RELATIVE LEFTMOST PATH PRINCIPLES AND OMEGA-MODEL REFLECTIONS OF TRANSFINITE INDUCTIONS

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Abstract. In this article, we give characterizations of Towsner's relative leftmost path principles in terms of omega-model reflections of transfinite inductions. In particular, we show that the omega-model reflection of Π^1_{n+1} transfinite induction is equivalent to the Σ^0_n relative leftmost path principle over RCA_0 for $n > 1$. As a consequence, we have that $\Sigma^0_{n+1}\mathsf{LPP}$ is strictly stronger than $\Sigma^0_n\mathsf{LPP}$.

§1. Introduction. In recent studies of reverse mathematics, not only the *big five* systems, but also other systems are focused on. For example, it is known that the strength of Kruskal's tree theorem is precisely between ATR_0 and Π^1_1-CA_0, and the best-known upper bound for Menger's theorem in graph theory is also between ATR_0 and Π^1_1-CA_0.

For studying intermediate Π^1_2 statements between ATR_0 and Π^1_1-CA_0, Towsner introduced new Π^1_2 principles which we call pseudo leftmost path principles[1] stating that any ill-founded tree has a *pseudo* leftmost path [8]. Here, a pseudo leftmost path is a path which may not be the actual leftmost path, but it behaves like a leftmost path in a restricted range. In [6], the author and Yokoyama generalized Towsner's work and gave some characterizations of $\{\sigma \in \Pi^1_2 : \Pi^1_1\text{-}\mathsf{CA}_0 \vdash \sigma\}$. They introduced *pseudo* hyperjumps and compared them with pseudo leftmost path principles.

The works in [1, 6] suggest that a finer analysis of the relationship between the n-th and $n+1$-th pseudo hyperjumps is also interesting. Since Towsner's Σ^0_n leftmost path principles approximate the single pseudo hyperjump, it seems helpful to study them for a finer analysis between pseudo hyperjumps.

In this article, we give new characterizations of pseudo leftmost path principles. We prove the following equivalences over RCA_0. For the detailed definitions, see Definitions 2.2, 3.2, and 3.5.

- The transfinite leftmost path principle (written as TLPP) and the single pseudo hyperjump with the base Π^1_1-TI (wirtten as $\beta^1_0\mathsf{RFN}(1;\Pi^1_1\text{-}\mathsf{TI})$) are equivalent.
- The arithmetic leftmost path principle (written as ALPP), the single pseudo hyperjump (written as $\beta^1_0\mathsf{RFN}(1;\top)$) and the omega-model reflection of Π^1_∞-TI (written as $\mathsf{RFN}(\Pi^1_\infty\text{-}\mathsf{TI})$) are mutually equivalent.

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[1] Originally they were called relative leftmost path principles.

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- The n-th pseudo hyperjump with the base Π^1_∞-TI (written as $\beta^1_0\mathsf{RFN}(n;\Pi^1_\infty\text{-TI})$) and the $n+1$-th pseudo hyperjump (written as $\beta^1_0\mathsf{RFN}(n+1;\top)$) are equivalent.
- The Σ^0_n leftmost path principle (written as $\Sigma^0_n\mathsf{LPP}$) and the omega-model reflection of Π^1_{n+1}-TI (written as $\mathsf{RFN}(\Pi^1_{n+1}\text{-TI})$ are equivalent for $n > 1$.

We note that these results help to reveal the proof-theoretic strength of pseudo leftmost path principles. We also note that the second clause is independently proved by Freund [1].

1.1. Structure of this article. In Section 2, we recall the notion of coded ω-models and the ω-model reflections. Then, we introduce the notion of realizers for coded ω-models, which is used to prove results in Section 3. In Section 3, we prove the main results. In Section 4, we rephrase the results in Section 3 from the point of view of Weihrauch reduction.

§2. Coded omega-models. In this section, we see some properties of coded ω-models. For the basic definitions, see also Simpson's textbook [5].

DEFINITION 2.1 (RCA_0). For a set X, we define the i-th segment X_i by $X_i = \{j : \langle i, j\rangle \in X\}$, where $\langle i, j\rangle$ denotes the standard pairing function from $\mathbb{N}^2$ to $\mathbb{N}$. Then, a set X can be regarded as a sequence $\langle X_i\rangle_{i\in\mathbb{N}}$. In this sense, we say a set A is contained in X (written $A \in X$) if $\exists i(A = X_i)$ holds.

When we identify a set $\mathcal{M}$ with the structure $(\mathbb{N}, \langle\mathcal{M}_i\rangle_{i\in\mathbb{N}})$, we call $\mathcal{M}$ a coded ω-model. If it is clear from the context, we sometimes omit the word *coded*, and simply say $\mathcal{M}$ is an ω-model. One of the important notions related to coded ω-models is the principle of ω-model reflection. We introduce two kinds of ω-model reflections.

DEFINITION 2.2. Let T be a recursively axiomatized theory. We define the ω-model reflection of T (written $\mathsf{RFN}(T)$) as the assertion that any set X is contained in an ω-model of $T + \mathsf{ACA}_0$.

It is easy to see that $\mathsf{RFN}(T)$ implies the consistency of $\mathsf{ACA}_0 + T$ over RCA_0. Therefore, sometimes $\mathsf{RFN}(T)$ is used for separating two theories.

PROPOSITION 2.3. *Let T and T' be recursive consistent extensions of ACA_0 such that T proves $\mathsf{RFN}(T')$. Then, T' does not prove T. If T proves T' in addition, then T is strictly stronger than T'.*

PROOF. If T' proved T, then T' would prove the consistency of T', a contradiction. ⊣

DEFINITION 2.4. Let Γ be a class of formulas. We define the Γ-reflection schema (written Γ-RFN) as the collection of

$$\forall X(\varphi(X) \to \exists\mathcal{M}\text{: a coded } \omega\text{-model } (\mathcal{M} \models \mathsf{ACA}_0 + \varphi(X)))$$

for all $\varphi \in \Gamma$.

The following characterization of Π^1_n-RFN is well-known.

THEOREM 2.5 (ACA_0). [2] *Let $n \in \omega$ such that $n > 0$. Then, Π^1_{n+1}-RFN is equivalent to Π^1_n-TI.*

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In the usual definition in the context of formal logics, the truth of a sentence σ over a structure $\mathcal{M}$ is defined by Tarski's truth definition. However, it is practically identified with the truth of the relativization $\sigma^{\mathcal{M}}$. We note that when working in a weak system like RCA_0, we need to distinguish between these two kinds of truth: $\sigma^{\mathcal{M}}$ and $\mathcal{M} \models \sigma$. For a detailed definition of $\mathcal{M} \models \sigma$, see [5].

Definition 2.6 (RCA_0). Let $\mathcal{M} = \langle \mathcal{M}_i \rangle_{i \in \mathbb{N}}$ be a sequence of sets. We say $\mathcal{M}$ is a jump ideal if

$$\forall i(\mathrm{TJ}(\mathcal{M}_i) \in \mathcal{M}) \wedge \forall i, j(\mathcal{M}_i \oplus \mathcal{M}_j \in \mathcal{M}) \wedge \forall i \forall X \leq_{\mathrm{T}} \mathcal{M}_i (X \in \mathcal{M}).$$

Here, $\mathrm{TJ}, \oplus$, and $\leq_{\mathrm{T}}$ denote the Turing jump, the Turing sum, and the Turing reduction, respectively.

For $S \subseteq \mathcal{P}(\omega)$, S is a jump ideal if and only if S is an ω-model of ACA_0. This fact is provable in ACA_0. However, a jump ideal may not be a model of ACA_0 in RCA_0 because RCA_0 does not ensure the existence of a (partial) truth valuation for a coded ω-model. We introduce realizers for jump ideals that make a jump ideal an ω-model of ACA_0 within RCA_0.

Definition 2.7 (RCA_0). Let $\mathcal{M}$ be a jump ideal, f_{TJ} be a function from $\mathbb{N}$ to $\mathbb{N}$ and $f_{\oplus}$ be a function from $\mathbb{N}^2$ to $\mathbb{N}$. We say the pair $(f_{\mathrm{TJ}}, f_{\oplus})$ realizes that $\mathcal{M}$ is a jump ideal (written $(f_{\mathrm{TJ}}, f_{\oplus}) \Vdash_{\mathcal{M}} \mathsf{ACA}_0$) if

$$\forall i(\mathcal{M}_{f_{\mathrm{TJ}}(i)} = \mathrm{TJ}(\mathcal{M}_i)) \wedge \forall i, j(\mathcal{M}_{f_{\oplus}(i,j)} = \mathcal{M}_i \oplus \mathcal{M}_j).$$

We say a jump ideal $\mathcal{M}$ is realizable if there is a realizer $(f_{\mathrm{TJ}}, f_{\oplus})$ for it.

Remark 2.8. We note that realizable jump ideals are essentially the same as effective ω-models of ACA_0 introduced in [7] in the context of Weihrauch degrees.

Definition 2.9. For each $n \in \omega$ such that $n > 0$, let π^0_n denote a Π^0_n-unversal formula. That is, for any Π^0_n formula θ whose Gödel number is $\ulcorner \theta \urcorner$, the following holds:

$$\mathsf{RCA}_0 \vdash \forall \vec{x}, \vec{X}(\theta(\vec{x}, \vec{X}) \leftrightarrow \pi^0_n(\ulcorner \theta \urcorner, \vec{x}, \vec{X}).$$

Let $n \in \omega$ such that $n > 0$. Then, RCA_0 proves that any realizable jump ideal has the Π^0_n truth valuation. More precisely, the following holds.

Proposition 2.10. *Let $n \in \omega$ such that $n > 0$. Then, RCA_0 proves the following. For any jump ideal $\mathcal{M}$ realized by f_{TJ} and $f_{\oplus}$, there is a $\{\top, \bot\}$-valued function $f \leq_{\mathrm{T}} \mathcal{M} \oplus f_{\mathrm{TJ}} \oplus f_{\oplus}$ such that*

$$f(e, x, i, j) = \top \leftrightarrow \pi^0_n(e, x, \mathcal{M}_i, \mathcal{M}_j).$$

Proof. We first note that RCA_0 proves that any $\Pi^0_n(X \oplus Y)$ set is many-one reducible to $\mathrm{TJ}^n(X \oplus Y)$ where TJ^n denotes the n-times iteration of the jump operator, and $\Pi^0_n(X \oplus Y)$ denotes the collection of Π^0_n-definable sets from an oracle $X \oplus Y$. More precisely, there is a primitive recursive function p such that the following holds in RCA_0. For any set X and Y such that $\mathrm{TJ}^n(X \oplus Y)$ exists, we have

$$\forall e, x(\pi^0_n(e, x, X, Y) \leftrightarrow p(e, x) \in \mathrm{TJ}^n(X \oplus Y)).$$

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We work in RCA_0 and show that any realizable jump ideal has the Π^0_n truth valuation. Let $\mathcal{M}$ be a jump ideal realized by $(f_{\mathrm{TJ}}, f_\oplus)$. Then, for any e, x, i, j, $\pi^0_n(e, x, \mathcal{M}_i, \mathcal{M}_j)$ is equivalent to $p(e, x) \in \mathcal{M}_{(f_{\mathrm{TJ}})^n(f_\oplus(i,j))}$. Therefore, the desired f is Δ^0_1-definable as follows:

$$f(e, x) = \top \leftrightarrow p(e, x) \in \mathcal{M}_{(f_{\mathrm{TJ}})^n f_\oplus(i,j)}.$$

This completes the proof. ⊣

For a class of formulas Γ, we define the class $\exists X\Gamma$ by $\{\exists X\varphi : \varphi \in \Gamma\}$. Similarly, we define $\exists f\,\Gamma$ by $\{\exists f \in \mathbb{N}^{\mathbb{N}}\varphi : \varphi \in \Gamma\}$. Here, $f \in \mathbb{N}^{\mathbb{N}}$ is the abbreviation of the Π^0_2 formula stating in RCA_0 that f is a function from $\mathbb{N}$ to $\mathbb{N}$. It is well-known that there exists a primitive recursive function p such that

$$\mathsf{RCA}_0 \vdash \exists X\pi^0_2(e, X) \leftrightarrow \exists f\pi^0_1(p(e), f).$$

Therefore, we may say that $\exists X\Pi^0_2$ is included in $\exists f\,\Pi^0_1$. Since $\exists f\,\Pi^0_1$ is a subset of $\exists X\Pi^0_2$, we may think that $\exists X\Pi^0_2$ and $\exists f\,\Pi^0_1$ are the same. Similarly, we may say that $\exists X\Pi^0_2$ and $\exists f\,\Pi^0_2$ are the same. Henceforth, we identify these three classes.

Remark 2.11. For $n > 0$, $\exists X\Pi^0_n$ is also written as $\Sigma^1_1(\Pi^0_n)$. Similarly, a formula of the form $\neg\varphi$ for some $\varphi \in \Sigma^1_1(\Pi^0_n)$ is said to be $\Pi^1_1(\Sigma^0_n)$. Formally, we inductively define $\Pi^1_m(\Gamma)$ and $\Sigma^1_m(\Gamma)$ for $\Gamma \in \{\Sigma^0_n, \Pi^0_n\}$ as follows:

$$\Sigma^1_{m+1}(\Gamma) = \exists X\Pi^1_n(\Gamma),$$
$$\Pi^1_{m+1}(\Gamma) = \forall X\Sigma^1_m(\Gamma).$$

Definition 2.12 (RCA_0). For a set X, we define the hyperjump $\mathrm{HJ}(X)$ of X by $\mathrm{HJ}(X) = \{(e, n) : \exists Y\pi^0_2(e, n, X, Y)\}$.

Remark 2.13. In the context of computability theory, $\mathrm{HJ}(X)$ is defined as a certain $\Pi^1_1(X)$-complete set. However, we prefer to define $\mathrm{HJ}(X)$ as a $\Sigma^1_1(\Pi^0_2)$-complete set over RCA_0 for technical reasons. We note that RCA_0 is not sufficient to prove that $\mathrm{HJ}(X)$ is Σ^1_1-complete. ACA_0 is needed to prove the Σ^1_1-completeness of hyperjumps.

Definition 2.14 (RCA_0). Let $\mathcal{M}$ be a coded ω-model. We say $\mathcal{M}$ is a β-model if

$$\exists X\pi^0_2(e, x, X, \mathcal{M}_i, \mathcal{M}_j) \leftrightarrow \mathcal{M} \models \exists X\pi^0_2(e, x, X, \mathcal{M}_k, \mathcal{M}_i)$$

holds.

Remark 2.15. In the usual context, a β-model is a Σ^1_1-elementary submodel of the ground model. Similarly to hyperjumps, RCA_0 is not enough to prove the Σ^1_1-elementarity of β-models.

It is well-known that the existence of the hyperjump of X is equivalent to the existence of a coded β-model containing X over ACA_0 [5]. If one reads this proof carefully, one can find that the construction of a coded β-model from a hyperjump involves information about Σ^1_1-truth of the constructed model. To clarify this fact, we introduce a realizer for coded models.

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DEFINITION 2.16. Let $\mathcal{M}$ be a coded ω-model. We say a function f is a realizer for $\exists X \Pi^0_2$ sentences true in $\mathcal{M}$ (written $f \Vdash_{\mathcal{M}} \exists X \Pi^0_2$) if $\exists j \pi^0_2(e, x, \mathcal{M}_i, \mathcal{M}_j) \to \pi^0_2(e, x, \mathcal{M}_i, \mathcal{M}_{f(e,x,i)})$.

As we will see in the proof of Lemma 3.17, $\mathrm{HJ}(X)$ computes a coded β-model $\mathcal{M}$ which contains X and has a computable realizer for $\exists X \Pi^0_2$ sentences.

In [7], the author and Yokoyama introduced a weaker variant of β-models called $\Delta^0_k\beta$-model to study pseudo leftmost path principles from the point of view of Weihrauch degrees.

DEFINITION 2.17. Let $n \in \omega$ such that $n > 0$. Let $\mathcal{M}$ be a jump ideal. We say $\mathcal{M}$ is a $\Delta^0_k\beta$-model if for any Π^0_2 formula φ with parameters from $\mathcal{M}$, the following holds.

$$\exists X \leq_{\mathrm{T}} \mathrm{TJ}^{k-1}(\mathcal{M})\varphi(X) \to \exists i \varphi(\mathcal{M}_i)$$

In the next section, we show the equivalence of $\Sigma^0_k\mathsf{LPP}$ and the existence of $\Delta^0_k\beta$-models with computable realizers.

§3. The equivalences. In this section, we show the equivalence of pseudo leftmost path principles and the ω-model reflection of transfinite inductions. We first introduce pseudo leftmost path principles.

DEFINITION 3.1. Let $T \subseteq \mathbb{N}^{<\mathbb{N}}$ be an ill-founded tree.

- We say a path f of T is the leftmost path (written $\mathrm{LP}(T, f)$) if f is the smallest in $[T]$ with respect to the lexicographic order. Here, $[T]$ denotes the set of paths of T.
- Let α be a well-order. We say a path f of T is a Σ^0_α-leftmost path if f is the smallest in $[T] \cap \{g : g \in \Sigma^0_\alpha(T \oplus f)\}$ with respect to the lexicographic order.

DEFINITION 3.2. [6, 8] We define $\Sigma^0_n\mathsf{LPP}$, ALPP, and TLPP as follows.

- $\Sigma^0_n\mathsf{LPP}$ is the assertion that any ill-founded tree has a Σ^0_n-leftmost path,
- ALPP is the assertion that any ill-founded tree has a Σ^0_ω-leftmost path,
- TLPP is the assertion that for any well-order α and ill-founded tree T, T has a Σ^0_α leftmost path.

We note that a path of a tree is a total function. Therefore, a path $g \in [T]$ is Σ^0_n-definable from $T \oplus f$ if and only if it is Δ^0_n-definable from $T \oplus f$.

DEFINITION 3.3. We say a path $f \in [T]$ is a Δ^0_k leftmost path if f is the smallest in $\{g \in [T] : g \leq_{\mathrm{T}} \mathrm{TJ}^{k-1}(T \oplus f)\}$ with respect to the lexicographic order. We define $\Delta^0_k\mathsf{LPP}$ as the assertion that any ill-founded tree has a Δ^0_k-leftmost path.

LEMMA 3.4. *Over* RCA_0, $\Sigma^0_k\mathsf{LPP}$ *is equivalent to* $\Delta^0_k\mathsf{LPP}$ *for* $k \in \omega$ *such that* $k > 0$.

PROOF. As we have mentioned, a path is Σ^0_k-definable if and only if it is Δ^0_k-definable. Therefore, a path is Σ^0_k-leftmost if and only if it is Δ^0_k-leftmost. ⊣

As it is proved in [7], $\Delta^0_k\mathsf{LPP}$ is equivalent to the assertion that any ill-founded tree has a path f that is the smallest in $\{g \in [T] : g \leq_{\mathrm{T}} \mathrm{TJ}^{k-1}(f)\}$. Thus, we do not distinguish between $\Sigma^0_k\mathsf{LPP}$, $\Delta^0_k\mathsf{LPP}$ and this assertion.

For studying Π^1_2 theorems above TLPP, the author and Yokoyama introduced the notion of pseudo hyperjumps in [6].

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DEFINITION 3.5. Let σ be a sentence and $n \in \omega$. We define the n-th pseudo hyperjump with the base σ as

$$\beta_0^1\mathsf{RFN}(n;\sigma) \equiv \forall X \exists \mathcal{M}(X \in \mathcal{M} \wedge \mathcal{M} \models \mathsf{ACA}_0 + \sigma + \exists \mathrm{HJ}^n(X)).$$

THEOREM 3.6 [6]. *Over* ACA_0,

$$\beta_0^1\mathsf{RFN}(1;\top) < \mathsf{TLPP} < \beta_0^1\mathsf{RFN}(1;\mathsf{ATR}_0) < \beta_0^1\mathsf{RFN}(2;\top).$$

Here, $T < T'$ *means that* T' *proves* T *and its consistency.*

We now provide a new characterization of TLPP via a pseudo hyperjump. First, we introduce some properties of TLPP.

LEMMA 3.7 (TLPP) [8, Theorem 4.6]. *For any ill-founded tree* T*, there exists an* ω*-model of* Σ_1^1-DC_0 *which believes that* T *has a leftmost path.*

LEMMA 3.8 (RCA_0) [6, Lemma 4.23]. TLPP *is equivalent to the assertion that for any well-order* α *and set* X*, there exists an* ω*-model* $\mathcal{M}$ *such that* $\alpha, X \in \mathcal{M}$ *and* $\mathcal{M}$ *believes* $\mathrm{HJ}(X)$ *exists and* $\mathcal{M}$ *is closed under* α*-times Turing jump.*

To connect leftmost paths and hyperjumps, we use the following lemma.

LEMMA 3.9. *Over* RCA_0*, we may assume that there exists a total Turing functional* $\Phi_T(X)$ *such that*

- $\Phi_T(X)$ *is a tree,*
- $\Phi_T(X)$ *has an infinite path which is uniformly computable from* X*,*
- *each path of* $\Phi_T(X)$ *computes* X*,*
- *if* $\mathrm{LP}(\Phi_T(X), f)$ *holds, then* $\mathrm{HJ}(X) \leq_\mathrm{T} f$*.*

For details, see [6, Section 4]. We note that in [6], the above properties are proved in ACA_0 rather than RCA_0. However, if one looks at the proofs carefully, one can find that these properties hold in RCA_0.

THEOREM 3.10. *Over* RCA_0, TLPP *is equivalent to* $\beta_0^1\mathsf{RFN}(1;\Pi_1^1\text{-}\mathsf{TI})$.

PROOF. Since Π_1^1-TI_0 is equivalent to Σ_1^1-DC_0 over ACA_0, it is enough to show the equivalence of TLPP and $\beta_0^1\mathsf{RFN}(1;\Sigma_1^1\text{-}\mathsf{DC})$.

We first show that TLPP implies $\beta_0^1\mathsf{RFN}(1;\Sigma_1^1\text{-}\mathsf{DC})$. Suppose TLPP. Let X be a set. Then, there exists an ω-model $\mathcal{M}$ of Σ_1^1-DC_0 such that $\mathcal{M}$ believes $\Phi_T(X)$ has a leftmost path. Therefore, $\mathcal{M}$ satisfies Σ_1^1-DC_0 and there exists the hyperjump of X.

We then show the converse direction by Lemma 3.8. Let α be a well-order and X be a set. Take an ω-model $\mathcal{M}$ such that $\alpha, X \in \mathcal{M}$ and $\mathcal{M} \models \Sigma_1^1$-$\mathsf{DC}_0$. Then, $\mathcal{M}$ is closed under α-times Turing jump. ⊣

REMARK 3.11. We note that Π_1^1-TI can be weaken to Montalbán's Jump Iteration JI [3] in the above proof.

COROLLARY 3.12. *Over* RCA_0, $\beta_0^1\mathsf{RFN}(1;\Pi_1^1\text{-}\mathsf{TI}) < \beta_0^1\mathsf{RFN}(1;\mathsf{ATR}_0)$.

PROOF. It is immediate from the equivalence of $\beta_0^1\mathsf{RFN}(1;\Pi_1^1\text{-}\mathsf{TI}_0)$ and TLPP. ⊣

We next show that both ALPP and the single pseudo hyperjump $\beta_0^1\mathsf{RFN}(1;\top)$ are characterized by the ω-model reflection of full transfinite induction. We note that this characterization is independently proved by Freund [1].

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THEOREM 3.13. *Over* RCA_0, *the following assertions are equivalent*:

1. ALPP,
2. $\beta^1_0\mathsf{RFN}(1)$,
3. $\mathsf{RFN}(\Pi^1_\infty\text{-}\mathsf{TI})$,
4. $\mathsf{RFN}(\mathrm{A}\Pi^1_1\text{-}\mathsf{TI})$, *where* $\mathrm{A}\Pi^1_1\text{-}\mathsf{TI}_0$ *is the class built from* Π^1_1 *with logical connectives and number quantifiers.*

PROOF. The equivalence of (1) and (2) is proved in [6], and the equivalence of (2) and (3) is proved in [1]. The implication of (3) to (4) is trivial. For the implication of (4) to (2), the same proof of the implication of (3) to (2) works. For this implication, see also [5, Theorem VII.2.18 and Corollary VII.2.21]. ⊣

The equivalence of (2) and (3) in the previous theorem can be extended to the n-th pseudo hyperjump level as follows.

THEOREM 3.14. *Over* RCA_0, $\beta^1_0\mathsf{RFN}(n+1)$ *is equivalent to* $\beta^1_0\mathsf{RFN}(n;\Pi^1_\infty\text{-}\mathsf{TI})$ *for any* $n \in \omega$ *such that* $n > 0$.

PROOF. Since the same proof for the previous theorem works, we give a brief proof. The implication from $\beta^1_0\mathsf{RFN}(n+1)$ to $\beta^1_0\mathsf{RFN}(n;\Pi^1_\infty\text{-}\mathsf{TI})$ is trivial. For proving $\beta^1_0\mathsf{RFN}(n+1)$ from $\beta^1_0\mathsf{RFN}(n;\Pi^1_\infty\text{-}\mathsf{TI})$, take an X and an ω-model $\mathcal{M}$ such that $X \in \mathcal{M} \wedge \mathcal{M} \models (\exists \mathrm{HJ}^n(X) \wedge \Pi^1_\infty\text{-}\mathsf{TI})$. Let $Y \in \mathcal{M}$ such that $\mathcal{M} \models Y = \mathrm{HJ}^n(X)$. Let $\mathcal{M}'$ be the collection of the definable-in-$\mathcal{M}$ subclasses of $\mathbb{N}$. Then, $\mathcal{M} \subseteq_\beta \mathcal{M}'$. Moreover, there is a set $Z \in \mathcal{M}'$ such that $Z = \{n : \mathcal{M} \models \theta_{\mathrm{HJ}}(n, Y)\}$, where θ_{HJ} is the Σ^1_1 formula defining hyperjumps. Then, $\mathcal{M}' \models Z = \mathrm{HJ}(Y) \wedge Y = \mathrm{HJ}^n(X)$, hence $\mathcal{M}' \models Z = \mathrm{HJ}^{n+1}(X)$. This completes the proof. ⊣

Combining Corollary 3.12 and the previous theorem, we have the following corollary.

COROLLARY 3.15. *Over* RCA_0,

$$\beta^1_0\mathsf{RFN}(1;\top) < \beta^1_0\mathsf{RFN}(1;\Pi^1_1\text{-}\mathsf{TI}) < \beta^1_0(1;\Sigma^1_1\text{-}\mathsf{TI}) < \beta^1_0\mathsf{RFN}(1;\Pi^1_\infty\text{-}\mathsf{TI}).$$

PROOF. We first remark that any of the above statements includes ACA_0. We also note that $\beta^1_0(1;\Sigma^1_1\text{-}\mathsf{TI})$ is equivalent to $\beta^1_0(1;\mathsf{ATR}_0)$ because $\Sigma^1_1\text{-}\mathsf{TI}$ and ATR_0 are equivalent over Σ^1_1-induction that is true in any coded ω-model. Thus, the inequality $\beta^1_0\mathsf{RFN}(1;\top) < \beta^1_0\mathsf{RFN}(1;\Pi^1_1\text{-}\mathsf{TI}) < \beta^1_0(1;\Sigma^1_1\text{-}\mathsf{TI})$ follows from Theorem 3.6 and Corollary 3.12. The inequality $\beta^1_0(1;\Sigma^1_1\text{-}\mathsf{TI}) < \beta^1_0\mathsf{RFN}(1;\Pi^1_\infty\text{-}\mathsf{TI})$ follows from Theorems 3.6 and 3.14. ⊣

We then show that the equivalence in Theorem 3.13 also holds at the level of $\Sigma^0_k\mathsf{LPP}$. For proving the equivalence of $\Sigma^0_k\mathsf{LPP}$ and $\mathsf{RFN}(\Pi^1_{k+1}\text{-}\mathsf{TI})$, we use $\Delta^0_k\beta$-models with computable realizers.

DEFINITION 3.16. Let $n \in \omega$ such that $n > 0$. We define $\mathrm{R}\Delta^0_n\beta\text{-}\mathsf{RFN}$ as the assertion that any set X is contained in a $\Delta^0_n\beta$-model $\mathcal{M}$ with total computable functions $f_{\mathrm{TJ}}, f_\oplus, f_{\Sigma^1_1}$ such that $(f_{\mathrm{TJ}}, f_\oplus) \Vdash_{\mathcal{M}} \mathsf{ACA}_0$ and $f_{\Sigma^1_1} \Vdash_{\mathcal{M}} \exists X \Pi^0_2$.

LEMMA 3.17. *For* $k \in \omega$ *such that* $k > 0$, $\Sigma^0_{k+1}\mathsf{LPP}$ *implies* $\mathrm{R}\Delta^0_{k+1}\beta\text{-}\mathsf{RFN}$ *over* RCA_0.

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PROOF. We first note that $\Sigma^0_{k+1}\mathsf{LPP}$ includes ACA^+_0. Thus, we may assume that any coded ω-model has a total valuation.

Let X be a set. Take a Δ^0_{k+1}-leftmost path f of $\Phi_T(X)$. Define an ω-model $\mathcal{M} = \Delta^0_{k+1}(\Phi_T(X) \oplus f) = \{A : A \leq_{\mathrm{T}} \mathrm{TJ}^k(\Phi_T(X) \oplus f)\}$. Then, $\mathcal{M} \models \mathsf{RCA}_0 + \mathrm{LP}(\Phi_T(X), f)$. Therefore, $\mathrm{HJ}^{\mathcal{M}}(X) \leq_{\mathrm{T}} f$ exists.

We perform the construction of [5, Lemma VII.2.9(19)] within $\mathcal{M}$. Then, we have a function $g \leq_{\mathrm{T}} \mathrm{HJ}^{\mathcal{M}}(X) \leq_{\mathrm{T}} f$ having the following condition.

$$\mathcal{M} \models \forall e, m(\exists h \in \mathbb{N}^{\mathbb{N}} \pi^0_1(e, m, (g)_{<\langle e,m\rangle}, h) \to \pi^0_1(e, m, (g)_{<\langle e,m\rangle}, g_{\langle e,m\rangle})).$$

Here, g_n denotes the n-th segment of g, that is, $g_n(x) = g(\langle n, x\rangle)$. Thus, g is identified the sequence $\langle g_n\rangle_n$. In addition, $g_{<n}$ denotes the sequence $\langle g_0 \ldots, g_{n-1}\rangle$.

We show that the sequence $\langle g_n\rangle_n$ is closed under Turing jumps. That is, for any n, there exists m such that $g_m = \mathrm{TJ}(g_n)$. To see this, take an n. Then, $g_n \leq_{\mathrm{T}} g \leq_{\mathrm{T}} f$ by definition. Thus, $\mathrm{TJ}(g_n) \leq_{\mathrm{T}} \mathrm{TJ}(f)$ and hence $\mathrm{TJ}(g_n) \in \mathcal{M}$. Since $\exists h(h = \mathrm{TJ}(g_n))$ is of the form $\exists h' \in \mathbb{N}^{\mathbb{N}}\Pi^0_1$, we have $g_{m_n} = \mathrm{TJ}(g_n)$ for some m_n. Moreover, the correspondence $f_{\mathrm{TJ}} : n \mapsto m_n$ is clearly computable. By a similar way, we have that there exists a computable correspondence $f_{\oplus} : (n, n') \mapsto m'(n, n')$ such that $g_{m'(n,n')} = g_n \oplus g'_n$. It is also proved by a similar way that $\langle g_n\rangle_n$ is closed under the Turing reduction. Therefore, $\langle g_n\rangle_n$ is a jump ideal realized by computable functions f_{TJ} and $f_{\oplus}$. In addition, $\langle g_n\rangle_n$ and $\mathcal{M}$ are clearly absolute for $\exists h \in \mathbb{N}^{\mathbb{N}}\Pi^0_1$ formulas.

We show that $\langle g_n\rangle_n$ is a $\Delta^0_{k+1}\beta$-model. Let $e, x, m \in \mathbb{N}$ and $h \leq_{\mathrm{T}} \mathrm{TJ}^k(\langle g_n\rangle_n)$ be such that $\pi^0_1(e, x, g_m, h)$. Then, $h \leq_{\mathrm{T}} \mathrm{TJ}^k(\Phi_T(X) \oplus f)$ because $\langle g_n\rangle_n \leq_{\mathrm{T}} f$. Hence, $h \in \mathcal{M}$. Thus we have $\mathcal{M} \models \exists h\pi^0_1(e, x, g_n, h)$. By the construction of $\langle g_n\rangle_n$, $\langle g_n\rangle_N \models \pi^0_1(e, x, g_m, g_{m'})$ for some m'. Consequently, we have that $\langle g_n\rangle_n$ is a $\Delta^0_{k+1}\beta$-model.

We then show that $\langle g_n\rangle_n$ has a computable realizer for $\exists h \in \mathbb{N}^{\mathbb{N}}\Pi^0_1$ formulas. We note that

$$\langle g_n\rangle_n \models \exists h\pi^0_1(e, m, g_{<\langle e,m\rangle}, h) \text{ if and only if } \langle g_n\rangle_n \models \pi^0_1(e, m, g_{<\langle e,m\rangle}, g_{\langle e,m\rangle})$$

holds. Hence, to decide either $\langle g_n\rangle_n \models \exists h\pi^0_1(e, m, g_{<\langle e,m\rangle}, h)$ or not, it is enough to check either $\langle g_n\rangle_n \models \pi^0_1(e, m, g_{<\langle e,m\rangle}, g_{\langle e,m\rangle})$ or not. Since $\langle g_n\rangle_n$ is a realizable jump ideal, it has a computable truth valuation for Π^0_1 formulas. Thus, we have a computable function which decides either $\langle g_n\rangle_n \models \pi^0_1(e, m, g_{<\langle e,m\rangle}, g_{\langle e,m\rangle})$ or not. This completes the proof. ⊣

LEMMA 3.18. *Let $\mathcal{M}$ be a $\Delta^0_k\beta$-model and $f_{\mathrm{TJ}}, f_{\oplus}, f_{\Sigma^1_1}$ be computable functions such that $(f_{\mathrm{TJ}}, f_{\oplus}) \Vdash_{\mathcal{M}} \mathsf{ACA}_0$ and $f_{\Sigma^1_1} \Vdash_{\mathcal{M}} \exists X\Pi^0_2$. Then, for any arithmetical formula $\varphi(n, X, Y)$ and $n, i \in \mathbb{N}$, the set $\{n : \mathcal{M} \models \exists Y\varphi(n, \mathcal{M}_i, Y)\}$ is $\mathcal{M}$-computable.*

PROOF. We may assume that φ is Π^0_2 because $\mathcal{M}$ is a model of ACA_0.

Let e be a code of φ. Since $f_{\Sigma^1_1} \Vdash_{\mathcal{M}} \exists X\Pi^0_2$, $\{n : \mathcal{M} \models \exists Y\varphi(n, \mathcal{M}_i, Y)\} = \{n : \varphi(n, \mathcal{M}_i, \mathcal{M}_{f_{\Sigma^1_1}(e,n,i))})\}$. Since $\mathcal{M}$ is a realizable jump ideal, there is a $(\mathcal{M} \oplus f_{\mathrm{TJ}} \oplus f_{\oplus})$-computable truth valuation for Π^0_2 formulas. Thus, $\{n : \varphi(n, \mathcal{M}_i, \mathcal{M}_{f_{\Sigma^1_1}(e,n,i)})\}$ is computable from $\mathcal{M} \oplus f_{\mathrm{TJ}} \oplus f_{\oplus} \oplus f_{\Sigma^1_1}$. Since the realizers are computable, $\{n : \varphi(n, \mathcal{M}_i, \mathcal{M}_{f_{\Sigma^1_1}(e,n,i)})\}$ is $\mathcal{M}$-computable. ⊣

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LEMMA 3.19. *Let $k \in \omega$ such that $k > 0$. Let $\mathcal{M}$ be a $\Delta^0_k\beta$-model having computable realizers $f_{\mathrm{TJ}}, f_{\oplus}, f_{\Sigma^1_1}$. Then, for any Σ^1_{k+1} formula $\varphi(n, X)$, the set $\{n : \mathcal{M} \models \varphi(n, \mathcal{M}_i)\}$ is Σ^0_k-definable from $\mathcal{M}$ and $\{n : \mathcal{M} \models \neg\varphi(n, \mathcal{M}_i)\}$ is Π^0_k-definable from $\mathcal{M}$.*

PROOF. It is an easy induction on k. ⊣

LEMMA 3.20 (ACA_0). *Let $k \in \omega$ such that $k > 0$. Let $\mathcal{M}$ be a $\Delta^0_k\beta$-model, f_{TJ}, $f_{\oplus}$ $f_{\Sigma^1_1}$ be computable realizers for $\mathcal{M}$. Then, $\mathcal{M} \models \Pi^1_{k+1}$-$\mathsf{TI}$.*

PROOF. Let $W \in \mathcal{M}$ be such that $\mathcal{M} \models \mathrm{WO}(W)$. Let $\varphi(i)$ be a Π^1_{n+1} formula such that $\mathcal{M}$ satisfies the antecedent of transfinite induction along W. That is,

$$\mathcal{M} \models \forall i \in |W|(\forall j <_W i\varphi(j) \to \varphi(i))$$

where $|W|$ denotes the field of W.

Assume that $\mathcal{M} \not\models \forall i \in |W|\varphi(i)$. Then, $\{i \in |W| : \neg\varphi(i)\}$ is nonempty and has no W-minimal element. In addition, $\{i \in |W| : \neg\varphi(i)\}$ is Σ^0_k-definable from $\mathcal{M}$. Thus, there exists a W-descending sequence from $\{i \in |W| : \neg\varphi(i)\}$ which is Δ^0_k definable from $\mathcal{M}$. That is, we have

$$\exists A \leq_{\mathrm{T}} \mathrm{TJ}^{k-1}(\mathcal{M})(A \text{ is a descending sequence of } W).$$

Since $\mathcal{M}$ is a $\Delta^0_k\beta$-model, $\mathcal{M} \models \neg\mathrm{WO}(W)$. However, this is a contradiction. ⊣

Now, we have the implication $\Sigma^0_{k+1}\mathsf{LPP} \to \mathsf{RFN}(\Pi^1_{k+2}$-$\mathsf{TI})$. For the converse, we consider a Δ^0_k analogue of the implication $\mathsf{RFN}(\mathrm{A}\Pi^1_1$-$\mathsf{TI}) \to \mathsf{ALPP}$. The point of this implication is that for a given coded ω-model $\mathcal{M}$ of $\mathrm{A}\Pi^1_1$-TI, the set of $\mathrm{A}\Pi^1_1$-definable subclasses $\{\{n : \mathcal{M} \models \theta(n)\} : \theta \in \mathrm{A}\Pi^1_1\}$ of $\mathcal{M}$ forms a β-extension of $\mathcal{M}$. For considering a Δ^0_k analogue, we begin with introducing the classes $\Sigma^0_k(\Pi^1_1)$, $\Pi^0_k(\Pi^1_1)$ and $\Delta^0_k(\Pi^1_1)$. Intuitively, $\Sigma^0_k(\Pi^1_1)$ is the class of formulas of the form $\varphi[t \in X/\theta(t)]$ for some Σ^0_k formula φ and Π^1_1 formula $\theta(x)$, $\Pi^0_k(\Pi^1_1)$ is its dual, and $\Delta^0_k(\Pi^1_1)$ is the intersection of $\Sigma^0_k(\Pi^1_1)$ and $\Pi^0_k(\Pi^1_1)$. Here, $\varphi[t \in X/\theta(t)]$ is the formula obtained from φ by replacing each occurrence of $t \in X$ with $\theta(t)$. Formally, we define these classes as follows.

DEFINITION 3.21. We say a set φ is $\Pi^0_1(\Pi^1_1)$ if it is of the form $\forall x\theta(x)$ for some formula θ which is built from Π^1_1 using logical connectives $\vee, \wedge, \neg$ and bounded quantifiers $\forall x < y, \exists x < y$. Similarly, a formula is $\Sigma^0_1(\Pi^1_1)$ if it is of the form $\exists x\theta(x)$ for some formula θ which is built from Π^1_1 using logical connectives $\vee, \wedge, \neg$ and bounded quantifiers $\forall x < y, \exists x < y$. For the general case, we say a formula is $\Pi^0_{k+1}(\Pi^1_1)$ if it is of the form $\forall x\theta(x)$ for some formula θ which is built from $\Pi^0_k(\Pi^1_1)$ using logical connectives $\vee, \wedge, \neg$ and bounded quantifiers $\forall x < y, \exists x < y$. We also say a formula is $\Sigma^0_{k+1}(\Pi^1_1)$ if it is of the form $\exists x\theta(x)$ for some formula θ which is built from $\Pi^0_k(\Pi^1_1)$ using logical connectives $\vee, \wedge, \neg$ and bounded quantifiers $\forall x < y, \exists x < y$.

We say a formula is $\Delta^0_n(\Pi^1_1)$ (in a fixed base theory T) if it belongs to both $\Pi^0_n(\Pi^1_1)$ and $\Sigma^0_n(\Pi^1_1)$.

In what follows, we show that the class of $\Delta^0_n(\Pi^1_1)$-definable sets forms a model of RCA_0 in a sense.

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LEMMA 3.22 ($\mathsf{RCA}_0 + \Sigma^1_\infty$-IND). *The class $\Sigma^0_n(\Pi^1_1)$ is closed under $\vee, \wedge$ and bounded quantifications and the existential number quantification. Similarly, $\Pi^0_n(\Pi^1_1)$ is closed under $\vee, \wedge$ and bounded quantifications and the universal number quantification.*

PROOF. Working in $\mathsf{RCA}_0 + \Sigma^1_\infty$-IND, the usual proof for the closure properties of Σ^0_n and Π^0_n works. ⊣

LEMMA 3.23 ($\mathsf{RCA}_0 + \Sigma^1_\infty$-IND). *The class $\Pi^0_n(\Pi^1_1)$ is a subclass of Π^1_{n+1}.*

PROOF. We prove by induction on n. We note that if a formula φ is built from Π^1_1 with $\wedge, \vee, \neg, \forall x < y, \exists x < y$, then φ is Δ^1_2 in $\mathsf{RCA}_0 + \Sigma^1_\infty$-IND. Thus, a $\Pi^0_1(\Pi^1_1)$ formula is of the form $\forall x \Delta^1_2$, and hence Π^1_2.

For the induction step, we can show that a formula built from $\Pi^0_n(\Pi^1_1)$ with $\wedge, \vee, \neg, \forall x < y, \exists x < y$ is Δ^1_{n+1}. Thus, we have that a $\Pi^0_{n+1}(\Pi^1_1)$ formula is Π^1_{n+1} formula. ⊣

LEMMA 3.24 ($\mathsf{RCA}_0 + \Sigma^1_\infty$-IND). *Let $\varphi(n)$ be a $\Sigma^0_k(\Pi^1_1)$ formula and $\psi(n)$ be a $\Pi^0_k(\Pi^1_1)$ formula. If $X = \{n : \varphi(n)\} = \{n : \psi(n)\}$ and $Y \leq_{\mathrm{T}} X$, then Y is $\Delta^0_n(\Pi^1_1)$-definable.*

PROOF. It is enough to show that if Y is Σ^0_1-definable from X then Y is $\Sigma^0_k(\Pi^1_1)$-definable, and if Y is Π^0_1-deinable from X then Y is $\Pi^0_k(\Pi^1_1)$-definable. This is immediate from induction on the construction of Σ^0_1/Π^0_1 formulas. ⊣

LEMMA 3.25 ($\mathsf{RCA}_0 + \Sigma^1_\infty$-IND). *Let $\varphi_X(n), \varphi_Y(n)$ be $\Sigma^0_k(\Pi^1_1)$ formulas and $\psi_X(n)$, $\psi_Y(n)$ be $\Pi^0_k(\Pi^1_1)$ formulas. If $X = \{n : \varphi_X(n)\} = \{n : \varphi_Y(n)\}$ and $Y = \{n : \varphi_Y(n)\} = \{n : \varphi_Y(n)\}$, then $X \oplus Y$ is $\Delta^0_k(\Pi^1_1)$-definable.*

PROOF. Recall that $X \oplus Y = \{2n : n \in X\} \cup \{2n+1 : n \in Y\} = \{2n : \varphi_X(n)\} \cup \{2n+1 : \varphi_Y(n)\}$. Define $\varphi(n) \equiv \exists m((m = 2n \wedge \varphi_X(m)) \vee (m = 2n+1 \wedge \varphi_Y(m))$. Since $\Sigma^0_k(\Pi^1_1)$ is closed under logical connectives and existential number quantification, φ is also $\Sigma^0_k(\Pi^1_1)$. Similarly, $\psi(n) \equiv \forall m((m = 2n \to \varphi_X(m)) \vee (m = 2n+1 \to \varphi_Y(m))$ is $\Pi^0_k(\Pi^1_1)$. Then, $X \oplus Y$ is defined by both φ and ψ. This completes the proof. ⊣

LEMMA 3.26 (ACA_0). *Let $\mathcal{M}$ and $\mathcal{N}$ be coded ω-models such that $\mathcal{M} \models \mathsf{ACA}_0$, $\mathcal{N} \models \mathsf{RCA}_0$ and $\forall X(X \in \mathcal{M} \to X \in \mathcal{N})$. Assume that for any linear order $L \in \mathcal{M}$, $\mathcal{M} \models \mathrm{WO}(L)$ if and only if $\mathcal{N} \models \mathrm{WO}(L)$. Then for any $\Sigma^1_1(\Pi^0_2)$ sentence σ with parameters from $\mathcal{M}$, $\mathcal{M} \models \sigma$ if and only if $\mathcal{N} \models \sigma$.*

PROOF. Write $\sigma \equiv \exists X \forall x \exists y \theta$ by a Σ^0_0 formula θ. It is enough to show that if $\mathcal{N} \models \sigma$ then $\mathcal{M} \models \sigma$.

By Skolemization and Kleene's normal form, there is a Σ^0_0 formula η such that $\sigma \leftrightarrow \exists X \exists f \in \mathbb{N}^{\mathbb{N}} \forall x \eta(X[x], x, f[x])$ is provable in RCA_0. Assume $\mathcal{N} \models \exists X \forall x \exists y \theta(X, x, y)$. Pick an X_0 and f_0 such that $\mathcal{N} \models \forall x \eta(X_0[x], x, f_0[x])$. We define a tree $T \subseteq (2 \times \mathbb{N})^{<\mathbb{N}}$ by $T = \{(\sigma, \tau) : \eta(\sigma, |\sigma|, \tau)\}$. Then, T is computable from parameters in σ. Thus, T is in $\mathcal{M}$ and $\mathbb{N}$.

We have $\mathcal{N} \models (X_0, f_0) \in [T]$. Thus, $\mathcal{N} \models \neg \mathrm{WO}(\mathrm{KB}(T))$ where $\mathrm{KB}(T)$ denotes the Kleene-Brouwer ordering of T. By assumption, $\mathcal{M} \models \neg \mathrm{WO}(\mathrm{KB}(T))$. Since $\mathcal{M}$ is a model of ACA_0, $\mathcal{M} \models [T] \neq \varnothing$. Thus, $\mathcal{M} \models \sigma$. ⊣

LEMMA 3.27 (RCA_0). $\mathsf{RFN}(\Delta^0_k(\Pi^1_1)\text{-}\mathsf{TI}_0)$ *implies* $\Delta^0_k\mathsf{LPP}$.

PROOF. Since $\mathsf{RFN}(\Delta^0_k(\Pi^1_1)\text{-}\mathsf{TI}_0)$ implies ACA^+_0, we may assume ACA^+_0.

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Let T be an ill-founded tree with a path f. Take an ω-model $\mathcal{M}$ such that $\mathcal{M} \models \Delta^0_k(\Pi^1_1)\text{-}\mathsf{TI}_0$ and $T, f \in \mathcal{M}$.

Let $\langle(\varphi_e(x), \psi_e(x))\rangle_e$ be an enumeration of $\Delta^0_k(\Pi^1_1)$-in-$\mathcal{M}$ formulas. That is, each φ_e is $\Sigma^0_k(\Pi^1_1)$ and ψ_e is $\Pi^0_k(\Pi^1_1)$ having the same parameters from $\mathcal{M}$, and

$$\mathcal{M} \models \forall x(\varphi_e(x) \leftrightarrow \psi_e(x)).$$

Define $\mathcal{N}_e = \{x : \mathcal{M} \models \varphi_e(x)\}$ and $\mathcal{N} = \langle \mathcal{N}_e \rangle_e$. Since $\mathcal{N}$ is closed under Turing reduction and Turing sum, $\mathcal{N}$ is an ω-model of RCA_0.

We show that $\mathcal{M} \models \mathrm{WO}(L)$ if and only if $\mathcal{N} \models \mathrm{WO}(L)$ for any linear order $L \in \mathcal{M}$. Let $L \in \mathcal{M}$ be a linear order. Assume $\mathcal{M} \models \mathrm{WO}(L)$ but $\mathcal{N} \not\models \mathrm{WO}(L)$. Then, there exists $Y \in \mathcal{N}$ such that $\varnothing \subsetneq Y \subseteq |L|$ and Y has no minimal element. Here, $|L|$ denotes the field of L. Let φ_e, ψ_e be the defining formulas of Y. That is,

$$\mathcal{M} \models \varphi_e \leftrightarrow \psi_e,$$
$$\forall x(x \in Y \leftrightarrow \mathcal{M} \models \varphi_e(x)).$$

We show that $\mathcal{M} \models \forall i \in |L|(\neg\varphi_e(i))$ by transfinite induction along L. Let $i \in |L|$ such that $\mathcal{M}$ satisfies $\forall j <_L i\, \neg\varphi_e(j)$. Then, $\forall j <_L i (j \notin Y)$. Since Y has no minimal element, $i \notin Y$ and hence $\mathcal{M}$ satisfies $\neg\varphi_e(i)$. We have $Y = \varnothing$, contradiction.

We note that $\mathcal{M}$ and $\mathcal{N}$ are $\exists f \Pi^0_2$ conservative by Lemma 3.26. Define a tree $S = \{\sigma : \mathcal{M} \models \exists f \in [T](\sigma \prec f)\}$. Then, $S = \{\sigma : \mathcal{N} \models \exists f \in [T](\sigma \prec f)\}$ by conservativity. In addition, $S \in \mathcal{N}$ because S is Σ^1_1 definable over $\mathcal{M}$. Let $\varphi(x)$ be a Σ^1_1 formula defining S. Then, there exists a $\Delta^0_k(\Pi^1_1)$ formula $\psi(x)$ such that $\mathrm{TJ}^{k-1}(S) = \{x : \mathcal{M} \models \psi(x)\}$. Thus, $\mathrm{TJ}^{k-1}(S)$ is in $\mathcal{N}$.

We note that S is an ill-founded pruned tree by definition. Therefore, S computes its leftmost path g. Then, $\mathrm{TJ}^{k-1}(g) \leq_{\mathrm{T}} \mathrm{TJ}^{k-1}(S)$ and hence g and $\mathrm{TJ}^{k-1}(g) \in \mathcal{N}$. We claim that g is a Δ^0_k-leftmost path of T. Indeed, if a path $h \in [T]$ satisfies $h \leq_{\mathrm{T}} \mathrm{TJ}^{k-1}(g)$, then $h \in \mathcal{N}$ and hence $h \in [S]$. Then, g is lexicographically smaller than h because g is the leftmost path of S. ⊣

THEOREM 3.28. *For $k \in \omega$ such that $k > 1$, the following assertions are equivalent over* RCA_0:

1. $\Sigma^0_k\mathsf{LPP}$,
2. $\mathrm{R}\Delta^0_k\beta\text{-}\mathsf{RFN}$,
3. *the ω-model reflection of* $\Pi^1_{k+1}\text{-}\mathsf{TI}$,
4. *the ω-model reflection of* $\Delta^0_k(\Pi^1_1)\text{-}\mathsf{TI}$.

PROOF. We note that each assertion includes ACA_0. Therefore, we may assume that we work in ACA_0.

The implication (1) → (2) is Lemma 3.17. The implication (2) → (3) is immediate form Lemma 3.20. The implication (3) → (4) follows from Lemma 3.23. The implication (4) → (1) is Lemma 3.27. ⊣

REMARK 3.29. We note that the implications (2) → (3) → (4) → (1) still hold for $n = 1$. We do not know either (1) → (2) holds for $n = 1$.

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REMARK 3.30. In [8], Towsner proved that

1. $\mathsf{RCA}_0 + \Sigma^0_{k+2}\mathsf{LPP}$ implies the existence of an ω-model of $\Pi^0_k(\Pi^1_1)$-TI (Theorem 4.3),
2. $\mathsf{RCA}_0 + \Pi^0_{k+2}(\Pi^1_1)$-$\mathsf{TI}$ proves $\Sigma^0_k\mathsf{LPP}$ (Theorem 6.1).

The previous theorem can be seen as a refinement of these results.

At the end of this section, we see some applications of the previous theorem.

COROLLARY 3.31. *Let $k \in \omega$ such that $k > 1$. Then $\Sigma^0_k\mathsf{LPP}$ is equivalent to $\mathsf{RFN}(\Pi^1_{k+2}\text{-}\mathsf{RFN}_0)$ over RCA_0.*

PROOF. It is immediate from the fact that Π^1_{k+1}-TI is equivalent to the Π^1_{k+2} reflection over RCA_0. ⊣

COROLLARY 3.32. *Let $n \in \omega$ such that $n > 0$. Then, $\Sigma^0_{n+1}\mathsf{LPP}$ proves the ω-model reflection of $\Sigma^0_n\mathsf{LPP}$ over RCA_0. Hence, $\Sigma^0_{n+1}\mathsf{LPP}$ is strictly stronger than $\Sigma^0_n\mathsf{LPP}$.*

PROOF. Since Π^1_{n+2}-RFN_0 proves a Π^1_{n+2} axiomatizable theory Π^1_{n+1}-RFN_0, it proves the ω-model reflection of Π^1_{n+1}-RFN_0. Hence, we have the following implications:

$$\Sigma^0_{n+1}\mathsf{LPP} \to \mathsf{RFN}(\Pi^1_{n+2}\text{-}\mathsf{RFN}_0) \to \mathsf{RFN}(\mathsf{RFN}(\Pi^1_{n+1}\text{-}\mathsf{RFN}_0)) \to \mathsf{RFN}(\Sigma^0_n\mathsf{LPP}). \quad \dashv$$

REMARK 3.33. Our $\mathsf{RFN}(\Pi^1_{n+2}\mathsf{RFN}_0)$ is the same as Probst's $\mathsf{p_1p_{n+2}}(\mathsf{ACA}_0)$. He studied the proof theoretic strength of $\mathsf{p_1p_{n+2}}(\mathsf{ACA}_0)$ in his habilitation thesis [4].

§4. Weihrauch degrees. In [7], the author and Yokoyama introduced a Weihrauch problem corresponding to $\Sigma^0_n\mathsf{LPP}$ and studied the computability-theoretic strength of it. In this section, we refine the results in [7]. We prove that $\Sigma^0_{n+1}\mathsf{LPP}$ is strictly stronger than $\Sigma^0_n\mathsf{LPP}$ in the sense of Weihrauch reducibility.

DEFINITION 4.1. A Weihrauch problem is a partial multivalued function on the Cantor space 2^ω.

We sometimes identify a Weihrauch problem P and a partial function P' from 2^ω to $\mathcal{P}(2^\omega)$ defined by $\mathsf{P}'(X) = \{Y \subseteq \omega : Y \text{ is an output of } \mathsf{P} \text{ at } X\}$. In this sense, we write $Y \in \mathsf{P}(X)$ to mean Y is an output of P at X.

DEFINITION 4.2. Let P and Q be Weihrauch problems. We say P is Weihrauch reducible to Q (written $\mathsf{P} \leq_{\mathrm{W}} \mathsf{Q}$) if there are (partial) Turing functionals Φ and Ψ such that

$$\forall X \in \mathrm{dom}(\mathsf{P})(\Phi(X)\downarrow \in \mathrm{dom}(\mathsf{Q}) \wedge \forall Y \in \mathsf{Q}(\Phi(X))(\Psi(X, Y)\downarrow \in \mathsf{P}(X))).$$

We write $\mathsf{P} <_{\mathrm{W}} \mathsf{Q}$ to mean $\mathsf{P} \leq_{\mathrm{W}} \mathsf{Q}$ but $\mathsf{Q} \not\leq_{\mathrm{W}} \mathsf{P}$.

We say P and Q are Weihrauch equivalent (written $\mathsf{P} \equiv_{\mathrm{W}} \mathsf{Q}$) if both of $\mathsf{P} \leq_{\mathrm{W}} \mathsf{Q}$ and $\mathsf{Q} \leq_{\mathrm{W}} \mathsf{P}$ hold.

DEFINITION 4.3. We define Weihrauch problems $\Sigma^0_n\mathsf{LPP}$, $\Delta^0_n\mathsf{LPP}$ and $\Delta^0_n\overline{\mathsf{LPP}}$ as follows. [l]$\Sigma^0_n\mathsf{LPP}$

Input An ill-founded tree $T \subseteq \omega^{<\omega}$ and a path f of T.

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Output A path g of T such that g is leftmost in $\Sigma^0_n(T \oplus f \oplus g)$.

[l]$\Delta^0_n\mathsf{LPP}$

Input An ill-founded tree $T \subseteq \omega^{<\omega}$ and a path f of T.

Output A path g of T such that g is leftmost in $\Delta^0_n(T \oplus f \oplus g)$.

[l]$\Delta^0_n\overline{\mathsf{LPP}}$

Input An ill-founded tree $T \subseteq \omega^{<\omega}$ and a path f of T.

Output A path g of T such that g is leftmost in $\Delta^0_n(g)$.

LEMMA 4.4. $\Sigma^0_n\mathsf{LPP} \equiv_{\mathrm{W}} \Delta^0_n\mathsf{LPP} \equiv_{\mathrm{W}} \Delta^0_n\overline{\mathsf{LPP}}$.

PROOF. As a path is Σ^0_n-definable if and only if it is Δ^0_n-definable, $\Sigma^0_n\mathsf{LPP}$ and $\Delta^0_n\mathsf{LPP}$ are the same. For the equivalence of $\Delta^0_n\mathsf{LPP}$ and $\Delta^0_n\overline{\mathsf{LPP}}$, see [7, Lemma 6.5] ⊣

As the same proof of Lemma 3.17, the following problem is Weihrauch reducible to $\Sigma^0_n\mathsf{LPP}$ for $n > 1$. [l]$\mathrm{R}\Delta^0_n\beta\mathsf{RFN}$

Input Any set X.

Output A $\Delta^0_n\beta$-model $\mathcal{M}$ and computable functions $f_{\mathrm{TJ}}, f_{\oplus}, f_{\Sigma^1_1}$ such that

$$X \in \mathcal{M}, \qquad (f_{\mathrm{TJ}}, f_{\oplus}) \Vdash_{\mathcal{M}} \mathsf{ACA}_0, \qquad f_{\Sigma^1_1} \Vdash_{\mathcal{M}} \exists X \Pi^0_2.$$

We see that the converse reduction also holds.

THEOREM 4.5. *For $n > 1$, $\Sigma^0_n\mathsf{LPP} \equiv_{\mathrm{W}} \mathrm{R}\Delta^0_n\beta\mathsf{RFN}$.*

PROOF. It is enough to show that $\Delta^0_n\overline{\mathsf{LPP}} \leq_{\mathrm{W}} \mathrm{R}\Delta^0_n\beta\mathsf{RFN}$ holds. Let T be a tree and f be its path. Let $\mathcal{M}, f_{\mathrm{TJ}}, f_{\oplus}, f_{\Sigma^1_1}$ be such that

$$T, f \in \mathcal{M}, \qquad (f_{\mathrm{TJ}}, f_{\oplus}) \Vdash_{\mathcal{M}} \mathsf{ACA}_0, \qquad f_{\Sigma^1_1} \Vdash_{\mathcal{M}} \exists X \Pi^0_2.$$

Define a subtree S of T as the set of σ in T such that $\mathcal{M}$ believes σ is extended to a path. That is, $S = \{\sigma \in T : \mathcal{M} \models \exists g \in [T](\sigma \prec g)\}$. Then, S is $\mathcal{M}$-computable by Lemma 3.18. Moreover, it is easy to see that there is a uniform procedure to make S from $\mathcal{M}, f_{\mathrm{TJ}}, f_{\oplus}, f_{\Sigma^1_1}$. We now have

- S is ill-founded because $f \in [S]$,
- S is pruned because of the definition, and
- $S = \{\sigma \in T : \exists g \in [T] \cap \Delta^0_n(\mathcal{M})(\sigma \prec g)\}$ because $\mathcal{M}$ is $\Delta^0_n\beta$-model.

Let $g \leq_{\mathrm{T}} S$ be the leftmost path of S. Then, g is leftmost in $\Delta^0_n(\mathcal{M})$. Since g is $\mathcal{M}$-computable, g is leftmost in $\Delta^0_n(g)$. This completes the proof. ⊣

We then show the separation of $\Sigma^0_{n+1}\mathsf{LPP}$ and $\Sigma^0_n\mathsf{LPP}$. For the separation, we use the ω-model reflection of a Weihrauch problem introduced in [7].

DEFINITION 4.6. Let P be a Weihrauch problem. We define the ω-model reflection of P as the following problem.

Input Any set X.

Output A sequence $\mathcal{M}$ and functions $f_{\mathrm{TJ}}, f_{\oplus}$ such that

$$X \in \mathcal{M} \,\&\, (f_{\mathrm{TJ}}, f_{\oplus}) \Vdash_{\mathcal{M}} \mathsf{ACA}_0 \,\&\,$$
$$\forall Y \in \mathcal{M} \cap \mathrm{dom}(\mathsf{P}) \exists Z \in \mathcal{M}(Z \in \mathsf{P}(X))$$

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We note that if there are arithmetical formulas θ, η such that $\mathrm{dom}(\mathsf{P}) = \{X : \theta(X)\}$ and $\mathsf{P}(X) = \{Y : \eta(X, Y)\}$, then the condition $\forall Y \in \mathcal{M} \cap \mathrm{dom}(\mathsf{P}) \exists Z \in \mathcal{M}(Z \in \mathsf{P}(X))$ is equivalent to $\mathcal{M} \models \mathsf{P}$. It is also proved that in that case, the ω-model reflection of the problem is strictly stronger than the problem. For details, see [7, Section 4]. We note that the input and output conditions of $\Sigma^0_n\mathsf{LPP}$ can be written by arithmetical formulas.

LEMMA 4.7 (ACA_0). *Let $\mathcal{M}$ be a $\Delta^0_{n+1}\beta$-model and $f_{\mathrm{TJ}}, f_{\oplus}, f_{\Sigma^1_1}$ be computable functions such that $(f_{\mathrm{TJ}}, f_{\oplus}) \Vdash_{\mathcal{M}} \mathsf{ACA}_0$ and $f_{\Sigma^1_1} \Vdash_{\mathcal{M}} \exists\Pi^0_2$. Then, $\mathcal{M}$ is a model of $\Sigma^0_n\mathsf{LPP}$.*

PROOF. Let $T \in \mathcal{M}$ be a tree such that $\mathcal{M} \models [T] \neq \varnothing$. Then, as in the proof of Theorem 4.5, there is an $\mathcal{M}$-computable function g such that $\forall f \leq_{\mathrm{T}} \mathrm{TJ}^n(g)(f \in [T] \to g \leq_l f)$. In particular, we have $\forall f \leq_{\mathrm{T}} \mathrm{TJ}^{n-1}(g)(f \in [T] \to g \leq_l f)$. Thus, we have

$$(*)\ \exists(g_0 \oplus g_1 \oplus \cdots \oplus g_n) \leq_{\mathrm{T}} \mathrm{TJ}^n(\mathcal{M})\Big[\bigwedge_{i<n} g_i = \mathrm{TJ}(g_{i-1}) \wedge \forall f \leq_{\mathrm{T}} g_{n-1}(f \in [T] \to g_0 \leq_{\mathrm{T}} f\Big].$$

by taking $g_i = \mathrm{TJ}^i(g)$. We note that the condition in the square bracket is Π^0_2 for $(g_0 \oplus g_1 \oplus \cdots \oplus g_n)$ because $g_n = \mathrm{TJ}(g_{n-1})$. Therefore, the above $(*)$ is of the form $\exists X \leq_{\mathrm{T}} \mathrm{TJ}^n(\mathcal{M})\theta(X)$ for a Π^0_2 formula θ. Since $\mathcal{M}$ is a $\Delta^0_{n+1}\beta$-model, we have

$$\mathcal{M} \models \exists g_0 \forall f \leq_{\mathrm{T}} \mathrm{TJ}^{n-1}(g_0)(f \in [T] \to g_0 \leq_{\mathrm{T}} f).$$

Thus, $\mathcal{M}$ believes that T has a Δ^0_n leftmost path. This completes the proof. ⊣

THEOREM 4.8. *For $n > 0$, $\Sigma^0_n\mathsf{LPP} <_{\mathrm{W}} \Sigma^0_{n+1}\mathsf{LPP}$.*

PROOF. By Theorem 4.5 and Lemma 4.7, the ω-model reflection of $\Sigma^0_n\mathsf{LPP}$ is Weihrauch reducible to $\Sigma^0_{n+1}\mathsf{LPP}$. ⊣

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