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A coding theoretic study of MLL proof nets

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In this paper we propose a novel approach for analysing proof nets of Multiplicative Linear Logic (MLL) using coding theory. We define families of proof structures called PS-families and introduce a metric space for each family. In each family:

(1) an MLL proof net is a true code element; and
(2) a proof structure that is not an MLL proof net is a false (or corrupted) code element.

The definition of our metrics elegantly reflects the duality of the multiplicative connectives. We show that in our framework one-error-detection is always possible but one-error-correction is always impossible. We also demonstrate the importance of our main result by presenting two proof-net enumeration algorithms for a given PS-family: the first searches proof nets naively and exhaustively without help from our main result, while the second uses our main result to carry out an intelligent search. In some cases, the first algorithm visits proof structures exponentially, while the second does so only polynomially.

1. Introduction

Linear Logic (Girard 1987) has been studied extensively for more than twenty years and may be regarded as providing some fundamental infrastructure for the study of linear phenomena in computer programming. For example, we showed in Matsuoka (2007) that a separation result holds in a linear system without deletion and duplication. Our concern in the current paper is the multiplicative fragment of Linear Logic without multiplicative constants (MLL for short), which forms the core part of Linear Logic.

On the other hand, coding theory is useful for real-world applications. Basically, coding theory is used to study ways of detecting and/or correcting data that may be true or false.

In this paper we propose a novel approach for analysing proof nets of Multiplicative Linear Logic (MLL) using coding theory. We define families of proof structures called PS-families and introduce a metric space for each family. In each family:

(1) An MLL proof net is a true code element, which is usually called a *codeword* in the literature of coding theory.
(2) A proof structure that is not an MLL proof net is a false (or corrupted) code element.

Figure 1 shows four MLL proof nets in a standard notation taken from Girard (1987). In our framework, proof nets (a) and (b) belong to the same family because if we forget $\otimes$ and $\parr$ symbols, they are the same (in fact, they are equal even without forgetting the $\otimes$ and $\parr$ symbols – we will discuss this later). However, proof net (c) does not belong to the

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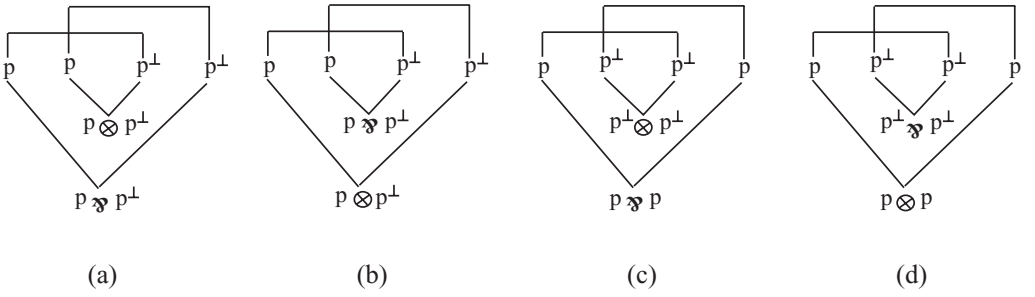


Fig. 1. An explanatory example

same family because when we forget its $\otimes$ and $\parr$ symbols, we cannot identify it with (a) or (b) because of the mismatch between the literals p and $p^{\perp}$. On the other hand, proof nets (c) and (d) do belong to the same family. Also, the distance between (c) and (d) is two because the logical connectives are different at two locations ($p@p$ and $p^{\perp}@p^{\perp}$), while the distance between (a) and (b) is zero because they are the same. The definition of our metrics provides an elegant reflection of the duality of the multiplicative connectives.

The main technical achievement of this paper is Theorem 3, which says that in our framework one-error-*detection* is always possible but one-error-*correction* is always impossible. Our proof of the theorem is interesting in the sense that a proof-theoretic property is proved using a graph-theoretic argument. We also demonstrate the importance of our main result by presenting two proof-net enumeration algorithms for a given PS-family: the first searches proof nets naively and exhaustively without using our main result; the second uses our main result to carry out an intelligent search. In some cases, the former algorithm visits proof structures exponentially, while the latter does so only polynomially.

The structure of the paper

Section 2 introduces basic properties of MLL proof nets. In particular, we define MLL proof nets and describe a sequentialisation theorem for them. We also introduce the notion of empires, which we will need when we come to prove the main theorems. Section 3 introduces the notion of PS-families and distances on them, and show that they are metric spaces. We then state some other basic properties of PS-families and our main theorems – most of the details of the proofs of the main theorems are given in appendices. We also provide a non-trivial example (Example 1). Finally, we describe and compare the two proof-net enumeration algorithms.

2. The MLL system

2.1. *The basic theory of MLL proof nets*

In this section, we describe multiplicative proof nets, which we also call *MLL proof nets*, or simply, *proof nets*. We will begin by defining MLL formulas. In this paper, we will only

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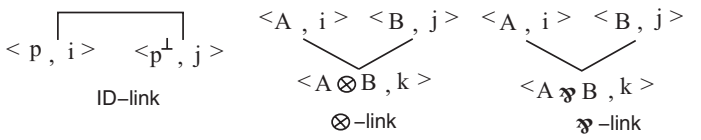


Fig. 2. MLL links.

consider MLL formulas with a single propositional variable p since this restriction has no significant effect on our main result. For the same reason, we will only consider ID-links with literal conclusions. Moreover, we will not consider cut-links or cut-elimination since our main result does not affect them.

Definition 1 (Literals). A literal is p or $p^{\perp}$. The positive literal is p and the negative literal is $p^{\perp}$.

Definition 2 (MLL formulas). An MLL formula (or simply a formula) F is a literal, or $F_1 \otimes F_2$ or $F_1 \parr F_2$, where F_1 and F_2 are MLL formulas – in the latter cases, F is called a $\otimes$-formula and a $\parr$-formula, respectively. We denote the set of all the MLL formulas by MLLFml.

Definition 3 (Negations of MLL formulas). Let F be an MLL formula. The negation $F^{\perp}$ of F is defined as follows according to the form of F:

— If F is p, then $F^{\perp} \equiv_{\text{def}} p^{\perp}$.
— If F is $p^{\perp}$, then $F^{\perp} \equiv_{\text{def}} p$.
— If F is $F_1 \otimes F_2$, then $F^{\perp} \equiv_{\text{def}} F_1^{\perp} \parr F_2^{\perp}$.
— If F is $F_1 \parr F_2$, then $F^{\perp} \equiv_{\text{def}} F_1^{\perp} \otimes F_2^{\perp}$.

So $F^{\perp}$ is actually an MLL formula too.

Definition 4 (Indexed MLL formulas). An indexed MLL formula is a pair $\langle F, i \rangle$, where F is an MLL formula and i is a natural number.

Figure 2 shows the links we use in this paper, where:

(1) in an ID-link, $\langle A, i \rangle$ and $\langle A^{\perp}, j \rangle$ are called the conclusions of the link; and
(2) in a $\otimes$-link or a $\parr$-link, $\langle A, i \rangle$ is called the left premise, $\langle B, j \rangle$ is called the right premise and $\langle A \otimes B, k \rangle$ or $\langle A \parr B, k \rangle$, respectively, is called the conclusion of the link.

We call links that are not ID-links *multiplicative links*.

Definition 5 (MLL proof structures). Let $\mathbb{F}$ be a finite set of MLL formula occurrences, that is, a finite set of indexed MLL formulas, and $\mathbb{L}$ be a finite set of MLL link occurrences such that for each $L \in \mathbb{L}$, the conclusions and the premises of L belong to $\mathbb{F}$. The pair $\Theta = \langle \mathbb{F}, \mathbb{L} \rangle$ is an MLL proof structure (or simply, a proof structure) if Θ satisfies the following conditions:

(1) For any $\langle F_0, i \rangle$ and $\langle F_0', j \rangle$ in $\mathbb{F}$, if $i = j$, then $F_0 = F_0'$ (that is, each element in $\mathbb{F}$ has a different index number).

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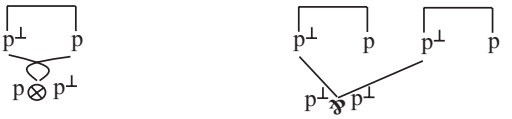


Fig. 3. Two examples of MLL proof structures.

(2) For each $F \in \mathbb{F}$ and each $L \in \mathbb{L}$, if F is a premise of L, then F is not a premise of any other link $L' \in \mathbb{L}$.
(3) For each $F \in \mathbb{F}$, there is a unique $L \in \mathbb{L}$ such that F is a conclusion of L.

Remark 1. For convenience, when we discuss proof structures or proof nets in the rest of the paper, we will omit the indices in cases where they are unimportant. Moreover, when we draw proof structures or proof nets, we will also omit indices since in such drawings the positions play the role of indices.

We say that in $\Theta = \langle \mathbb{F}, \mathbb{L} \rangle$, a formula occurrence $F \in \mathbb{F}$ is a conclusion of Θ if F is not a premise of L for any $L \in \mathbb{L}$.

It is well known that a proof structure does not necessarily correspond to a sequent calculus proof. For example, the two MLL proof structures in Figure 3 do not have any corresponding sequent calculus proofs. The following definition provides a judgement as to whether a proof structure corresponds to a sequent calculus proof or not.

Definition 6 (Sequentialisability). An MLL proof structure $\Theta = \langle \mathbb{F}, \mathbb{L} \rangle$ is sequentialisable if any of the following conditions holds:

(1) $\mathbb{L} = \{L\}$ and L is an ID-link.
(2) There is a $\parr$-link $L \in \mathbb{L}$ such that the conclusion $A \parr B$ of L is a conclusion of Θ and $\langle \mathbb{F} - \{A \parr B\}, \mathbb{L} - \{L\} \rangle$ is sequentialisable.
(3) There is a $\otimes$-link $L \in \mathbb{L}$ and there are two subsets $\mathbb{F}_1$ and $\mathbb{F}_2$ of $\mathbb{F}$ and two subsets $\mathbb{L}_1$ and $\mathbb{L}_2$ of $\mathbb{L}$ such that:
   (a) the conclusion $A \otimes B$ of L is a conclusion of Θ,
   (b) $\mathbb{F} = \mathbb{F}_1 \uplus \mathbb{F}_2 \uplus \{A \otimes B\}$,
   (c) $\mathbb{L} = \mathbb{L}_1 \uplus \mathbb{L}_2 \uplus \{L\}$, and
   (d) $\langle \mathbb{F}_1, \mathbb{L}_1 \rangle$ or $\langle \mathbb{F}_2, \mathbb{L}_2 \rangle$ is an MLL proof structure and sequentialisable,

   where $\uplus$ denotes the disjoint union operator.

Definition 7 (MLL proof nets). An MLL proof structure Θ is an MLL proof net if Θ is sequentialisable.

We will now, following Girard (1996), give a graph-theoretic characterisation of MLL proof nets. The characterisation, called the *sequentialisation theorem*, was first proved in Girard (1987) and then improved in Danos and Regnier (1989). In order to characterise the MLL proof nets among the MLL proof structures, we will introduce *Danos–Regnier graphs* (DR-graphs, for short). Let Θ be an MLL proof structure. We assume that we are given a function S from the set of occurrences of $\parr$-links in Θ to $\{0, 1\}$. Such a function

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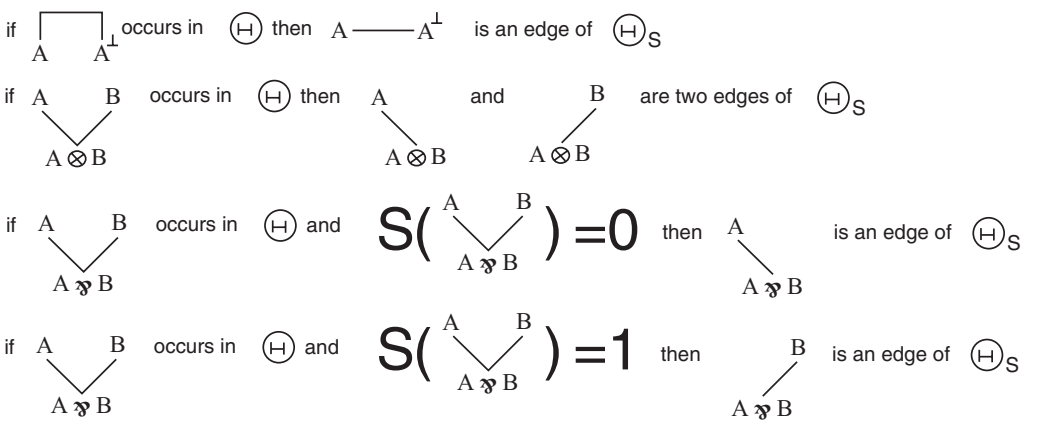


Fig. 4. The rules for generating the edges of a Danos–Regnier graph Θ_S.

is called a *DR-switching* for Θ. Then the Danos–Regnier graph Θ_S for Θ and S is an undirected graph such that:

(1) the nodes are all the formula occurrences in Θ; and
(2) the edges are generated by the rules of Figure 4.

We will also use the alternative notation $S(\Theta)$ for the Danos–Regnier graph Θ_S.

Theorem 1 (Sequentialisation Theorem). An MLL proof structure Θ is an MLL proof net if and only if for each switching function S for Θ, the Danos–Regnier graph Θ_S is acyclic and connected.

2.2. *Empires*

In this section we introduce the notion of *empires* from Girard (2006). We will need this notion when we establish our main result.

Given a proof structure $\Theta = \langle \mathbb{F}_\Theta, \mathbb{L}_\Theta \rangle$, we will use the notation $\mathrm{fml}(\Theta) \equiv_{\mathrm{def}} \mathbb{F}_\Theta$.

Definition 8 (Empires). To define the empire of a formula A in a proof net $\Theta = \langle \mathbb{F}, \mathbb{L} \rangle$ (denoted by $e_\Theta(A)$), we first let S be a DR-switching for Θ and define an undirected maximally connected graph $(\Theta_S)^A$ (or simply ${\Theta_S}^A$) as follows:

(1) If there is a link $L \in \mathbb{L}$ such that A is a premise of L, then $(\Theta_S)^A$ is the maximally connected graph including A obtained from Θ_S by deleting e, where e is the edge between A and the conclusion of L in Θ_S.
(2) Otherwise, $(\Theta_S)^A = \Theta_S$.

Then we define

$$e_\Theta(A) \equiv_{\mathrm{def}} \bigcap_{S \text{ is a DR-switching for } \Theta} \mathrm{fml}({\Theta_S}^A).$$

A DR-switching S for Θ is principal for A if $e_\Theta(A) = \mathrm{fml}({\Theta_S}^A)$. It is well known that any formula A in Θ has a principal switching for itself (Girard 1996; 2006).

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From the definition, it is obvious that $A \in e_\Theta(A)$. Although the empire $e_\Theta(A)$ is defined as a set of formula occurrences, by considering the set $\mathbb{L}_{e_\Theta(A)}$of links whose conclusions and premises are all included in $e_\Theta(A)$, the empire $e_\Theta(A)$ can also be considered as the pair $\langle e_\Theta(A), \mathbb{L}_{e_\Theta(A)}\rangle$. In the following, we identify $e_\Theta(A)$ with this pair. It is well known that $e_\Theta(A)$ is an MLL proof net and the greatest sub-proof net of Θ having A as a conclusion (see, for example, Bellin and van de Wiele (1995)).

In the rest of this section we will just list those results for empires that we will need in the following – the proofs can be found in Girard (1987), Bellin and van de Wiele (1995), Girard (1996) and Girard (2006).

Proposition 1. Let $B \in e_\Theta(A)$ and $L \in \mathbb{L}_{e_\Theta(A)}$ be such that the conclusion of L is B. Then if B' is a premise or a conclusion of L, then $B' \in e_\Theta(A)$.

Proposition 2. If $B \notin e_\Theta(A)$ and $A \notin e_\Theta(B)$, then $e_\Theta(A) \cap e_\Theta(B) = \varnothing$.

Proposition 3. If $B \notin e_\Theta(A)$ and $A \in e_\Theta(B)$, then A is not a conclusion of $e_\Theta(B)$ and $e_\Theta(A) \subsetneq e_\Theta(B)$.

Proposition 4. Let Θ be an MLL proof net including the $\otimes$-link

$$L : \frac{A \quad B}{A \otimes B}.$$

Then $e_\Theta(A) \cap e_\Theta(B) = \varnothing$.

Proposition 5. Let Θ be an MLL proof net including the $\parr$-link

$$L : \frac{A \quad B}{A \parr B}.$$

Then $e_\Theta(A) = e_\Theta(B)$.

Lemma 1 (Splitting Lemma). Let Θ be an MLL proof net whose conclusions do not include any $\parr$-formulas. Then there is a conclusion

$$L : \frac{A \quad B}{A \otimes B}$$

in Θ such that

$$\mathrm{fml}(\Theta) = \{A \otimes B\} \uplus e_\Theta(A) \uplus e_\Theta(B).$$

3. Our framework, the main result and its application

3.1. *Our framework*

We will begin by defining some families of proof-structures called PS-families. Informally, saying that two proof structures Θ_1 and Θ_2 belong to the same family means that Θ_2 can be obtained from Θ_1 by replacing several $\otimes$-links by $\parr$-links, or *vice versa*. We will define such families using graph isomorphisms on directed graphs in a mathematically rigorous way. We will also define our equality on proof structures and our metric over proof structures belonging to the same PS-family in a similar manner. It might be thought

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that the definitions given in this subsection are rather cumbersome, but they are required to cover some subtle points, which is why we insist on a rigorous approach – we will discuss this further at the end of the subsection.

A function $\mathrm{top}_{\otimes\parr} : \mathrm{MLLFml} \to \{p, p^\perp, \otimes, \parr\}$ is defined as follows:

$$\begin{aligned}\mathrm{top}_{\otimes\parr}(p) &= p\\ \mathrm{top}_{\otimes\parr}(p^\perp) &= p^\perp\\ \mathrm{top}_{\otimes\parr}(A \otimes B) &= \otimes\\ \mathrm{top}_{\otimes\parr}(A \parr B) &= \parr.\end{aligned}$$

Definition 9 (Labelled directed graphs). Let $\mathbb{L}_V$ and $\mathbb{L}_E$ be sets. A labelled directed graph with labels $\mathbb{L}_E$ (or $\mathbb{L}_V$ and $\mathbb{L}_E$) is a tuple $\langle V, E, \ell_E : E \to \mathbb{L}_E\rangle$ (or $\langle V, E, \ell_V : V \to \mathbb{L}_V, \ell_E : E \to \mathbb{L}_E\rangle$, respectively) where V and E are sets with two functions $\mathrm{src} : E \to V$ and $\mathrm{tgt} : E \to V$.

In the following, we will assume $\mathbb{L}_V = \{p, p^\perp, \otimes, \parr\}$ and $\mathbb{L}_E = \{\mathbf{L}, \mathbf{R}, \mathbf{ID}\}$.

We will now define two translations from proof structures to labelled directed graphs: the first forgets the information of logical connectives and the second retains the information about the top connectives of formulas.

Definition 10 (Labelled directed graphs induced by proof structures). Let $\Theta = \langle \mathbb{F}, \mathbb{L}\rangle$ be a proof structure and $\rho : \mathrm{MLLFml} \to \mathbb{L}_V$. A labelled directed graph:

$$\begin{aligned}G(\Theta) &= \langle V, E, \ell_E : E \to \{\mathbf{L}, \mathbf{R}, \mathbf{ID}\}\rangle\\ G^{\otimes\parr}(\Theta) &= \langle V, E, \ell_V : V \to \mathbb{L}_V, \ell_E : E \to \{\mathbf{L}, \mathbf{R}, \mathbf{ID}\}\rangle\end{aligned}$$

is defined from Θ as follows:

(1) We have:

$$\begin{aligned}V &= \{i \mid \langle A, i\rangle \in \mathbb{F}\}\\ \ell_V &= \{\langle i, \mathrm{top}_{\otimes\parr}(A)\rangle \mid \langle A, i\rangle \in \mathbb{F}\}.\end{aligned}$$

Since each formula occurrence has a unique index in Θ, it is easy to see that V is bijective to $\mathbb{F}$.

(2) E and ℓ_E is the least set satisfying the following conditions:

— If $L \in \mathbb{L}$ is an ID-link occurrence with conclusions $\langle p, i\rangle$ and $\langle p^\perp, j\rangle$, then there is an edge $e \in E$ such that

$$\begin{aligned}\mathrm{src}(e) &= i\\ \mathrm{tgt}(e) &= j\\ \ell_E(e) &= \mathbf{ID}.\end{aligned}$$

— If $L \in \mathbb{L}$ is a $\otimes$-link or $\parr$-link occurrence with the form

$$\frac{\langle A, i\rangle \quad \langle B, j\rangle}{\langle A \otimes B, k\rangle}$$

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or

$$\frac{\langle A, i\rangle \quad \langle B, j\rangle}{\langle A \parr B, k\rangle},$$

respectively, then there are two edges $e_1 \in E$ and $e_2 \in E$ such that

$$\begin{aligned} \mathrm{src}(e_1) &= i \\ \mathrm{tgt}(e_1) &= k \\ \mathrm{src}(e_2) &= j \\ \mathrm{tgt}(e_2) &= k \\ \ell_E(e_1) &= \mathbf{L} \\ \ell_E(e_2) &= \mathbf{R}. \end{aligned}$$

The next definition is a slight extension of the standard definition of graph isomorphisms.

Definition 11 (Graph isomorphisms on labelled directed graphs). Let G_i be a labelled directed graph $\langle V_i, E_i, \ell_{V_i}, \ell_{E_i}\rangle$ or $\langle V_i, E_i, \ell_{E_i}\rangle$ for $i = 1, 2$. Then a graph homomorphism from G_1 to G_2 is a pair $\langle h_V : V_1 \to V_2, h_E : E_1 \to E_2\rangle$ satisfying the following conditions:

(1) For any $e \in E_1$, we have $h_V(\mathrm{src}(e)) = \mathrm{src}(h_E(e))$ and $h_V(\mathrm{tgt}(e)) = \mathrm{tgt}(h_E(e))$.
(2) If ℓ_{V_1} and ℓ_{V_2} are specified, then for any $v \in V_1$, we have $\ell_{V_1}(v) = \ell_{V_2}(h_V(v))$.
(3) For any $e \in E_1$, we have $\ell_{E_1}(e) = \ell_{E_2}(h_E(e))$.

The graph homomorphism $\langle h_V, h_E\rangle$ is a graph isomorphism if $h_V : V_1 \to V_2$ and $h_E : E_1 \to E_2$ are both bijections (in which case, we write $\langle h_V, h_E\rangle : G_1 \simeq G_2$).

Definition 12 (PS-families). Let Θ_1 and Θ_2 be proof structures. Then $\Theta_1 \sim \Theta_2$ if there is a graph isomorphism $\langle h_V : V_1 \to V_2, h_E : E_1 \to E_2\rangle$ from $G(\Theta_1) = \langle V_1, E_1, \ell_{E_1}\rangle$ to $G(\Theta_2) = \langle V_2, E_2, \ell_{E_2}\rangle$. It is obvious that $\sim$ is an equivalence relation. Therefore, for a given proof structure Θ, we can define the equivalence class $[\Theta]$ such that $\Theta' \in [\Theta]$ if and only if $\Theta \sim \Theta'$. Then we say $[\Theta]$ is a *PS-family* of Θ. We also say that Θ belongs to the PS-family $[\Theta]$.

Remark 2. We define a PS-family as an equivalence class generated by the relation $\sim$. Of course, an alternative would be to define a PS-family as an MLL proof structure in which all the occurrences of multiplicative links are of the form

$$\frac{A \quad B}{A @ B}$$

instead of $\otimes$- and $\parr$-links, where @ is a new symbol, but this is really just a matter of taste.

We denote PS-families by $\mathscr{F}, \mathscr{F}_0, \mathscr{F}_1$, and so on. Given a PS-family $\mathscr{F}$, we will now introduce a metric $d_{\mathscr{F}}$ on $\mathscr{F}$.

Definition 13. Let $\mathscr{F}$ be a PS-family. We assume that two MLL proof structures Θ_1 and Θ_2 belong to $\mathscr{F}$. So by definition we have at least one graph isomorphism $\langle h_V, h_E\rangle$ from

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$G(\Theta_1)$ to $G(\Theta_2)$. Let

$$G^{\otimes\parr}(\Theta_1) = \langle V_1, E_1, \ell_{V_1}, \ell_{E_1}\rangle$$
$$G^{\otimes\parr}(\Theta_2) = \langle V_2, E_2, \ell_{V_2}, \ell_{E_2}\rangle.$$

Then $d_{\mathscr{F}}(\Theta_1, \Theta_2) \in \mathbb{N}$ is defined as follows:

$$d_{\mathscr{F}}(\Theta_1, \Theta_2) = \min\Big\{ |\ \{v_1 \in V_1 \mid \ell_{V_2}(h_V(v_1)) \neq \ell_{V_1}(v_1)\}\ | \ \Big|\ \langle h_V, h_E\rangle : G(\Theta_1) \simeq G(\Theta_2)\Big\}.$$

Before proving that $\langle \mathscr{F}, d_{\mathscr{F}}\rangle$ is a metric space, we must define an equality between two MLL proof structures since the statement concerns the equality on $\mathscr{F}$.

Definition 14 (Equality on MLL proof structures). Let Θ_1 and Θ_2 be proof structures, and let

$$G^{\otimes\parr}(\Theta_1) = \langle V_1, E_1, \ell_{V_1}, \ell_{E_1}\rangle$$
$$G^{\otimes\parr}(\Theta_2) = \langle V_2, E_2, \ell_{V_2}, \ell_{E_2}\rangle.$$

Then $\Theta_1 = \Theta_2$ if there is a graph isomorphism $\langle h_V : V_1 \to V_2, h_E : E_1 \to E_2\rangle$ from $G^{\otimes\parr}(\Theta_1)$ to $G^{\otimes\parr}(\Theta_2)$.

It is obvious that $=$ is an equivalence relation.

Proposition 6. The pair $\langle \mathscr{F}, d_{\mathscr{F}}\rangle$ is a metric space under the equality $=$ of Definition 14.

Proof. The non-negativity and symmetry of $d_{\mathscr{F}}$ are obvious, as is the formula

$$\Theta_1 = \Theta_2 \Rightarrow d_{\mathscr{F}}(\Theta_1, \Theta_2) = 0.$$

We will now prove that

$$d_{\mathscr{F}}(\Theta_1, \Theta_2) = 0 \Rightarrow \Theta_1 = \Theta_2.$$

Let

$$G^{\otimes\parr}(\Theta_1) = \langle V_1, E_1, \ell_{V_1}, \ell_{E_1}\rangle$$
$$G^{\otimes\parr}(\Theta_2) = \langle V_2, E_2, \ell_{V_2}, \ell_{E_2}\rangle.$$

Since Θ_1 and Θ_2 belong to the same PS-family $\mathscr{F}$, there is a graph isomorphism $\langle h_V : V_1 \to V_2, h_E : E_1 \to E_2\rangle$ from $G(\Theta_1)$ to $G(\Theta_2)$. This means that both $h_V : V_1 \to V_2$ and $h_E : E_1 \to E_2$ are bijections and $\langle h_V, h_E\rangle$ satisfies Conditions (1) and (3) of Definition 11. On the other hand, since $d_{\mathscr{F}}(\Theta_1, \Theta_2) = 0$, we can find a graph isomorphism $\langle h_V, h_E\rangle : G(\Theta_1) \simeq G(\Theta_2)$ with the additional property that for any $v \in V_1$, we have $\ell_{V_1}(v) = \ell_{V_2}(h_V(v))$. So, since $\langle h_V, h_E\rangle$ satisfies Condition (2) of Definition 11, it is a graph isomorphism from $G^{\otimes\parr}(\Theta_1)$ to $G^{\otimes\parr}(\Theta_2)$. By Definition 14, we then obtain $\Theta_1 = \Theta_2$.

In order to prove the triangle inequality on $d_{\mathscr{F}}$, we need the following claim, which says that any intermediate proof structure between two proof structures belonging to the same PS-family has the expected distance property.

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Claim 1. Let Θ_1 and Θ_2 be two proof structures belonging to the same PS-family $\mathscr{F}$ and let $\langle h_V, h_E\rangle : G(\Theta_1) \simeq G(\Theta_2)$ be a graph isomorphism such that

$$d_{\mathscr{F}}(\Theta_1, \Theta_2) = \mid \{v_1 \in V_1 \mid \ell_{V_2}(h_V(v_1)) \neq \ell_{V_1}(v_1)\} \mid .$$

We define

$$V_h = \{v_1 \in V_1 \mid \ell_{V_2}(h_V(v_1)) \neq \ell_{V_1}(v_1)\}.$$

Let V' be a subset of V_h and Θ_0 be the proof structure obtained from Θ_1 by replacing the $\otimes$-link or $\parr$-link corresponding to v by a $\parr$-link or $\otimes$-link, respectively, for each $v \in V'$. Then $d_{\mathscr{F}}(\Theta_1, \Theta_0) = |V'|$.

Proof of Claim 1. We assume that $d_{\mathscr{F}}(\Theta_1, \Theta_0) < |V'|$. Then we have

$$\langle h_V^0, h_E^0\rangle : G(\Theta_1) \simeq G(\Theta_0)$$

such that

$$d_{\mathscr{F}}(\Theta_1, \Theta_0) = \mid \{v_1 \in V_1 \mid \ell_{V_0}(h_V^0(v_1)) \neq \ell_{V_1}(v_1)\} \mid,$$

where

$$G^{\otimes\parr}(\Theta_0) = \langle V_0, E_0, \ell_{V_0}, \ell_{E_0}\rangle.$$

On the other hand,

$$\langle h_V, h_E\rangle : G(\Theta_1) \simeq G(\Theta_2)$$

can be decomposed into

$$\langle h_V^{10}, h_E^{10}\rangle : G(\Theta_1) \simeq G(\Theta_0)$$
$$\langle h_V^{02}, h_E^{02}\rangle : G(\Theta_0) \simeq G(\Theta_2)$$

(that is, $\langle h_V, h_E\rangle = \langle h_V^{02}, h_E^{02}\rangle \circ \langle h_V^{10}, h_E^{10}\rangle$) such that

$$V' = \{v_1 \in V_1 \mid \ell_{V_0}(h_V^{10}(v_1)) \neq \ell_{V_1}(v_1)\}$$

and

$$V_h - V' = \{v_1 \in V_1 \mid \ell_{V_0}(h_V^{10}(v_1)) = \ell_{V_1}(v_1) \wedge \ell_{V_2}(h_V^{02}(h_V^{10}(v_1))) \neq \ell_{V_1}(v_1)\}.$$

So we have

$$|V_h - V'| = \mid \{v_0 \in V_0 \mid \ell_{V_2}(h_V^{02}(v_0)) \neq \ell_{V_0}(v_0)\} \mid .$$

So

$$\langle h_V^{02} \circ h_V^0, h_E^{02} \circ h_E^0\rangle : G(\Theta_1) \simeq G(\Theta_2)$$

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and thus

$$\begin{aligned}
d_{\mathscr{F}}(\Theta_1,\Theta_2) &\leqslant |\,\{v_1 \in V_1 \mid \ell_{V_2}((h_V^{02}\circ h_V^0)(v_1)) \neq \ell_{V_1}(v_1)\}\,| \\
&\leqslant |\,\{v_1 \in V_1 \mid \ell_{V_0}(h_V^0(v_1)) \neq \ell_{V_1}(v_1)\}\,| \\
&\quad + |\,\{v_1 \in V_1 \mid \ell_{V_2}(h_V^{02}(h_V^0(v_1))) \neq \ell_{V_0}(h_V^0(v_1))\}\,| \qquad \text{(by set inclusion)} \\
&= |\,\{v_1 \in V_1 \mid \ell_{V_0}(h_V^0(v_1)) \neq \ell_{V_1}(v_1)\}\,| \\
&\quad + |\,\{v_0 \in V_0 \mid \ell_{V_2}(h_V^{02}(v_0)) \neq \ell_{V_0}(v_0)\}\,| \qquad (h_V^0 \text{ is bijective}) \\
&= d_{\mathscr{F}}(\Theta_1,\Theta_0) + |V_h - V'| \\
&< |V'| + |V_h - V'| \\
&= |V_h| \\
&= d_{\mathscr{F}}(\Theta_1,\Theta_2),
\end{aligned}$$

which gives a contradiction, so we have proved Claim 1. □

Using the claim, we can now prove the triangle inequality for $d_{\mathscr{F}}$ in a similar way to the proof for the set of all the binary words with a fixed length. Let Θ_0, Θ_1 and Θ_2 be proof structures belonging to the same PS-family $\mathscr{F}$. We consider the following cases:

Case 1. There exists a graph isomorphism $\langle h_V, h_E\rangle : G(\Theta_1) \simeq G_2(\Theta_2)$ such that $d_{\mathscr{F}}(\Theta_1,\Theta_2) = |V_h|$, where $V_h = \{v_1 \in V_1 \mid \ell_{V_2}(h_V(v_1)) \neq \ell_{V_1}(v_1)\}$, and there exists $V' \subseteq V_h$ such that Θ_0 is obtained from Θ_1 by replacing the $\otimes$-link or $\parr$-link corresponding to v by a $\parr$-link or $\otimes$-link, respectively, for each $v \in V'$:

In this case, by the claim above, we have $d_{\mathscr{F}}(\Theta_1,\Theta_0) = |V'|$. By the same reasoning, we have $d_{\mathscr{F}}(\Theta_2,\Theta_0) = |V_h - V'|$. Hence

$$\begin{aligned}
d_{\mathscr{F}}(\Theta_1,\Theta_2) &= |V_h| \\
&= |V'| + |V_h - V'| \\
&= d_{\mathscr{F}}(\Theta_1,\Theta_0) + d_{\mathscr{F}}(\Theta_0,\Theta_2).
\end{aligned}$$

Case 2. Otherwise, there exists a graph isomorphism $\langle h_V, h_E\rangle : G(\Theta_1) \simeq G_2(\Theta_2)$ such that $d_{\mathscr{F}}(\Theta_1,\Theta_2) = |V_h|$, where $V_h = \{v_1 \in V_1 \mid \ell_{V_2}(h_V(v_1)) \neq \ell_{V_1}(v_1)\}$, and there are $V' \subseteq V_h$ and non-empty $V'' \subseteq V_1 - V_h$ such that Θ_0 is obtained from Θ_1 by replacing the $\otimes$-link or $\parr$-link corresponding to v by a $\parr$-link or $\otimes$-link, respectively, for each $v \in V' \uplus V''$.

We can choose such $\langle h_V, h_E\rangle$, V' and V'' such that $|V''|$ is the minimum. Then

$$\begin{aligned}
d_{\mathscr{F}}(\Theta_1,\Theta_0) &= |V'| + |V''| \\
d_{\mathscr{F}}(\Theta_2,\Theta_0) &= |V' - V_h| + |V''|.
\end{aligned}$$

Since V'' is non-empty, we have

$$d_{\mathscr{F}}(\Theta_1,\Theta_2) < |V'| + |V_h - V'| + 2|V''| = d_{\mathscr{F}}(\Theta_1,\Theta_0) + d_{\mathscr{F}}(\Theta_0,\Theta_2),$$

which completes the proof of Proposition 6. □

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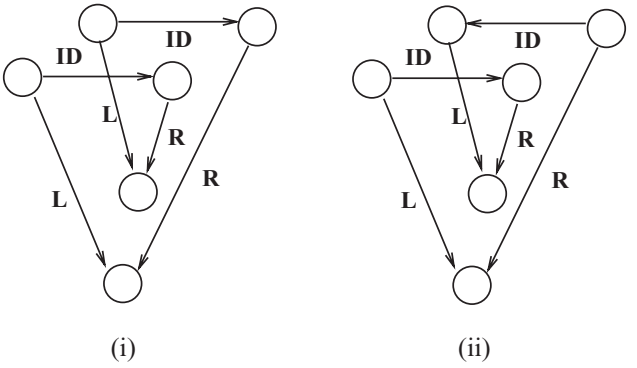


Fig. 5. The induced directed graphs from (i) Θ_1 and Θ_2 and (ii) Θ_3.

Remark 3. The proof nets in Figures 7 and 8 provide some examples. In particular, Θ, Θ_1 and Θ' are a typical case where the equality holds since

$$d(\Theta, \Theta_1) + d(\Theta_1, \Theta') = 2 + 2 = 4 = d(\Theta, \Theta').$$

Θ, Θ_2 and Θ' provide another example where the equality holds. On the other hand, Θ, Θ_1 and Θ_2 are a typical case where the inequality holds because both $d(\Theta_1, \Theta_2) = 2$ and $d(\Theta, \Theta_2) = 2$.

We can justify the definitions above using Figure 1. Let Θ_1, Θ_2 and Θ_3, respectively, be the proof nets (a), (b) and (c) of Figure 1. Then $\Theta_1 \sim \Theta_2$, since $G(\Theta_1)$ and $G(\Theta_2)$ are graph-isomorphic to the directed graph (i) in Figure 5. But note that there are two graph isomorphisms $\{\otimes \mapsto \otimes, ⅋ \mapsto ⅋\}$ and $\{\otimes \mapsto ⅋, ⅋ \mapsto \otimes\}$ between $G(\Theta_1)$ and $G(\Theta_2)$. In the former, we can identify Θ_1 with Θ_2, while in the latter, there are two discrepancies with respect to the multiplicative nodes. Then, since the min operator in the definition of our metric (Definition 13) means we choose $\{\otimes \mapsto \otimes, ⅋ \mapsto ⅋\}$ rather than $\{\otimes \mapsto ⅋, ⅋ \mapsto \otimes\}$, we have $d_{\mathscr{F}}(\Theta_1, \Theta_2) = 0$, and thus $\Theta_1 = \Theta_2$. Note that in order to determine the distance uniquely, we need the min operator for the definition of $d_{\mathscr{F}}(\Theta_1, \Theta_2)$. In addition, since we have graph isomorphisms between $G(\Theta_1)$ and $G(\Theta_2)$, we know Θ_1 and Θ_2 belong to the same PS-family. But Θ_1 and Θ_3 (and Θ_2 and Θ_3) do not belong to the same PS-family, since $G(\Theta_3)$ is graph-isomorphic to the directed graph (ii) in Figure 5, and (i) and (ii) are not graph-isomorphic.

Note that the information about the directions of the edges labelled with **ID** is essential because without it the two graphs of Figure 5 would be isomorphic.

3.2. *Basic results*

We begin this section by slightly extending Corollary 17.1 of Girard (2006, Subsection 11.A.2) – we are grateful to an anonymous referee of an early version of this paper for suggesting this extension.

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Proposition 7. Let $\Theta = \langle \mathbb{F}_\Theta, \mathbb{L}_\Theta \rangle$ be an MLL proof net and $\mathbb{L}_\Theta^{\mathrm{ID}}$, $\mathbb{L}_\Theta^{\otimes}$ and $\mathbb{L}_\Theta^{\parr}$, respectively, be the set of ID-links, $\otimes$-links and $\parr$-links in $\mathbb{L}$, and let con_Θ be the set of conclusions in $\mathbb{F}_\Theta$. Then $|\mathrm{con}_\Theta| + |\mathbb{L}_\Theta^{\parr}| = |\mathbb{L}_\Theta^{\mathrm{ID}}| + 1$ and $|\mathbb{L}_\Theta^{\mathrm{ID}}| - |\mathbb{L}_\Theta^{\otimes}| = 1$.

Proof. The proof is by induction on $|\mathbb{L}_\Theta|$:

— $|\mathbb{L}_\Theta| = 1$:

We have $|\mathbb{L}_\Theta^{\mathrm{ID}}| = |\mathbb{L}| = 1$, $|\mathrm{con}_\Theta| = 2$ and $|\mathbb{L}_\Theta^{\parr}| = |\mathbb{L}_\Theta^{\otimes}| = 0$, so the statement is obvious.

— $|\mathbb{L}_\Theta| > 1$:

We consider two subcases:

(1) Θ includes a $\parr$-formula as a conclusion:

We choose one of the $\parr$-link $L_\parr$ among these $\parr$-links. Let $\Theta_0 = \langle \mathbb{F}_{\Theta_0}, \mathbb{L}_{\Theta_0} \rangle$ be the same as Θ but with $L_\parr$ removed. Since Θ_0 is also an MLL proof net (because otherwise Θ is not an MLL proof net), by the induction hypothesis,

$$|\mathrm{con}_{\Theta_0}| + |\mathbb{L}_{\Theta_0}^{\parr}| = |\mathbb{L}_{\Theta_0}^{\mathrm{ID}}| + 1$$
$$|\mathbb{L}_{\Theta_0}^{\mathrm{ID}}| - |\mathbb{L}_{\Theta_0}^{\otimes}| = 1.$$

which gives the claim since

$$|\mathbb{L}_\Theta^{\mathrm{ID}}| = |\mathbb{L}_{\Theta_0}^{\mathrm{ID}}|$$
$$|\mathrm{con}_\Theta| = |\mathrm{con}_{\Theta_0}| - 1$$
$$|\mathbb{L}_\Theta^{\parr}| = |\mathbb{L}_{\Theta_0}^{\parr}| + 1$$
$$|\mathbb{L}_\Theta^{\otimes}| = |\mathbb{L}_{\Theta_0}^{\otimes}|.$$

(2) Θ does not include any $\parr$-formula as a conclusion:

In this case $|\mathbb{L}_\Theta^{\otimes}|$ must be greater than 0. Then, by the Splitting lemma (Lemma 1), we have a $\otimes$-conclusion $A \otimes B$ and its $\otimes$-link $L_{A\otimes B}$ in Θ such that Θ is decomposed into

$$\Theta_1 = e_\Theta(A)$$
$$\Theta_2 = e_\Theta(B)$$

and the $\otimes$-link $L_{A\otimes B}$. By the induction hypothesis,

$$|\mathrm{con}_{\Theta_1}| + |\mathbb{L}_{\Theta_1}^{\parr}| = |\mathbb{L}_{\Theta_1}^{\mathrm{ID}}| + 1$$
$$|\mathbb{L}_{\Theta_1}^{\mathrm{ID}}| - |\mathbb{L}_{\Theta_1}^{\otimes}| = 1$$
$$|\mathrm{con}_{\Theta_2}| + |\mathbb{L}_{\Theta_2}^{\parr}| = |\mathbb{L}_{\Theta_2}^{\mathrm{ID}}| + 1$$
$$|\mathbb{L}_{\Theta_2}^{\mathrm{ID}}| - |\mathbb{L}_{\Theta_2}^{\otimes}| = 1$$

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hold. Moreover,

$$\begin{aligned}|\mathbb{L}^{\mathrm{ID}}_{\Theta}| &= |\mathbb{L}^{\mathrm{ID}}_{\Theta_1}| + |\mathbb{L}^{\mathrm{ID}}_{\Theta_2}|\\ |\mathrm{con}_{\Theta}| &= |\mathrm{con}_{\Theta_1}| + |\mathrm{con}_{\Theta_2}| - 1\\ |\mathbb{L}^{\parr}_{\Theta}| &= |\mathbb{L}^{\parr}_{\Theta_1}| + |\mathbb{L}^{\parr}_{\Theta_2}|\\ |\mathbb{L}^{\otimes}_{\Theta}| &= |\mathbb{L}^{\otimes}_{\Theta_1}| + |\mathbb{L}^{\otimes}_{\Theta_2}| + 1,\end{aligned}$$

so the claim holds. □

Corollary 1. Let $\mathscr{F}$ be a PS-family and Θ_1 and Θ_2 be MLL proof nets belonging to $\mathscr{F}$. Then the number of $\otimes$-links (or $\parr$-links) occurring in Θ_1 is the same as in Θ_2.

Proof. Since Θ_1 and Θ_2 are members of $\mathscr{F}$, we have $|\mathrm{con}_{\Theta_1}| = |\mathrm{con}_{\Theta_2}|$ and $|\mathbb{L}^{\mathrm{ID}}_{\Theta_1}| = |\mathbb{L}^{\mathrm{ID}}_{\Theta_2}|$. Therefore, by Proposition 7, $|\mathbb{L}^{\otimes}_{\Theta_1}| = |\mathbb{L}^{\otimes}_{\Theta_2}|$ and $|\mathbb{L}^{\parr}_{\Theta_1}| = |\mathbb{L}^{\parr}_{\Theta_2}|$. □

Then, from Corollary 1, it is easy to get the following proposition, which is stated in Girard (2006, Subsection 11.3.3).

Proposition 8. Let Θ be an MLL proof net, and

$$L_{\otimes} : \frac{A \quad B}{A \otimes B}$$

or

$$L_{\parr} : \frac{C \quad D}{C \parr D},$$

respectively, be a $\otimes$-link or $\parr$-link in Θ. Let Θ' be the same proof structure as Θ except that $L_{\otimes}$ or $L_{\parr}$ is replaced by

$$L'_{\parr} : \frac{A \quad B}{A \parr B}$$

or

$$L'_{\otimes} : \frac{C \quad D}{C \otimes D},$$

respectively. Then Θ' is not an MLL proof net.

The following corollary is also obvious, and says that if a PS-family $\mathscr{F}$ has n MLL proof nets, then $\mathscr{F}$ can be used as a *one-error-detecting code system* with n different code elements (see Appendix A).

Corollary 2. Let Θ_1 and Θ_2 be MLL proof nets belonging to the same PS-family $\mathscr{F}$. Then $d_{\mathscr{F}}(\Theta_1, \Theta_2) \geqslant 2$.

We will now define an important notion, which we will use in the next subsection.

Definition 15 ($\otimes$-$\parr$-exchange). Let Θ be a proof structure and

$$L_{\otimes} : \frac{A \quad B}{A \otimes B}$$

and

$$L_{\parr} : \frac{C \quad D}{C \parr D},$$

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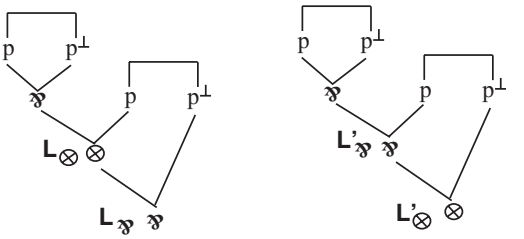


Fig. 6. A counterexample.

respectively, be a ⊗–link and a ⅋-link in Θ. Then, we define $\mathrm{ex}_{\otimes\parr}(\Theta, L_\otimes, L_\parr)$ to be the proof structure obtained from Θ by simultaneously replacing $L_\otimes$ by

$$L'_\parr : \frac{A \quad B}{A \parr B}$$

and $L_\parr$ by

$$L'_\otimes : \frac{C \quad D}{C \otimes D}.$$

Then $\mathrm{ex}_{\otimes\parr}(\Theta, L_\otimes, L_\parr)$ is called a ⊗-⅋-exchange of Θ by $L_\otimes$ and $L_\parr$.

More generally, when $\langle L_{\otimes_1}, \ldots, L_{\otimes_{\ell_1}} \rangle$ is a list of ⊗-links and $\langle L_{\parr_1}, \ldots, L_{\parr_{\ell_2}} \rangle$ a list of ⅋-links in Θ,

$$\mathrm{ex}_{\otimes\parr}(\Theta, \langle L_{\otimes_1}, \ldots, L_{\otimes_{\ell_1}} \rangle, \langle L_{\parr_1}, \ldots, L_{\parr_{\ell_2}} \rangle)$$

is defined to be a proof structure obtained from Θ by simultaneously replacing $L_{\otimes_1}, \ldots, L_{\otimes_{\ell_1}}$ by the list of ⅋-links $L'_{\parr_1}, \ldots, L'_{\parr_{\ell_1}}$ and $L_{\parr_1}, \ldots, L_{\parr_{\ell_2}}$ by the list of ⊗-links $L'_{\otimes_1}, \ldots, L'_{\otimes_{\ell_2}}$.

It is obvious that Θ and $\mathrm{ex}_{\otimes\parr}(\Theta, L_\otimes, L_\parr)$ belong to the same PS-family. Moreover,

$$\mathrm{ex}_{\otimes\parr}(\mathrm{ex}_{\otimes\parr}(\Theta, L_\otimes, L_\parr), L'_\otimes, L'_\parr)$$

is Θ. Then for each pair of proof structures Θ_1 and Θ_2, we define a relation $\Theta_1 \Leftrightarrow \Theta_2$ if there are a ⊗-link $L_\otimes$ and a ⅋-link $L_\parr$ in Θ_1 such that Θ_2 is $\mathrm{ex}_{\otimes\parr}(\Theta_1, L_\otimes, L_\parr)$. So, from the observation above, $\Leftrightarrow$ is a symmetric relation. On the other hand, if Θ is an MLL proof net and $\Theta \Leftrightarrow \Theta'$, then Θ' is not always an MLL proof net – Figure 6 shows an example of this. Theorem 2 describes a necessary and sufficient condition for Θ' to be an MLL proof net.

Note that for convenience, we do not assume for the general ⊗-⅋-exchange

$$\mathrm{ex}_{\otimes\parr}(\Theta, \langle L_{\otimes_1}, \ldots, L_{\otimes_{\ell_1}} \rangle, \langle L_{\parr_1}, \ldots, L_{\parr_{\ell_2}} \rangle),$$

that each of the elements of $\langle L_{\otimes_1}, \ldots, L_{\otimes_{\ell_1}} \rangle$ (or $\langle L_{\parr_1}, \ldots, L_{\parr_{\ell_2}} \rangle$) does not appear in Θ, unlike the assumption made for λ-calculus substitution. Also note that Proposition 8 states that when Θ is an MLL proof net and

$$L_\otimes : \frac{A \quad B}{A \otimes B}$$

or

$$L_\parr : \frac{C \quad D}{C \parr D}$$

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appears in Θ, we have that $\mathrm{ex}_{\otimes\parr}(\Theta, \langle L_{\otimes}\rangle, \langle\rangle)$ or $\mathrm{ex}_{\otimes\parr}(\Theta, \langle\rangle, \langle L_{\parr}\rangle)$, respectively, is not an MLL proof net (although they do belong to the same PS-family as Θ).

Moreover, it is easy to see from Corollary 1 that if Θ_1 and Θ_2 are MLL proof nets that belong to the same PS-family, then there is a sequence of proof structures $\Theta'_1, \ldots, \Theta'_k$ $(k \geqslant 0)$ such that $\Theta_1 \Leftrightarrow \Theta'_1 \Leftrightarrow \cdots \Leftrightarrow \Theta'_k \Leftrightarrow \Theta_2$. Theorem 3, which does not appear to be a trivial result, says that we can always find such a sequence $\Theta'_1, \ldots, \Theta'_k$ such that each element Θ'_i $(1 \leqslant i \leqslant k)$ is an **MLL proof net**.

3.3. *The main theorem*

In this section, we ask 'when is error-correcting possible in our framework?' – the answer is that error-correcting is *always impossible*, and Corollary 3 says that it is always impossible even for one-error-correction.

Before we come to that result, we will state a characterisation of the condition $d_{\mathscr{F}}(\Theta_1, \Theta_2) = 2$, where Θ_1 and Θ_2 are proof nets belonging to $\mathscr{F}$.

Theorem 2. Let Θ be an MLL proof net and

$$L_{1\otimes} : \frac{A \quad B}{A \otimes B}$$

and

$$L_{2\parr} : \frac{C \quad D}{C \parr D},$$

respectively, be $\otimes$- and $\parr$-links in Θ . Then $\mathrm{ex}_{\otimes\parr}(\Theta, L_{1\otimes}, L_{\parr 2})$ is an MLL proof net if and only if one of the followings holds in Θ:

(1) C is a conclusion of $e_\Theta(A)$ and D is a conclusion of $e_\Theta(B)$; or
(2) D is a conclusion of $e_\Theta(A)$ and C is a conclusion of $e_\Theta(B)$.

Proof. See Appendix B. □

We will need the following lemma to prove our main result, Theorem 3.

Lemma 2 (Main Lemma). Let Θ be an MLL proof net with a conclusion $C_0 \parr D_0$ with the $\parr$-link

$$L_{\parr 0} : \frac{C_0 \quad D_0}{C_0 \parr D_0}.$$

Let $m_\otimes$ and $m_\parr$, respectively, be the numbers of all the $\otimes$-links and $\parr$-links occurring in Θ. We assume that there are m_1 $\otimes$-links

$$L_{\otimes 1} : \frac{A_1 \quad B_1}{A_1 \otimes B_1}, \ldots, L_{\otimes m_1} : \frac{A_{m_1} \quad B_{m_1}}{A_{m_1} \otimes B_{m_1}}$$

and m_2 $\parr$-links

$$L_{\parr 1} : \frac{C_1 \quad D_1}{C_1 \parr D_1}, \ldots, L_{\parr m_2} : \frac{C_{m_2} \quad D_{m_2}}{C_{m_2} \parr D_{m_2}}$$

in Θ, where $0 \leqslant m_1 \leqslant m_\otimes$ and $0 \leqslant m_2 \leqslant m_\parr - 1$. We also assume:

(*) For each i, j $(1 \leqslant i \leqslant m_1, 0 \leqslant j \leqslant m_2)$, we have $\Theta_{i,j} = \mathrm{ex}_{\otimes\parr}(\Theta, L_{\otimes i}, L_{\parr j})$ is not an MLL proof net.

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We define Θ' by

$$\Theta' \equiv_{\text{def}} \text{ex}_{\otimes\parr}(\Theta, \langle L_{\otimes 1}, \ldots, L_{\otimes m_1}\rangle, \langle L_{\parr 0}, L_{\parr 1}, \ldots, L_{\parr m_2}\rangle).$$

Then, if Θ' is an MLL proof net, there is an $i\,(1 \leqslant i \leqslant m_1)$ such that

$$\Theta'' \equiv_{\text{def}} \text{ex}_{\otimes\parr}(\Theta, \langle L_{\otimes 1}, \ldots, \widehat{L_{\otimes i}}, \ldots, L_{\otimes m_1}\rangle, \langle L_{\parr 1}, \ldots, L_{\parr m_2}\rangle)$$

is an MLL proof net, where $\widehat{(-)}$ means that it is absent.

Proof. See Appendix C. □

Theorem 3 (Main Theorem). Let Θ and Θ' be two MLL proof nets belonging to the same PS-family $\mathscr{F}$. Then there is $n \in \mathbb{N}$ and a sequence of MLL proof nets $\Theta_1, \ldots, \Theta_n$ such that

$$\Theta \Leftrightarrow \Theta_1 \Leftrightarrow \cdots \Leftrightarrow \Theta_n \Leftrightarrow \Theta'.$$

Proof. We will prove the statement by induction on $|\mathbb{L}_\Theta|$, where $|\mathbb{L}_\Theta|$ is the number of link occurrences in Θ.

— $|\mathbb{L}_\Theta| = 1$:

In this case $\Theta = \Theta'$ must be an ID-link, so we can set $n = 0$.

— $|\mathbb{L}_\Theta| > 1$:

To show a contradiction, we assume that we do not have such a sequence of MLL proof nets between Θ and Θ' for any $n \in \mathbb{N}$. Moreover, we can choose two MLL proof nets Θ and Θ' in $\mathscr{F}$ such that there is no MLL proof net Θ'' such that $d(\Theta, \Theta'') < d(\Theta, \Theta')$ and $d(\Theta'', \Theta') < d(\Theta, \Theta')$ since that is sufficient to prove the statement. We will now derive a contradiction using the induction hypothesis. From Corollary 1, we can easily deduce that $d_{\mathscr{F}}(\Theta, \Theta')$ is even, that is, $d_{\mathscr{F}}(\Theta, \Theta') = 2m$. In addition, there are m $\otimes$-links

$$L_{\otimes 1} : \frac{A_1 \quad B_1}{A_1 \otimes B_1}, \ldots, L_{\otimes m} : \frac{A_m \quad B_m}{A_m \otimes B_m}$$

and m $\parr$-links

$$L_{\parr 1} : \frac{C_1 \quad D_1}{C_1 \parr D_1}, \ldots, L_{\parr m} : \frac{C_m \quad D_m}{C_m \parr D_m}$$

in Θ such that Θ' is

$$\text{ex}_{\otimes\parr}(\Theta, \langle L_{\otimes 1}, \ldots, L_{\otimes m}\rangle, \langle L_{\parr 1}, \ldots, L_{\parr m}\rangle).$$

Let $\Theta_{i,j}\,(1 \leqslant i, j \leqslant m)$ be $\text{ex}_{\otimes\parr}(\Theta, L_{\otimes i}, L_{\parr j})$. Then our assumption means that $\Theta_{i,j}$ is not an MLL proof net for any $i, j\,(1 \leqslant i, j \leqslant m)$ (this assumption is used in the proof of Lemma 2 in Appendix C).

We consider the two cases:

(a) Θ includes a $\parr$-formula $C \parr D$ as a conclusion:

We choose such a $\parr$-link

$$L_{\parr} : \frac{C \quad D}{C \otimes D}.$$

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There are two subcases to consider:

(i) $C \parr D$ is not $C_j \parr D_j$ for any $j\,(1 \leqslant j \leqslant m)$:

Let Θ_0 be the same as Θ but with $L_{\parr}$ eliminated. Then we can apply the induction hypothesis to Θ_0 and a subproof net of Θ',

$$\Theta'_0 = \mathrm{ex}_{\otimes\parr}(\Theta_0, \langle L_{\otimes 1}, \ldots, L_{\otimes m}\rangle, \langle L_{\parr 1}, \ldots, L_{\parr m}\rangle).$$

So we have a sequence of MLL proof nets $\Theta_{01}, \ldots, \Theta_{0n}$ such that

$$\Theta_0 \Leftrightarrow \Theta_{01} \Leftrightarrow \cdots \Leftrightarrow \Theta_{0n} \Leftrightarrow \Theta'_0,$$

and we can then easily construct the desired sequence of MLL proof nets between Θ and Θ'.

(ii) $C \parr D$ is $C_{j_0} \parr D_{j_0}$ for some $j_0\,(1 \leqslant j_0 \leqslant m)$:

In this case, from Lemma 2, there is $i\,(1 \leqslant i \leqslant m)$ such that

$$\Theta'' = \mathrm{ex}_{\otimes\parr}(\Theta, \langle L_{\otimes 1}, \ldots, \widehat{L_{\otimes i}}, \ldots, L_{\otimes m}\rangle, \langle L_{\parr 1}, \ldots, \widehat{L_{\parr j_0}}, \ldots, L_{\parr m}\rangle)$$

is an MLL proof net. But $d(\Theta, \Theta'') < d(\Theta, \Theta')$, which gives a contradiction.

(b) The conclusions of Θ do not include any $\parr$-formulas:

By the Splitting lemma (Lemma 1), we have a $\otimes$-conclusion $A \otimes B$ and its $\otimes$-link $L_{A\otimes B}$ in Θ such that Θ is decomposed into $e_\Theta(A)$, $e_\Theta(B)$ and $\otimes$-link $L_{A\otimes B}$. There are two subcases to consider:

(i) $A \otimes B$ is not $A_i \otimes B_i$ for any $i\,(1 \leqslant i \leqslant m)$:

Let $\mathscr{L}_{\otimes m}$ and $\mathscr{L}_{\parr m}$ be $\langle L_{\otimes 1}, \ldots, L_{\otimes m}\rangle$ and $\langle L_{\parr 1}, \ldots, L_{\parr m}\rangle$, respectively. We can apply the induction hypothesis to $e_\Theta(A)$ and $e_\Theta(B)$ and the subproof nets of Θ',

$$\mathrm{ex}_{\otimes\parr}(e_\Theta(A), \mathscr{L}_{\otimes m}, \mathscr{L}_{\parr m})$$

and

$$\mathrm{ex}_{\otimes\parr}(e_\Theta(B), \mathscr{L}_{\otimes m}, \mathscr{L}_{\parr m}),$$

respectively. We can then construct the desired sequence of MLL proof nets between Θ and Θ' in a similar manner to (a)-(i).

(ii) $A \otimes B$ is $A_i \otimes B_i$ for some $i\,(1 \leqslant i \leqslant m)$:

We can find a DR-switching S' for Θ' such that $S'(\Theta')$ is disconnected since the $\otimes$-link $L_{\otimes i}$ is replaced by a $\parr$-link $L'_{\parr i}$. So Θ' is not an MLL proof net, which gives a contradiction. □

When a PS-family $\mathscr{F}$ has at least two MLL proof nets, we define the distance $d(\mathscr{F})$ for $\mathscr{F}$ itself in the usual way:

$$d(\mathscr{F}) = \min\{d_{\mathscr{F}}(\Theta_1, \Theta_2) \mid \Theta_1, \Theta_2 \in \mathscr{F} \wedge (\Theta_1 \text{ and } \Theta_2 \text{ are MLL proof nets}) \wedge \Theta_1 \neq \Theta_2\}.$$

The following corollary follows easily from Theorem 3.

Corollary 3. For any PS-family $\mathscr{F}$, if the number of MLL proof nets in $\mathscr{F}$ is equal to or greater than 2, then $d(\mathscr{F}) = 2$.

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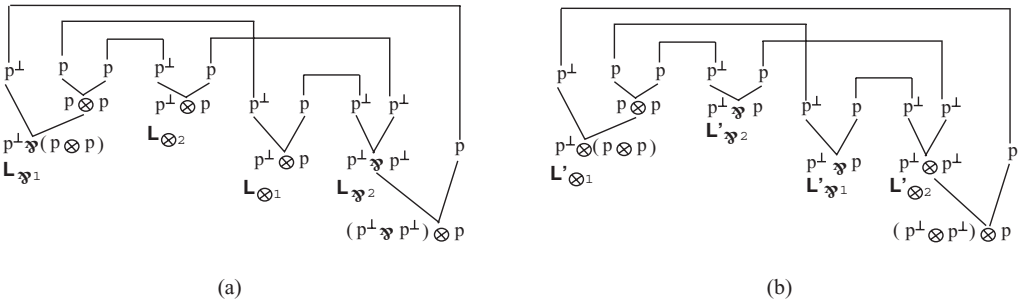


Fig. 7. MLL proof nets (a) Θ and (b) Θ'.

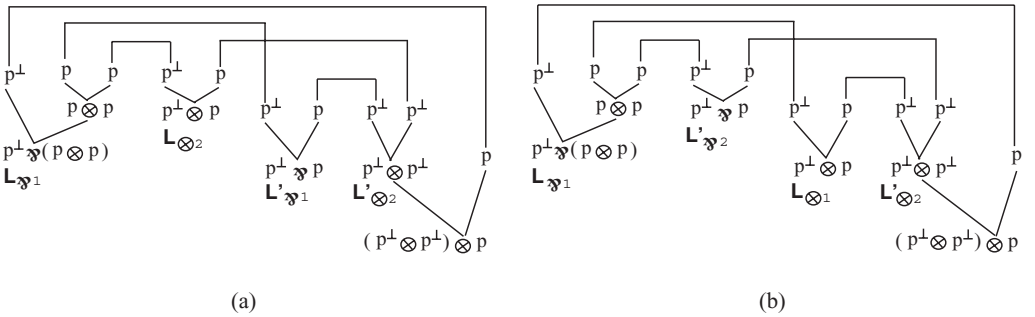


Fig. 8. MLL proof nets (a) Θ_1 and (b) Θ_2.

Corollary 3 means that one-error-correction is impossible for any PS-family of MLL.

Example 1. Our proof of Theorem 3 states that when Θ and Θ' are MLL proof nets belonging to the same PS-family $\mathscr{F}$ and $d_{\mathscr{F}}(\Theta, \Theta') \geqslant 2$, we can always find an MLL proof net Θ'' such that $d_{\mathscr{F}}(\Theta, \Theta'') = 2$ and $d_{\mathscr{F}}(\Theta'', \Theta') = d_{\mathscr{F}}(\Theta, \Theta') - 2$.

As an example, consider the two MLL proof nets Θ in Figure 7(a) and Θ' in Figure 7(b), which belong to the same PS-family $\mathscr{F}$, and for which we have

$$d_{\mathscr{F}}(\Theta, \Theta') = 4.$$

Now consider the proof net Θ_1 in Figure 8(a). We have $\Theta_1 = \mathrm{ex}_{\otimes\parr}(\Theta, L_{\otimes 1}, L_{\parr 2})$ (and $\Theta = \mathrm{ex}_{\otimes\parr}(\Theta_1, L'_{\otimes 2}, L'_{\parr 1})$). Moreover, we find

$$d_{\mathscr{F}}(\Theta, \Theta_1) = 2$$
$$d_{\mathscr{F}}(\Theta_1, \Theta') = 2.$$

However, this Θ_1 is not unique. For example, if we consider Θ_2 in Figure 8 (b), we have $\Theta_2 = \mathrm{ex}_{\otimes\parr}(\Theta, L_{\otimes 2}, L_{\parr 2})$ (and $\Theta = \mathrm{ex}_{\otimes\parr}(\Theta_2, L'_{\otimes 2}, L'_{\parr 2})$). Incidentally, the PS-family has nine MLL proof nets.

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3.4. *Two algorithms for proof-net enumeration*

It is interesting to consider how many MLL proof nets a given PS-family has. In the following, we discuss the problem of enumerating all the (cut-free) MLL proof nets for a given PS-family. In order to demonstrate the importance of our main result, Theorem 3, we will present two proof-net enumeration algorithms: the first searches proof nets naively and exhaustively without using our main result, while the second uses our main result to carry out an intelligent search. We will start with a definition.

Definition 16 (PS-connected). Let $\mathscr{F}$ be a PS-family. Let $\Theta_{\otimes}$ be the unique proof structure of $\mathscr{F}$ such that the set of DR-switchings for $\Theta_{\otimes}$ is a singleton. Then the unique element S of the singleton is an empty assignment of $\parr$-links to $\{0,1\}$. We say that $\mathscr{F}$ is PS-connected if the DR-graph $S(\Theta_{\otimes})$ is connected.

The following proposition provides a characterisation of when a given PS-family has at least one MLL proof net.

Proposition 9. Let $\mathscr{F}$ be a PS-family. Then $\mathscr{F}$ does not have any MLL proof nets if and only if $\mathscr{F}$ is not PS-connected.

Proof.

If direction:

This is obvious from the definition of PS-connectedness.

Only if direction:

We will prove by induction on the number n of the multiplicative links in $\mathscr{F}$ that if $\mathscr{F}$ is PS-connected, then $\mathscr{F}$ has at least one MLL proof net:

— $n = 0$:

Since $\mathscr{F}$ is PS-connected, $\mathscr{F}$ must be the singleton consisting of exactly one ID-link, so $\mathscr{F}$ has exactly one MLL proof net.

— $n > 0$:

We consider the two cases:

– There is an element Θ of $\mathscr{F}$ such that two disjoint proof structures Θ_1 with a conclusion A and Θ_2 with a conclusion B are obtained from Θ by removing one multiplicative link

$$L : \frac{A \quad B}{A@B}$$

of Θ and its conclusion $A@B$:

So $A@B$ must be a conclusion of Θ. Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be the PS-families including Θ_1 and Θ_2, respectively. Both $\mathscr{F}_1$ and $\mathscr{F}_2$ are PS-connected since $\mathscr{F}$ is PS-connected. Therefore, by the induction hypothesis, $\mathscr{F}_1$ and $\mathscr{F}_2$ have MLL proof nets Θ'_1 and Θ'_2, respectively. We now let Θ' be the proof structure obtained from Θ'_1 and Θ'_2 by connecting A and B through the $\otimes$-link

$$L' : \frac{A \quad B}{A \otimes B}.$$

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It is then obvious that Θ' is an MLL proof net and an element of $\mathscr{F}$.

– Otherwise, there is an element Θ of $\mathscr{F}$ such that one proof structure Θ_0 with conclusions A and B is obtained from Θ by removing one multiplicative link

$$L : \frac{A \quad B}{A@B}$$

of Θ and its conclusion $A@B$, and, furthermore, $A@B$ must be a conclusion of Θ:

Let $\mathscr{F}_0$ be the PS-family including Θ_0. Then $\mathscr{F}_0$ must be PS-connected. Therefore, by the induction hypothesis, $\mathscr{F}_0$ has an MLL proof net Θ'_0. We now let Θ' be the proof structure obtained from Θ'_0 by connecting A and B through the $\parr$-link

$$L' : \frac{A \quad B}{A \parr B}.$$

It is then obvious that Θ' is an MLL proof net and an element of $\mathscr{F}$. □

Remark 4. Note that the 'only-if' part of the proof of Proposition 9 also provides an algorithm that constructs an MLL proof net from a given connected PS-family. Of course, the resulting MLL proof net belongs to the given PS-family.

We will now present two algorithms for proof-net enumeration: a naive one without using Theorem 3, and an improved one using the theorem.

Algorithm 1 (Naive version). Let $\mathscr{F}$ be a given PS-family. The output is a list of proof nets PNList.

(1) If $\mathscr{F}$ is not connected, then there is no proof net in $\mathscr{F}$ by Proposition 9. In the following we assume that $\mathscr{F}$ is connected.
(2) By Remark 4, we can find a proof net Θ in $\mathscr{F}$.
(3) Then, by Corollary 1, the number of $\otimes$-links (or $\parr$-links) of any proof net belonging to $\mathscr{F}$ is always the same. Let $\{\Theta_1, \ldots, \Theta_\ell\}$ be all the MLL proof structures in $\mathscr{F}$ such that the number of $\otimes$-links in each member is the same as that of proof net Θ.
(4) For each $i\,(1 \leqslant i \leqslant \ell)$, we check whether Θ_i is a proof net – we can do this in linear time with respect to the number of multiplicative links of Θ_i (Guerrini 1999). If Θ_i is a proof net, we put Θ_i in PNList.

Algorithm 2 (Improved version).

(1) This step is the same as Algorithm 1 (1)
(2) This step is the same as Algorithm 1 (2) and means we can find a proof net Θ_0 in $\mathscr{F}$.
(3) We put Θ_0 in PNList, and then call Neighbor_Search (Θ_0) defined in Table 1.

Proposition 10. Given a connected PS-family $\mathscr{F}$, Algorithm 2 enumerates all the proof nets in $\mathscr{F}$.

Proof. As we discussed earlier, we can find a proof net Θ in $\mathscr{F}$. So we define

$$\mathrm{Reach}_\Theta = \{\Theta_n \mid \Theta \leftrightarrows \Theta_1 \leftrightarrows \cdots \leftrightarrows \Theta_n \text{ for some } n \geqslant 0 \wedge \forall i\,(0 \leqslant i \leqslant n).\Theta_i \text{ is a proof net}\}.$$

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```
procedure Neighbor_Search (Θ)
begin
    Let L⊗1, ..., L⊗m1 (respectively, L⅋1, ..., L⅋m2 ) be all the ⊗-links (respectively, ⅋-links) in Θ.
    Then let Θi,j be ex⊗⅋(Θ, L⊗i, L⅋j) for each i, j (1 ⩽ i ⩽ m1, 1 ⩽ j ⩽ m2).
    for i = 1 to m1 and j = 1 to m2
    begin
        If Θi,j is a proof net and Θi,j is not in PNList then
        begin
            Θi,j is added to PNList;
            call Neighbor_Search (Θi,j )
        end
    end
end
```

Table 1. Neighbor_Search (Θ_0)

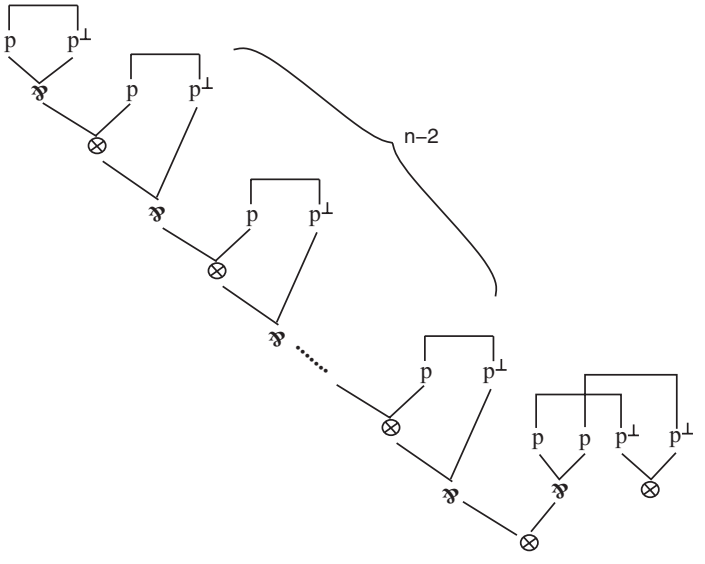


Fig. 9. An example.

Theorem 3 says that Reach_Θ is the set of all proof nets in Θ, so Algorithm 2 lists Reach_Θ. □

Example 2. In order to show that Algorithm 2 really is an improvement on Algorithm 1, we consider the PS-family $\mathscr{F}$ that includes the proof net Θ of Figure 9. Note that the number of ⊗-links is the same as the number of ⅋-links, *viz*. n. Then the number of proof structures with exactly n ⊗-links (and n ⅋-links) is, by Stirling's formula,

$$\binom{2n}{n} \sim \frac{2^{2n}}{\sqrt{\pi n}}.$$

So the number of proof structures we must check exponentially increases in Algorithm 1. On the other hand, Algorithm 2 only checks $2n^2 - 1$ proof structures.

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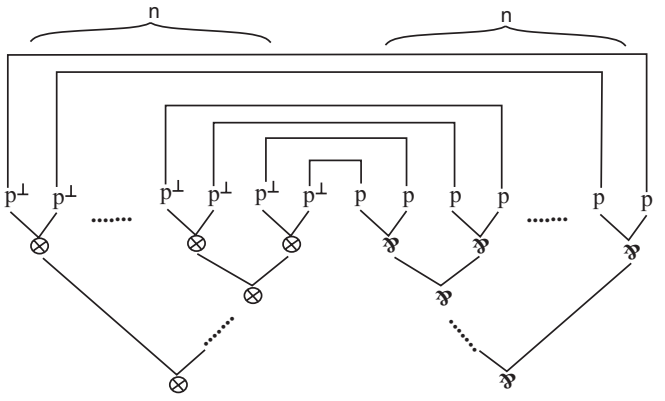


Fig. 10. Another example.

Remark 5. Note, however, that Algorithm 2 does not belong to the complexity class **P**. For example, we consider the PS-family $\mathscr{F}$ that includes the proof net Θ of Figure 10. While the PS-family has 2^n MLL proof nets, the size of the PS-family, that is, the number of multiplicative links it has is $4n-2$. This means that there are PS-families such that the number of MLL proof nets is an exponential function of their size.

4. Conclusion

In this paper, we have introduced the notion of a PS-family over MLL proof structures and the metric spaces associated with them. Moreover, we have proved that in the case where a PS-family has more than two MLL proof nets, the distance of the PS-family is 2.

We have also presented two proof-net enumeration algorithms, and shown that the algorithm using our main result is superior to the one that does not use it.

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Appendix A. Codes over binary words

In this appendix, we will present some basic and elementary information about codes over binary finite words – further detail can be found in any textbook on coding theory, such as Baylis (1998) or MacWilliams and Sloane (1993). The purpose of the appendix is to help the non-expert reader understand the current paper more easily by comparing with the standard theory, and can be ignored by anyone already familiar with the area.

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Definition 17 (Binary finite words). A binary word w with length $n (\in \mathbb{N})$ is an element of $\{0,1\}^n$. For each $i (1 \leqslant i \leqslant n)$, $w[i]$ $(\in \{0,1\})$ denotes the ith element of w.

Definition 18 (Distance for binary words with the same length). Let w_1 and w_2 be binary words with the same length n. The distance between w_1 and w_2, written $d(w_1, w_2)$, is defined as follows:

$$d(w_1, w_2) = |\{w_1[i] \in \{0,1\} \mid 1 \leqslant i \leqslant n \wedge w_1[i] \neq w_2[i]\}|.$$

For example, $d(00110, 10011) = 3$.

Definition 19 (Code over words with length n). A code C over words with length n is a subset of $\{0,1\}^n$. An element of C is called a *codeword*. The distance for C is defined as follows:

$$d(C) = \min\{d(w_1, w_2) \mid w_1, w_2 \in C \wedge w_1 \neq w_2\}.$$

Example 3 (Hamming $\langle 7,4 \rangle$ code). The Hamming $\langle 7,4 \rangle$ code C is the subset of $\{0,1\}^7$ such that $w \in \{0,1\}^7$ is an element of C if and only if w satisfies the following three equations (where $\oplus$ denotes 'exclusive or'):

$$w[1] \oplus w[2] \oplus w[4] \oplus w[5] = 0$$
$$w[2] \oplus w[3] \oplus w[4] \oplus w[6] = 0$$
$$w[1] \oplus w[3] \oplus w[4] \oplus w[7] = 0.$$

An easy calculation shows that $|C| = 16$ and $d(C) = 3$. As a result, the Hamming $\langle 7,4 \rangle$ code is *one-error-correcting* because when a given $w \in \{0,1\}^7$, if $d(w, w') = 1$ for some $w' \in C$, then for any $w''(\neq w') \in C$, we have $d(w, w'') > 1$. Therefore, we can judge that w is w' with one error. Moreover, the Hamming $\langle 7,4 \rangle$ code is *two-error-detecting* because for a given $w \in \{0,1\}^7$, if $d(w, w') = 2$ for some $w' \in C$, then w is not a codeword. Therefore, we can judge that w has exactly two errors. But since there may be a different codeword $w''' \in C$ from w' such that $d(w, w''') = 2$, we cannot judge that w is w' with two errors.

On the other hand, in the Hamming $\langle 7,4 \rangle$ code C, we cannot do both one-error-correcting and two-error-detecting at the same time because there are $w_1, w_2 \in C$ and $w' \in \{0,1\}^7 - C$ such that $d(w_1, w') = 2$ and $d(w', w_2) = 1$. Therefore, we cannot decide whether w' is w_1 with two errors or w_2 with one error. We have to decide whether we adopt the one-error-correcting or two-error-detecting interpretation. If we adopt the one-error-correcting interpretation, then w' is w_2 with one error. If we adopt the two-error-detecting interpretation, then w' has two errors, but we cannot say w' is w_1 with two errors for the reasons given in the previous paragraph.

Appendix B. Proof of Theorem 2.

We begin by defining a notion we will use in the proof.

Definition 20. Let Θ be an MLL proof net, S be a DR-switching for Θ and $A, B \in \mathrm{fml}(\Theta)$. Then there is a unique path θ from A to B in Θ_S. We say that θ *passes immediately above*

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or adjacent to A (respectively, *B*) if θ includes a formula C such that there is a unique link L whose conclusion is A (respectively, B) and C is a premise or another conclusion of L. We say that θ *passes immediately below A* (respectively, B) if θ includes a formula C such that there is a unique link L whose premise is A (respectively, B) and C is the conclusion of L.

Proof of Theorem 2. Let Θ' be $\mathrm{ex}_{\otimes\parr}(\Theta, L_{1\otimes}, L_{2\parr})$ and $L'_{1\parr}$ and $L'_{2\otimes}$ be

$$\frac{A \quad B}{A \parr B}$$

and

$$\frac{C \quad D}{C \otimes D},$$

respectively.

If direction:

We consider two cases:

(1) C is a conclusion of $e_\Theta(A)$ and D is a conclusion of $e_\Theta(B)$:

Let S' be a DR-switching for Θ'. We assume that $\Theta'_{S'}$ has a cycle or is disconnected. We consider two subcases:

(a) S' selects A in $L'_{1\parr}$:

By the assumption on $\Theta'_{S'}$:

(i) there is a cycle including $C \otimes D$ in $\Theta'_{S'}$; or

(ii) A and B are disconnected in $\Theta'_{S'}$.

So we let S be the DR-switching for Θ such that S is the same as S' except that S chooses the left or the right premise of $L_{2\parr}$ and the domain of S does not include $L'_{1\parr}$. Then there are two unique paths θ_1 and θ_2 in Θ_S from A to C and from B to D, respectively. From our assumption about C and D, it is easy to see that all the indexed formulas in θ_1 and θ_2 are included in $e_\Theta(A)$ and $e_\Theta(B)$, respectively. In particular:

– θ_1 passes immediately above or adjacent to both A and C; and

– θ_2 passes immediately above or adjacent to both B and D.

Moreover, by our assumption and Proposition 4, we obtain $e_\Theta(A) \cap e_\Theta(B) = \varnothing$. Therefore, if we consider θ_1 and θ_2 to be two sets of indexed formulas, θ_1 and θ_2 are disjoint. Also, the two paths θ_1 and θ_2 in Θ_S are preserved in $\Theta'_{S'}$ because neither θ_1 nor θ_2 includes $A \otimes B$ or $C \parr D$. Hence, if we let $(\theta_2)^r$ be the reverse of θ_2, then $\theta_1, C \otimes D, (\theta_2)^r$ is the unique path from A to B in $\Theta'_{S'}$, so case (i) holds since case (ii) is impossible.

If $\Theta'_{S'}$ has a cycle π, one of the following conditions must be satisfied:

(a.1) The cycle π in $\Theta'_{S'}$ includes $C, C \otimes D, D$:

Since $C \in e_\Theta(A)$, $D \in e_\Theta(B)$ and $e_\Theta(A) \cap e_\Theta(B) = \varnothing$, π must include at least one indexed formula from each of the following three types of indexed formulas except $C, C \otimes D, D$:

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(I) indexed formulas from $e_\Theta(A)$ different from A;

(II) indexed formulas from $e_\Theta(B)$ different from B; and

(III) indexed formulas that are not included in $e_\Theta(A) \cup e_\Theta(B)$.

Let E be an indexed formula of type (I) that is included in π, and F be an indexed formula of type (II) that is included in π. Then there is a path τ_1 from A to E in $\Theta'_{S'}$ such that all the indexed formulas in τ_1 are included in $e_\Theta(A)$ and τ_1 passes immediately above or adjacent to A. Similarly, there is a path τ_2 from B to F in $\Theta'_{S'}$ such that all the indexed formulas in τ_2 are included in $e_\Theta(B)$ and τ_2 passes immediately above or adjacent to B. On the other hand, since π has indexed formulas of type (III), there is a subpath π' of π from E to F such that π' includes at least one indexed formula that is not included in $e_\Theta(A) \cup e_\Theta(B)$.
Since Θ is an MLL proof net, Θ_S must be acyclic and connected. However, we have the cycle $A \otimes B, \tau_1, \pi', (\tau_2)^r, A \otimes B$ in Θ_S, which gives a contradiction.

(a.2) The cycle π in $\Theta'_{S'}$ includes C and $C \otimes D$, but does not include D:

In this case there is the subpath π_0 of π from C to $C \otimes D$ in $\Theta'_{S'}$ such that π_0 passes immediately above or adjacent to C and immediately below $C \otimes D$. We let the DR-switching S for Θ select C in $L_{2\parr}$. Since Θ is an MLL proof net, Θ_S must be acyclic and connected. But since π_0 in $\Theta'_{S'}$ survives in Θ_S, we know Θ_S has a cycle, which gives a contradiction.

(a.3) The cycle π in $\Theta'_{S'}$ includes D and $C \otimes D$, but does not include C:

This is similar to (a.2) except that we let the DR-switching S for Θ select D in $L_{2\parr}$.

(b) S' selects B in $L'_{1\parr}$:

This is similar to case (a).

(2) D is a conclusion of $e_\Theta(A)$ and C is a conclusion of $e_\Theta(B)$:

This is similar to case (1).

Only if direction:

To show a contradiction, we suppose that Θ and $\Theta'(= \mathrm{ex}_{\otimes\parr}(\Theta, L_{1\otimes}, L_{\parr 2}))$ are proof nets, but neither Condition (1) nor (2) of the statement of the theorem holds. Basically, we will find a DR-switching S' for Θ' such that $S'(\Theta')$ has a cycle. We prove this by case analysis:

(1) $C \parr D \in e_\Theta(A)$:

By Proposition 1, $C \in e_\Theta(A)$ and $D \in e_\Theta(A)$. Let S be a principal DR-switching for A in Θ. Without loss of generality, we can assume that S selects C in $L_{2\parr}$. Since Θ_S is acyclic and connected, there are two unique paths θ_1 from A to C and θ_2 from A to D in $(\Theta_S)^A$ such that both θ_1 and θ_2 pass immediately above or adjacent to A. Moreover, since $e_\Theta(A) \cap e_\Theta(B) = \varnothing$ and all the formulas in θ_1 and θ_2 are included in $e_\Theta(A)$, neither θ_1 nor θ_2 includes B. There are two subcases to consider:

(a) θ_1 passes immediately below $C \parr D$:

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In this case there is either a cycle including $C, C \parr D$ in Θ_S or, when we let $S_{\mathbf{R}}$ be the DR-switching obtained from S by selecting D in $L_{2\parr}$, there is a cycle including $D, C \parr D$ in $\Theta_{S_{\mathbf{R}}}$.

(b) Otherwise:

In this case, neither θ_1 nor θ_2 includes $C \parr D$. Therefore θ_1 (respectively, θ_2) passes immediately above or adjacent to C (respectively, D). So let S' be the same as S except that the $\parr$-switch for $L_{2\parr}$ is deleted and the $\parr$-switch for $L'_{1\parr}$ selects A or B. Since $e_\Theta(A) \cap e_\Theta(B) = \varnothing$ and all the formulas in θ_1 and θ_2 are included in $e_\Theta(A)$, both θ_1 and θ_2 in $S(\Theta)$ survive in $S'(\Theta')$. So we find a cycle $\theta_1, C \otimes D, (\theta_2)^r$ in $S'(\Theta')$.

(2) $C \parr D \in e_\Theta(B)$:

This is similar to case (1).

(3) $C \parr D \notin e_\Theta(A)$ and $C \parr D \notin e_\Theta(B)$: We consider two subcases:

(a) $e_\Theta(A \otimes B) \cap e_\Theta(C \parr D) = \varnothing$:

Let $S_{\mathbf{L}}$ be a DR-switching for Θ selecting C in $L_{2\parr}$. Then there is a unique path θ from D to $C \parr D$ in $\Theta_{S_{\mathbf{L}}}$ with length > 1. Let $S'_{\mathbf{L}}$ be the same as $S_{\mathbf{L}}$ except that the $\parr$-switch for $L_{2\parr}$ is deleted and the $\parr$-switch for $L'_{1\parr}$ selects A (or B). Then $\Theta'_{S'_{\mathbf{L}}}$ has a cycle θ, D, which gives a contradiction.

(b) $e_\Theta(A \otimes B) \cap e_\Theta(C \parr D) \neq \varnothing$:

By Proposition 2, $C \parr D \in e_\Theta(A \otimes B)$ or $A \otimes B \in e_\Theta(C \parr D)$. We consider further subcases:

(b.1) $C \parr D \in e_\Theta(A \otimes B)$:

Since neither (1) nor (2) of the statement of the theorem holds, one of the following must hold:

(b.1.1) Neither C nor D is a conclusion of $e_\Theta(A)$:

In this case, since $C \parr D \notin e_\Theta(A)$, we have $C \notin e_\Theta(A)$ and $D \notin e_\Theta(A)$. Let $S_{\mathbf{L}A}$ be a principal switching for A and Θ such that $S_{\mathbf{L}A}$ selects C in $L_{2\parr}$. Then there are two unique paths θ_1 from A to C and θ_2 from A to D in $\Theta_{S_{\mathbf{L}A}}$ such that both θ_1 and θ_2 pass immediately below A. Let S' be the same as $S_{\mathbf{L}A}$ except that the $\parr$-switch for $L_{2\parr}$ is deleted and the $\parr$-switch for $L'_{1\parr}$ selects A (or B). Then $\Theta'_{S'}$ has a cycle $\theta_1, C \otimes D, (\theta_2)^r$, which gives a contradiction.

(b.1.2) C is neither a conclusion of $e_\Theta(A)$ nor a conclusion of $e_\Theta(B)$:

Since $C \parr D \notin e_\Theta(A)$ (respectively, $C \parr D \notin e_\Theta(B)$), we can easily see that $C \notin e_\Theta(A)$ (respectively, $C \notin e_\Theta(B)$) since if $C \in e_\Theta(A)$ (respectively, $C \in e_\Theta(B)$), then C is a conclusion of $e_\Theta(A)$ (respectively, $e_\Theta(B)$). Since $e_\Theta(A) \cap e_\Theta(B) = \varnothing$, there is a principal switching S_{AB} for both A and B. Since $C \notin e_\Theta(A)$ and $C \notin e_\Theta(B)$, there is a unique path θ_1 from $A \otimes B$ to C in $\Theta_{S_{AB}}$ such that θ_1 passes immediately below $A \otimes B$. If $D \in e_\Theta(A)$, then there is a unique path θ_2 from A to D in $\Theta_{S_{AB}}$ such that θ_2 passes immediately above or adjacent to A. Let S' be the same as S_{AB}

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except that the $\parr$-switch for $L_{2\parr}$ is deleted and the $\parr$-switch for $L'_{1\parr}$ selects A. Then, since θ_1 and θ_2 is preserved in $\Theta'_{S'}$, we know $\Theta'_{S'}$ has a cycle $A\parr B, A, \theta_2, D, C \otimes D, C, (\theta_1)^r, A, A\parr B$. If $D \in e_\Theta(B)$, a similar thing happens. So we assume that $D \notin e_\Theta(A)$ and $D \notin e_\Theta(B)$. Then there is a unique path θ_3 from $A \otimes B$ to D in $\Theta_{S_{AB}}$ such that θ_3 passes immediately below $A \otimes B$. Then, since θ_1 and θ_3 are preserved in $\Theta'_{S'}$, we know $\Theta'_{S'}$ has a path $\theta = A\parr B, \theta_1, C, C \otimes D, D, (\theta_3)^r, A\parr B$. Since $L_{2\parr}$ is a $\parr$-link, either θ_1 or θ_3 does not include $C\parr D$ in $\Theta_{S_{AB}}$, so θ must be a cycle in $\Theta'_{S'}$.

(b.1.3) Neither C nor D is a conclusion of $e_\Theta(B)$:

This is similar to (b.1.1).

(b.1.4) D is neither a conclusion of $e_\Theta(A)$ nor a conclusion of $e_\Theta(B)$:

This is similar to (b.1.2).

(b.2) $C\parr D \notin e_\Theta(A \otimes B)$ and $A \otimes B \in e_\Theta(C\parr D)$:

By Proposition 3, $A \otimes B$ is not a conclusion of $e_\Theta(C\parr D)$ and $e_\Theta(A \otimes B) \subsetneq e_\Theta(C\parr D)$. So we can find a DR-switching S' for Θ' such that $S'(\Theta')$ has a cycle including $C \otimes D$. We will prove this claim in the following.

Let $S_{A\otimes B}$ be a principal switching for $A \otimes B$ in $e_\Theta(C\parr D)$. Then we can obtain a principal switching $S_{C\parr D}$ for $C\parr D$ in Θ by extending $S_{A\otimes B}$. Then the unique path θ from C to D in $S_{C\parr D}(\Theta)$ includes neither $A, A \otimes B, A$ nor $B, A \otimes B, A$, because in order for θ to include $A, A \otimes B, A$ or $B, A \otimes B, A$, we know θ must enter $e_\Theta(A \otimes B)$ from a conclusion of $e_\Theta(A \otimes B)$ other than $A \otimes B$. But this is impossible because $S_{C\parr D}$ is an extension of $S_{A\otimes B}$ that is a principal switching for $A \otimes B$. There are three cases for θ from C to D to consider:

(b.2.1) θ includes neither A, B, nor $A \otimes B$:

Let S' be the same as $S_{C\parr D}$ except that the $\parr$-switch for $L_{2\parr}$ is deleted and the $\parr$-switch for $L'_{1\parr}$ selects A or B. Since θ from C to D in $S_{C\parr D}(\Theta)$ survives in $S'(\Theta')$, we know $S'(\Theta')$ has a cycle $\theta, C \otimes D, C$.

(b.2.3) θ includes $A, A \otimes B$ or $A \otimes B, A$:

Let S' be the same as $S_{C\parr D}$ except that the $\parr$-switch for $L_{2\parr}$ is deleted and the $\parr$-switch for $L'_{1\parr}$ selects A. Since θ from C to D in $S_{C\parr D}(\Theta)$ survives in $S'(\Theta')$, we know $S'(\Theta')$ has a cycle $\theta, C \otimes D, C$.

(b.2.3) θ includes $B, A \otimes B$ or $A \otimes B, B$:

This is similar to (b.2.2). □

Appendix C. Proof of Lemma 2.

Proof of Lemma 2. Let Θ_0 be the MLL proof net obtained from Θ by deleting

$$L_{\parr 0} : \frac{C_0 \quad D_0}{C_0 \parr D_0},$$

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and let Θ'_0 be

$$\mathrm{ex}_{\otimes\parr}(\Theta_0, \langle L_{\otimes 1}, \ldots, L_{\otimes m_1}\rangle, \langle L_{\parr 1}, \ldots, L_{\parr m_2}\rangle).$$

If $m_1 = 0$, it is easy to see that there is a DR-switching S'_0 for Θ'_0 such that there is a path θ' from C_0 to D_0 in $S'_0(\Theta'_0)$, so $S'_0(\Theta')$ has a cycle.

So we now assume $m_1 > 0$, and consider cases for m_2:

— $m_2 = 0$:

Since $m_1 > 0$, if Θ' is a proof net, we must have $m_1 = 1$. Then by the condition (∗) of the statement of the lemma, Θ' is not an MLL proof net.

— $m_2 > 0$:

It is obvious that Θ'_0 is not an MLL proof net since if Θ'_0 were a proof net, Θ' would not be a proof net. Hence, there is a DR-switching S' for Θ'_0 such that $S'(\Theta'_0)$ has a cycle or is disconnected. If $S'(\Theta'_0)$ has a cycle, we are done since $S'(\Theta')$ also has a cycle. If $S'(\Theta'_0)$ has more than two maximally connected components, we are done since $S'(\Theta')$ is disconnected. Therefore, we can assume that $S'(\Theta'_0)$ has exactly two maximally connected components in which each component does not have any cycle for any DR-switching S' for Θ'_0 (note that the number of the edges of $S'(\Theta'_0)$ is always the same for any DR-switching S' for Θ'_0). Let $M_{1S'}$ and $M_{2S'}$ be two maximally connected components of $S'(\Theta'_0)$ that have no cycles. Note that if $M_{1S'}$ or $M_{2S'}$ includes both C_0 and D_0, then $S'(\Theta_0)$ has a cycle including $C_0, C_0 \otimes D_0, D_0$, and we are done.

In the following we assume that $M_{1S'}$ (respectively, $M_{2S'}$) includes C_0 (respectively, D_0) for any DR-switching S' for Θ'_0.

Let S' be a DR-switching for Θ'_0. We say that a $\parr$-link

$$L'_{E\parr F} : \frac{E \quad F}{E \parr F}$$

in Θ'_0 is *separated* at S' if one maximally connected component has E and the other F in $S'(\Theta'_0)$. Then we say that a $\parr$-link $L'_{E\parr F}$ is separated if $L'_{E\parr F}$ is separated at S' for any DR-switching S' for Θ'_0. Then any $\parr$-link $L'_{E\parr F}$ in Θ'_0 is classified as follows:

(i) for any DR-switching S', we have $E, F \in M_{1S'}$ or $E, F \in M_{2S'}$;

(ii) $L'_{E\parr F}$ is separated;

(iii) for some DR-switching S'_a, we have $E, F \in M_{1S'_a}$ or $E, F \in M_{2S'_a}$, and for some DR-switching S'_b, we have $L'_{E\parr F}$ is separated at S'_b.

Note that it may happen that $E, F \in M_{1S'_a}$ and $E, F \in M_{2S'_b}$ for two different S'_a and S'_b in (i) and (iii). Also note that it may happen that $E \in M_{1S'_a}$ and $F \in M_{2S'_a}$ for some S'_a, and $F \in M_{1S'_b}$ and $E \in M_{2S'_b}$ for some S'_b in (iii).

The following claim states a useful sufficient condition for $\parr$-links with type (i).

Claim 2. Let

$$L'_{E_0\parr F_0} : \frac{E_0 \quad F_0}{E_0 \parr F_0}$$

be a $\parr$-link in Θ'_0. Then if there are a DR-switching S' for Θ'_0 and a path θ from C_0 or D_0 to $E_0 \parr F_0$ in $S'(\Theta')$ passing immediately below $E_0 \parr F_0$, then $L'_{E_0\parr F_0}$ has type (i).

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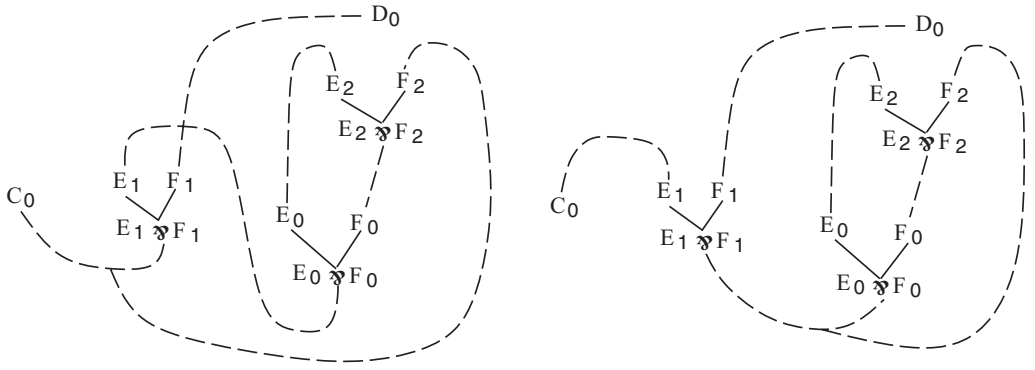


Fig. 11. Two configurations of Θ'_0.

Proof of Claim 2. We suppose the claim is false and show a contradiction. Without loss of generality, we can assume that there is a path from C_0 to $E_0 ⅋ F_0$ in $S'(\Theta')$ passing immediately below $E_0 ⅋ F_0$. If $L'_{E_0 ⅋ F_0}$ is not of type (i), without loss of generality, we can say that there are two $⅋$-links $L'_{E_1 ⅋ F_1}$ and $L'_{E_2 ⅋ F_2}$ in Θ'_0 such that Θ'_0 has one of the two configurations shown in Figure 11 (it is easy to show that any other configuration for Θ'_0 is impossible). But if Θ'_0 had such a configuration, we could find a DR-switching S'' such that $S''(\Theta'_0)$ has a cycle, which completes the proof of Claim 2. □

By Claim 2, if

$$L'_{E_0 ⅋ F_0} : \frac{E_0 \quad F_0}{E_0 ⅋ F_0}$$

has type (ii) or (iii), then for any DR-switching S' for Θ'_0, when θ is a path from E_0 or F_0 to C_0 or D_0 in $S'(\Theta'_0)$, we have θ must pass immediately above or adjacent to E_0 or F_0.

Claim 3. For any DR-switching S' for Θ'_0, there is $i\,(1 \leqslant i \leqslant m_1)$ such that $A_i \in M_{1S'}$ and $B_i \in M_{2S'}$, or $B_i \in M_{1S'}$ and $A_i \in M_{2S'}$.

Proof of Claim 3. If the claim is false, we could find a DR-switching S for Θ such that $S(\Theta)$ is disconnected, which gives a contradiction. □

Claim 4. There is a separated $⅋$-link in Θ'_0.

Proof of Claim 4. We suppose the claim is false and show a contradiction. Hence there is no $⅋$-link in Θ'_0 with type (ii). By Claim 3, we can choose one $⅋$-link

$$L'_{E_0 ⅋ F_0} : \frac{E_0 \quad F_0}{E_0 ⅋ F_0}$$

in Θ'_0 with type (iii). We have a DR-switching S'_0 for Θ'_0 such that E_0 and F_0 belong to different maximally connected components in $S'_0(\Theta'_0)$. Then, since $L'_{E_0 ⅋ F_0}$ is not separated, there is a DR-switching S'_1 for Θ'_0 such that $E_0, F_0 \in M_{1S'_1}$ or $E_0, F_0 \in M_{2S'_1}$. We will now prove the following subclaim.

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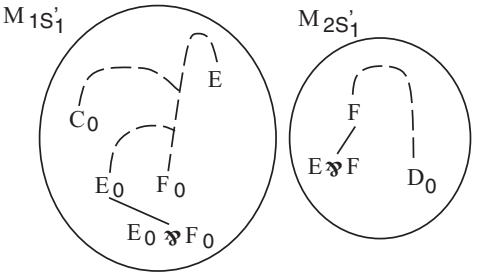


Fig. 12. $S'_1(\Theta'_0)$.

Subclaim 1. Let $L'_{E_0 \otimes F_0}$ and S'_0 be as above. Then there are DR-switching S'_1 for Θ'_0 and a $\otimes$-link $L'_{E_1 \otimes F_1}$ in Θ'_0 such that:

(a) $E_0, F_0 \in M_{1S'_1}$ or $E_0, F_0 \in M_{2S'_1}$;

(b) a path θ from E_0 or F_0 to $E_1 \otimes F_1$ in $S'_1(\Theta'_0)$ passes immediately above or adjacent to E_0 or F_0 and immediately below $E_1 \otimes F_1$; and

(c) $L'_{E_1 \otimes F_1}$ is separated at both S'_0 and S'_1.

Proof of Subclaim 1. We can assume $E_0 \in M_{1S'_0}$ and $F_0 \in M_{2S'_0}$ without loss of generality. Let S'_1 be a DR-switching for Θ'_1 such that $L'_{E_0 \otimes F_0}$ is not separated at S'_1 and the number of the different $\mathbf{L}/\mathbf{R}$-choices of $\otimes$-premises between S'_1 and S'_0 is *minimal*. We can assume $E_0, F_0 \in M_{1,S'_1}$ without loss of generality.
Then there is a $\otimes$-link

$$L'_{E_1 \otimes F_1} : \frac{E_1 \quad F_1}{E_1 \otimes F_1}$$

such that:

(A) $S'_0(L'_{E_1 \otimes F_1}) \neq S'_1(L'_{E_1 \otimes F_1})$ (we can assume S'_1 chooses E_1 in $L'_{E_1 \otimes F_1}$ without loss of generality);

(B) there is a unique path θ from F_0 to $E_1 \otimes F_1$ in $S'_1(\Theta'_0)$ that passes immediately above or adjacent to F_0 and immediately below $E_1 \otimes F_1$; and

(C) $E_1 \in M_{1S'_1}$ and $F_1 \in M_{2S'_1}$ (hence, $L'_{E_1 \otimes F_1}$ is separated at S'_1).

If this were not the case, then for any $\otimes$-link

$$L'_{E \otimes F} : \frac{E \quad F}{E \otimes F}$$

such that $S'_0(L'_{E \otimes F}) \neq S'_1(L'_{E \otimes F})$ in Θ'_0, if $E \in M_{1S'_1}$ and $F \in M_{2S'_1}$, then the unique path θ' from F_0 to E in $S'_1(\Theta'_0)$ passing immediately above or adjacent to F_0 would have to pass immediately above or adjacent to E. So there would have to be a $\otimes$-link $L'_{E \otimes F}$ such that $S'_0(L'_{E \otimes F}) \neq S'_1(L'_{E \otimes F})$ in order for $L'_{E_0 \otimes F_0}$ to be separated at S'_0, but not separated at S'_1. Moreover, there would need to be exactly one such $\otimes$-link $L'_{E \otimes F}$ because of the minimality assumption of S'_1. So $S'_1(\Theta'_0)$ would have the form shown in Figure 12 (the other cases are impossible by Claim 2). But then we would have

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$F_0 \in M_{1,S'_0}$ in $S'_0(\Theta'_0)$, which gives a contradiction. Therefore there is a $\parr$-link $L'_{E_1 \parr F_1}$ satisfying (A), (B) and (C).

We now prove that this $\parr$-link $L'_{E_1 \parr F_1}$ is separated at S'_0. We define a set $\mathbb{L}_{S'_0,S'_1}$ of $\parr$-links in Θ'_0 as follows:

$$\mathbb{L}_{S'_0,S'_1} = \{L'_{E' \parr F'} \mid S'_1(L'_{E' \parr F'}) \neq S'_0(L'_{E' \parr F'})\}.$$

By the minimality of S'_1, for any $\parr$-link $L'_{E' \parr F'}$ in $\mathbb{L}_{S'_0,S'_1}$ except for $L'_{E_1 \parr F_1}$, the conclusion $E' \parr F'$ of $L'_{E' \parr F'}$ must be included in the path θ from F_0 to $E_1 \parr F_1$ in $S'_1(\Theta'_0)$. Therefore, since $E_1 \in M_{1S'_1}$ and $F_1 \in M_{2S'_1}$, we have $E_1 \in M_{1S'_0}$ and $F_1 \in M_{2S'_0}$.

We have now proved that $L'_{E_1 \parr F_1}$ is separated at S'_0; we will go on to prove a stronger statement.

So $E', F' \in M_{2S'_0}$ holds. Note that since $F_0, F_1 \in M_{2S'_0}$ and S'_0 chooses F_1 in $L'_{E_1 \parr F_1}$, all the formulas in θ must be included in $M_{2S'_0}$. Moreover, if E' and F' were included in different components in $S'_0(\Theta'_0)$, we could choose $L'_{E' \parr F'}$ instead of $L'_{E_1 \parr F_1}$. We pick one of these $\parr$-links $L'_{E' \parr F'}$ such that $L'_{E' \parr F'}$ is the nearest to F_0 in θ. Then, if θ includes the subpath $E', E' \parr F'$ in $S'_1(\Theta'_0)$, there is a unique path θ_1 from F_0 to F' in $M_{2S'_0}$ passing immediately above or adjacent to both F_0 and F'. Let S'_2 be the same as S'_1 except that S'_2 chooses F' in $L'_{E' \parr F'}$. Then S'_2 is strictly nearer to S'_0 than S'_1, and there is a unique path θ' from F_0 to $E_1 \parr F_1$ including the subpath $\theta_1, F', E' \parr F'$ in $S'_2(\Theta'_0)$ such that θ' passes immediately above or adjacent to F_0 and immediately below $E_1 \parr F_1$, so $L'_{E_0 \parr F_0}$ is not separated at S'_2. This contradicts the minimality of S'_1. On the other hand, if θ includes the subpath $E' \parr F', E'$ in $S'_1(\Theta'_0)$, there is a unique path θ_2 from F' to E' in $M_{2S'_0}$ passing immediately above or adjacent to both E' and F'. Let S'_2 be the same as S'_1 except that S'_2 chooses F' in $L'_{E' \parr F'}$. Then S'_2 is strictly nearer to S'_0 than S'_1 and there is a unique path θ' from F_0 to $E_1 \parr F_1$ including the subpath $E' \parr F', F', \theta_2, E'$ in $S'_2(\Theta'_0)$ such that θ' passes immediately above or adjacent to F_0 and immediately below $E_1 \parr F_1$, so $L'_{E_0 \parr F_0}$ is not separated at S'_2. This contradicts the minimality of S'_1. So, in fact, $\mathbb{L}_{S'_0,S'_1} = \{L'_{E_1 \parr F_1}\}$. Therefore, it is obvious that $L'_{E_1 \parr F_1}$ is separated at S'_0. This completes the proof of Subclaim 1. □

$S'_1(\Theta'_0)$ satisfying the subclaim above is shown, without loss of generality, in Figure 13 (a). The $\parr$-link $L'_{E_1 \parr F_1}$ has type (ii) or (iii). If it has type (ii), we are done. If $L'_{E_1 \parr F_1}$ is not separated, then we can use the subclaim to find a $\parr$-link

$$L'_{E_2 \parr F_2} : \frac{E_2 \quad F_2}{E_2 \parr F_2}$$

and a DR-switching S'_2 for Θ'_0 such that $L'_{E_2 \parr F_2}$ is separated at S'_1 and S'_2. Figure 13 (b) shows $S'_2(\Theta'_0)$ without loss of generality. We then note that the differences between S'_0 and S'_2 are $L'_{E_1 \parr F_2}$ and $L'_{E_2 \parr F_2}$. Therefore, $L'_{E_2 \parr F_2}$ is not only separated at S'_2 and S'_1, but also separated at S'_0. If $L'_{E_2 \parr F_2}$ is not separated, we repeat the procedure above. Since Θ'_0 has a finite number of multiplicative links, we can always find a separated $\parr$-link in Θ'_0. This completes the proof of Claim 4. □

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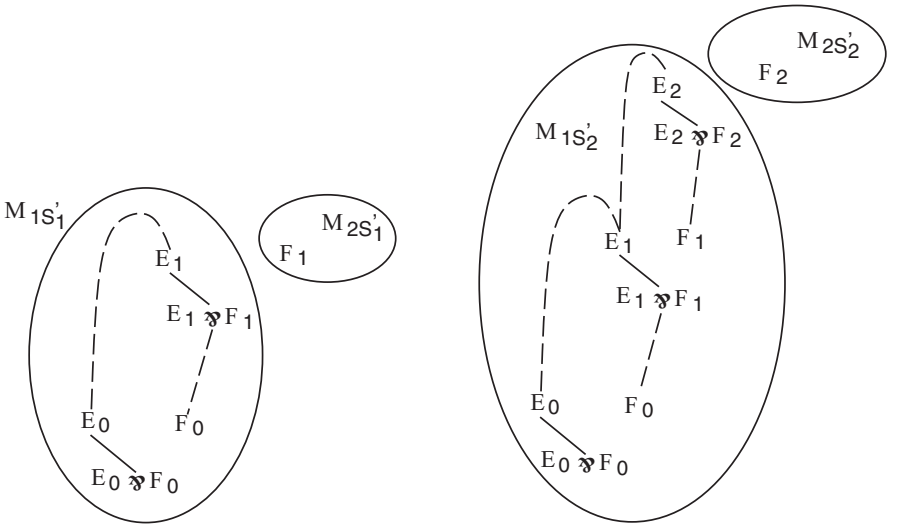


Fig. 13. (a) $S'_1(\Theta'_0)$ and (b) $S'_2(\Theta'_0)$.

If there is a separated link

$$L'_{\parr i} : \frac{A_i \quad B_i}{A_i \parr B_i}$$

for some $i\,(1 \leqslant i \leqslant m_1)$, we are done because the following claim, which states a kind of stability for separated $\parr$-links, holds.

Claim 5. Let

$$L'_{E_0 \parr F_0} : \frac{E_0 \quad F_0}{E_0 \parr F_0}$$

in Θ'_0 be separated (that is, type (ii)). Then for any DR-switching S'_1 and S'_2 for Θ'_0, if $E_0 \in M_{1S'_1}$, then $E_0 \in M_{1S'_2}$.

Proof of Claim 5. We assume the claim is false and show a contradiction. We can assume without loss of generality that there is a DR-switching S'_1 for Θ'_0 such that $E_0 \in M_{1S'_1}$. By our assumption, there is a DR-switching S'_2 for Θ'_0 such that $E_0 \in M_{2S'_1}$. Moreover, we can choose such an S'_2 so that the number of the different $\mathbf{L}/\mathbf{R}$-choices of $\parr$-premises between S'_1 and S'_2 is minimal. For convenience, we will continue to write S'_2 for such a minimal S'_2. So, using similar reasoning to that in the proof of the subclaim of Claim 4, there are two $\parr$-links

$$L'_{E'_1 \parr F'_1} : \frac{E'_1 \quad F'_1}{E'_1 \parr F'_1}$$

and

$$L'_{E'_2 \parr F'_2} : \frac{E'_2 \quad F'_2}{E'_2 \parr F'_2}$$

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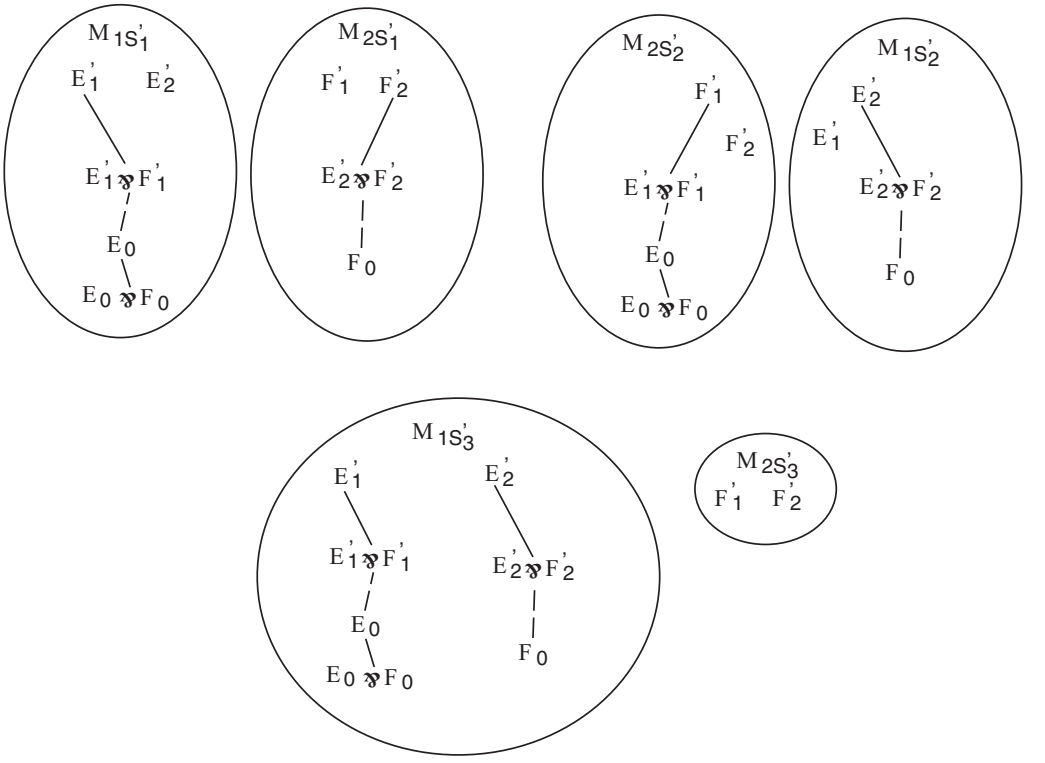


Fig. 14. $S'_1(\Theta'_0)$, $S'_2(\Theta'_0)$, and $S'_3(\Theta'_0)$.

in Θ'_0 satisfying following conditions:

(A) there is a unique path from E_0 (respectively, F_0) to $E'_1 \parr F'_1$ (respectively, $E'_2 \parr F'_2$) that passes immediately above or adjacent to E_0 (respectively, F_0) and immediately below $E'_1 \parr F'_1$ (respectively, $E'_2 \parr F'_2$);

(B) $E'_1, E'_2 \in M_{1S'_1}$ and $F'_1, F'_2 \in M_{2S'_1}$;

(C) the differences between S'_1 and S'_2 are $L'_{E'_1 \parr F'_1}$ and $L'_{E'_2 \parr F'_2}$; and

(D) as a result of (C), we have $E'_1, E'_2 \in M_{1S'_2}$ and $F'_1, F'_2 \in M_{2S'_2}$ (see Figure 14).

Let S'_3 be the same as S'_2 except that S'_3 chooses E'_1 in $L'_{E'_1 \parr F'_1}$. But then $E_0, F_0 \in M_{1S'_3}$ (see Figure 14), and since $L'_{E_0 \parr F_0}$ has type (ii), we get a contradiction. This completes the proof of Claim 5. □

It is obvious that

$$L'_{\otimes 0} : \frac{C_0 \quad D_0}{C_0 \otimes D_0}$$

and $L'_{\parr i}$ satisfy the condition of Theorem 2 in Θ'. Hence

$$\mathrm{ex}_{\otimes \parr}(\Theta', L'_{\otimes 0}, L'_{\parr i}) = \mathrm{ex}_{\otimes \parr}(\Theta, \langle L_{\otimes 1}, \ldots, \widehat{L_{\otimes i}}, \ldots, L_{\otimes m_1} \rangle, \langle L_{\parr 1}, \ldots, L_{\parr m_2} \rangle)$$

is an MLL proof net.

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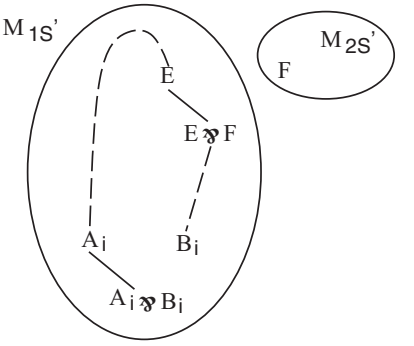


Fig. 15. $S'(\Theta'_0)$.

So we now suppose we have no separated links with the form

$$L'_{\parr i} : \frac{A_i \quad B_i}{A_i \parr B_i}.$$

So we have a separated $\parr$-link

$$L'_{E \parr F} : \frac{E \quad F}{E \parr F},$$

a $\parr$-link

$$L'_{\parr i} : \frac{A_i \quad B_i}{A_i \parr B_i}$$

with type (iii), and a DR-switching S' such that:

(a) $L'_{E \parr F}$ is not $L'_{\parr i'}$ for some i' $(1 \leqslant i' \leqslant m_1)$,

(b) $S'(\Theta'_0)$ has, without loss of generality, the form shown in Figure 15; and

(c) for any DR-switching S'' for Θ'_0, if $S''(\Theta'_0)$ has the form shown in Figure 15, then there is no $A_{i'} \parr B_{i'}$ $(1 \leqslant i' \leqslant m_1)$ in the path from B_i to $E \parr F$ in $S''(\Theta'_0)$ such that the subpath of the path from B_i to $A_{i'} \parr B_{i'}$ passes immediately below $A_{i'} \parr B_{i'}$.

Since Θ_0 is a proof net, $S(\Theta_0)$ does not have a cycle including $E_i, E_i \parr F$ and $A_i, A_i \otimes B_i, B_i$ for any DR-switching S. So for any DR-switching S' for Θ'_0, if $S'(\Theta'_0)$ has the form shown in Figure 15, then there is a $\otimes$-link

$$L'_{\otimes j} : \frac{C_j \quad D_j}{C_j \otimes D_j}$$

for some j $(1 \leqslant j \leqslant m_2)$ such that $S'(\Theta'_0)$ has the form shown in Figure 16. We pick one pair S' and $L'_{\otimes j}$ from the DR-switchings S' and $\otimes$-links

$$L'_{\otimes j} : \frac{C_j \quad C_j}{C_j \otimes C_j}$$

such that there is no

$$L'_{\otimes j'} : \frac{C_{j'} \quad D_{j'}}{C_{j'} \otimes D_{j'}} \; (1 \leqslant j' \leqslant m_2)$$

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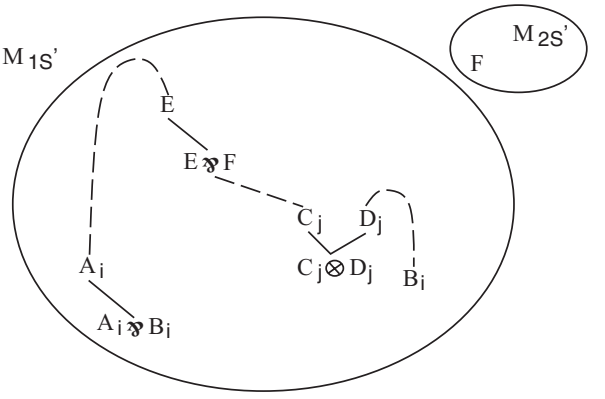


Fig. 16. $S'(\Theta'_0)$.

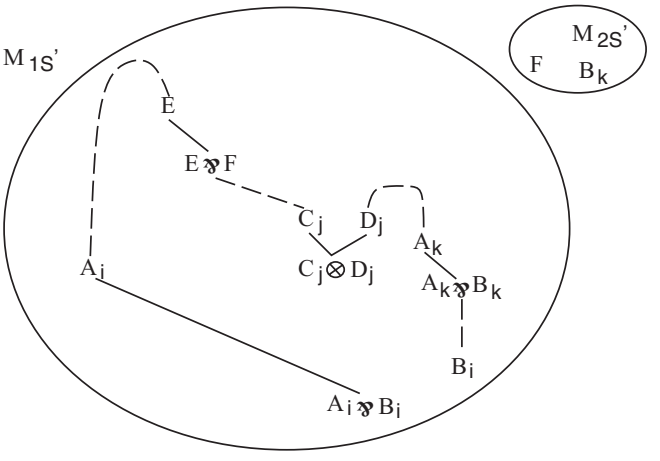


Fig. 17. $S'(\Theta'_0)$ (Case 1).

satisfying the following conditions:

(I) there is S'' for Θ'_0 such that $S''(\Theta'_0)$ has the form shown in Figure 16 with $L'_{\otimes j'}$ instead of $L'_{\otimes j}$; and

(II) $S'(\Theta'_0)$ has the path from $C_{j'} \otimes D_{j'}$ to $C_j \otimes D_j$ passing immediately above $C_{j'} \otimes D_{j'}$ and immediately below $C_j \otimes D_j$ (this means that below the chosen $L'_{\otimes j}$, there is no $\otimes$-links with the same property).

So if $L'_{\otimes i}$ and $L'_{\otimes j}$ satisfy the condition of Theorem 2 in Θ', we are done. So we suppose they do not satisfy them and show a contradiction. Therefore, $A_i \notin e_{\Theta'}(C_j)$ or $B_i \notin e_{\Theta'}(D_j)$.

First we assume $B_i \notin e_{\Theta'}(D_j)$. Then there are a $\otimes$-link

$$L'_{\otimes k} : \frac{A_k \quad B_k}{A_k \otimes B_k}$$

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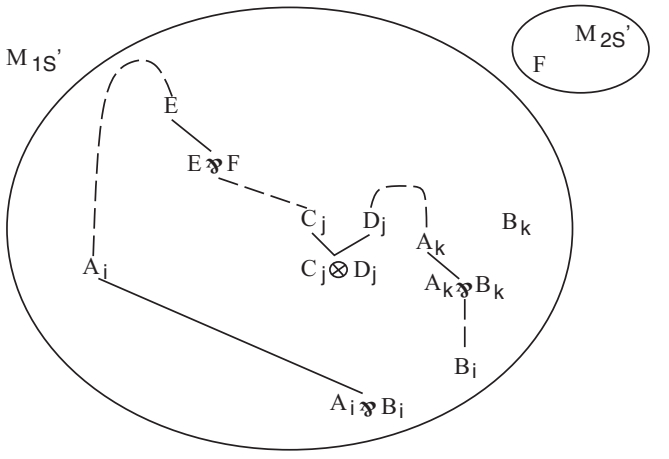


Fig. 18. $S'(\Theta'_0)$ (Case 2).

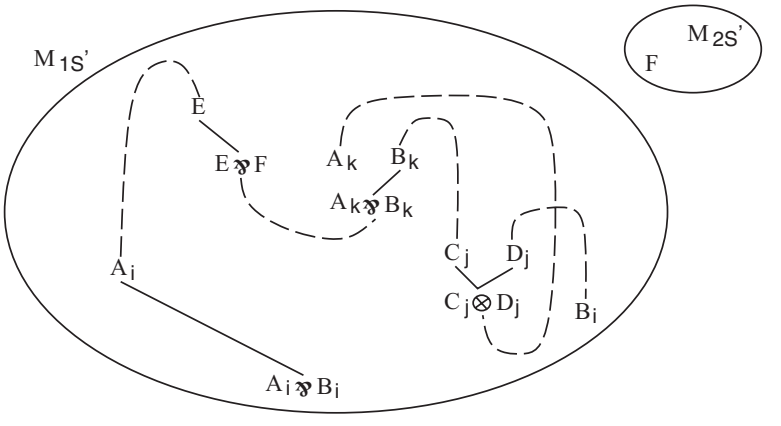


Fig. 19. $S'(\Theta'_0)$ (Case 1).

for some $k\,(1 \leqslant k \leqslant m_1)$ and a DR-switching S' for Θ'_0 such that $S'(\Theta'_0)$ has, without loss of generality, the form shown in Figure 17 or Figure 18. But this contradicts assumption (c).

So we now assume $B_i \in e_{\Theta'}(D_j)$, but $A_i \notin e_{\Theta'}(C_j)$. Then there are a $\parr$-link

$$L'_{\parr k} : \frac{A_k \quad B_k}{A_k \parr B_k}$$

for some $k\,(1 \leqslant k \leqslant m_1)$ and a DR-switching S' for Θ'_0 such that $S'(\Theta'_0)$ has, without loss of generality, the form shown in Figure 19, Figure 20 or Figure 21. We consider each of these possibilities in turn:

– $S'(\Theta'_0)$ has the form shown in Figure 19:

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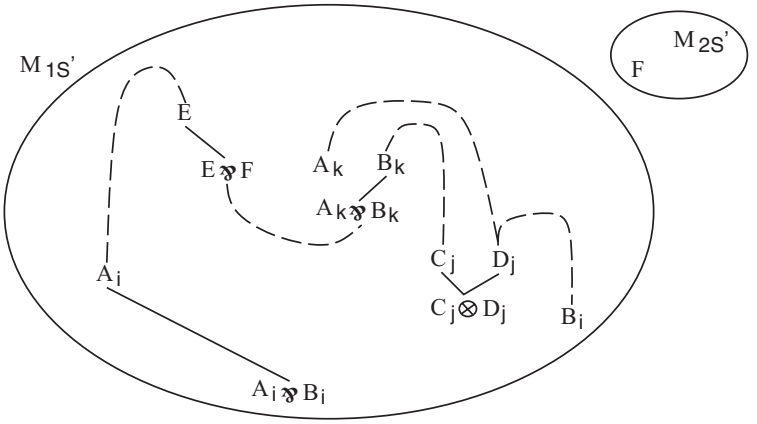


Fig. 20. $S'(\Theta'_0)$ (Case 2).

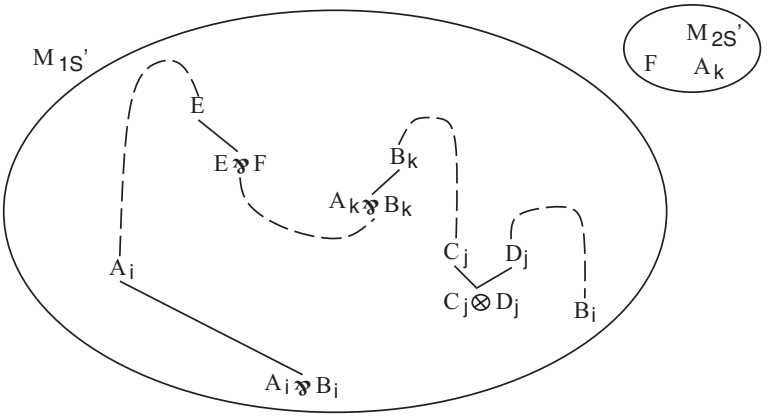


Fig. 21. $S'(\Theta'_0)$ (Case 3).

In order to avoid there being a DR-switching S for Θ such that $S(\Theta)$ has a cycle including $A_i, E, E \parr F, A_k \otimes B_k, A_k, C_j \parr D_j, D_j, B_i, A_i \otimes B_i$, there must be a $\otimes$-link

$$L'_{\otimes m} : \frac{C_m \quad D_m}{A_m \otimes B_m}$$

for some $m\,(1 \leqslant m \leqslant m_2)$ such that $S'(\Theta'_0)$ has the form shown in Figure 22. So we let S'' be the same as S' except that S'' chooses A_k in $L'_{\parr k}$. When we consider the path from B_i to A_i in $S''(\Theta'_0)$, we get that $L'_{\otimes m}$ plays the role of $L'_{\otimes j}$ in S'' instead of S'. But $S'(\Theta'_0)$ has a path from $C_m \parr D_m$ to $C_j \parr D_j$ passing immediately above $C_m \parr D_m$ and immediately below $C_j \parr D_j$, which contradicts our choice of $L'_{\otimes j}$.

– $S'(\Theta'_0)$ has the form shown in Figure 20:

We choose $L'_{\parr k}$ and S' such that there is no $\parr$-link $L'_{\parr k'}$ with the same configuration as $L'_{\parr k}$ in $S'(\Theta'_0)$ such that $L'_{\parr k'}$ is nearer to $L'_{\otimes j}$ than $L'_{\parr k}$. We are done if $L'_{\otimes j}$ and $L'_{\parr k}$ satisfy the condition of Theorem 2 in Θ', so we suppose they do not. Then

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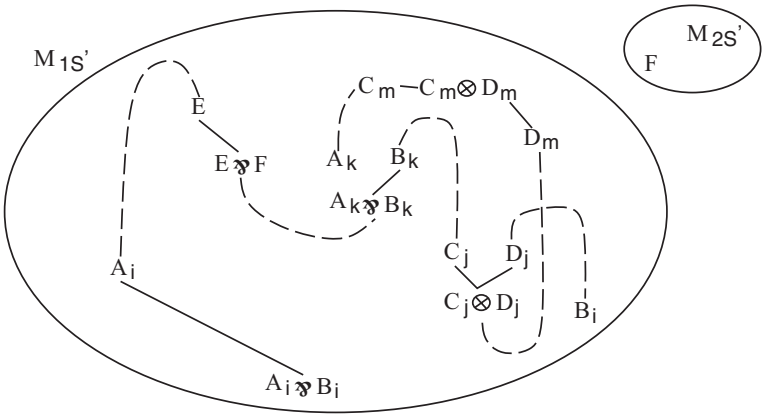


Fig. 22. $S'(\Theta'_0)$ (Case 1).

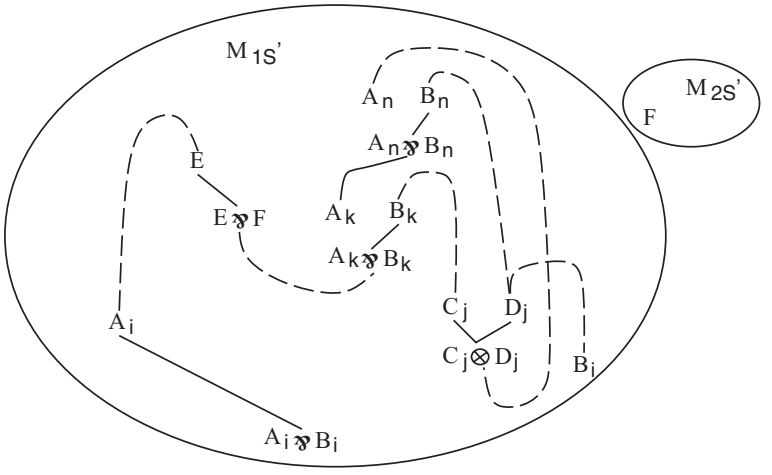


Fig. 23. $S'(\Theta'_0)$ (Case 2).

there must be a ⅋-link $L'_{⅋n}$ such that $S'(\Theta'_0)$ has the form shown in Figure 23 because the other possibilities are impossible by our choice of $L'_{⅋k}$ and S'. Then, to avoid $S(\Theta)$ having a cycle, there must be a ⊗-link

$$L'_{\otimes m} : \frac{C_m \quad D_m}{A_m \otimes B_m}$$

for some $m\,(1 \leqslant m \leqslant m_2)$ such that $S'(\Theta'_0)$ has the form shown in Figure 24. Then we can derive a contradiction in the same way as in the previous case.

– $S'(\Theta'_0)$ has the form shown in Figure 21:

Since $L'_{E⅋F}$ is a separated link, that is, it has type (ii), $S'(\Theta'_0)$ has the form shown in Figure 25 by Claim 2. Let S'' be the same as S' except that S' chooses A_k in $L'_{⅋k}$. Then $S''(\Theta'')$ has a cycle including $C_0 \otimes D_0, C_0, E, E⅋F, A_k⅋B_k, A_k, D_0$. □

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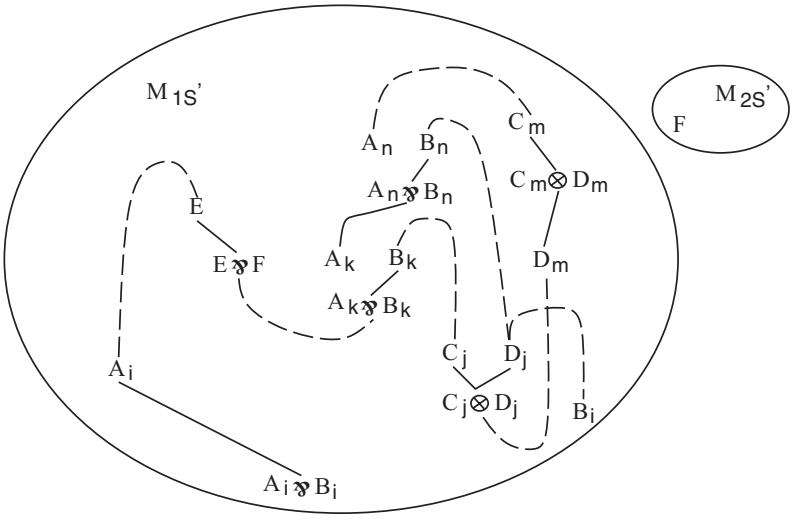


Fig. 24. $S'(\Theta'_0)$ (Case 2).

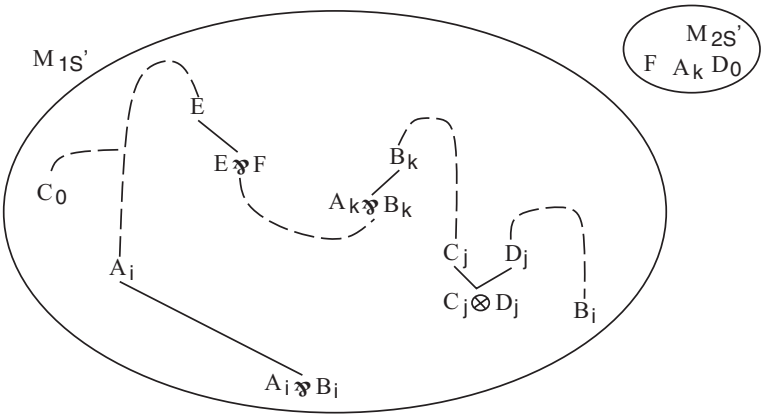


Fig. 25. $S'(\Theta'_0)$ (Case 3).

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