STRONGLY INCREASING SEQUENCES

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Abstract. Using a variation of Woodin's $\mathbb{P}_{\max}$ forcing, we force over a model of the Axiom of Determinacy to produce a model of ZFC containing a very strongly increasing sequence of length ω_2 consisting of functions from ω to ω. We also show that there can be no such sequence of length ω_4.

§1. Introduction. Given functions f, g from ω to the ordinals, a relation R in $\{<, >, \leq, \geq, =\}$ and $n \in \omega$, we write

- $fR_n g$ to mean that $f(m)Rg(m)$ for all $m \in \omega \setminus n$;
- $fR^* g$ to mean that $\{n \in \omega : \neg(f(n)Rg(n))\}$ is finite.

This article concerns well-ordered sequences of functions from the integers to the ordinals, which are increasing in the following senses.

DEFINITION 1. Given ordinals η and γ, we say that a sequence $\langle f_\alpha : \alpha < \gamma \rangle$ of functions from ω to η is *strongly increasing* if

1. for all $\alpha < \beta < \gamma$, $f_\alpha <^* f_\beta$;
2. for each limit ordinal $\beta < \gamma$, there exist a club $C_\beta \subseteq \beta$ and an $n_\beta < \omega$ such that, for all α in C_β, $f_\alpha <_{n_\beta} f_\beta$.

DEFINITION 2. We say that a strongly increasing sequence $\langle f_\alpha : \alpha < \gamma \rangle$ is *very strongly increasing* if

1. n_β in part (2) of Definition 1 can be chosen to be 0 for each limit ordinal β of uncountable cofinality;
2. for each ordinal of the form $\beta + \omega < \gamma$, and each $n \in \omega$, there is an α in the interval $[\beta + 1, \beta + \omega)$ such that $f_\alpha \leq_n f_{\beta+\omega}$ and $f_\alpha(m) = 0$ for all $m < n$.

The ordinal γ is called the *length* of $\langle f_\alpha : \alpha < \gamma \rangle$, and a sequence is said to have *successor length* if its length is a successor ordinal.

REMARK 3. Condition (2) of Definition 2 is designed to make condition (2) of Definition 1 hold automatically when β has countable cofinality. In addition, it ensures that whenever $\bar{f} = \langle f_\alpha : \alpha < \gamma \rangle$ is very strongly increasing and γ has countable cofinality and is not of the form $\beta + \omega$, there exists a function f_γ such that $\langle f_\alpha : \alpha \leq \gamma \rangle$ is very strongly increasing; moreover, f_γ can be any function f

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such that $f_\alpha <^* f$ for all $\alpha < \gamma$. Extending a very strongly increasing sequence $\langle f_\alpha : \alpha \leq \gamma \rangle$ to one of the form $\langle f_\alpha : \alpha \leq \gamma + \omega \rangle$ can be done by letting $f_{\gamma+\omega}(n)$ be $f_\gamma(n) + n$ for each $n \in \omega$, and letting $f_{\gamma+k}(n)$(for all $k, n \in \omega$ with k positive) be 0 if $n \leq k$ and $f_\gamma(n) + k$ otherwise.

REMARK 4. If $\langle f_\alpha : \alpha < \gamma \rangle$ is a strongly increasing sequence, then there is a sequence $\langle f'_\alpha : \alpha < \gamma \rangle$ satisfying condition (2) of Definition 2 such that $f_\alpha =^* f'_\alpha$ for all $\alpha < \gamma$.

Our interest in strongly increasing sequences is partially motivated by two longstanding open problems:

1. Is the Chang's Conjecture variant $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_2, \aleph_1)$ consistent?
2. Is there consistently a pair of inner models $V \subseteq W$ of ZFC such that $(\aleph_{\omega+1})^V = (\aleph_2)^W$.

In Section 2, we review some of what is known about these questions and show that a positive answer to either would entail the consistency of the existence of an ordinal $\eta < \omega_2$ and a strongly increasing sequence of length ω_2 consisting of functions from ω to η. We also show that there can exist no such sequence of length ω_n for any $4 \leq n < \omega$.

In Section 3, we prove the main result of our article (Theorem 22), using a variant of Woodin's $\mathbb{P}_{\max}$ forcing to establish the consistency of the existence of a strongly increasing sequence of length ω_2 consisting of functions from ω to ω. The following is a simplified version of the theorem.

THEOREM 5. *Suppose that $V = L(A, \mathbb{R})$, for some $A \subseteq \omega^\omega$, and that AD^+ holds. Then there is a forcing extension in which ZFC holds and there exists a very strongly increasing ω_2-sequence of functions from ω to ω.*

The axiom AD^+ is a strengthening of the Axiom of Determinacy due to Woodin (see [9]). The theorem implies the weaker version where $A = \emptyset$ and AD^+ is replaced with AD. The partial order used in the proof is a variation of Woodin's $\mathbb{P}_{\max}$ forcing [14]. The use of the hypothesis $\mathsf{AD} + V = L(\mathbb{R})$ and the fact that ZFC holds in the corresponding extension are both part of the standard $\mathbb{P}_{\max}$ machinery.

§2. Chang's conjecture and strongly increasing sequences. In this section we motivate our result by linking it to some well-known questions about variants of Chang's Conjecture and collapsing successors of singular cardinals. None of the results in this section is essentially new,[1] but we include them for completeness. Some of the results have not been stated before in the precise forms we need.

We begin by noting the following consequence of Remark 3, which can be proved by induction on β.

LEMMA 6. *If F is a very strongly increasing sequence in ω^ω, γ and β are ordinals with $\beta < \omega_2$ and $\gamma + 1$ is the length of F, then there is a very strongly increasing sequence in ω^ω of length $\gamma + \beta$ extending F.*

In particular, there is always a strongly increasing sequence in ω^ω of length ω_1. We next show that instances of Chang's Conjecture at singular cardinals of countable

[1] See, e.g., [1, 11] or [12] for related results in slightly different contexts.

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cofinality entail the existence of more interesting strongly increasing sequences. Recall that the Chang's Conjecture principle

$$(\kappa, \lambda) \twoheadrightarrow (\delta, \gamma)$$

says that whenever M is a structure of cardinality κ over some countable language, and B is a subset of M of cardinality λ, then M has an elementary substructure X of cardinality δ such that $|X \cap B| = \gamma$. We use equivalent formulations of these principles below.

LEMMA 7. *Suppose that μ is a singular cardinal of countable cofinality, $\kappa < \mu$ is a cardinal, and $(\mu^+, \mu) \twoheadrightarrow (\kappa^+, \kappa)$ holds. Then there exist $\eta < \kappa^+$ and a strongly increasing sequence $\langle f_\alpha : \alpha < \kappa^+ \rangle$ of functions from ω to η.*

PROOF. Let $\langle \mu_i : i < \omega \rangle$ be an increasing sequence of regular cardinals, cofinal in μ. It is straightforward to recursively construct a strongly increasing sequence $\vec{g} = \langle g_\beta : \beta < \mu^+ \rangle$ of functions in $\prod_{i<\omega} \mu_i$, letting n_β be the least i such that $\mu_i > \mathrm{cof}(\beta)$, for each limit ordinal β. Let θ be a sufficiently large regular cardinal, and use $(\mu^+, \mu) \twoheadrightarrow (\kappa^+, \kappa)$ to find an elementary substructure $N \prec (H(\theta), \in, \mu^+, \mu, \vec{g})$ such that $|N \cap \mu^+| = \kappa^+$ and $|N \cap \mu| = \kappa$. Since N contains a surjection from μ to each element of $\mu^+ \cap N$, $\mathrm{otp}(N \cap \mu^+) = \kappa^+$.

By elementarity and the fact that $\vec{g}$ is strongly increasing, we know that, for every limit ordinal $\delta \in N \cap \mu^+$, there exist a club $C_\delta \subseteq \delta$ and an $n_\delta < \omega$ such that $C_\delta \in N$ and, for all $\beta \in C_\delta$, we have $g_\beta <_{n_\delta} g_\delta$. Let M be the transitive collapse of N, and let $\pi : N \to M$ be the collapse map.

CLAIM 8. *Suppose that $\delta \in N \cap \mu^+$ is a limit ordinal. Then $\pi(C_\delta)$ is closed in its supremum.*

PROOF. Let $\bar{\beta} < \pi(\delta)$ be such that $\sup(\pi(C_\delta) \cap \bar{\beta}) = \bar{\beta}$. Let $\beta = \pi^{-1}(\bar{\beta})$, and note that β is a limit ordinal. If $\alpha < \beta$, then $\pi(C_\delta) \cap (\pi(\alpha), \bar{\beta}) \neq \emptyset$, so $C_\delta \cap (\alpha, \beta) \neq \emptyset$. It follows that $N \models$ "$C_\delta \cap \beta$ is unbounded in β," so, by elementarity, C_δ is in fact unbounded in β and, since C_δ is closed in δ, we have $\beta \in C_\delta$. But then $\bar{\beta} \in \pi(C_\delta)$, as desired. ⊣

Since $\omega \subseteq N$, it follows that for all $\beta \in N \cap \mu^+$, we have $\mathrm{range}(g_\beta) \subseteq N$. Let $\langle \beta_\alpha : \alpha < \kappa^+ \rangle$ enumerate $M \cap \mu^+$ in increasing order. Define a sequence $\vec{f} = \langle f_\alpha : \alpha < \kappa^+ \rangle$ of functions from ω to $\pi(\mu)$ by letting $f_\alpha(n) = \pi(g_{\beta_\alpha}(n))$ for all $\alpha < \kappa^+$ and all $n < \omega$. Claim 8 and the fact that $\pi(\mu) < \kappa^+$ imply that $\vec{f}$ is strongly increasing, as witnessed by n_{β_α} and $\pi(C_{\beta_\alpha})$ for each limit ordinal $\alpha < \kappa^+$.

In [10], Levinski et al. prove, assuming the consistency of a certain large cardinal hypothesis, the consistency of a number of Chang's Conjecture variants involving singular cardinals, most notably $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$. However, in any instance $(\mu^+, \mu) \twoheadrightarrow (\kappa^+, \kappa)$ known to be consistent in which μ is a singular cardinal, we have $\mathrm{cf}(\mu) = \mathrm{cf}(\kappa)$. This leads to the following natural folklore question.

QUESTION 9. *Is $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_2, \aleph_1)$ consistent*?

A closely related question, raised by Bukovský and Copláková-Hartová in [3], is the following.

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QUESTION 10. *Is there consistently a pair of inner models $V \subseteq W$ of* ZFC *such that* $(\aleph_{\omega+1})^V = (\aleph_2)^W$?

To see one connection between these two questions, observe that, if

$$(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_2, \aleph_1)$$

and there is a Woodin cardinal, then there is a V-generic filter G for Woodin's stationary tower forcing such that $(\aleph_{\omega+1})^V = (\aleph_2)^{V[G]}$ (see [7]).

Essentially the same argument as that given in the proof of Lemma 7 yields the following result, which states that a positive answer to Question 10 would entail the existence of interesting strongly increasing sequences.

LEMMA 11. *Suppose that $V \subseteq W$ are inner models of* ZFC, *$n < \omega$, and $(\aleph_{\omega+1})^V = (\aleph_{n+1})^W$. Then, in W, there is $\eta < \omega_{n+1}$ and a strongly increasing sequence $\langle f_\alpha : \alpha < \omega_{n+1} \rangle$ of functions from ω to η.*

PROOF SKETCH As in the proof of Lemma 7, in V we can construct a strongly increasing sequence $\vec{f} = \langle f_\alpha : \alpha < \omega_{\omega+1} \rangle$ in $\prod_{i<\omega} \omega_i$. In W, we have $(\omega_{\omega+1})^V = \omega_{n+1}$, and $\vec{f}$ remains a strongly increasing sequence of functions. Thus, letting $\eta = (\omega_\omega)^V < (\omega_{n+1})^W$, we see that $\vec{f}$ is as desired. ⊣

For more information and partial progress on Questions 9 and 10, we refer the reader to [4, 11].

For the next result, we need some notions from PCF theory [12]. The following is a special case of a more general definition.

DEFINITION 12. Suppose that $\vec{f} = \langle f_\alpha : \alpha < \gamma \rangle$ is a $<^*$-increasing sequence of functions from ω to the ordinals. A function g from ω to the ordinals is an *exact upper bound* (eub) for $\vec{f}$ if:

- g is an *upper bound*, i.e., for every $\alpha < \gamma$, we have $f_\alpha <^* g$;
- for every function h such that $h <^* g$, there is $\alpha < \gamma$ such that $h <^* f_\alpha$.

An ordinal $\beta < \gamma$ is called *good* for $\vec{f}$ if $\mathrm{cf}(\beta) > \omega$ and there is an eub h for $\vec{f} \restriction \beta = \langle f_\alpha : \alpha < \beta \rangle$ such that $\mathrm{cf}(h(i)) = \mathrm{cf}(\beta)$ for all but finitely many $i < \omega$.

Recall that S^λ_κ denotes the set of ordinals below λ of cofinality κ. The following theorem, due in a much more general form to Shelah [12], is a basic result in PCF theory. For a proof, we refer the reader to [5, Theorem 10.1].

THEOREM 13. *Suppose that $\kappa < \lambda$ are uncountable regular cardinals and that $\vec{f} = \langle f_\alpha : \alpha < \lambda \rangle$ is a $<^*$-increasing sequence of functions from ω to the ordinals. If there are stationarily many $\beta \in S^\lambda_\kappa$ such that β is good for $\vec{f}$, then $\vec{f}$ has an eub h such that $\mathrm{cf}(h(i)) > \kappa$ for all $i < \omega$.*

We can now prove the following result bounding the lengths of strongly increasing sequences of functions.

THEOREM 14. *Suppose that λ is an uncountable regular cardinal and $\varepsilon < \lambda^{+3}$. Then there is no strongly increasing sequence $\vec{f} = \langle f_\alpha : \alpha < \lambda^{+3} \rangle$ of functions from ω to ε.*

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PROOF. Suppose for the sake of contradiction that $\vec{f} = \langle f_\alpha : \alpha < \lambda^{+3} \rangle$ is a strongly increasing sequence of functions from ω to ε. Fix a club-guessing sequence $\langle C_\xi : \xi \in S^{\lambda^{++}}_\lambda \rangle$, i.e., a sequence such that each C_ξ is a club in ξ of order type λ and, for every club $D \subseteq \lambda^{++}$, there is a $\xi \in S^{\lambda^{++}}_\lambda$ such that $C_\xi \subseteq D$ (see Claim 2.14B of Section III(2) of [12], for instance, for a proof that this can be done).

Our goal is to show that there are stationarily many elements of $S^{\lambda^{+3}}_\lambda$ that are good for $\vec{f}$. To this end, fix a club $C \subseteq \lambda^{+3}$. We now construct an increasing, continuous sequence $\langle \alpha_\eta : \eta < \lambda^{++} \rangle$ of elements of C. Begin by letting $\alpha_0 = \min(C)$. If $\eta < \lambda^{++}$ is a limit ordinal, then $\alpha_\eta = \sup\{\alpha_\zeta : \zeta < \eta\}$. Suppose now that $\eta < \lambda^{++}$ and $\langle \alpha_\zeta : \zeta \leq \eta \rangle$ has been constructed. We show how to obtain $\alpha_{\eta+1}$.

For each $\xi \in S^{\lambda^{++}}_\lambda$, ask whether there exist $\beta < \lambda^{+3}$ and $n < \omega$ such that, for all $\zeta \in C_\xi \cap (\eta + 1)$, we have $f_{\alpha_\zeta} <_n f_\beta$. Note that if β has this property, then so does every γ with $\beta \leq \gamma < \lambda^{+3}$. If the answer is yes, then choose an ordinal $\alpha^\xi_\eta \in C \setminus (\alpha_\eta + 1)$ witnessing this. If the answer is no, then let $\alpha^\xi_\eta = \min(C \setminus (\alpha_\eta + 1))$. Let $\alpha_{\eta+1} = \sup\{\alpha^\xi_\eta : \xi \in S^{\lambda^{++}}_\lambda\}$.

Let $\beta = \sup\{\alpha_\eta : \eta < \lambda^{++}\}$. By the fact that $\vec{f}$ is strongly increasing, we can find a club $D \subseteq \beta$ and a natural number n such that, for all $\alpha \in D$, we have $f_\alpha <_n f_\beta$. Let $E = \{\eta < \lambda^{++} : \alpha_\eta \in D\}$, and note that E is club in λ^{++}. Fix $\xi \in S^{\lambda^{++}}_\lambda$ such that $C_\xi \subseteq E$. For $\eta \in C_\xi$, let $\eta^\dagger$ denote $\min(C_\xi \setminus (\eta + 1))$.

Suppose $\eta \in C_\xi$. When defining $\alpha_{\eta+1}$, the answer to the question asked about C_ξ was "yes," as witnessed by β. Therefore, $\alpha_{\eta+1}$ was chosen to be large enough so that, for some $n < \omega$ and all $\zeta \in C_\xi \cap (\eta + 1)$, we have $f_{\alpha_\zeta} <_n f_{\alpha_{\eta+1}}$. The same obviously holds for all η' in the interval (η, λ^{++}). In particular, there is a natural number n_η such that, for all $\zeta \in C_\xi \cap (\eta + 1)$, we have $f_{\alpha_\zeta} <_{n_\eta} f_{\alpha_{\eta^\dagger}}$.

Since λ is regular and uncountable, we can fix a natural number n and an unbounded set $A \subseteq C_\xi$ such that, for all $\eta \in A$, we have $n_\eta = n$. Let $B = \{\alpha_{\eta^\dagger} : \eta \in A\}$. Then B is unbounded in α_ξ and, for all $\zeta < \eta$, both in A, we have $f_{\alpha_{\zeta^\dagger}} <_n f_{\alpha_{\eta^\dagger}}$. But now, if g is a function from ω to the ordinals such that, for all $n \leq m < \omega$, we have $g(m) = \sup\{f_{\alpha_{\eta^\dagger}}(m) : \eta \in A\}$, it is easily verified that g is an eub for $\vec{f} \restriction \alpha_\xi$. But then g witnesses that α_ξ is good for $\vec{f}$. Moreover, by construction, $\alpha_\xi \in C$. Since C was arbitrary, we have shown that there are stationarily many elements of $S^{\lambda^{+3}}_\lambda$ that are good for $\vec{f}$.

By Theorem 13, it follows that there is an eub h for $\vec{f}$ such that $\mathrm{cf}(h(i)) > \lambda$ for all $i < \omega$.

CLAIM 15. *$\mathrm{cf}(h(i)) \geq \lambda^{+3}$ for all but finitely many $i < \omega$.*

PROOF. If not, then there exist $k \in \{1, 2\}$ and an unbounded $A \subseteq \omega$ such that, for all $i \in A$, we have $\mathrm{cf}(h(i)) = \lambda^{+k}$. For each $i \in A$, let $\{\delta^i_\eta : \eta < \lambda^{+k}\}$ enumerate, in increasing fashion, a set of ordinals cofinal in $h(i)$. For each $\eta < \lambda^{+k}$, define a function h_η from ω to the ordinals by letting $h_\eta(i) = \delta^i_\eta$ if $i \in A$ and $h_\eta(i) = 0$ otherwise. For each $\eta < \lambda^{+k}$, we have $h_\eta <^* h$, so, since h is an eub for $\vec{f}$, there is $\beta_\eta < \lambda^{+3}$ such that $h_\eta <^* f_{\beta_\eta}$. Let $\gamma = \sup\{\beta_\eta : \eta < \lambda^{+k}\}$. Since $k < 3$, we have $\gamma < \lambda^{+3}$. Therefore, for all $\eta < \lambda^{+k}$, we have $h_\eta <^* f_\gamma$. Fix an unbounded $B \subseteq \lambda^{+k}$

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and an $n < \omega$ such that, for all $\eta \in B$, we have $h_\eta <_n f_\gamma$. But then, for all $i \in A \setminus n$, we must have $f_\gamma(i) \geq \sup\{\delta^i_\eta : \eta \in B\} = h(i)$, contradicting the fact that h is an upper bound for $\vec{f}$. ⊣

But this claim immediately contradicts the fact that $\vec{f}$ is a sequence of functions from ω to ε and $\varepsilon < \lambda^{+3}$. This is because, by the claim, we must have $h(i) > \varepsilon$ for all but finitely many $i < \omega$. But then the constant function, taking value ε, witnesses that h fails to be an eub.

The results in this section lead to the following corollary. We note that clauses (2) and (3) of the corollary were already known via slightly different proofs (see, e.g., [4, 11]).

COROLLARY 16. *Suppose that* $3 \leq n < \omega$.

1. *If* $\eta < \omega_{n+1}$, *then there is no strongly increasing sequence* $\langle f_\alpha : \alpha < \omega_{n+1} \rangle$ *of functions from* ω *to* η.
2. $(\aleph_{\omega+1}, \aleph_\omega) \not\twoheadrightarrow (\aleph_{n+1}, \aleph_n)$.
3. *There are no inner models* $V \subseteq W$ *of* ZFC *such that* $(\aleph_{\omega+1})^V = (\aleph_{n+1})^W$.

It also follows that the only regular cardinals that can possibly be lengths of strongly increasing sequences from ω^ω are $\aleph_n$ for $0 \leq n \leq 3$. We have seen that there are always such sequences of length $\aleph_0$ and $\aleph_1$. We will prove, in Section 3, the consistency of the existence of a strongly increasing sequence of length $\aleph_2$. The question about the consistency of the existence of a strongly increasing sequence of length $\aleph_3$ remains open.

§3. Consistency via a $\mathbb{P}_{\max}$ variation. In this section we use a natural variation of Woodin's partial order $\mathbb{P}_{\max}$ to force over a model of $\mathsf{AD}^+ + \exists A \subseteq \mathbb{R}\, V = L(A, \mathbb{R})$ to produce a model of ZFC with a very strongly increasing sequence in ω^ω of length ω_2. We refer the reader to [9] for background on AD^+. Since $L(\mathbb{R}) \models \mathsf{AD} \to \mathsf{AD}^+$ (by a result of Kechris), our result applies to models of AD of the form $L(\mathbb{R})$. We refer the reader to [14] for background on $\mathbb{P}_{\max}$, especially Chapter 4 and Section 9.2. The article [8] may also be helpful.

Conditions in our partial order $\mathbb{P}$ are triples (M, F, a) such that

- M is a countable transitive model of $\mathsf{ZFC} + \mathsf{MA}_{\aleph_1}$ which is iterable with respect to $\mathrm{NS}^M_{\omega_1}$;
- F is in M a very strongly increasing sequence of successor length;
- a is an element of $\mathcal{P}(\omega_1)^M$ such that, for some $x \in \mathcal{P}(\omega)^M$, $\omega_1^M = \omega_1^{L[a,x]}$.

The order is: $(M, F, a) < (N, H, b)$ if there exists in M an iteration

$$j : (N, \mathrm{NS}^N_{\omega_1}) \to (N^*, \mathrm{NS}^{N^*}_{\omega_1})$$

such that

- $j(b) = a$;
- $\mathrm{NS}^{N^*}_{\omega_1} = \mathrm{NS}^M_{\omega_1} \cap N^*$;
- $j(H)$ is a proper initial segment of F.

The iterations referred to in the definition above are repeated generic elementary embeddings induced by forcing with $\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1}$; a model is iterable if this process

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always results in well-founded models (see Definition 3.5 of [14]). The map j in the definition of the order embeds N elementarily into N^*; N^* need not be an elementary substructure of M. However, since N^* and M have the same ω_1, and agree about stationarity for subsets of ω_1^M in N^*, $j(H)$ is a very strongly increasing sequence in M. The requirement above that F properly extends $j(H)$ simplifies the arguments below (by removing trivial cases), and adds no new complications, by Lemma 6. Note that j will be an element of $H(\aleph_2)^M$; moreover, the ordinal height of N^* will be less than ω_2^M.

REMARK 17. If (M, F, a) is a condition in our partial order $\mathbb{P}$, then $\langle (M, \mathrm{NS}_{\omega_1}^M), a\rangle$ is a condition in $\mathbb{P}_{\max}$. Conversely, if $\langle (M, \mathrm{NS}_{\omega_1}^M), a\rangle$ is a $\mathbb{P}_{\max}$ condition, and $M \models$ "F is a very strongly increasing sequence of successor length," then (M, F, a) is a condition in $\mathbb{P}$. If (M, F, a) and (N, G, b) are conditions in $\mathbb{P}$ such that $\langle (M, \mathrm{NS}_{\omega_1}^M), a\rangle < \langle (N, \mathrm{NS}_{\omega_1}^N), b\rangle$ (as $\mathbb{P}_{\max}$ conditions, as witnessed by an embedding j), then there is an F' in M such that $(M, F', a) < (N, G, b)$ (as conditions in $\mathbb{P}$); for instance F' can be an extension of $j(G)$ formed by adding one additional member. Moreover, by Lemma 6, F' can be chosen in M to have any desired length below ω_2^M. Finally, the requirement that $M \models \mathsf{MA}_{\aleph_1}$ and the conditions on the set $a \in \mathcal{P}(\omega_1)^M$ imply that the order on each comparable pair of $\mathbb{P}$-conditions is witnessed by a unique iteration (this follows immediately from the corresponding fact for $\mathbb{P}_{\max}$).

REMARK 18. Much of the standard $\mathbb{P}_{\max}$ machinery can be applied directly to the partial order $\mathbb{P}$. In particular, AD^+ implies the following facts about $\mathbb{P}$, each of which can be derived from Theorem 9.31 of [14] and Remark 17 (the first follows from the second). The assertion that $(N, \mathrm{NS}_{\omega_1}^N)$ is A-iterable means that $A \cap N \in N$, and, whenever $j : (N, \mathrm{NS}_{\omega_1}^N) \to (N^*, \mathrm{NS}_{\omega_1}^*)$ is an iteration, $j(A \cap N) = A \cap N^*$.

- For each set $x \in H(\aleph_1)$, there exists a $\mathbb{P}$-condition (M, F, a) with x in $H(\aleph_1)^M$.
- For each $\mathbb{P}$ condition (M, F, a) and each $A \subseteq \omega^\omega$ there is a $\mathbb{P}$-condition $(N, F', b) < (M, F, a)$ such that
  - $(N, \mathrm{NS}_{\omega_1}^N)$ is A-iterable;
  - $\langle H(\aleph_1)^N, \in, A \cap N\rangle \prec \langle H(\aleph_1), \in, A\rangle$.

In conjunction with the previous two remarks, the proof of Theorem 4.43 of [14] gives the ω-clousure of $\mathbb{P}$. We sketch the proof since the same situation appears again in the proof of Theorem 20.

LEMMA 19. *Every descending ω-sequence in $\mathbb{P}$ has a lower bound.*

PROOF. (Sketch) Suppose that $p_i = (M_i, F_i, a_i)$ $(i \in \omega)$ are the members of a descending ω-sequence in $\mathbb{P}$. By the first part of Remark 18 we can work inside a countable transitive model N such that $\langle p_i : i \in \omega\rangle \in H(\aleph_1)^N$ and $(N, \mathrm{NS}_{\omega_1}^N)$ is iterable. As in the proof of Theorem 4.43 of [14], by combining the iterations witnessing the order on the p_i's we get a sequence of $\mathbb{P}$-conditions $(\hat{M}_i, \hat{F}_i, \hat{a}_i)$ $(i \in \omega)$ such that, for each $i \in \omega$,

- $(\hat{M}_i, \mathrm{NS}_{\omega_1}^{\hat{M}_i})$ is an iterate of $(M_i, \mathrm{NS}_{\omega_1}^{M_i})$ and $\hat{F}_i$ and $\hat{a}_i$ are the corresponding images of F_i and a_i, respectively,
- $\omega_1^{\hat{M}_i} = \omega_1^{\hat{M}_0}$,

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- $\mathrm{NS}_{\omega_1}^{\hat{M}_i} = \mathrm{NS}_{\omega_1}^{\hat{M}_{i+1}} \cap \hat{M}_i$,
- $\hat{F}_i$ is a proper initial segment of $\hat{F}_{i+1}$, and
- $\hat{a}_i = \hat{a}_0$.

One can then iterate the sequence $\langle \hat{M}_i : i \in \omega \rangle$ (as in Definition 4.15 of [14]) in N to produce a sequence of iterations

$$j_i : (\hat{M}_i, \mathrm{NS}_{\omega_1}^{\hat{M}_i}) \to (M_i^*, \mathrm{NS}_{\omega_1}^{M_i^*}) \quad (i \in \omega)$$

such that

- each M_i^* is correct (in N) about stationary subsets of ω_1;
- $\bigcup_{i \in \omega} j_i(\hat{F}_i)$ (which we will call F) is (in N) a very strongly increasing sequence in ω^ω whose length is a limit ordinal of countable cofinality;
- each $j_i(\hat{a}_i)$ is the same set (which we will call a).

There exists by Remark 3 a very strongly increasing sequence F^* in N which extends F by the addition of one function. Then (M, F^*, a) is a lower bound for $\langle p_i : i \in \omega \rangle$. ⊣

The remaining argument concerns descending ω_1-sequences in $\mathbb{P}$, from the point of view of some countable transitive model. The main point of this argument is to prove the second item from Remark 21 below: in the $\mathbb{P}$-extension, every subset of ω_1 arises from an iteration of a model coming from a condition in the generic filter (the model that we are working in can be taken to satisfy the second part of Remark 18 with respect to some set of reals A coding a $\mathbb{P}$-name for a subset of ω_1). Less importantly for us, the same construction is also used to show that the nonstationary ideal is saturated in the $\mathbb{P}$-extension.

As in Lemma 5.2 of [8], we phrase the construction of our descending ω_1-sequence in terms of a game, where player I takes care of certain steps (typically, meeting each member of some $\aleph_1$-sized collection of dense sets) which are repeated without change in our context.

Given a condition (M, F, a), the ω_1-*sequence game* for $p = (M, F, a)$ is the game of length ω_1 where players I and II pick the members of a descending ω_1-sequence of conditions $p_\alpha = (M_\alpha, F_\alpha, a_\alpha)$ $(\alpha < \omega_1)$ from $\mathbb{P}$, where $p_0 = p$, player I picks p_α for all successor ordinals α and player II picks p_α for all limit ordinals α. We have in addition that for each $\alpha < \omega_1$, letting $j_{\alpha,\alpha+1}$ be the embedding of M_α into $M_{\alpha+1}$ witnessing that $p_{\alpha+1} < p_\alpha$, $F_{\alpha+1}$ properly extends $j_{\alpha,\alpha+1}(F_\alpha)$. Such a sequence induces (via composing the embeddings $j_{\alpha,\alpha+1}$) an elementary embedding k_α of each model M_α into a structure M_α^* of cardinality $\aleph_1$, and a sequence $F^* = \langle f_\alpha^* : \alpha < \gamma \rangle$ which is the union of the sequences $k_\alpha(F_\alpha)$. Player II wins a run of the game if and only if

- each M_α^* is correct about stationary subsets of ω_1;
- there is a very strongly increasing sequence in ω^ω of successor length extending F^*.

The last condition above is equivalent to the existence of a function $f \in \omega^\omega$ such that $\{\alpha < \gamma : f_\alpha <_0 f\}$ contains a club subset of γ (which will have cofinality $\aleph_1$).

Lemma 20 below is the only new ingredient in adapting the standard $\mathbb{P}_{\max}$ machinery to give the desired consistency result. The lemma is an adaptation of

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Lemma 4.46 of [14]; the steps involving the sequences σ and the use of Hechler and club-shooting forcing are the only new elements. For those unfamiliar with $\mathbb{P}_{\max}$, the most mysterious part of the proof may be the iteration of the sequence $\langle \dot{M}_{\beta,i} : i < \omega \rangle$. For details on this sort of construction (which was already sketched in the proof of Lemma 19) we refer the reader to Corollary 4.20 of [14]. According to the standard arguments, the lemma is applied inside a countable transitive model M which is correct about $\mathbb{P}$ (i.e., such that

$$\langle H(\aleph_1)^M, \in, A \rangle \prec \langle H(\aleph_1), \in, A \rangle$$

for some set of reals A coding $\mathbb{P}$, and typically a $\mathbb{P}$-name for a subset of ω_1) and which contains a Woodin cardinal, as given for instance by Theorem 9.30 of [14]. The forcing in the proof of Lemma 20 (Hechler forcing followed by adding a club subset of ω_1) is then followed by a two-stage forcing to produce a condition in $\mathbb{P}$ (a lower bound for the ω_1^M-sequence constructed), first collapsing the Woodin cardinal to be ω_2 to make the nonstationary ideal on ω_1 precipitous, and then forcing with a c.c.c. forcing (which preserves precipitousness) to make $\mathsf{MA}_{\aleph_1}$ hold (see, for instance, the top of page 628 in [14]).

We will use the following objects from the theory of cardinal characteristics of the continuum. The cardinal characteristic $\mathfrak{d}$ is defined to be the least cardinal κ such that there is a set $A \subseteq \omega^\omega$ of cardinality κ such that for every $f \in \omega^\omega$ there is a $g \in A$ such that $f <^* g$ (see [2]). We let $(\mathbb{H}, \leq_{\mathbb{H}})$ denote Hechler forcing [6], where

- $\mathbb{H}$ is the set of pairs (s, P) such that $s \in \omega^{<\omega}$ and $P \in [\omega^\omega]^{<\aleph_0}$;
- $(s, P) \leq_{\mathbb{H}} (s', P')$ if s extends s', P contains P' and $s(n) > f(n)$ for all $f \in P'$ and $n \in |s| \setminus |s'|$.

The partial order $\mathbb{H}$ adds an element of ω^ω which is $<^*$-above each element of ω^ω in the ground model. Moreover, $\mathbb{H}$ is c.c.c. (in fact, σ-centered) as conditions with the same first coordinate are compatible, so it preserves stationary subsets of ω_1. It follows that, for any cardinal κ, MA_κ implies that $\mathfrak{d} > \kappa$.

LEMMA 20. *For each condition $p = (M, F, a)$ in $\mathbb{P}$, there is a strategy for player II in the ω_1-sequence game for p such that each run of the game in V according to this strategy is winning for player II in some forcing extension which preserves the stationarity of each stationary subset of ω_1 in each model M_α^* arising from the run of the game.*

PROOF. A run of the game builds a descending ω_1 sequence of $\mathbb{P}$ conditions $p_\alpha = (M_\alpha, F_\alpha, a_\alpha)$, with (stationary set preserving, but not elementary) embeddings $j_{\alpha,\beta} \colon M_\alpha \to M_\beta$. Let γ_α be such that F_α has length $\gamma_\alpha + 1$ in M_α. The definition of the order on $\mathbb{P}$ requires that $\gamma_\beta > j_{\alpha,\beta}(\gamma_\alpha)$ for all $\alpha < \beta$.

We show how to play for player *II*, i.e., how to choose M_β, F_β, and a_β for each limit ordinal β, assuming that p_α $(\alpha < \beta)$ have already been chosen. Let us say that a countable limit ordinal β is *relevant* if β is the supremum of $\{\omega_1^{M_\alpha} : \alpha < \beta\}$. At the end of the game, the set of relevant ordinals will be a club subset of ω_1. For limit ordinals β which are not relevant, player *II* can let p_β be any lower bound for $\{p_\alpha : \alpha < \beta\}$ (lower bounds exist by Lemma 19).

As we carry out our construction, by some bookkeeping we associate to each pair (σ, A), where $\sigma \in \omega^{<\omega}$ and A is, in some M_α a stationary subset of $\omega_1^{M_\alpha}$, a stationary

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set $B_{\sigma,A} \subseteq \omega_1^V$, in such a way that the associated sets $B_{\sigma,A}$ are disjoint for distinct pairs (σ, A). We show how to play for II in such a way that, for all relevant limit ordinals β, if β is in $B_{\sigma,A}$, for some A which is a stationary subset of $\omega_1^{M_{\alpha'}}$ for some $\alpha' < \beta$, then β is in the induced image of A (i.e., $j_{\alpha',\alpha}(A)$) for all $\alpha \in [\beta, \omega_1]$ (note that $\omega_1^{M_\beta}$ will be greater than β) and σ is an initial segment of the first member of F_β which is not in the images of the preceding F_α's (i.e., the member of F_β indexed by the supremum of $\{j_{\alpha,\beta}(\gamma_\alpha) : \alpha < \beta\}$).

For each relevant limit ordinal β, in round β the triples $(M_\alpha, F_\alpha, a_\alpha)$ $(\alpha < \beta)$ will have been chosen, along with the maps $j_{\alpha,\delta}$ $(\alpha \leq \delta < \beta)$ witnessing that this is a descending sequence of conditions. Choosing a cofinal ω-sequence $\langle \alpha_i : i \in \omega \rangle$ in β, and composing the maps $j_{\alpha_i,\alpha_{i+1}}$ $(i \in \omega)$, we get a sequence $\langle \hat{M}_{\beta,i} : i < \omega \rangle$, where each $\hat{M}_{\beta,i}$ is the corresponding image of M_{α_i}. Let $\hat{J}_{\alpha_i,\beta}$ be the induced embedding of M_{α_i} into $\hat{M}_{\beta,i}$, for each $i < \omega$. Then $\omega_1^{\hat{M}_{\beta,i}} = \beta$ for all $i < \omega$, and each $\hat{M}_{\beta,i}$ is a subset of the corresponding $\hat{M}_{\beta,i+1}$, and correct in $\hat{M}_{\beta,i+1}$ about stationary subsets of its ω_1. Fix a countable transitive model M_β of ZFC with $\langle \hat{M}_{\beta,i} : i < \omega \rangle$, $\langle p_\alpha : \alpha < \beta \rangle$, and $\{\alpha_i : i \in \omega\}$ in $H(\aleph_1)^{M_\beta}$, and such that $(M_\beta, \mathrm{NS}_{\omega_1}^{M_\beta})$ is iterable.

Working in M_β, iterate $\langle \hat{M}_{\beta,i} : i < \omega \rangle$ (building an embedding j_β^* of $\bigcup_{i\in\omega} \hat{M}_{\beta,i}$ into M_β) in such a way that

- if β is in $B_{\sigma,A}$, for some $\sigma \in \omega^{<\omega}$ and some A which is a stationary subset of $\omega_1^{M_{\alpha'}}$ for some $\alpha' < \beta$, then the corresponding image of A ($\hat{J}_{\alpha_i,\beta}(j_{\alpha',\alpha_i}(A))$ for the least i such that $\alpha_i \geq \alpha'$) is in the first filter in the iteration (this ensures that $\beta \in j_\beta^*(\hat{J}_{\alpha_i,\beta}(j_{\alpha',\alpha_i}(A)))$);
- each model $j_\beta^*(\hat{M}_{\beta,i})$ is correct in M_β about stationary subsets of ω_1.

Having constructed j_β^*, extend the union of the sets $j_\beta^*(\hat{J}_{\alpha_i,\beta}(F_{\alpha_i}))$ (the length of which has countable cofinality, since each $\gamma_{\alpha_{i+1}}$ is greater than $j_{\alpha_i,\alpha_{i+1}}(\gamma_{\alpha_i})$) with one element having σ as an initial segment (which can be done by Remark 3), and let F_β be this extension. Let a_β be the common value of $j_\beta^*(\hat{J}_{\alpha_i,\beta}(a_{\alpha_i}))$ $(i < \omega)$. This completes the choice of M_β, F_β, and a_β.

Having constructed the entire run of the game, letting a be the union of the sets a_α $(\alpha < \omega_1)$, there is for each $\alpha < \omega_1$ a unique iteration j_{α,ω_1} of $(M_\alpha, \mathrm{NS}_{\omega_1}^{M_\alpha})$ sending a_α to a. Let γ_α^* denote the image of γ_α under this iteration, and let F_α^* be the corresponding image of F_α. Noting that the sequences F_α^* extend one another, let $F^* = \bigcup_{\alpha<\omega_1} F_\alpha^*$ and let γ^* be the length of F^*. For each countable limit ordinal β, let $\eta_\beta = \sup\{\gamma_\alpha^* : \alpha < \beta\}$. The set $\{\eta_\beta : \beta < \omega_1\}$ is closed below its supremum γ, which has cofinality ω_1.

Force (over the model we have been working in, which we call V) with Hechler forcing to add a $g \in \omega^\omega$ dominating each element of $\omega^\omega \cap V$ mod-finite. This forcing is c.c.c., and therefore preserves stationary sets.

We claim that, in $V[g]$, for each A which is, for some $\alpha < \omega_1$, a stationary subset of $\omega_1^{M_\alpha}$ in M_α, the set of $\beta \in j_{\alpha,\omega_1}(A)$ such that g dominates $f_{\eta_\beta}^*$ everywhere is stationary. To see that this is the case, consider a Hechler condition (σ, P) and an A. In any club $C \subseteq \omega_1$, we can find a $\beta \in C \cap B_{\sigma,A}$ such that

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$$\sup\{\omega_1^{M_\alpha} : \alpha < \beta\} = \beta.$$

Then $(\sigma, P \cup \{f^*_{\eta_\beta}\}) \leq (\sigma, P)$. Since (by the construction above) σ is an initial segment of $f^*_{\eta_\beta}$, $(\sigma, P \cup \{\beta\})$ forces that $g(n) \geq f^*_{\eta_\beta}(n)$ will hold for all $n \in \omega$ (i.e., $g \geq_0 f^*_{\eta_\beta}$).

Finally, force to shoot a club E (via the standard forcing with countable conditions) through the set of $\beta < \omega_1$ such that $g \geq_0 f^*_{\eta_\beta}$. By the previous paragraph, this forcing preserves the stationarity of each set $j_{\alpha,\omega_1}(A)$. It follows that in $V[g][E]$, letting $g'(n) = g(n) + 1$ for all $n \in \omega$, $F^* \cup \{(\gamma^*, g')\}$ witnesses that the run of the ω_1-sequence game just produced is winning for player II. ⊣

Remark 21. Lemma 20 and standard $\mathbb{P}_{\max}$ arguments (see Theorems 9.32 and 9.34 of [14]) give the following.

1. If $G \subseteq \mathbb{P}$ is a filter, and $A_G = \bigcup\{a : (M, F, a) \in G\}$, then for each $p = (M, F, a) \in G$ there is a unique iteration $j_{p,G}$ of $(M, \mathrm{NS}^M_{\omega_1})$ sending a to A_G.
2. If W is an inner model of AD^+ and $G \subseteq \mathbb{P}$ is W-generic, then every element of $\mathcal{P}(\omega_1)^{W[G]}$ is an element of $j_{p,G}[\mathcal{P}(\omega_1)^M]$ for some $p = (M, F, a)$ in G.
3. If $A \subseteq \mathbb{R}$ is such that $L(A, \mathbb{R}) \models \mathsf{AD}^+$, then forcing with $\mathbb{P}$ over $L(A, \mathbb{R})$ does not add ω-sequences from the ground model, preserves ω_1 and ω_2 as cardinals, and makes the cardinal Θ of the ground model the ω_3 of the extension.

Remark 21 and standard arguments give the main theorem of this section.

Theorem 22. *Suppose that $A \subseteq \mathbb{R}$ is such that $L(A, \mathbb{R}) \models \mathsf{AD}^+$, and $G \subseteq \mathbb{P}$ is $L(A, \mathbb{R})$-generic. Then $L(A, \mathbb{R})[G]$ satisfies* ZFC *along with the following statements*:

1. $2^{\aleph_0} = \aleph_2$;
2. $\exists B \subseteq \omega_2\ V = L[B]$;
3. NS_{ω_1} *is saturated*;
4. $\bigcup\{j_{p,G}(F) : p = (M, F, a) \in G\}$ *is a very strongly increasing sequence of length* ω_2.

Proof. (Sketch) We sketch the proof of part (4); the other parts follow from standard $\mathbb{P}_{\max}$ arguments. For each condition $p = (M, F, a)$ in the generic filter G, the elementarity of $j_{p,G}$ gives that $j_{p,G}(F)$ is a very strongly increasing sequence. The order on $\mathbb{P}$ implies that the union of these sequences $j_{p,G}(F)$ is a very strongly increasing sequence of length at most ω_2^V. To see that the length is in fact ω_2^V, fix $\gamma < \omega_2^V$, a prewellording $\leq$ of ω_1^V of length γ and, by part (2) of Remark 21, a condition $p = (M, F, a) \in G$ and a prewellordering $\leq_0$ of ω_1^M in M such that $\leq = j_{p,G}(\leq_0)$. Lemma 6 and the genericity of G imply that there exists a condition $p' = (M', F', a') < (M, F, a)$ in G (with the order witnessed by an iteration j of $(M, \mathrm{NS}^M_\omega)$) such that the length of F' is greater than the length of the wellordering $j(\leq_0)$. The elementarity of $j_{p',G}$ then implies that $j_{p',G}(F')$ has length greater than that of $\leq = j_{p',G}(j(\leq_0))$. ⊣

Remark 23. One could naturally try to reproduce the result proved here using an iterated forcing consisting alternately of Hechler forcing and adding club subsets of ω_1. One issue with this approach is that if $F = \{f_\alpha : \alpha < \omega_1\} \subseteq \omega^\omega$ is

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$\leq^*$-unbounded, and $g \in \omega^\omega$ is added by a c.c.c. forcing over V, then there are cofinally many $m \in \omega$ such that the set $\{\alpha < \omega_1 : f_\alpha(m) \geq g(m)\}$ is stationary (in V). If F is the range of a cofinal subsequence of some strongly increasing sequence (whose index set is closed in the length of the sequence) then the stationarity of some if these sets will have to be destroyed into order to use g to continue the sequence. The argument from Lemma 20 avoids this issue by requiring preservation only of the sets $j_{\alpha,\omega_1}(A)$ coming from the models in the sequence being built. That is, it is not necessary to preserve the stationarity of every stationary subset of ω_1 in V. The Semi-Properness Iteration Lemma on pages 485–486 of [13] offers a parallel degree of freedom, as (using the notation there) for each successor j the quotient P_j/P_i is required to be semi-proper only for arbitrarily large nonlimit $i < j$.

We conclude with a couple of remaining open questions. We first remark that the $\mathbb{P}_{\max}$ machinery is well-suited for constructing models with strongly increasing sequences of length ω_2 but does not seem to be readily adaptable to construct models with longer strongly increasing sequences. Corollary 16 implies that there cannot exist strongly increasing sequences of length ω_4 of functions from ω to ω, but the corresponding question about ω_3 remains open.

QUESTION 24. *Is it consistent with* ZFC *that there exists a strongly increasing sequence of length ω_3 consisting of functions from ω to ω?*

Lastly, we forced over a model satisfying $\mathrm{AD} + V = L(\mathbb{R})$ to obtain our main consistency result. We do not know if these hypotheses are optimal, or even if the existence of a strongly increasing ω_2-sequence of functions from ω to ω carries any large cardinal strength at all. The approach suggested by Remark 23 may answer the following question.

QUESTION 25. *What is the consistency strength of "*ZFC+ *there exists a strongly increasing ω_2-sequence of functions from ω to ω"?*

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