha_1\alpha_2 M_b^{(4321)} = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 0 & 1 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

If this is followed by a path beginning with $A1 \to A3$, then one of the previous cases implies that we will reach a positive matrix. The only other possibility is to alternate with the $A1 \to A1$ loop. But, the square of the above matrix is positive, and thus this case also yields a positive matrix.

As a consequence, we record the following. *Any path that begins with the α_1-loop and increases depth will yield a positive matrix.* In particular, in the consequent analysis, we never have to consider the case that the α_1-loop is inserted in the middle of some path—as then the resulting matrix will be positive upon increasing depth enough.

Paths that begin with the loop $A1 \to A2 \to A1$.

(1) If the loop is followed by the α_1–loop, then the above shows that the result will turn positive.

(2) If we follow by $A1 \to A3$, the resulting matrix is

$$\alpha_2 M_b^{(4321)}\alpha_2 M_a^{(4321)} = \begin{pmatrix} 2 & 5 & 4 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 3 & 3 & 3 \end{pmatrix},$$

which is almost positive. Similarly, if we follow the loop $A1 \to A2 \to A1$ several times, we obtain a matrix of the same form (first and last row positive; second and third as above).

In consequence, *any path that begins with the $(A1 \to A2 \to A1)$–loop and increases depth will yield an almost positive matrix.*

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Paths that begin with $A1 \to A2 \to A3 \to B1$. Here, we start with the matrix

$$\alpha_2 M_a^{(4321)} \alpha_4 = \begin{pmatrix} 2 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix},$$

which is followed by $B1 \to B3$, leading to

$$\alpha_2 M_a^{(4321)} \alpha_4 \beta_2 M_b^{(4321)} = \begin{pmatrix} 6 & 5 & 6 & 2 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 3 & 3 & 4 & 1 \end{pmatrix}.$$

From here, there are several possibilities.

(1) Returning to $A1$, leading to

$$\alpha_2 M_a^{(4321)} \alpha_4 \beta_2 M_b^{(4321)} \beta_4 = \begin{pmatrix} 6 & 5 & 11 & 7 \\ 0 & 1 & 2 & 1 \\ 1 & 1 & 3 & 1 \\ 3 & 3 & 7 & 4 \end{pmatrix},$$

which is weakly positive.

(2) Following $B3 \to B5 \to B1$, leading to a positive matrix.

(3) Following $B3 \to B5 \to C1$, leading to a weakly positive matrix.

Paths that begin with $A1 \to A2 \to A3 \to A5 \to D1$. This corresponds to the matrix

$$\alpha_2 M_a^{(4321)} \alpha_3 M_b^{(4321)} \alpha_5 = \begin{pmatrix} 4 & 5 & 4 & 2 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 3 & 2 & 1 \end{pmatrix}.$$

There are several possibilities.

(1) Following with $D1 \to D2 \to B1$ yields a weakly positive matrix.

(2) Following with $D1 \to D2 \to C1$ yields an almost positive matrix.

(3) Following with $D1 \to D3 \to D1 \to D2 \to C1$ (inserting the $D1$ loop into the previous case) leads to

$$\alpha_2 M_a^{(4321)} \alpha_3 M_b^{(4321)} \alpha_5 (M_b^{(4321)} \gamma_8) M_a^{(4321)} \gamma_5 = \begin{pmatrix} 6 & 13 & 16 & 12 \\ 1 & 3 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 3 & 7 & 8 & 6 \end{pmatrix}.$$

Following with $C1 \to C3$ leads to a weakly positive matrix. Following with $C1 \to C2 \to B1$ leads to a positive matrix. Since both of these lead to positive matrices, the proof is complete.

(4) Following with $D1 \to D3 \to A1$ yields an almost positive matrix.

Paths that begin with $A1 \to A2 \to A3 \to A5 \to A1$. Note that the matrix describing $A5 \to A1$ is, in fact, the same as $A1 \to A1$, and thus this follows from the first case.

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9. *Extending to n-IETs*

The goal of this section is to extend Theorem 4.1 to the general.

THEOREM 9.1. *Let π be a non-degenerate permutation for n-IETs, where $n \geq 4$. The set of uniquely ergodic n-IETs with permutation π is path-connected.*

The main tool in the proof of this is the following object.

Definition 9.2. An IET S is called a *secret ℓ-IET at level k* if there exists a non-negative vector v with at most ℓ non-zero entries and a matrix of Rauzy induction $M(T, r)$ with $r \leq k$ so that $L(S) = M(T, r)v$.

In other words, a secret 4-IET at level zero is an n-IET which has at most four intervals of non-zero length. A general secret 4-IET is one that is of this form after applying some number of Rauzy steps†.

The first main goal in the proof of Theorem 9.1 is the following result.

PROPOSITION 9.3. *Suppose that π is a non-degenerate permutation on $n \geq 4$ letters. Then the set of uniquely ergodic secret 4-IETs at level zero with permutation π is path-connected.*

The proof of this proposition relies on a combinatorial argument which roughly states that the different copies of 4-IET space in the space of all n-IETs 'intersect nicely enough', so one can concatenate the paths given by Theorem 4.1.

We begin by introducing some notation to describe how π acts on subsets of $\{1, \ldots, d\}$. Let $(p_1, \ldots, p_4)$ be a 4-tuple of numbers, labelled so that $p_1 < p_2 < p_3 < p_4$. Applying π to the p_i maps them to numbers $\pi(p_i)$, which may not be in order. Let $\pi|(p_1, p_2, p_3, p_4)$ be the permutation on the numbers $\{1, 2, 3, 4\}$, which describes how the order of p_i is rearranged by π. We call $\pi|(p_1, p_2, p_3, p_4)$ the *permutation π restricted to* $(p_1, \ldots, p_4)$.

To give a simple example, consider $\pi = (3142)$: then $\pi|(3, 4) = (12)$ and $\pi|(1, 3) = (21)$.

We say that a permutation π *acts as a rotation* on a pair (p_1, p_2) of symbols if $\pi|(p_1, p_2) = (21)$.

We define the *distance* of a k-tuple $(q_1, \ldots, q_k)$ and an ℓ-tuple $(p_1, \ldots, p_\ell)$ to be

$$\Big(\min_i\{q_i\} - \max_i\{p_i\}, \min_i\{\pi(q_i)\} - \max_i\{\pi(p_i)\}\Big),$$

ordered by lexicographic order (note that this is not a metric, and entries may be negative!).

From now on, we assume that π is a non-degenerate permutation.

LEMMA 9.4. *For any i, either there exists $r < i$ so that $\pi(r) > \pi(i)$ or there exists $r > i$ so that $\pi(r) < \pi(i)$.*

Proof. If $i = \pi(i)$ this simply follows by the irreducibility of π.

If $i \neq \pi(i)$, then this follows from the pigeonhole principle: indeed if $i < \pi(i)$, then there exists $r > i$ so that $\pi(r) < \pi(i)$ because π is a bijection. □

† Or the generalization of Rauzy induction for minimal IETs, where we formally choose a step a or b on a fail plane of usual Rauzy induction; see §2.1.

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LEMMA 9.5. *Let $P = (p_1, p_2)$ and $Q = (q_1, q_2)$ with $p_1 < q_i$ for $i = 1, 2$. Assume that their distance is greater than zero in both coordinates and $\pi|(p_1, p_2) = (21)$.*

Then there exists $P' = (x, p_i)$, where $i \in \{1, 2\}$ so that $\pi|(x, p_i) = (21)$ and the distance between P' and Q is less than the distance between P and Q.

Proof. We may assume that $p_1 < p_2$.

Case 1. If $\pi(p_1) > p_2$, then there exists $j > p_2$ so that $\pi(j) < \pi(p_1)$. Indeed, there are $\pi(p_1) - 1$ different j so that $\pi(j) < \pi(p_1)$, and $m - p_2 > m - \pi(p_1)$ different j so that $j > p_2$ and p_1 is in neither of these sets. Thus there exists a j that is in both of these sets. Then $P' = (p_1, j)$ is the desired pair.

Case 2. $\pi(p_1) \leq p_2$. By the irreducibility of π, there exists $j \in \{1, \ldots, p_2\}$ with $\pi(j) > p_2 \geq \pi(p_1) > \pi(p_2)$ (where, in the last step, we have used that π acts as a rotation on (p_1, p_2)). Let $P' = (j, p_2)$. □

Definition 9.6. Let T be an n-IET. We say it is *supported* on $(p_1, \ldots, p_r)$ if $L(T)_j = 0$ for all $j \notin \{p_1, \ldots, p_r\}$.

LEMMA 9.7. *Every uniquely ergodic secret 4-IETs at level zero is connected to a uniquely ergodic secret 2-IET at level zero by a path of uniquely ergodic secret 4-IETs at level zero.*

Proof. Let T be a secret 4-IET at level zero supported on (a subset of) (a, b, c, d). If $\pi|(a, b, c, d) \in \mathcal{R}(4321)$, since there exists a pair, say, (a, b), so that $\pi|(a, b) = (21)$, so there exists a uniquely ergodic secret 2-IET in this Rauzy class. Theorem 4.1 completes the lemma in this case. Otherwise, there is a 3-IET or a rotation that is equal to T (as a function). If it is a rotation, $L(T)$ is just a partition of 2-intervals which are rationally independent. We can keep the lengths to these two intervals constant, while changing the partition so that eventually there are only two non-zero terms†. Similarly, if there is a 3-IET, we may assume that only three of the lengths are non-zero and the resulting permutation is (321)‡. So $(1 - L_1)/(1 + L_2) \notin \mathbb{Q}$ and T is given by an induced map of rotation by $(1 - L_1)/(1 + L_2)$ on an interval (see the comments after Theorem 1.1). If $(1 - L_1)/(1 + L_2) = (L_2 + L_3)/(1 + L_2) < 1/2$, we may continuously change (L_1, L_2, L_3) in Δ_2, keeping $(1 - L_1)/(1 + L_2)$ constant, to reach $(L_1', L_2', 0)$. Similarly if $(1 - L_1)/(1 + L_2) > 1/2$, we may continuously change (L_1, L_2, L_3) in Δ_2, keeping $(1 - L_1)/(1 + L_2)$ constant, to reach $(0, L_2', L_3')$. □

LEMMA 9.8. *Let p, q be so that $\pi|(p, q) = (21)$. Then there exists a, b so that $\pi|(a, b, p, q) \in \mathcal{R}(4321)$.*

Proof. This will follow from an extensive case distinction. We assume throughout that $p < q$.

† For example, if $\pi(T) = (4123)$ and $L(T) = (\alpha/3, (\alpha/3)(\alpha/3), 1 - \alpha)$, consider the line segment of IETs S_t with $\pi(S_t) = (4123)$ and $L(S_t) = (\alpha/3 + 2t\alpha, \alpha/3 - t\alpha, \alpha/3 - t\alpha, 1 - \alpha)$.
‡ (312) and (231) are rotations.

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Case 1a. There is a symbol, j, between p, q which is not mapped between $\pi(q), \pi(p)$. Choose the largest such j. Since π is non-degenerate, there exists k so that either $j < k < q$ and $\pi(k)$ is not between $\pi(j)$ and $\pi(q)$ or k is not between j and q but $\pi(k)$ is between $\pi(j)$ and $\pi(q)$. $\pi|(p, q, j, k) \in \mathcal{R}(4321)$.

The case where there exists $p < j < q$ and $\pi(q) < \pi(p) < \pi(j)$ is similar.

Case 1b. There is a symbol, j, not between p, q which is mapped between $\pi(q), \pi(p)$. This is analogous to Case 1a.

Case 2. The symbols between p, q are exactly the ones mapped between $\pi(q), \pi(p)$. First, assume that there actually exists a j between p and q. By the irreducibility of π, there exists $a > q$ so that $\pi(a) < \pi(q)$ or $a < p$ so that $\pi(a) > \pi(p)$†. In either case, $\pi|(p, j, q, a) = (4321)$.

Finally, assume that no symbol is between p and q and no symbol is between $\pi(q)$ and $\pi(p)$. Similarly to the above, we obtain a so that either $a > q$ and $\pi(a) < \pi(q)$ or $a < p$ and $\pi(a) > \pi(p)$. If we may choose a so that p, q, a and $\pi(a), \pi(p), \pi(q)$ have no symbols between them, we may apply this argument again (to p, q, a) and obtain a similar symbol b and $\pi|(a, b, p, q) = (4321)$. Consider one of the remaining cases: there exists $a < b < p < q$ and $\pi(a) > \pi(p) > \pi(q)$ and we assume that a is maximal with this property. So $\pi(b) < \pi(a)$ and $\pi|(a, b, p, q) \in \{(4321), (2431)\}$, both of which are in $\mathcal{R}(4321)$. The other cases are similar. □

COROLLARY 9.9. *Given a pair p, q so that $\pi|(p, q) = (21)$, any two uniquely ergodic IETs supported on (p, q) are connected by paths of uniquely ergodic secret 4-IETs at level zero.*

Proof. By the previous Lemma, there exists a, b so that $\pi|(a, b, p, q) \in \mathcal{R}(4321)$. By Theorem 4.1, the set in question can be connected by paths made up of uniquely ergodic IETs supported on (a, b, p, q). □

LEMMA 9.10. *If there exists a, j, p so that $\pi|(a, j) = (21) = \pi|(j, p)$, then the uniquely ergodic IETs supported on (a, j) are connected to the uniquely ergodic IETs supported on (j, p) by paths of uniquely ergodic secret 4-IETs at level zero.*

Proof. Case 1. $\pi|(a, j, p) \neq (321)$. By the previous corollary, we may assume that the uniquely ergodic IETs are both rotations by the same number α. Now assuming that $a < p$, we either have $j < a < p$ and $\pi(a) < \pi(p) < \pi(j)$ or $a < p < j$ and $\pi(j) < \pi(a) < \pi(p)$. Thus, if we have an IET supported on (a, j), we have a path of IETs T_t supported on (a, p, j) so that $L(T_t)_a + L(T_t)_p$ is constant and $L(T_t)_j$ is constant. This connects it to our IET supported on (j, p) by a one parameter family of IETs that are all rotations by the same number.

Case 2. $\pi|(a, j, p) = (321)$. We may also assume that $a < j < p$. By Lemma 9.4, there exists $x < a$ so that $\pi(x) > \pi(a) > \pi(j)$ or $x > j$ so that $\pi(x) < \pi(j) < \pi(a)$. In the first case, $\pi|(a, j, x, p) \in \mathcal{R}(4321)$, so Theorem 4.1 implies the result. In the second

† Indeed, we may treat the 'intervals' between p, q and $\pi(q), \pi(p)$ as single symbols and appeal to Lemma 9.4.

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case, choose the minimal such x and there are two possibilities where $\pi|(a, j, p, x) \notin \mathcal{R}(4321)$: $a < j < x < p$ and $\pi(p) < \pi(j) < \pi(x) < \pi(a)$ or $a < j < p < x$ and $\pi(p) < \pi(x) < \pi(j) < \pi(a)$. Since $\pi|(a, j, x, y)$ is non-degenerate, there exists k with $j < k < x$ and $\pi(k)$ not between $\pi(j)$ and $\pi(x)$. Consider (a, k, x, p). In the second case, start with a uniquely ergodic IET supported on (j, p) and transfer the length from p to x. This terminates in a uniquely ergodic IET supported on (j, x) which is path-connected to any uniquely ergodic IET supported on (a, j) by Theorem 4.1. The case of $a < p < j$ and $j < p < a$ are similar. This handles the cases up to renaming a and p. □

Proof of Proposition 9.3. By Lemma 9.7, we may assume that our pair of IETs are both supported on two intervals. By Lemma 9.4, if there are (p, q) and (a, b) where π restricts to both of them as (21) and their distance is greater than zero in both coordinates, we may replace (p, q) by (p, x), where the distance from (p, x) to (a, b) is less than the distance from (p, q) to (a, b) and $\pi|(p, x) = (21)$. By the previous lemma, the uniquely ergodic IETs supported on (p, x) are connected by paths of uniquely ergodic IETs to the uniquely ergodic IETs supported on (p, q). Iterating this, we may assume that one coordinate of the distance between (p, q) and (a, b) is at most zero and one is at least zero. If it is zero in either coordinate, the previous lemma implies the proposition. Otherwise, $\pi|(a, b, p, q)$ is an irreducible permutation of four intervals and so $\pi|(a, b, p, q) \in \mathcal{R}(4321)$ or $\mathcal{R}(4231)$. In the first case, by Theorem 4.1, a uniquely ergodic IET supported on (a, b) is connected by a path of uniquely ergodic secret 4-IETs at level zero to any uniquely ergodic IET supported on (p, q). In the other case, there exists $k, \ell \in \{a, b, p, q\}$ so that $k < \ell$ and $\pi(k) < \pi(\ell)$†. Since π acts as a rotation on (a, b) and (p, q), without loss of generality, $j = a$ and $k = p$. Since π is non-degenerate (as an n-IET), there exists j so that either $a < j < p$ and $\pi(j)$ is not between $\pi(a)$ and $\pi(p)$ or $\pi(a) < \pi(j) < \pi(p)$ and j is not between $\pi(a)$ and $\pi(p)$. Also $\pi|(a, j)$ or $\pi|(j, p)$ is (21). Assuming the former, $\pi|(a, j, p, q) \in \mathcal{R}(4321)$, and by Theorem 4.1 an ergodic rotation supported on (p, q) is connected by a path of secret 4-IETs at level zero to an ergodic rotation supported on (a, j). By the previous lemma, this is connected by a path of uniquely ergodic secret 4-IETs at level zero to a uniquely ergodic IET supported on (a, b). The other case is similar. □

COROLLARY 9.11. *The set of uniquely ergodic secret* 4*-IETs at level (at most) k is path-connected for any k.*

Proof. This is analogous to the proof of Corollary 7.2. Indeed, consider $M(T, k)$ and the uniquely ergodic secret 4-IETs with permutation $\pi(T)$ and length vector given by $M(T, k)v$, where at most four entries of v are non-zero. If v_1 and v_2 are two such vectors, we can connect the (uniquely ergodic) IETs with permutation $\pi(R^k T)$ and length vectors v_1, v_2 by secret 4-IETs at level zero. Considering the (projective) image of this path under $M(T, k)$, we have a path in the set of uniquely ergodic 4-IETs at level k. There exist choices of v with at most four non-zero entries so that this $M(T, k)v$ gives the length of a uniquely ergodic IET and there exists w so that $M(T, k-1)w = M(T, k)v$ and w has at

† By direct check of [**D**, Figure 3.2].

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most four non-zero entries. So the uniquely ergodic secret 4-IETs with length vector given by $M(T, k)v$ are path-connected to the set of uniquely ergodic secret 4-IETs with length vector given by $M(T, k-1)v$. Iterating this, we see that the uniquely ergodic secret 4-IETs at level k with length vector given by $M(T, k)v$ is path-connected by secret 4-IETs at level k to uniquely ergodic secret 4-IETs at level zero. Since $M(T, k)$ is an arbitrary matrix of Rauzy induction, we obtain the corollary. □

The final ingredient to the proof of Theorem 9.1 is the following lemma.

LEMMA 9.12. *Let T be an IET so that $R^k(T)$ is defined for all k. Let S_1, S_2 be secret 4-IETs at level r which are not secret at level k. Let S_1, $S_2 \in M(T, r)\Delta$. Further, assume that $M(R^{k+1}(T), r-k-1)$ is a positive matrix. Then S_1, S_2 are path connected by uniquely ergodic IETs in $M(T, k)\Delta$.*

The idea of this proof is that we build paths connecting $R^k S_1$ to $R^k S_2$ and the path we are interested in is the (projective) image of this path under $M(T, k) = M(S_i, k)$.

Proof. Because $M(R^k T, r-k-1)$ is positive, it follows that if T' is an IET with $L(T') = M(T, r)v$ and $\pi(T') = \pi(T)$, then $R^i(T')$ is defined for $i \leq k$†. Let $S_i' = R^k(S_i)$ and P be the path given by Corollary 9.11 connecting S_1' and S_2'. $M(T, k)P$ is the path connecting S_1 and S_2 contained in $M(T, k)\Delta$. □

Proof of Theorem 9.1. If S, E have R^k defined for all k, then the theorem follows analogously to the proof of Theorem 4.1 via Lemma 7.3. In this case, the p_i are secret 4-IETs at level i contained in $M(S, i)\Delta$ or $M(E, i)\Delta$. The paths P_i connecting p_i and p_{i+1} are contained in $M(S, m_i)$ or $M(E, q_i)$, where the m_i or q_i is given by Lemma 9.12. The m_i and q_i go to infinity with i. Since S and E are uniquely ergodic and have Rauzy induction defined for all k, $M(S, i)\Delta$ and $M(E, i)\Delta$ contract to S and E, respectively. This verifies Lemma 7.3.

Otherwise, assume that S does not have all powers of Rauzy induction defined. We may, without loss of generality, assume that E does. We prove the theorem by assuming that we have inductively proved Theorem 9.1 for all $4 \leq m < n$. There exists an interval J so that the induced map $S|_J$ is a uniquely ergodic IET on $k < d$ symbols. There is a $d \times k$ matrix M so that $L(S) = ML(S|_J)$, where, in an abuse of notation, $L(S|_J)$ is a vector with k entries. Because every irreducible permutation has a pair where it acts irreducibly, we have a secret 4-IET, S', of the form Mv for some v whose non-zero entries are a subset of $L(S|_J)$. By induction, the k-IET $S|_J$ can be joined by a path of uniquely ergodic IETs to S'. Applying M to this path connecting $S|_J$ to S', we obtain a path connecting S to a secret 4-IET (with length vector is given by $ML(S')$). By the first case, E is path-connected by uniquely ergodic IETs to a secret 4-IET of some depth and this secret 4-IET is path-connected by uniquely ergodic IETs to every uniquely ergodic secret 4-IET. We have just shown that one of these is path-connected to S by a path of uniquely ergodic IETs. □

† Indeed, $M(T, r)v$ is in the interior of $M(T, k)$, which is cut out by fail planes of Rauzy induction for $i \leq k$.

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A. *Appendix. Composite paths.*

In Figure 4

$\alpha 1$: $(4132) \to (3142) \to (3142) \to (4132)$

$\alpha 2$: $(4132) \to (4213) \to (4321) \to (2431) \to (3241) \to (4321)$

$\alpha 3$: $(2431) \to (3241) \to (3241) \to (4321)$

$\alpha 4$: $(2431) \to (2413) \to (2431)$

$\alpha 5$: $(4132) \to (4213) \to (4321)$.

In Figure 5

$\beta 1$: $(2431) \to (2413) \to (2413) \to (2431)$

$\beta 2$: $(2431) \to (3241) \to (4321) \to (4132) \to (4213) \to (4321)$

$\beta 3$: $(4132) \to (4213) \to (4213) \to (4321)$

$\beta 4$: $(4132) \to (3142) \to (4132)$

$\beta 5$: $(2431) \to (3241) \to (4321)$.

In Figure 6

$\gamma 1$: $(2431) \to (3241) \to (3241) \to (4321)$

$\gamma 2$: $(2431) \to (2413) \to (2431)$

$\gamma 3$: $(4132) \to (3142) \to (3142) \to (4132)$

$\gamma 4$: $(4132) \to (4213) \to (4321)$

$\gamma 5$: $(2431) \to (3241) \to (4321)$

$\gamma 6$: $(2431) \to (2413) \to (2413) \to (2431)$

$\gamma 7$: $(4132) \to (3142) \to (4132)$

$\gamma 8$: $(4132) \to (4213) \to (4213) \to (4321)$.

In Figure 7

$\xi 1$: $(4213) \to (4213) \to (4321)$

$\xi 2$: $(2413) \to (2413) \to (2431) \to (2413) \to (2413) \to (2431)$

$\xi 3$: $(2431) \to (3241) \to (3241) \to (4321) \to (4132) \to (4213) \to (4321)$

$\xi 4$: $(2431) \to (2413) \to (2413) \to (2431)$

$\xi 5$: $(3241) \to (3241) \to (4321)$

$\xi 6$: $(3142) \to (3142) \to (4132) \to (3142) \to (3142) \to (4132)$

$\xi 7$: $(4132) \to (4213) \to (4213) \to (4321) \to (2431) \to (3241) \to (4321)$

$\xi 8$: $(4132) \to (3142) \to (3142) \to (4132)$.

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