10 Polytopes

In Chapter 9, we learned that the individual terms in a BCFW expansion of an N^{k-2}MHV n-point superamplitude are residues of a cyclically invariant integral-formula in the Grassmannian $\mathrm{Gr}(k,n)$. In this language, we found that the different BCFW representations of the 6-point NMHV amplitude $A_6[1^+2^-3^+4^-5^+6^-]$ are related by a simple contour deformation. This was encoded in the 6-term identity (9.50). In *momentum supertwistor space* [60], the 6-term identity is promoted to the 5-bracket relation

$$[2,3,4,6,1]+[2,3,4,5,6]+[2,4,5,6,1]=[3,1,6,5,4]+[3,2,1,6,5] \\ +[3,2,1,5,4]\,. \tag{10.1}$$

We have already encountered this version of the 6-term identity in (5.63) when we discussed the equivalence of the BCFW recursion relations based on $[2,3\rangle$ and $[3,2\rangle$ super-shifts. The 5-brackets were defined in Section 5.4 as

$$[i,j,k,l,m]\equiv\frac{\delta^4\big(\chi_{iA}\langle jklm\rangle+\text{cyclic}\big)}{\langle ijkl\rangle\langle jklm\rangle\langle klmi\rangle\langle lmij\rangle\langle mijk\rangle}\,, \tag{10.2}$$

with 4-brackets $\langle ijkl\rangle\equiv\epsilon_{IJKL}Z_i^IZ_j^JZ_k^KZ_l^L$ involving the bosonic components $Z_i^I=(|i\rangle,[\mu_i|)$ of the $SU(2,2|4)$ momentum supertwistors $\mathcal{Z}_i^{\mathsf{A}}\equiv(|i\rangle^{\dot a}\,,[\mu_i|^a\mid\chi_{iA})$, $\mathsf{A}=(\dot a,a,A)$. The $[\mu_i|$ are defined by the incidence relations (5.44).

Recall that momentum conservation is automatic for momentum twistors, so the 6-term identity (10.1) must hold for any six momentum twistors; it is not specific to the NMHV 6-point amplitude but is an intrinsic property of the 5-brackets (10.2). Hence it seems worthwhile to try to understand the structure of (10.1) better. Using cyclic and reflection symmetry of the 5-brackets, we can rewrite (10.1) as

$$[1,2,3,4,5]-[2,3,4,5,6]+[3,4,5,6,1]-[4,5,6,1,2] \\ +[5,6,1,2,3]-[6,1,2,3,4]=0\,. \tag{10.3}$$

Pretend for a moment that we do not know what the 5-brackets are. Consider a fully antisymmetric 5-bracket $\langle i,j,k,l,m\rangle$ defined as the contraction of five 5-component vectors $\mathsf{Z}_i^{\mathcal{I}}$ with a 5-index Levi-Civita tensor. Such an object would satisfy the 6-term Schouten identity

$$\langle1,2,3,4,5\rangle\,\mathsf{Z}_6^{\mathcal{I}}-\langle2,3,4,5,6\rangle\,\mathsf{Z}_1^{\mathcal{I}}+\langle3,4,5,6,1\rangle\,\mathsf{Z}_2^{\mathcal{I}} \\ -\langle4,5,6,1,2\rangle\,\mathsf{Z}_3^{\mathcal{I}}+\langle5,6,1,2,3\rangle\,\mathsf{Z}_4^{\mathcal{I}}-\langle6,1,2,3,4\rangle\,\mathsf{Z}_5^{\mathcal{I}}=0\,. \tag{10.4}$$

Since this looks quite similar to (10.3), including relative signs, we might be tempted to think that (10.3) somehow arises as a Schouten identity. This is of course too

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speculative: the 5-brackets $[i, j, k, l, m]$ really represent rational functions of the 4-component momentum twistors Z_i, not some 5-index objects contracted with a 5-index Levi-Civita tensor. However, we can entertain the idea a little further. Could the 5-bracket (10.2) be written in terms of some new 5-vectors? The fermionic variables χ_{iA} appear in (10.2) on a different footing than their bosonic counterparts. In the name of democracy, let us define the following purely bosonic 5-component vector

$$\mathsf{Z}_i^{\mathcal{I}} = \begin{pmatrix} Z_i^I \\ \chi_i \cdot \psi \end{pmatrix}, \qquad \mathcal{I} = 1, \ldots, 5\,, \tag{10.5}$$

where $\chi_i \cdot \psi = \chi_i^A \psi_A$ is $SU(4)$-invariant and ψ_A is an auxiliary Grassmann variable common for all external particles $i = 1, 2, \ldots, n$. If we define $\langle i, j, k, l, m\rangle$ as the contraction of five of these 5-vectors with a 5-indexed Levi-Civita tensor, then they will satisfy the Schouten identity (10.4) – but that is not what we are after, so read on.

To write the 5-bracket $[i, j, k, l, m]$ in terms of the 5-vectors (10.5), we must remove the auxiliary variable ψ_A. Since it is fermionic, this can be done via a Grassmann-integration: one finds that the 5-bracket (10.2) can be written as

$$[i, j, k, l, m] = \frac{1}{4!}\int d^4\psi \, \frac{\langle i, j, k, l, m\rangle^4}{\langle 0, i, j, k, l\rangle\langle 0, j, k, l, m\rangle\langle 0, k, l, m, i\rangle\langle 0, l, m, i, j\rangle\langle 0, m, i, j, k\rangle}\,, \tag{10.6}$$

where $\langle ijklm\rangle \equiv \epsilon_{IJKLM} Z_i^I Z_j^J Z_k^K Z_l^L Z_m^M$ and we have introduced the auxiliary reference 5-vector

$$\mathsf{Z}_0^{\mathcal{I}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{10.7}$$

The representation (10.6) is certainly not just contractions of five 5-vectors with a Levi-Civita tensor, so the origin of the identity in (10.3) is not a Schouten identity. But let us not give up just yet, for it will be worthwhile to examine (10.6) further.

Since the integral $\int d^4\psi$ is universal for all 5-brackets, we ignore it for the time being and focus on the integrand of (10.6). Each $\mathsf{Z}_i^{\mathcal{I}}$ appears an equal number of times in the numerator and the denominator, so the integrand is invariant under $\mathsf{Z}_i^{\mathcal{I}} \to t_i\, \mathsf{Z}_i^{\mathcal{I}}$ for each $i = 1, 2, \ldots, n$. In other words, the 5-vectors $\mathsf{Z}_i^{\mathcal{I}}$ appear projectively in (10.6), and therefore we can think of the $\mathsf{Z}_i^{\mathcal{I}}$ as homogeneous coordinates of points in projective space $\mathbb{CP}^4$. The presence of the reference vector $\mathsf{Z}_0^{\mathcal{I}}$ in the denominator breaks projective invariance, but only at this particular point.

There is an analogous case where we have encountered something similar. The momentum twistors Z_i^I in Section 5.4 are defined projectively and are elements in $\mathbb{CP}^3$. The map in Figure 5.1 shows how to relate momentum twistors Z_i^I with points y_i in dual space. Specifically, the distance between two points y_i and y_j in dual space is

$$y_{ij}^2 = \frac{\langle i-1, i, j-1, j\rangle}{\langle i-1, i\rangle\langle j-1, j\rangle} = \frac{\langle i-1, i, j-1, j\rangle}{\langle I_0, i-1, i\rangle\langle I_0, j-1, j\rangle}\,. \tag{10.8}$$

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The first equality is simply (5.52): it has a momentum twistor 4-bracket in the numerator and regular angle brackets in the denominator. In the second equality we have rewritten the denominator in a more suggestive form involving only 4-brackets, at the cost of introducing a reference bi-twistor I_0^{IJ} defined as

$$I_0^{IJ} = \begin{pmatrix} 0 & 0 \\ 0 & \epsilon_{\dot{a}\dot{b}} \end{pmatrix} . \tag{10.9}$$

In the literature, I_0 is often referred to as the ***infinity twistor***,[1] and its role is to break $SL(4)$ conformal invariance and provide a preferred metric for the definition of distance.

The expression (10.8) is similar to the integrand of (10.6): both are projectively defined, except for the reference bi-twistor/vector. The reference bi-twistor appears twice in the denominator of (10.8), reflecting the fact that this expression gives the distance between the two points i and j. An analogous expression involving three points and a reference vector appearing thrice defines the area of a triangle. And so on. The appearance of the reference vector $\mathsf{Z}_0^{\mathcal{I}}$ five times in the denominator of (10.6) gives us a hint that *the rational integrand in (10.6) is the volume of a geometric figure defined by five points in* $\mathbb{CP}^4$*!* In the following, we pursue the interpretation of 5-brackets as volumes of simplices and their sum – the superamplitudes – as volumes of ***polytopes***. Definitions and explanations follow next.

10.1 Volume of an n-simplex in $\mathbb{CP}^n$

Let us begin with the concepts of polytopes and simplices before reconnecting with the motivation above.

Polytopes and simplices: definitions and examples

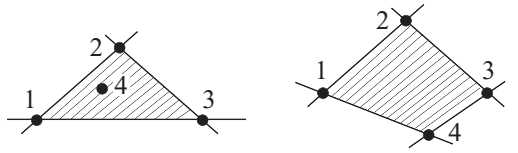
We are all familiar with polygons: triangles, squares (or more generally quadrilaterals), pentagons, hexagons, chiliagons, star-shapes, etc. These are figures in the plane bounded by a finite number of straight line-segments. Their 3-dimensional analogues – tetrahedrons, cubes, prisms, dodecahedrons, etc – are solids whose faces are polygons. The n-dimensional versions of polygons and polyhedrons are called ***polytopes*** or ***n-polytopes***. A 2-polytope is a polygon and a 3-polytope is a polyhedron.

A simplex is in a sense the simplest example of an polytope. To define it, recall first that a ***convex set*** C (in, for example, $\mathbb{R}^n$ or $\mathbb{CP}^n$) has the property that the line segment between any two points in C lies entirely in C. In the plane, triangles are convex, but star-shaped polygons are not. Given a set of points S, the ***convex hull*** of S is the intersection of all convex sets containing S. Examples from the plane: 1) the convex hull of a circle is the closed disk bounded by the circle; 2) the convex hull of three points is a triangle. Adding

[1] The infinity twistor in (10.9) corresponds to a flat space metric. For AdS_4 it is given as $I_0^{IJ} = \begin{pmatrix} \epsilon_{ab}\Lambda & 0 \\ 0 & \epsilon_{\dot{a}\dot{b}} \end{pmatrix}$, where Λ is the cosmological constant.

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a fourth point that lies inside the triangle, the convex hull of the four points is the same triangle. A fourth point outside the triangle (but in the same plane) gives a convex hull that is a convex quadrilateral, or in pictures:



(10.10)

An ***n-simplex*** is the convex hull of a set of $n+1$ points. Examples:

$$\begin{aligned} &0\text{-simplex} = \text{a point},\\ &1\text{-simplex} = \text{line segment},\\ &2\text{-simplex} = \text{triangle},\\ &3\text{-simplex} = \text{tetrahedron}. \end{aligned} \tag{10.11}$$

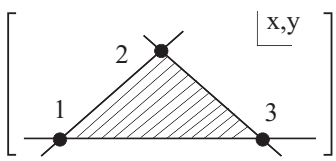
An n-simplex is bounded by $n+1$ $(n-1)$-simplices which intersect each other in $\binom{n+1}{2}$ $(n-2)$-simplices. For $n+1$ generic points in $\mathbb{R}^n$, an n-simplex has an n-dimensional volume. (For $\mathbb{CP}^n$ it will be n-complex dimensional.) The volume of a polytope can be calculated by "tessellating" it into simplices, whose volumes are easier to calculate.

Now that we know what simplices and polytopes are, let us progress towards understanding how the integrand in (10.6) represents the volume of a 4-simplex in $\mathbb{CP}^4$, as claimed. As a warm-up, we begin in 2-dimensions with a 2-simplex (a triangle).

Area of a 2-simplex in $\mathbb{CP}^2$

The area of a triangle in a 2-dimensional plane can be computed as

$$\text{Area}\left[\;\right] = \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}, \tag{10.12}$$



where the (x_i, y_i) are the coordinates of the three vertices.

▶ **Exercise 10.1**

If the area formula (10.12) is not familiar, you should derive it by showing that it is equivalent to the "$\frac{1}{2}$ × base × height" formula that was imprinted on your brain in elementary school.

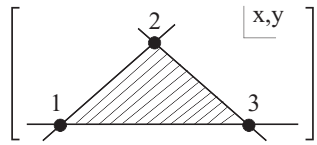
The 1s in the last row of (10.12) are redundant as we can write the same formula as a sum of the 2×2 minors. In physics, when faced with a redundancy we can choose to eliminate it or promote it to a feature. Choosing the latter, we define three 3-vectors along with a reference vector:

$$\mathsf{W}_{iI} = \begin{pmatrix} x_i \\ y_i \\ 1 \end{pmatrix}, \qquad \mathsf{Z}_0^I = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad I = 1, 2, 3\,. \tag{10.13}$$

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The area can now be written

$$\text{Area}\left[\;\raisebox{-0.5em}{}\;\right] = \frac{1}{2}\,\frac{\langle 1,2,3\rangle}{(\mathsf{Z}_0\cdot\mathsf{W}_1)(\mathsf{Z}_0\cdot\mathsf{W}_2)(\mathsf{Z}_0\cdot\mathsf{W}_3)}\,, \tag{10.14}$$



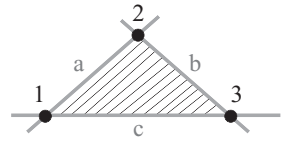
where the 3-bracket is the contraction of a 3-index Levi-Civita tensor with the three W_i vectors: $\langle 1,2,3\rangle = \epsilon^{IJK}\mathsf{W}_{1I}\mathsf{W}_{2J}\mathsf{W}_{3K}$. Using (10.13), the $\langle 1,2,3\rangle$-numerator exactly equals the 3×3-determinant in (10.12), so you might consider the trivial dot-products $\mathsf{Z}_0\cdot\mathsf{W}_i = \mathsf{Z}_0^I\mathsf{W}_{iI} = 1$ in the denominator a provocation of your sense of humor. However, written in the form (10.14), the redundancy has been promoted to projective symmetry: the new area-formula (10.14) is invariant under scalings $\mathsf{W}_i \to t_i\,\mathsf{W}_i$. When the 3-vectors W_i are "gauge fixed" to the canonical form in (10.13), we immediately recover the original area formula. Since the triangle vertices are specified in terms of projectively defined 3-vectors, we can think of the triangle as an object in $\mathbb{CP}^2$ and the W_is as the homogeneous coordinates of the vertices.

The area formula in (10.14) involves an antisymmetric 3-bracket as well as the inner product of 3-vectors. To make contact with (10.6) and (10.8), we would like to have a representation that is given solely in terms of 3-brackets. To achieve this, it is useful to characterize the triangle by its edges instead of its vertices. We define a "dual space" whose points Z_a^I are associated with lines in W-space: a given line is defined as the set of points W_I satisfying the incidence relations

$$\mathsf{Z}^I\mathsf{W}_I = 0\,. \tag{10.15}$$

Since Z^I is a vector in the 2-dimensional space $\mathbb{CP}^2$, the constraint indeed defines a 1-dimensional subspace, i.e. a line.

Now, to define the triangle in terms of three lines in dual space, note that each W_{iI} is characterized by lying simultaneously on two lines. Labeling the three edges of the triangle as a, b, and c, the vertex W_{1I} is the intersection of lines a and c, so that $\mathsf{Z}_a^I\mathsf{W}_{1I} = \mathsf{Z}_c^I\mathsf{W}_{1I} = 0$. These two constraints are easily solved and we have

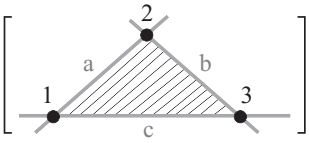


$$\to \begin{array}{l} \mathsf{W}_1 = \langle *, \mathsf{Z}_c, \mathsf{Z}_a\rangle \\ \mathsf{W}_2 = \langle *, \mathsf{Z}_a, \mathsf{Z}_b\rangle \\ \mathsf{W}_3 = \langle *, \mathsf{Z}_b, \mathsf{Z}_c\rangle\,, \end{array} \tag{10.16}$$

where the $*$ indicates the free index, e.g. $\mathsf{W}_{1I} = \langle *, \mathsf{Z}_c, \mathsf{Z}_a\rangle = \epsilon_{IJK}\mathsf{Z}_c^J\mathsf{Z}_a^K$.

Plugging the map (10.16) into (10.14), we find that the area is now given as

$$\text{Area}\left[\;\raisebox{-0.5em}{}\;\right] = \frac{1}{2}\,\frac{\langle a,b,c\rangle^2}{\langle 0,b,c\rangle\langle 0,a,b\rangle\langle 0,c,a\rangle} \equiv \big[a,b,c\big]\,, \tag{10.17}$$



where $\langle a,b,c\rangle = \epsilon_{IJK}\mathsf{Z}_a^I\mathsf{Z}_b^J\mathsf{Z}_c^K$ and "0" indicates the reference vector Z_0^I introduced in (10.13).

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▶ **Exercise 10.2**

Show that (10.17) follows from (10.14).

As advertised earlier, we now see that the "volume" (i.e. area) of a 2-simplex is given by a rational function whose denominator is the product of all 3-brackets involving two of the edge variables Z_i^I, $i = a, b, c$, and a reference vector Z_0^I. Requiring projective invariance for each Z_i^I with $i = a, b, c$ uniquely fixes the numerator. For later convenience, we have introduced the notation $[a, b, c]$ to denote the volume (10.17). Note that the area-formula (10.17) comes with an "orientation" in the sense that $[a, b, c]$ is fully antisymmetric in a, b, c.

Volume of an n-simplex in $\mathbb{CP}^n$

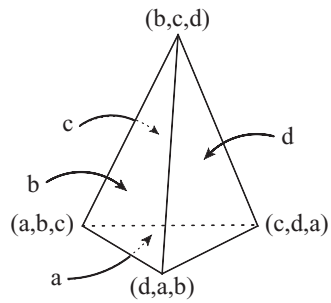
The expression in (10.17) can be generalized to the volume of an n-simplex in $\mathbb{CP}^n$: we denote it by an antisymmetric $(n+1)$-bracket

$$[\mathsf{Z}_{i_1}, \ldots \mathsf{Z}_{i_{n+1}}] = \frac{1}{n!} \frac{\langle i_1, i_2, \ldots, i_{n+1}\rangle^n}{\langle 0, i_1, \ldots, i_n\rangle\langle 0, i_2, \ldots, i_{n+1}\rangle \cdots \langle 0, i_{n+1}, i_1, \ldots, i_{n-1}\rangle}, \quad (10.18)$$

where the angle-brackets are the contractions of $n+1$ $\mathbb{CP}^n$-vectors Z_i with an $(n+1)$-index Levi-Civita. The $n+1$ variables $\mathsf{Z}_{i_1}, \ldots, \mathsf{Z}_{i_{n+1}} \in \mathbb{CP}^n$ carry the information about the $n+1$ boundaries of the n-simplex as follows. For a given vector Z_i^I, the set of W_Is satisfying the incidence relation $\mathsf{Z}_i^I \mathsf{W}_I = 0$ span an $(n-1)$-dimensional subspace of $\mathbb{CP}^n$. These subspaces contain the $n+1$ faces of the n-simplex; each face is an $(n-1)$-simplex.

In the example of the 2-simplex in $\mathbb{CP}^2$, the 3 Z_is label the 1-dimensional lines a, b, c bounding the triangle. Each pair of lines intersect in a point that is a vertex of the triangle: we can label the vertices (a, b), (b, c), and (c, a). They are defined in terms of the Z_is in (10.16) and the denominator of the volume formula (10.17) is the dot-product of all vertex point vectors with the reference vector.

As a second example, consider a 3-simplex (tetrahedron) in $\mathbb{CP}^3$. The four 4-component homogeneous coordinates of (dual) $\mathbb{CP}^3$ – Z_a^I, Z_b^I, Z_c^I, Z_d^I – have 2-dimensional orthogonal complements spanned by the W_Is satisfying $\mathsf{Z}_i^I \mathsf{W}_I = 0$. Pairwise, these generic 2-planes intersect in a line: this gives six lines, (a, b), (b, c), etc., that define the 1-simplex edges of the tetrahedron. Three generic 2-planes in $\mathbb{CP}^3$ intersect in a point: this defines the four vertices of the tetrahedron and we label them (a, b, c), (b, c, d), (c, d, a), and (d, a, b), as illustrated here:



(10.19)

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Just as in the case of the triangle, the denominator of the volume formula (10.18) is the product of each vertex coordinate dotted into the reference vector Z_0. The numerator compensates the scaling of each "face"-variable $\mathsf{Z}_a^I, \mathsf{Z}_b^I, \mathsf{Z}_c^I, \mathsf{Z}_d^I$ to make the volume formula projective.

Now, here comes the highlight of your day: for $n = 4$ the 5-bracket volume expression (10.18) for a 4-simplex is identical to the integrand in the amplitude 5-bracket expression (10.6)! This verifies our statement at the beginning of the chapter that the rational function in (10.6) is indeed the volume of a 4-simplex in $\mathbb{CP}^4$. The denominator factors in (10.18) involve the five vertices of the 4-simplex. Since an NMHV tree superamplitude is a sum of 5-brackets, we are led to view the amplitude as a volume of a polytope in $\mathbb{CP}^4$. We realize this expectation in the next section and discuss its consequences.

10.2 NMHV tree superamplitude as the volume of a polytope

The simplest NMHV case is the 5-point (anti-MHV)superamplitude

$$A_5^{\text{NMHV}}[1,2,3,4,5] \;=\; A_5^{\text{MHV}} \times \big[1,2,3,4,5\big]. \tag{10.20}$$

Thus, up to the MHV factor, A_5^{NMHV} is the volume of a 4-simplex in $\mathbb{CP}^4$.

Next, for the NMHV 6-point superamplitude, consider the $[2,3\rangle$ super-BCFW representation on the LHS of (10.1):

$$A_6^{\text{NMHV}}[1,2,3,4,5,6] \;\propto\; \underbrace{\big[2,3,4,6,1\big]}_{\langle 4,6,1,2\rangle,\ \langle 2,3,4,6\rangle} + \underbrace{\big[2,3,4,5,6\big]}_{\langle 6,2,3,4\rangle,\ \langle 4,5,6,2\rangle} + \underbrace{\big[2,4,5,6,1\big]}_{\langle 2,4,5,6\rangle,\ \langle 6,1,2,4\rangle}. \tag{10.21}$$

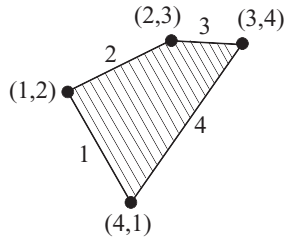
Apart from the overall MHV factor, the 6-point NMHV superamplitude is the sum of the volumes of three 4-simplices in $\mathbb{CP}^4$; we expect this to be the volume of a polytope obtained by somehow gluing the three simplices together. But how exactly does this work? To address this question, it is useful to examine the poles in the 5-brackets.

Recall from (5.50) and (5.52) that momentum twistor 4-brackets $\langle i-1, i, j-1, j\rangle$ in the denominator give local poles, whereas other 4-brackets, such as $\langle 1,2,4,6\rangle$, give spurious "non-local" poles. Examining the denominator terms of the 5-brackets in (10.21), we find that each of them has two spurious poles; they are listed under each 5-bracket. The spurious poles come in pairs – for example $\langle 4,6,1,2\rangle$ and $\langle 6,1,2,4\rangle$ in the first and third 5-brackets – and cancel in the sum (10.21), as required by locality of the physical amplitude. In the geometric description of a 5-bracket as a 4-simplex in $\mathbb{CP}^4$, each of the five factors in the denominator of the volume-expression (i.e. the 5-bracket) is determined by a vertex of the associated 4-simplex. In particular, spurious poles must be associated with vertices in $\mathbb{CP}^4$ that somehow "disappear" from the polytope whose volume equals the sum of the simplex-volumes in (10.21). We now discuss how the "spurious" vertices disappear in the sum of simplices. Let us start in $\mathbb{CP}^2$ where the polytopes are easier to draw.

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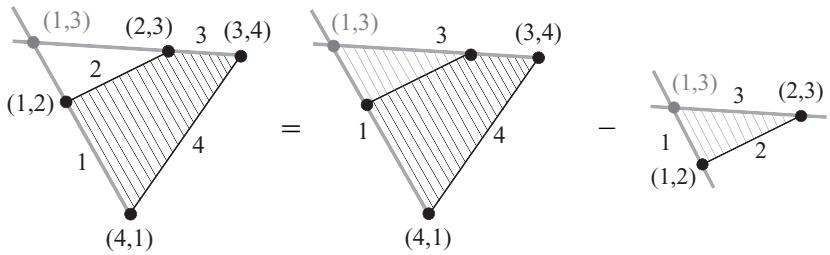
Polytopes in $\mathbb{CP}^2$

In 2-dimensions, consider the 4-edge polytope



. (10.22)

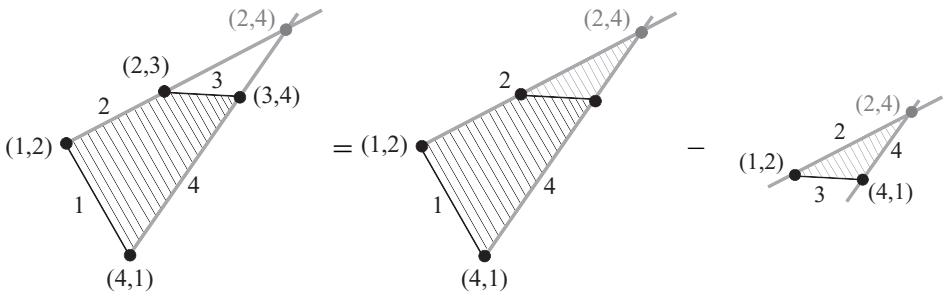
All vertices for this $\mathbb{CP}^2$ "amplitude" are defined by adjacent edges and are in this sense local. We would like to compute the area of the 2-polytope (10.22) using 2-simplex volumes $[a, b, c]$. There are several different ways to do this, corresponding to different triangulations of the polytope. As an example, introduce a "non-local" point (1, 3) as the intersection of lines 1 and 3. The resulting triangulation is



$$= [4, 1, 3] - [2, 1, 3]$$
$$= [4, 1, 3] + [1, 2, 3]. \quad (10.23)$$

The area of the 4-edge polytope is given by the difference of two triangular areas. The non-local vertex (1, 3) appears in both triangles. Comparing the last two lines, the sign of the 3-bracket indicates the orientation of the triangle with respect to a particular predetermined ordering of all edges (or, in higher dimensions, boundaries).

It is useful to also consider another triangulation, so introduce the point (2, 4):



$$= [1, 2, 4] - [2, 4, 3]$$
$$= [1, 2, 4] + [2, 3, 4]. \quad (10.24)$$

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The two triangulations (10.23) and (10.24) compute the same area ("amplitude"), so we have a $\mathbb{CP}^2$ version of the identity (10.1), namely $[4, 1, 3] + [1, 2, 3] = [1, 2, 4] + [2, 3, 4]$ which can also be written

$$[2, 3, 4] - [1, 3, 4] + [1, 2, 4] - [1, 2, 3] = 0\,. \tag{10.25}$$

▶ **Exercise 10.3**

Suppose the 4-vertex polytope in the example above was not convex as drawn in (10.24): show that the volume of a non-convex 4-vertex polytope can also be written $[1, 2, 4] + [2, 3, 4]$.

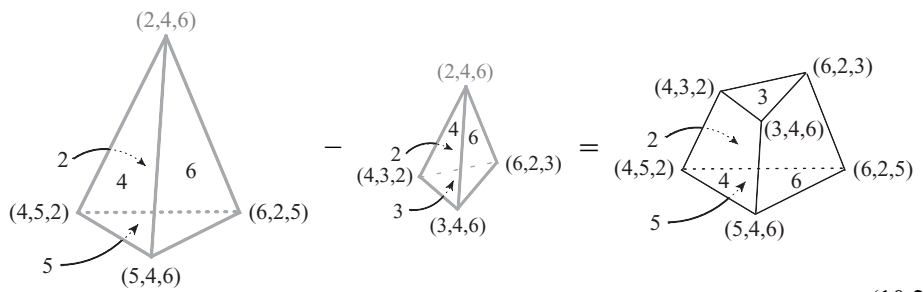
Polytopes in $\mathbb{CP}^4$

Extending the simple $\mathbb{CP}^2$ example to $\mathbb{CP}^4$, one finds that the BCFW representation of a 6-point NMHV tree superamplitude corresponds to a triangulation of the associated polytope by introduction of three new auxiliary vertices. This allows one to use the given external data, the boundaries of the polytope, to efficiently construct the corresponding triangulation. Efficiency here means using a minimum number of 4-simplices; we come back to this point in Section 10.4. Different BCFW constructions simply correspond to different choices of auxiliary vertices. As an example, the auxiliary vertices for the BCFW representation in (10.21) are (2, 4, 5, 6), (6, 1, 2, 4) and (2, 3, 4, 6). Note that these exactly label the spurious poles in (10.21).

Let us now see how the removal of an auxiliary vertex works in $\mathbb{CP}^4$. Since it can be slightly challenging to draw a 4-dimensional object on paper, we go to the 3d boundary of the 4-dimensional polytope. Specifically, at the 3d boundary defined by $\mathsf{Z}_1 \cdot \mathsf{W} = 0$, only the two simplices [2, 3, 4, 6, 1] and [2, 4, 5, 6, 1] in (10.21) contribute, and their projections to the boundary are the tetrahedrons defined by the faces Z_2, Z_3, Z_4, and Z_6 and, respectively, by Z_2, Z_4, Z_5, and Z_6. The two boundary tetrahedrons share the non-local vertex (1, 2, 4, 6), which we simply label (2, 4, 6) on the boundary. Since this vertex does not appear in other terms of (10.21), we should be able to visualize its cancellation geometrically on the boundary defined by Z_1. On the 3d subspace, the superamplitude contains the combination

$$[2, 4, 5, 6, 1] + [2, 3, 4, 6, 1] \xrightarrow{\mathsf{Z}_1 \text{ bdr}} [6, 2, 4, 5] - [6, 2, 4, 3] = \text{vol(bdr polytope)}\,. \tag{10.26}$$

On the RHS we have arranged the common faces, Z_6, Z_2, and Z_4, to appear in the same order to facilitate the geometrical interpretation. The pictorial representation of (10.26) is



(10.27)

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The "non-local" auxiliary vertex $(2, 4, 6)$ indeed "cancels" in the sum and leaves behind the volume of the 3-dimensional polytope with five faces and six *local* vertices! To see that the remaining vertices are local, remember that we are in the subspace $\mathsf{Z}_1 \cdot W = 0$, so each vertex is really represented as $(1, *, *, *)$ in $\mathbb{CP}^4$. Thus each of the six vertices in (10.27), namely

$$(1, 2, 3, 4), \quad (1, 2, 3, 6), \quad (1, 3, 4, 6), \quad (1, 4, 5, 2), \quad (1, 4, 5, 6), \quad (1, 2, 5, 6)\,, \tag{10.28}$$

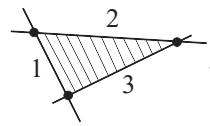
involves two pairs of adjacent labels, and therefore – by (5.52) – they correspond to local poles. Thus we conclude that on the subspace $\mathsf{Z}_1 \cdot W = 0$, which involves only two of the simplices in (10.21), the amplitude is free of non-local vertices. One can similarly understand the cancellation of the two other spurious poles in (10.21).

We have found that the 6-point NMHV tree superamplitude is given by the volume of a polytope in $\mathbb{CP}^4$. It is defined as the sum of the three 4-simplices in (10.21) and its six boundaries are in 1-to-1 correspondence with the momentum supertwistors Z_i^I, $i = 1, \ldots, 6$. Different BCFW representations correspond to the different tessellations of the polytope into 4-simplices; each representation requires introduction of "spurious" vertices and the associated spurious poles cancel because they are absent in the original polytope. The vertices of the polytopes are all local and can be characterized as the nine quadruple intersections $(i, i+1, j, j+1)$ of the six boundaries determined by Z_i^I.

The polytope interpretation of the amplitudes was first presented by Hodges [24] with the goal of geometrizing the cancellation of spurious poles in the BCFW expansion. Building on Hodges' work, the authors of [130] constructed the representation of the NMHV superamplitude where both dual superconformal symmetry and locality are manifest.

10.3 The boundary of simplices and polytopes

We have studied the volumes of simplices and polytopes, but their boundaries also have interesting properties. Let us again start with a simple triangle in $\mathbb{CP}^2$,

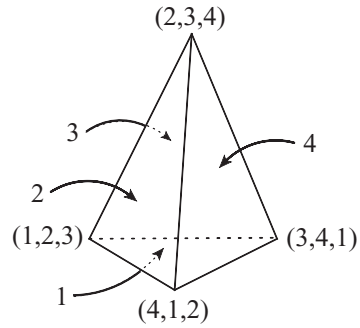


. (10.29)

The subspace defined by $\mathsf{Z}_1 \cdot \mathsf{W} = 0$ contains part of the boundary of the triangle, namely the line segment bounded by the intersections of lines 2 and 3 with line 1. The length of the line segment is just the projection to the subspace defined by Z_1, namely $[2, 3]$. Note that since the "volumes" (i.e. lengths) of the line segments are defined with a choice of sign, we have to pick an orientation for each: here and in the following, we pick the orientation of the faces to point into the volume of the polytope that they are bounding. With this choice of orientation, the circumference is $[12] + [23] + [31]$.

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To see a little more structure, consider the tetrahedron in $\mathbb{CP}^3$,



. (10.30)

The volume is $[1, 2, 3, 4]$. The 2-plane defined by Z_1 contains the face bounded by the intersections of the plane 1 with the planes 2, 3, and 4. Therefore the area of this face is $[2, 3, 4]$. Keeping careful track of the orientations of the faces, we find that the total area of boundary of the tetrahedron is $[2, 3, 4] + [1, 4, 3] + [2, 4, 1] + [2, 1, 3]$.

We can summarize the results for the boundary "volumes" so far as

$$\begin{aligned} &\text{bdr of } \mathbb{CP}^2 \text{ triangle}: \quad \partial\big[1,2,3\big] = \big[2,3\big] - \big[1,3\big] + \big[1,2\big], \\ &\text{bdr of } \mathbb{CP}^3 \text{ tetrahedron}: \quad \partial\big[1,2,3,4\big] = \big[2,3,4\big] - \big[1,3,4\big] + \big[1,2,4\big] - \big[1,2,3\big]. \end{aligned} \tag{10.31}$$

This motivates us to define the ***boundary operation*** for any $(n-1)$-simplex:

$$\partial\big[123\ldots n\big] = \sum_{i=1}^{n} (-1)^{i+1} \big[1,2,\ldots,i-1,i+1,\ldots,n\big]. \tag{10.32}$$

► **Exercise 10.4**

The boundary of a boundary vanishes, so our definition (10.32) had better yield a nilpotent operator, $\partial^2 = 0$. Show that the action of ∂^2 on any n-simplex is zero.

At this stage you might have noticed the similarity between the RHS of the tetrahedron boundary identity (10.31) and the $\mathbb{CP}^2$ vanishing identity (10.25). This is easy to understand: in $\mathbb{CP}^2$, we cannot construct a 3-simplex $[1, 2, 3, 4]$ with a 3d volume, so in particular the boundary of such a formal object must vanish:

$$\mathbb{CP}^2: \quad 0 = \partial\big[1,2,3,4\big] = \big[2,3,4\big] - \big[1,3,4\big] + \big[1,2,4\big] - \big[1,2,3\big]. \tag{10.33}$$

This gives another geometric interpretation of the $\mathbb{CP}^2$ BCFW identity (10.25).

Similarly, a 5-simplex $\big[1, 2, 3, 4, 5, 6\big]$ in $\mathbb{CP}^4$ has vanishing boundary:

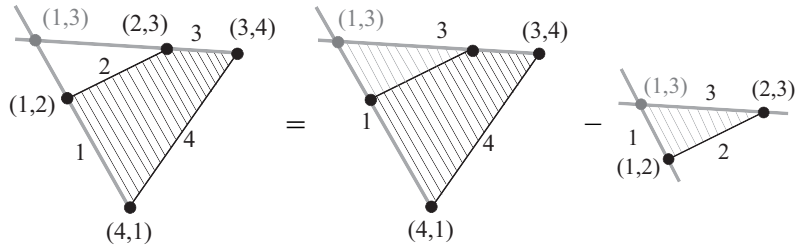
$$\begin{aligned} \mathbb{CP}^4: \quad 0 = \partial\big[1,2,3,4,5,6\big] \equiv \big[2,3,4,5,6\big] - \big[3,4,5,6,1\big] + \big[4,5,6,1,2\big] \\ - \big[5,6,1,2,3\big] + \big[6,1,2,3,4\big] - \big[1,2,3,4,5\big]. \end{aligned} \tag{10.34}$$

The RHS is exactly the 6-term identity (10.3) which originated from the equivalence of different super-BCFW shifts (10.1). This was the identity that motivated our study at the

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beginning of the chapter: we understand now that it is not a Schouten identity, but that it has an interpretation as the vanishing boundary of a formal 5-simplex in $\mathbb{CP}^4$.

Let us now take a look at the action of the boundary operation on a superamplitude. As per usual, we start with $\mathbb{CP}^2$ to get intuition for the problem. Consider the triangulation used in (10.23) to calculate the volume of a 4-sided polygon in $\mathbb{CP}^2$,



$$= [4,1,3] - [2,1,3]\,. \tag{10.35}$$

We apply the boundary operator to each 2-simplex and find:

$$\begin{aligned} \partial[4,1,3] &= [1,3] - [4,3] + [4,1]\,, \\ \partial[2,1,3] &= [1,3] - [2,3] + [2,1]\,. \end{aligned} \tag{10.36}$$

The RHS of each equation is the circumference of the respective triangles. Let us try to interpret $\partial\big([4,1,3] - [2,1,3]\big)$. It contains $\big([4,1] - [2,1]\big)$: this is a difference of lengths of two line segments in the subspace defined by Z_3 and thus it is the length of the side labeled 3 in the 4-sided polygon on the LHS of (10.35). Similarly, $\big([4,3] - [2,3]\big)$ is the length of side 1 of the polygon. Now we are left with two terms that are both labeled $[1,3]$ in (10.36). It is tempting to cancel these two terms, but this is not quite correct: $[1,3]$ in $\partial[4,1,3]$ lives in the subspace defined by Z_4, while $\partial[2,1,3]$ is in the Z_2-subspace. So the two $[1,3]$s are the lengths of the sides 2 and 4 in the polygon (10.35). Why does their difference show up in $\partial\big([4,1,3] - [2,1,3]\big)$ instead of their sum? Easy: that is because they have the opposite orientations: in our conventions, side 4 in the big triangle in (10.35) is oriented to point into the polygon, but side 2 in the small triangle points out of the 4-sided polygon. Flipping the orientation and labeling the 2-brackets by the subspace Z_i they live on, we see that the difference of the two terms in (10.36) exactly calculates the circumference of the 4-sided polygon on the LHS of (10.35):

$$\partial\big([4,1,3] - [2,1,3]\big) = \big([4,3] - [2,3]\big)_{\mathsf{Z}_1} + [3,1]_{\mathsf{Z}_2} + \big([4,1] - [2,1]\big)_{\mathsf{Z}_3} + [1,3]_{\mathsf{Z}_4}\,. \tag{10.37}$$

The boundary operator ∂ was introduced [130] as a formal operation useful for studying the cancellation of spurious poles in the BCFW expansion, without emphasis on the interpretation as the "boundary volume" we have presented here. So let us now comment on the application of ∂ in [130]. Note that for a simplex, there is a unique point "opposite" each face: in particular in the triangles in (10.35) the point labeled $(1,3)$ is the non-local "spurious" point that sits across from the line segments $[1,3]_{\mathsf{Z}_2}$ and $[3,1]_{\mathsf{Z}_4}$, respectively. So since the two $[1,3]$s define the same point $(1,3)$, one can in a *vertex-interpretation* of the boundary operation cancel them in $\partial([4,1,3] - [2,1,3])$: one can think of this as the cancellation of the spurious point in the $(1,3)$ in this particular triangulation. To distinguish

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the vertex-interpretation from the boundary volume, we include a V (for vertex) with each term; then we write

$$\partial\big([4,1,3]-[2,1,3]\big) = V\big[3,4\big]+V\big[4,1\big]+V\big[2,3\big]+V\big[1,2\big]\,. \tag{10.38}$$

Note how each term on the RHS is of the form $V[i,i+1]$ indicating that the polytope has only local vertices; the non-local vertex $V[1,3]$ canceled. This is the interpretation of the boundary operation given in [130].

► Exercise 10.5

As an example in $\mathbb{CP}^3$, consider the dissection (10.27) of the 5-faced polytope into two tetrahedrons. Keep careful track of the orientations of the boundaries to show that $\partial([6,2,4,5]-[6,2,4,3])$ calculates the surface area of the 5-faced 3-polytope on the RHS of (10.27). Next use the vertex-interpretation discussed above to show that the spurious poles cancel. Lift the example back to $\mathbb{CP}^4$ (remember that (10.27) was the projection on the subspace defined by Z_1) to see that each boundary vertex term is of the form $V[i,i+1,j,j+1]$ as in (10.28).

Enough of toy-examples! Let us compute the boundary of the NMHV 6-point tree superamplitude in the BCFW representation

$$A_6^{\text{NMHV}}[1,2,3,4,5,6] = A_6^{\text{MHV}}\times\Big(\big[1,3,4,5,6\big]+\big[3,5,6,1,2\big]+\big[5,1,2,3,4\big]\Big)\,. \tag{10.39}$$

Using the vertex-interpretation, ∂ acts on the first two 4-simplices to give

$$\begin{aligned}\partial\Big(\big[1,3,4,5,6\big]+\big[3,5,6,1,2\big]\Big) = {}& V\big[3,4,5,6\big]+V\big[4,5,6,1\big]+V\big[6,1,3,4\big]\\ &+V\big[5,6,1,2\big]+V\big[2,3,6,1\big]+V\big[2,3,5,6\big]\\ &+V\big[1,3,4,5\big]+V\big[1,2,3,5\big]\,.\end{aligned} \tag{10.40}$$

All vertices on the RHS are local except the last two. Including the boundary of the third 5-bracket in (10.39), the non-local vertices cancel completely and we have

$$\begin{aligned}&\partial\Big(\big[1,3,4,5,6\big]+\big[3,5,6,1,2\big]+\big[5,1,2,3,4\big]\Big)\\ &\qquad=\sum_{i=1}^{6}V\big[i,i+1,i+2,i+3\big]+\sum_{i=1}^{3}V\big[i,i+1,i+3,i+4\big]\,,\end{aligned} \tag{10.41}$$

where, as usually, the arguments are understood cyclically. This shows that, indeed, the boundary of the polytope that corresponds to our tree superamplitude A_6^{NMHV} contains only local vertices. The power of the boundary operation is that it makes the cancellation of spurious points clear without a need to draw any polyhedrals.

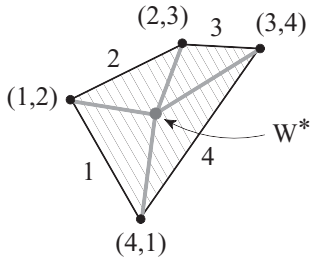
► Exercise 10.6

How many local vertices are there in the polytope corresponding to the 7-point NMHV tree superamplitude?

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10.4 Geometric aftermath

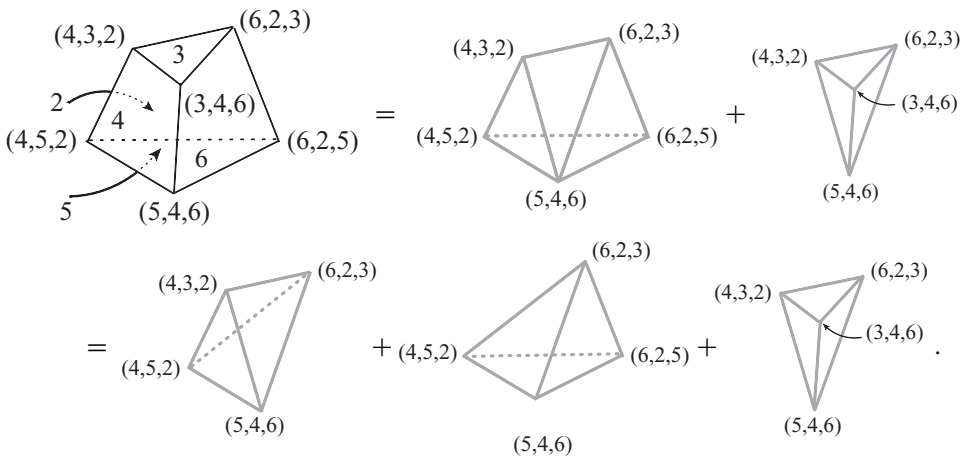
The BCFW triangulation is an efficient representation of the tree-level NMHV superamplitudes in the sense that it involves only relatively few terms. However, we can imagine other triangulations. For example, consider the 4-sided polygon in $\mathbb{CP}^2$. Introducing an auxiliary point W^* inside the polygon, we can triangulate it as



$$\mathsf{W}^* = \sum_{i=1}^{4} \frac{1}{2} \frac{\langle \mathsf{W}^*, \mathsf{W}_{(i-1,i)}, \mathsf{W}_{(i,i+1)} \rangle}{(\mathsf{Z}_0 \cdot \mathsf{W}^*)\langle 0, i-1, i\rangle\langle 0, i, i+1\rangle} \,. \tag{10.42}$$

This gives a 4-term expression for the volume of the polygon, as opposed to the 2-term BCFW triangulations in (10.23) or (10.24). In this sense, BCFW is more efficient. The representation (10.42) may remind you of another representation of scattering amplitudes, namely the CSW expansion (or MHV vertex expansion) of Section 3.4. It is actually not quite the same; CSW in momentum twistor space involves a reference supertwistor $\mathcal{Z}^* = (0, |X], 0)$ instead of W^*; see [131] for details.

As yet another way to calculate the amplitudes, we might ask if there is a triangulation that does not give spurious poles? A prescription for such a representation was given in [130] for the tree-level NMHV superamplitudes. To give a hint of how it works, consider the boundary $\mathsf{Z}_1 \cdot \mathsf{W} = 0$ that we also analyzed in (10.27). Instead of the 2-term triangulation applied in (10.27) at the cost of a non-local vertex, we can triangulate the 5-sided polytope as



(10.43)

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The manifestly local tessellation of the polytope gives more terms than the BCFW representation. You can find the general expression in [130].

We have argued that each n-point tree-level NMHV superamplitude of $\mathcal{N} = 4$ SYM can be interpreted as the volume of a polytope in $\mathbb{CP}^4$. It should be rather obvious by now that the reverse is not true: not all polytopes in $\mathbb{CP}^4$ correspond to tree-level superamplitudes in $\mathcal{N} = 4$ SYM. An example is the polytope obtained by gluing together two of the three simplices in the BCFW representation (10.39): that is a perfectly fine polytope, but it has non-local poles so does not correspond to a physical amplitude. The color-ordering plays a key role in interpreting the superamplitudes polytopes. It is of course an interesting question if this geometric picture can be extended beyond the leading color level – or if other polytopes might have interpretations in terms of scattering processes.

The current discussion of tree superamplitudes utilizes the dual superconformal invariance of planar $\mathcal{N} = 4$ SYM, focusing on the 5-brackets invariants. Since the tree amplitudes of pure Yang–Mills theory can be projected out from $\mathcal{N} = 4$ SYM superamplitudes, a similar analysis can be applied to pure YM as well. In fact, it was in pure Yang–Mills theory that Hodges first realized the polytope picture [24].

The polytope interpretation described here is valid for NMHV n-point tree superamplitudes as well as 1-loop n-point MHV integrands in planar $\mathcal{N} = 4$ SYM [130]. The generalization is not obvious. Tree-level N^KMHV superamplitudes involve sums of products of K 5-brackets, so a geometric interpretation in terms of simplex-volumes is not straightforward. There is nonetheless a geometrization of all N^KMHV tree superamplitudes and loop-integrands in planar $\mathcal{N} = 4$ SYM: it goes under the name of the ***amplituhedron*** [132]. This is a polytope defined in a space whose coordinates are a union of momentum supertwistors and Grassmannian coordinates that extend the $\chi_i \cdot \psi$-construction in (10.5). For tree-level NMHV, the amplituhedron reduces to a dual of the polytope discussed here, with vertices and faces interchanged. The amplituhedron description makes locality manifest, while unitarity is an emergent property. BCFW arises as a particular triangulation. Loops appear from integrating out pairs of "hidden" points, in a somewhat similar way to the description of the loop-integrands in Chapter 7.

It is curious that in connection to amplitudes, polytopes can appear in different guises. An example, different from our discussion so far, is the observation [133] that a 1-loop box integral can be interpreted as the volume of a tetrahedron in AdS_5. The vertices of the tetrahedron are the four dual region variables y_i (in the embedding formalism) and the edges are geodesics in AdS_5. Since all $\mathcal{N} = 4$ SYM 1-loop amplitudes are given by an expansion in box integrals, the amplitudes can be interpreted as sums of volumes of such AdS_5 tetrahedrons, weighted by the appropriate box-coefficients. Non-planar 1-loop amplitudes can be given as linear combinations of planar ones, so the same conclusion extends to non-planar amplitude as well [134].

We have encountered many different representations of the tree amplitude and have now seen a unifying geometric interpretation. But there is yet another representation of amplitudes that we had a glimpse of in Section 2.5, namely the "BCJ representation" in which the color- and kinematic-structures enter on a dual basis. For such representations

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the amplitude can be manifestly local. We discuss BCJ further in Chapter 13, but note here that the color-ordering is crucial for the relation between polytopes and amplitudes: it allows us to relate the polytope to the momentum space representation of an amplitude and this is key for the statements about locality. Polytope or amplituhedron interpretations for the planar BCJ representation, non-planar, or non-color-ordered amplitudes are still open problems.

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