The surfaces whose prime-sections contain a g_3^1. By J. Bronowski, B.A., Jesus College.

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1. The surfaces whose prime-sections are hyperelliptic curves of genus p have been classified by G. Castelnuovo*. If $p > 1$, they are the surfaces which contain a (rational) pencil of conics, which traces the g_2^1 on the prime-sections. Thus, if we exclude ruled surfaces, they are rational surfaces. The supernormal surfaces are of order $4p + 4$ and lie in space $[3p + 5]$. The minimum directrix curve to the pencil of conics—that is, the curve of minimum order which meets each conic in one point—may be of any order k, where $0 \leqslant k \leqslant p + 1$. The prime-sections of these surfaces are conveniently represented on the normal rational ruled surfaces, either by quadric sections, or by quadric sections residual to a generator, according as k is even or odd.

The problem suggests itself, similarly to classify the surfaces whose prime-sections are curves of genus p which contain a g_3^1. If $p > 4$, it is clear that the g_3^1 on the prime-sections of such a surface is traced by a (rational) pencil of curves; which must therefore be cubic curves. We shall confine ourselves to surfaces which are not ruled, and which are without singularity. Then two cases arise: that in which the curves of the pencil are twisted, rational cubics; and that in which they are plane, elliptic cubics. The surfaces G of the first type are rational. The surfaces F of the second type are, in general, irrational.

2. We discuss, first, the rational surfaces G. Since a surface G contains a pencil of rational curves, it can be represented on the plane in such a way that this pencil is represented by the pencil of lines through a point X of the plane. Thus, let the prime-sections of the surface G, of sectional genus p, of order m, and lying in space $[r]$, be represented in the plane by curves of order n having one $(n-3)$-ple base-point X, and having other base-points of total multiplicities Σt. The twisted cubics of G are then represented by the lines through X. If K' is any directrix curve to the pencil of cubics on G, of order k', then K' is represented in the plane by a curve of order N having an $(N-1)$-ple point at X, and having points of total multiplicities ΣT at the other base-points (each T being, in fact, 0 or 1), where

$$3N - 3 + n - \Sigma tT = k'.$$

* G. Castelnuovo, *Rend. di Palermo*, 4 (1890), 73.

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Now $\quad p = 2n - 5 - \frac{1}{2}\Sigma t(t-1), \qquad m = 6n - 9 - \Sigma t^2,$

and, if we assume that the base-points offer independent conditions,

$$r = 4n - 3 - \tfrac{1}{2}\Sigma t(t+1).$$

Hence $r = 2p + 7 + \frac{1}{2}\Sigma t(t-3), \qquad m = 3p + 6 + \frac{1}{2}\Sigma t(t-3).$

Since no t can be greater than 3 (else the join of the corresponding base-point to X forms part of every curve of the system), it follows that the maximum values of r, m are

$$r = 2p + 7, \qquad m = 3p + 6;$$

and that these values are attained only when every $t = 3$. Thus the supernormal surfaces G of sectional genus p which contain a rational pencil of twisted cubics are of order $3p + 6$ and lie in space $[2p + 7]$. But the order of the minimum directrix curve to the pencil of cubics of a supernormal surface G is not arbitrary. For it follows from the foregoing equations (when each $t = 3$) that the order k' of any directrix curve on G is subject to the condition

$$k' + p \equiv 1 \pmod 3. \qquad (1)$$

The minimum systems of plane curves which represent the prime-sections of supernormal surfaces G are now easily enumerated. For these systems have, in addition to the base-point X, either one distinct triple base-point, or a number of triple base-points all lying in the first neighbourhood of X. Systems of the first kind exist only for even values of p; and they then consist of curves of order $\frac{1}{2}p + 4$ which have a $(\frac{1}{2}p + 1)$-ple base-point X, and a distinct triple base-point Y. The corresponding surface G contains a single infinity of minimum directrix curves, each of order $\frac{1}{2}p + 1$, represented by the lines through Y. Systems of the second kind exist for all values of p, and consist of curves of order $p - k + 2$ which have a $(p - k - 1)$-ple base-point X, and $\frac{1}{3}(p - 2k - 1)$ triple base-points lying in the first neighbourhood of X, where k is the order of the minimum directrix curve on the corresponding surface G, and is subject to the conditions (1) and $0 \leqslant k \leqslant \frac{1}{2}(p - 1)$. This minimum directrix curve on such a surface G is unique, and is represented by the neighbourhood of X.

The foregoing results may also be obtained by putting the cubics of a surface G in correspondence with the generators of an appropriate normal rational ruled surface. The prime-sections of the surfaces G are, in fact, conveniently represented on the normal rational ruled surfaces by the sections by cubic primals, residual to no, one, or two generators according as $p \equiv 1$, 2, or 0 (mod. 3).

3. Before discussing the general surfaces F, which contain a rational pencil of elliptic cubics, we examine those exceptional surfaces F' which are rational. Let the prime-sections of such

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a rational surface F', of sectional genus p, of order m, and lying in space $[r]$, be represented in the plane by a system S of curves of order n, in such a way that the pencil of elliptic cubics on F' is represented by a pencil R of cubics passing through nine associated points*. Let the system S have base-points of total multiplicities Σt at these nine points, and $\Sigma t'$ elsewhere; these base-points being assumed to offer independent conditions. Since the cubics of the pencil R represent cubics on F',

$$3n - \Sigma t = 3.$$

Therefore
$$r = p + 3n - \Sigma t - \Sigma t' - 1 = p + 2 - \Sigma t',$$
$$m = 2p + 3n - \Sigma t - \Sigma t' - 2 = 2p + 1 - \Sigma t'.$$

Hence the maximum values of r, m are

$$r = p + 2, \qquad m = 2p + 1;$$

and these values are attained only when every $t' = 0$. Moreover, these surfaces for which each $t' = 0$ are the only normal rational surfaces F' without singularity. For (as may be verified from the representation) if such a surface is projected from a point of itself, it acquires an elliptic double line; if it is projected from one of its tangent planes, it acquires a node whose neighbourhood is elliptic; and so on. Thus the normal rational surfaces F' of sectional genus p, without singularity, which contain a rational pencil of elliptic cubics are of order $2p + 1$ and lie in space $[p + 2]$.

The values of n, Σt for the systems S which represent the prime-sections of such normal rational surfaces F' are therefore given by

$$n^2 - \Sigma t^2 = 2p + 1, \qquad 3n - \Sigma t = 3.$$

To solve these indeterminate equations, write

$$p = 3i + a \quad (a = -1, 0, \text{or } 1), \qquad n = 3i + y,$$
$$t_0 = i + s_0, \quad t_1 = i + s_1, \quad \ldots, \quad t_8 = i + s_8.$$

The equations then become

$$y^2 - \Sigma s^2 = 2a + 1 \quad (a = -1, 0, \text{or } 1), \qquad 3y - \Sigma s = 3.$$

These equations have five solutions, subject to the condition $s_j + s_k + s_l \leqslant y$, which specifies that the systems S shall be of

* If v is a prime number, then a pencil of elliptic curves of order v on a rational surface is represented in the plane, either by a pencil R of cubics, or by a Halphen pencil of curves of order $3v$ with nine v-ple base-points (not in general position). In the present case, however, $v = 3$, it is easily shown that the Halphen pencil gives rise to surfaces having a unode: they are in fact surfaces discussed by P. Du Val, *Proc. London Math. Soc.* (2), 35 (1933), 1. Hence the only non-singular surfaces F' arise from the pencil R of cubics given in the text.

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minimum order. They give five projectively distinct systems S, which have been classified*; namely,

when $p \equiv 3i-1$, the system $c^{3i+1}(i-1, i^7, i+1)$;

when $p \equiv 3i$, the systems $c^{3i+1}(i^9)$ and $c^{3i+3}(i-1, i, i+1^7)$;

when $p \equiv 3i+1$, the systems $c^{3i+3}(i^3, i+1^6)$ and $c^{3i+6}(i-1, i+2^8)$.

The minimum directrix curves to the pencils of cubics on the corresponding surfaces F' are respectively: a unique curve of order $i-1$; nine curves of order i; a unique curve of order $i-1$; three curves of order i; and a unique curve of order $i-1$. All these curves are represented by the neighbourhoods of the base-points of least multiplicity in the system S.

4. We proceed to the general, irrational surfaces F which contain a rational pencil of elliptic cubics. It is convenient to divide these surfaces into two types: surfaces $F_{(1)}$ which lie on a cubic primal; and surfaces $F_{(2)}$ which do not lie on a cubic primal. Namely, the planes of the cubics of any surface F generate a rational three-fold E. A surface $F_{(1)}$ is the section of E by an appropriate cubic primal, residual to a number of planes of E. A surface $F_{(2)}$ is the section of E by an appropriate primal of order M, residual to a surface which meets every plane of E in a curve of order $M-3$, and to a number of planes of E.

If we confine ourselves to supernormal surfaces F (which, as in § 3, are the only normal surfaces F without singularity), then E may be taken to be a normal rational three-fold E^q generated by a pencil of planes, which lies in space $[q+2]$. Now let $F_{(1)}$ be the section of such a normal three-fold E^q by a cubic primal which contains d planes of E^q. Then the sectional genus p and the order m of $F_{(1)}$ are

$$p = 3q - 2d - 2, \quad m = 3q - d;$$

where, in order that $F_{(1)}$ (and its prime-sections) shall be normal,

$$d \leqslant q-1; \text{ say, } \quad d = q-1-b \qquad (q-1 \geqslant b \geqslant 0).$$

Thus there exist normal surfaces $F_{(1)}$ of sectional genus p, without singularity, only in the spaces of dimension

$$r = p + 2 - 2b \qquad (r-3 \geqslant b \geqslant 0);$$

and the order of any such surface is

$$m = \tfrac{1}{2}p + \tfrac{3}{2}r - 2 = 2p + 1 - 3b.$$

The three-fold E^q is characterised† by the order $h \leqslant \frac{2}{3}q$ of its minimum directrix ruled surface H (which meets every plane of

* J. Bronowski, *Proc. Camb. Phil. Soc.*, 29 (1933), 482–3.
† C. Segre, *Atti Acc. Torino*, 21 (1885), 95.

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E in a generator), and by the order $c \leqslant \frac{1}{3}q$ of the minimum directrix curve on this rational ruled surface H. The prime-sections of E^q are then conveniently represented* in three-dimensional space by the (ruled) surfaces of order q which have a fixed line L for $(q-1)$-ple line, and which have $q-1$ fixed lines L', each concurrent with L, for simple lines; these lines being in such positions that there exists a ruled surface H' (representing H) of order h and with a minimum directrix curve of order c, which contains L as $(h-1)$-ple line and L' as generators. Therefore a surface $F_{(1)}$ is represented upon a surface $f_{(1)}$ in three-dimensional space, whose order is $3q-d=2p+1-3b$, which has L for $(3q-d-3)$-ple line, that is for $(2p-2-3b)$-ple line, and which has each of the lines L' for triple line. The prime-sections of $F_{(1)}$ are represented upon $f_{(1)}$ by the curves of order $3q-d=2p+1-3b$ which are the residual sections of $f_{(1)}$ by the above ruled surfaces of order $q=p-2b$; where throughout $r-3 \geqslant b \geqslant 0$.

A directrix curve K', of order k', to the pencil of cubics on $F_{(1)}$ is represented by a directrix curve, of the same order k', to the pencil of cubics on $f_{(1)}$ (cut out by the planes through L). But the surfaces $F_{(1)}$, $f_{(1)}$ may, in fact, contain no directrix curves to their pencils of cubics. For such a directrix curve on $F_{(1)}$, if it exists, is also a directrix curve to the planes of E. Now (as may be verified from the representation) the freedom of directrix curves of order k' to the planes on E^q is

$$3k'-q+2.$$

The number of conditions that such a curve shall lie on a cubic primal containing $d \leqslant q-1$ planes of E is

$$3k'-d+1,$$

which, except when $d=q-1$, is always greater than the freedom of such curves on E. Thus, when $d<q-1$, that is when $b>0$, the most general surface $F_{(1)}$, namely the section of E by a general cubic primal through $d=q-1-b$ planes of E, contains no directrix curve to its pencil of cubics. But, for every value of $b>0$, there also exist surfaces $F_{(1)}$ which contain one, or more, directrix curves of any assigned order k. For such a surface is obtained by taking the section of a three-fold E which contains one, respectively ∞^1, directrix curves of order k by a cubic primal which, in addition to containing d planes of E, contains this one, respectively several, directrix curves of order k.

When $d=q-1$, that is when $b=0$, the surface $F_{(1)}$ always contains a finite number of directrix curves of every order $k' \geqslant \frac{1}{3}(q-1)$, that is of every order $k' \geqslant \frac{1}{3}(p-1)$. We shall show in § 6 that it follows that every surface $F_{(1)}$ for which $b=0$ is

* F. Palatini, *Giorn. di mat.*, 38 (1900), 315.

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rational, and is therefore one of the five surfaces F' which have been enumerated in § 3. Thus if we attempt to construct a surface $F_{(1)}$ with $b = 0$ which shall have directrix curves other than those given in § 3 (for example, directrix curves of lower orders), we obtain a surface which is either singular or reducible.

5. Finally, we consider the section $F_{(2)}$ of a normal three-fold E^q, in space $[q+2]$, by a primal of order M which contains d planes of E, and which contains a surface U which meets every plane of E in a curve of order $M-3$. Let the order of the surface U be u, where

$$q(M-2)-d > u > \tfrac{1}{2}q(M-3).$$

Then the sectional genus p and the order m of $F_{(2)}$ are

$$p = q(2M-3)-2(u+d+1), \quad m = qM-u-d.$$

Moreover, we can depress the upper bound of u; for, in order that $F_{(2)}$ shall not lie on a cubic primal, it is necessary that

$$u+d \leqslant q(M-3)-1;$$

say,

$$u+d = q(M-3)-1-a \qquad (a \geqslant 0).$$

Then there exist normal surfaces $F_{(2)}$ of sectional genus p, without singularity, only in the spaces of dimension

$$r = \tfrac{1}{3}p+2-\tfrac{2}{3}a \qquad (a \geqslant 0,\ 2a \equiv p \bmod 3);$$

and the order of any such surface is

$$m = \tfrac{1}{2}p+\tfrac{3}{2}r-2 = p+1-a.$$

Instead of this notation, we can also employ the notation of § 4. Namely, write

$$u+d = q(M-2)-1-b \qquad (b \geqslant q).$$

Then, for the normal surfaces $F_{(2)}$,

$$\left.\begin{aligned} r &= p+2-2b \\ m &= \tfrac{1}{2}p+\tfrac{3}{2}r-2 = 2p+1-3b \end{aligned}\right\}, \qquad (2)$$

where $b \geqslant r-2$. Comparing with § 4, we see that the formulae (2) hold for all surfaces F; the surfaces $F_{(1)}$ being those for which $r-3 \geqslant b \geqslant 0$; the surfaces $F_{(2)}$ those for which $b \geqslant r-2$.

With this notation, we obtain a representation of the surfaces $F_{(2)}$ which is identical with the representation of the surfaces $F_{(1)}$ of § 4. Namely, every surface F is represented upon a corresponding surface f in three-dimensional space, whose order is $2p+1-3b$, which has a $(2p-2-3b)$-ple line L, and which has $r-3 = p-1-2b$ triple lines L' each concurrent with L, the arrangement of the lines L, L' being that given in § 4. The prime-

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sections of F are represented upon f by curves of order $2p+1-3b$, which are the residual sections of f by the ruled surfaces of order $p-2b$ which have the line L for $(p-1-2b)$-ple line, and which have each of the lines L' for a generator. The surfaces $F_{(1)}$ are those represented upon surfaces $f_{(1)}$ for which $r-3 \geqslant b \geqslant 0$; the surfaces $F_{(2)}$ are those represented upon surfaces $f_{(2)}$ for which $b \geqslant r-2$.

Like the surfaces $F_{(1)}$, $f_{(1)}$, the surfaces $F_{(2)}$, $f_{(2)}$ in general contain no directrix curves to their pencils of plane cubics. But, for every value of b, there also exist surfaces $F_{(2)}$, $f_{(2)}$ which contain one or more directrix curves of any assigned order k. Every surface $F_{(2)}$, $f_{(2)}$ is irrational.

6. Since every surface F contains a pencil of elliptic curves, the canonical system of every surface F is compounded with this pencil*, as will also appear immediately, from the consideration of the representative surfaces f. Such surfaces are characterised by the values of three invariants: p_g, p_n, and the *determinant* of the surface.

We calculate p_g not for the surfaces F but for the birationally equivalent surfaces f. The canonical system on f is cut out by surfaces of order $(2p+1-3b)-4$ which contain the line L as a line of multiplicity $(2p-2-3b)-1$, and which contain each of the $p-1-2b$ lines L', concurrent with L, as double lines. That is, these surfaces consist of the $p-1-2b$ fixed planes LL', each counted twice; and of $b-1$ variable planes through L. Hence the freedom p_g-1 of the canonical systems on f and on F is $b-1$; whence $p_g = b$.

To calculate p_n, we first calculate the values of the invariants I, ω for the surfaces F. Consider, first, the pencil of cubics on a surface $F_{(1)}$. This is a pencil of curves of genus 1, without base-points. To find the number of cubics of the pencil which have a double point, project all the cubics on to the plane of one of them, that is, on to a general plane of the three-fold E^q. There is thus obtained in this plane a rational ∞^1 of cubics, of index $q-d = q-(q-1-b) = b+1$; the number of cubics of such a system which have double points is $12(b+1)$. This, it follows, is the number of cubics of the pencil on $F_{(1)}$ which have double points. Hence the value of I for a surface $F_{(1)}$ is $12(b+1)-4 = 12b+8$. Similarly we obtain the value of I for any surface F, and it is $12b+8$. Next, the grade of the canonical system on F (which contains no exceptional curves) is $\omega-1$; since the system is compounded with a pencil, this grade is zero, and hence $\omega = 1$. Substituting in Noether's formula

$$I + \omega = 12p_n + 9,$$

* F. Enriques, *Rend. Lincei* (5), 23 (1914), 206, 291.

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we thus obtain the value $p_n = b$. Thus the surfaces F are all regular surfaces, having $p_g = p_n = b$. In particular*, the rational surfaces F', which have $b = 0$, have of course $p_g = p_n = 0$.

The third invariant of the surfaces F is their determinant, the least number of points in which a curve of the surface meets each cubic of the pencil lying on the surface. This number is, in general, three: this being the number of points in which a prime-section meets each cubic. But, as we have seen, for every value of p, b there exist surfaces F which have the determinant one; namely the surfaces which contain directrix curves to their pencil of cubics. There exist no surfaces F which have the determinant two. For if a surface F contains a bisecant curve to its pencil of cubics, then those chords of this curve which fall into the planes of the three-fold E generate a rational ruled surface; which meets F residually in a common directrix curve. Hence F also contains a directrix curve to its pencil of cubics, and its determinant is one. Thus for every value of $b > 0$ there exist two essentially distinct types of surfaces F, which cannot be transformed birationally one into the other: surfaces F which have $p_g = p_n = b$ and the determinant three; and surfaces F which have $p_g = p_n = b$ and the determinant one. When $b = 0$, we have shown in § 4 that the surfaces F always contain directrix curves, and therefore have the determinant one. Since every such surface also has $p_g = p_n = 0$, it follows that it is rational. Hence the only non-singular, irreducible surfaces F for which $b = 0$ are the five rational surfaces F' given in § 3.

* The interesting surfaces F which have $b = 1$ have been discussed by P. Du Val, *Rend. Lincei* (6), 15 (1932), 276.

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