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Remarks on structurally stable proper foliations

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Let M be a closed manifold of dimension 3 and let $\mathrm{Fol}(M)$ be the space of codimension one C^∞-foliations on M. A foliation $\mathscr{F} \in \mathrm{Fol}(M)$ is said to be C^r-structurally stable if there exists a neighbourhood V of $\mathscr{F}$ in $\mathrm{Fol}(M)$ in the (Epstein) C^r-topology such that every foliation $\mathscr{G} \in V$ is topologically conjugate to $\mathscr{F}$, through a homeomorphism near to the identity. Some background on the problem of structural stability of foliations can be found in [**8**]. In this paper we shall be concerned with proper foliations, i.e. foliations all of whose leaves are proper.

A necessary condition for the C^1-structural stability of a proper foliation has been obtained by C. Bonatti in [**1**], in the more general context of foliations with Morse singularities. The work of Bonatti motivates the following definition. To simplify, M will be supposed orientable and all foliations will be assumed transversely orientable (and hence orientable).

Definition. A proper foliation $\mathscr{F}$ on M is called a Bonatti foliation if:

(i) every leaf $L \in \mathscr{F}$ has genus 0 or 1; the number of leaves of genus 1 is finite;
(ii) the leaves of genus 0 are without holonomy;
(iii) the leaves of genus 1 have cyclic hyperbolic holonomy.

A C^1-structurally stable proper foliation is necessarily a Bonatti foliation [**1**]. The converse is not true, see for example the case $M = \mathbf{T}^3$ analyzed in [**8**]. However, we prove that the converse is true modulo a perturbation of the foliation, small in the C^0-topology.

Theorem 1. *Every Bonatti foliation can be C^0-approximated by Bonatti foliations which are C^1-structurally stable.*

The main tool, as in [**8**], is the work of N. Kopell on commuting diffeomorphisms of the interval [**7**].

Recall that to every leaf L of a proper foliation $\mathscr{F}$ one may assign a natural number $lev(L)$, called level of L [**2**]: $lev(L) = 0 \Leftrightarrow L$ is compact, and $lev(L) = k+1$ if

$$\forall L' \subset \bar{L} \backslash L \; lev(L') \leqslant k \quad \text{and} \quad lev(L) \neq h \; \forall h \in \{0, \ldots, k\}.$$

The depth of $\mathscr{F}$ is then defined by $dp(\mathscr{F}) = Sup_{L \in \mathscr{F}} lev(L) \in \mathbf{N} \cup \{+\infty\}$.

For a Bonatti foliation $\mathscr{F}$ it is easily checked that leaves of genus 0 have maximal level, leaves of genus 1 have non-maximal level, and $dp(\mathscr{F}) < +\infty$. As a consequence of the previous theorem we will obtain:

Corollary. *For every $k \in \mathbf{N}$ there exists a C^1-structurally stable proper foliation $\mathscr{F}$ (on some 3-manifold M, depending on k) such that $dp(\mathscr{F}) = k$.*

It would be interesting to prove that the approximations in Theorem 1 can be done in the C^1-topology; this is possible, for instance, in the case $M = \mathbf{T}^3$ [**8**]. We give here a result in that direction.

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Let S_k be the sphere with k holes and let $M = S_k \times \mathbf{S}^1$. Let $\mathscr{F}$ be a foliation on M obtained by turbulizing along the boundary the obvious fibration $M \to \mathbf{S}^1$. We call $(M, \mathscr{F})$ a generalized Reeb component. The following generalizes the cases $k = 1$ and $k = 2$ analysed in [**8**].

THEOREM 2. *Let $(M_0, \mathscr{F}_0)$ be a foliated 3-manifold and let $(M, \mathscr{F}) \subset (M_0, \mathscr{F}_0)$ be a generalized Reeb component. Then it is possible to approximate $\mathscr{F}_0$, in the C^∞-topology, with foliations $\mathscr{F}'_0$ such that*: (i) *M is saturated by $\mathscr{F}'_0$ and $\mathscr{F}'_0|_M$ is a generalized Reeb component*; (ii) *$\mathscr{F}'_0$ is C^2-structurally stable near M (i.e there exists a neighbourhood U of M such that $\mathscr{F}'_0|_U$ is C^2-structurally stable*).

Our proof of Theorem 2 gives only the C^2-structural stability; however, it seems possible to choose the approximations $\mathscr{F}'_0$ in such a way that they are C^1-structurally stable near M (as in the case $k = 2$). It seems possible also that Theorem 2 extends to cover the case where M is any S_k-bundle over $\mathbf{S}^1$, not necessarily trivial.

The existence of a structurally stable foliation on a 3-manifold M imposes some restrictions on the topology of M [**8**]. A property weaker than structural stability is structural semistability: $\mathscr{F} \in \mathrm{Fol}(M)$ is C^r-structurally semistable if there exists a C^r-neighbourhood U of $\mathscr{F}$ in $\mathrm{Fol}(M)$ such that for every $\mathscr{G} \in U$ there exists a continuous surjective map $h: M \to M$ (near to id) sending leaves of $\mathscr{G}$ into leaves of $\mathscr{F}$.

Using arguments similar to those employed in Theorem 1, we prove

THEOREM 3. *Every closed orientable 3-manifold M admits a C^1-structurally semistable proper foliation.*

Proof of Theorem 1. We start by recalling some structurally stable proper foliations on $\mathbf{T}^2 \times [0,1]$ [**7**, **8**].

Let $Diff^\infty_+([0,1])$ be the space of orientation preserving C^∞-diffeomorphisms of $[0,1]$. There exists an open and dense subset $\mathscr{Z} \subset Diff^\infty_+([0,1])$, in the C^1-topology, such that every $f \in \mathscr{Z}$ is hyperbolic and for every $f \in \mathscr{Z}$ there are C^1-neighbourhoods U of f and V of id in $Diff^\infty_+([0,1])$ with the following property:

$$Z(\tilde{f}) \cap V = \{id\} \quad \forall \tilde{f} \in U,$$

where $Z(\tilde{f})$ denotes the centralizer of $\tilde{f}$ in $Diff^\infty_+([0,1])$. See appendix and [**7**].

Let α, β be generators of $\pi_1(\mathbf{T}^2) = \mathbf{Z} \oplus \mathbf{Z}$ and let

$$\rho: \pi_1(\mathbf{T}^2) \to Diff^\infty_+([0,1])$$

be a representation such that:

$$\rho(\alpha) = f \in \mathscr{Z} \quad \text{and} \quad \rho(\beta) = id.$$

Then the proper foliation $\mathscr{H}_f$ on $\mathbf{T}^2 \times [0,1]$ obtained by suspending ρ is C^1-structurally stable.

Let now $\mathscr{F}$ be a Bonatti foliation on M. We shall use the following 'reduction' of $\mathscr{F}$ to a simpler foliation ([**1**], lemma 3).

Let $T_1, \ldots, T_n$ be the toral leaves of $\mathscr{F}$. Because the holonomy of every T_j is cyclic and hyperbolic, we may find for every $j = 1, \ldots, n$ an embedding $e_j: \mathbf{T}^2 \times [-1,1] \to M$ such that $e_j(\mathbf{T}^2 \times \{0\}) = T_j$ and $e_j(\mathbf{T}^2 \times \{t\})$ is transverse to $\mathscr{F}$ for every $t \neq 0$. We remove from M all the images $e_j(\mathbf{T}^2 \times (-1,1))$, obtaining a manifold with boundary M' (perhaps non-connected) equipped with a foliation $\mathscr{F}'$ transverse to $\partial M'$.

On every component of $\partial M'$, a foliation by circles is induced by $\mathscr{F}'$. We glue to

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every such component a solid torus $\mathbf{D}^2 \times \mathbf{S}^1$ equipped with the trivial foliation by discs. The result is a closed manifold M_1 together with a foliation $\mathscr{F}_1$. This foliation is still a Bonatti foliation, and moreover

$$dp(\mathscr{F}_1) = dp(\mathscr{F}) - 1.$$

Denote by A_j, $j = 1, \ldots, n$, the components of $\partial M'$ arising from $e_j(\mathbf{T}^2 \times \{1\})$ and by B_j, $j = 1, \ldots, n$, those arising from $e_j(\mathbf{T}^2 \times \{-1\})$. Denote by P_j the solid tori glued along A_j, $j = 1, \ldots, n$, and by Q_j those glued along B_j, $j = 1, \ldots, n$. Let γ_j be the circle $\{0\} \times \mathbf{S}^1 \subset \mathbf{D}^2 \times \mathbf{S}^1 \simeq P_j$, and η_j the circle $\{0\} \times \mathbf{S}^1 \subset \mathbf{D}^2 \times \mathbf{S}^1 \simeq Q_j$. Then the foliation $\mathscr{F}$ can be reconstructed from $\mathscr{F}_1$ through a removal of tubular neighbourhoods V_j, U_j of γ_j, η_j, a turbulization along $\partial V_j, \partial U_j$, and identifications $\partial V_j \simeq \partial U_j$, $j = 1, \ldots, n$.

We repeat the above constructions obtaining foliations $\mathscr{F}_1, \mathscr{F}_2, \ldots, \mathscr{F}_N$, with $dp(\mathscr{F}_N) = 0$, *i.e.* $\mathscr{F}_N$ is a finite collection of trivial foliations by spheres of $\mathbf{S}^2 \times \mathbf{S}^1$ (N is the depth of $\mathscr{F}$). Every $\mathscr{F}_j$ is a Bonatti foliation. The foliation $\mathscr{F}_k$ can be reconstructed from $\mathscr{F}_{k+1}$ through a removal of tubular neighbourhoods of closed transversals, turbulization along the boundary, and identifications of boundary components.

We modify the foliation $\mathscr{F}_{N-1}$ in the following way: we 'thicken' every toral leaf $T \in \mathscr{F}_{N-1}$ to a saturated set $\hat{T} \simeq \mathbf{T}^2 \times [0, 1]$, equipped with a foliation $\mathscr{H}_f$, $f \in \mathscr{Z}$, of the type described before. This is possible because T has hyperbolic cyclic holonomy. Denote by $\hat{\mathscr{F}}_{N-1}$ the result of such a modification; everything can be done in such a way that $\hat{\mathscr{F}}_{N-1}$ is near to $\mathscr{F}_{N-1}$ in the C^0-topology.

Now we perform on $\hat{\mathscr{F}}_{N-1}$ the same operations which transform $\mathscr{F}_{N-1}$ into $\mathscr{F}_{N-2}$ (removal of tubular neighbourhoods of closed transversals, turbulization, identifications of boundary components), obtaining a foliation $\tilde{\mathscr{F}}_{N-2}$. Again, we modify $\tilde{\mathscr{F}}_{N-2}$ through a thickening of the toral leaves, obtaining a foliation $\hat{\mathscr{F}}_{N-2}$ near to $\mathscr{F}_{N-2}$ in the C^0-topology. It would be sufficient to thicken only those toral leaves of $\tilde{\mathscr{F}}_{N-2}$ which arise from the transformation $\hat{\mathscr{F}}_{N-1} \to \tilde{\mathscr{F}}_{N-2}$, but remark that every toral leaf of $\tilde{\mathscr{F}}_{N-2}$ arises from such a transformation (by construction of the $\mathscr{F}_j$).

After a finite number of steps we finally obtain a foliation $\hat{\mathscr{F}}$, near to $\mathscr{F}$ in the C^0-topology, which is clearly a Bonatti foliation. We claim that $\hat{\mathscr{F}}$ is C^1-structurally stable.

The claim is proved by recurrence on $\hat{\mathscr{F}}_0 = \hat{\mathscr{F}}, \hat{\mathscr{F}}_1, \ldots, \hat{\mathscr{F}}_{N-1}, \hat{\mathscr{F}}_N = \mathscr{F}_N$ (it is coherent to set $\hat{\mathscr{F}}_N = \mathscr{F}_N$). The foliation $\hat{\mathscr{F}}_N$ is C^1-structurally stable by Reeb's stability theorem. Suppose that $\hat{\mathscr{F}}_k$ is C^1-structurally stable ($1 \leqslant k \leqslant N$); we have to prove that $\hat{\mathscr{F}}_{k-1}$ is also C^1-structurally stable.

Let $\mathscr{G}_{k-1}$ be a foliation C^1-near to $\hat{\mathscr{F}}_{k-1}$. Let $\hat{T} \simeq \mathbf{T}^2 \times [0, 1]$ be a thickened torus of $\hat{\mathscr{F}}_{k-1}$. Then (by Hirsch's theorem, [**6**]) $\mathscr{G}_{k-1}$ has a saturated set $\hat{S} \simeq \mathbf{T}^2 \times [0, 1]$ near $\hat{T}$, and $\mathscr{G}_{k-1}|_{\hat{S}}$ is C^1-near to $\hat{\mathscr{F}}_{k-1}|_{\hat{T}}$. The choice of $\hat{\mathscr{F}}_{k-1}|_{\hat{T}}$ implies that $\mathscr{G}_{k-1}|_{\hat{S}}$ is C^0-conjugate to $\hat{\mathscr{F}}_{k-1}|_{\hat{T}}$.

In particular, the two toral leaves of $\mathscr{G}_{k-1}$ composing $\partial \hat{S}$ have hyperbolic cyclic holonomy, as the two toral leaves of $\hat{\mathscr{F}}_{k-1}$ composing $\partial \hat{T}$. It follows that $\mathscr{G}_{k-1}$ may be obtained from a foliation $\mathscr{G}_k$ C^1-near to $\hat{\mathscr{F}}_k$ through the same operations which transform $\hat{\mathscr{F}}_k$ into $\hat{\mathscr{F}}_{k-1}$ (compare [**8**], lemma 1.4). Because $\hat{\mathscr{F}}_k$ is C^1-structurally stable, $\mathscr{G}_k$ is C^0-conjugate to $\hat{\mathscr{F}}_k$ and hence $\mathscr{G}_{k-1}|_{\hat{S}^c}$ is C^0-conjugate to $\hat{\mathscr{F}}_{k-1}|_{\hat{T}^c}$. Together with the C^0-conjugation between $\mathscr{G}_{k-1}|_{\hat{S}}$ and $\hat{\mathscr{F}}_{k-1}|_{\hat{T}}$, this gives the C^1-structural stability of $\hat{\mathscr{F}}_{k-1}$.

Proof of the corollary. The approximations founded in the theorem respect the

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depth of the foliation, hence it is sufficient to construct for every $k \in \mathbf{N}$ a Bonatti foliation $\mathscr{F}_k$ with $dp(\mathscr{F}_k) = k$.

We may set $\mathscr{F}_0 =$ trivial foliation of $\mathbf{S}^2 \times \mathbf{S}^1$, $\mathscr{F}_1 =$ foliation of $\mathbf{T}^3$ obtained by suspending $\rho : \pi_1(\mathbf{T}^2) \to Diff_+^\infty(\mathbf{S}^1)$, $\rho(\beta) = id$, $\rho(\alpha) =$ diffeomorphism with two hyperbolic fixed points. The foliation $\mathscr{F}_k$, $k \geqslant 2$, is constructed from $\mathscr{F}_{k-1}$ as follows.

Take two parallel closed curves γ, η transverse to $\mathscr{F}_{k-1}$ and intersecting all the toral leaves of $\mathscr{F}_{k-1}$ (these curves clearly exist for $\mathscr{F}_1$). Remove tubular neighbourhoods U and V of γ and η, and turbulize along ∂U and ∂V, but in opposite directions, so that there exists a segment transverse to the foliation and joining ∂U with ∂V. Denote by $\mathscr{F}_k$ the double of the foliation so obtained; it is a Bonatti foliation. Moreover $\mathscr{F}_k$ has only two toral leaves and it admits two parallel closed transversals intersecting these toral leaves, hence the construction can be iterated.

Because $dp(\mathscr{F}_k) = dp(\mathscr{F}_{k-1}) + 1$, the corollary is proved.

Remark. A Bonatti foliation $\mathscr{F}$ on M does not admit closed transversals homotopic to zero and hence the growth of any leaf is dominated by the growth of $\pi_1(M)$; recall that for a leaf L of a proper foliation $lev(L) = k \Leftrightarrow L$ has polynomial growth of degree k [**2**]. It follows that if $\pi_1(M)$ has polynomial growth of degree k then every Bonatti foliation on M has depth less than or equal to k.

On the other hand, the arguments of [**5**, sect. 4], show that if $M \neq \mathbf{S}^2 \times \mathbf{S}^1$ admits a Bonatti foliation (which can be assumed without Reeb components, [**8**]) then either $\pi_1(M)$ contains a free subgroup on two generators ($\Rightarrow \pi_1(M)$ has exponential growth) or M is a $\mathbf{T}^2$-bundle over $\mathbf{S}^1$ with monodromy $\left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} -1 & m \\ 0 & -1 \end{smallmatrix}\right)$, $m \in \mathbf{Z}$ (a priori, M could be also a union of two nontrivial $[0, 1]$-bundles over the Klein bottle glued along their boundary tori, but then the hyperbolicity of the toral leaves implies that M is still a $\mathbf{T}^2$-bundle with monodromy $\left(\begin{smallmatrix} -1 & m \\ 0 & -1 \end{smallmatrix}\right)$, $m \in \mathbf{Z}$). Bonatti foliations on these manifolds are easily classified [**5**, **8**]. It follows that if $\mathscr{F}$ is a Bonatti foliation on M without Reeb components and with $dp(\mathscr{F}) \geqslant 2$ (or with Reeb components and $dp(\mathscr{F}) \geqslant 3$), then $\pi_1(M)$ contains $\mathbf{Z} * \mathbf{Z}$.

Proof of Theorem 2. We may assume that the toral leaves $T_1, \ldots, T_k$ which compose ∂M have hyperbolic holonomy.

Let $\Gamma \subset M = S_k \times \mathbf{S}^1$ be an embedded circle, transverse to $\mathscr{F}$ and given by a fibre of the fibration $M \to S_k$. The circle Γ can be identified with the space of leaves of the foliation $\mathscr{F}|_{int\, M}$. In particular, every diffeomorphism of Γ can be extended to a diffeomorphism of $int M$ which preserves $\mathscr{F}|_{int\, M}$. On the other hand, by Sternberg's linearization theorem and by the fact that a germ of diffeomorphism of $\mathbf{R}$ commuting with a hyperbolic linear one is also linear [**7**, **9**], it follows that the set of orientation preserving diffeomorphisms of Γ which extend to $\mathscr{F}$-preserving diffeomorphisms of $T_j \cup int M$ is reduced to a subgroup $R_j \subset Diff_+^\infty(\Gamma)$ isomorphic to $\mathbf{S}^1$ (in a neighbourhood of L_j the foliation is given by the suspension of a germ of a hyperbolic action of $\mathbf{Z} \oplus \mathbf{Z}$ on $\mathbf{R}$). More precisely, we may choose diffeomorphisms

$$\psi_j : \mathbf{S}^1 \to \Gamma$$

such that $\psi_j^{-1} \circ R_j \circ \psi_j = Rot(\mathbf{S}^1) =$ rigid rotations of the circle.

These 'linearizing' diffeomorphisms are uniquely defined up to composition with rigid rotations of $\mathbf{S}^1$ (if ψ_j and $\bar{\psi}_j$ are two such diffeomorphisms then

$$\psi_j^{-1} \circ \bar{\psi}_j \in Rot(\mathbf{S}^1)).$$

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The diffeomorphisms

$$\eta_j = \psi_j^{-1} \circ \psi_{j+1} \in Diff_+^\infty(\mathbf{S}^1), \quad j = 1, \ldots, k-1$$

are hence uniquely defined up to the following equivalence relation:

$$\{\eta_1, \ldots, \eta_{k-1}\} \sim \{\bar{\eta}_1, \ldots, \bar{\eta}_{k-1}\} \quad \text{if} \quad \exists r_1, \ldots, r_k \in Rot(\mathbf{S}^1) \text{ such that}$$
$$\bar{\eta}_j = r_j \circ \eta_j \circ r_{j+1}^{-1}, \quad \forall j = 1, \ldots, k-1.$$

It is not difficult to see that every collection $\{\eta_1, \ldots, \eta_{k-1}\} \subset Diff_+^\infty(\mathbf{S}^1)$ can be realized by some foliation $\mathscr{F}$. The collection $\{\eta_1, \ldots, \eta_{k-1}\}$ (modulo the above equivalence relation) must be understood as a generalization of Mather invariant for interval diffeomorphisms [**10**, ch. VI], to which it reduces for $k = 2$.

Let now $\tilde{\mathscr{F}}_0$ be a C^2-near to $\mathscr{F}_0$. By Hirsch's theorem we may suppose that M is saturated also by $\tilde{\mathscr{F}}_0$; $\tilde{\mathscr{F}} \overset{\text{def}}{=} \tilde{\mathscr{F}}_0|_M$ is still transverse to the circle Γ, and the structural stability of $\mathscr{F}_0$ near M is equivalent to that of $\mathscr{F}$.

Let $V \subset S_k$ be an open collar of ∂S_k, let $\hat{S}_k = S_k \backslash V$ and let $\hat{M} = \hat{S}_k \times \mathbf{S}^1 \subset M$. We may assume that $\Gamma \subset \hat{M}$. The boundary of $\hat{M}$ is formed by tori $\hat{T}_1, \ldots, \hat{T}_k$ ($\hat{T}_j$ is near to T_j) and every $\hat{T}_j$ is transverse to $\mathscr{F}$. The foliation $\mathscr{F}$ induces on every $\hat{T}_j$ a foliation by circles. We may assume that $\tilde{\mathscr{F}}$ is also transverse to every $\hat{T}_j$, and then $\tilde{\mathscr{F}}$ induces on every $\hat{T}_j$ a foliation C^∞ conjugate to a linear one (Sternberg's theorem: the holonomy of $\tilde{\mathscr{F}}$ along T_j can be linearized).

More precisely, let γ_j be an element of $\pi_1(\hat{S}_k)$ (as a base point we take the point of $\hat{S}_k$ corresponding to Γ), freely homotopic to the component of $\partial \hat{S}_k$ corresponding to $\hat{T}_j$. Let $\tilde{\rho}(\gamma_j) \in Diff_+^\infty(\Gamma)$ be the holonomy of $\tilde{\mathscr{F}}$ along γ_j, which is a diffeomorphism C^2-near to the identity. The diffeomorphism $\psi_j^{-1} \circ \tilde{\rho}(\gamma_j) \circ \psi_j \in Diff_+^\infty(\mathbf{S}^1)$ is conjugate to a rigid rotation of angle α_j, R_{α_j}, through a diffeomorphism $\phi_j \in Diff_+^\infty(\mathbf{S}^1)$:

$$\psi_j^{-1} \circ \tilde{\rho}(\gamma_j) \circ \psi_j = \phi_j \circ R_{\alpha_j} \circ \phi_j^{-1}.$$

The angle α_j is small, and ϕ_j is near to id but only in the C^1-topology, because the diffeomorphism which linearizes a hyperbolic diffeomorphism C^2 near to a linear one is only C^1 near to id [**9**]. In other words, there exists on Γ a new coordinate $\tilde{\psi}_j : \mathbf{S}^1 \to \Gamma$, which linearizes $\tilde{\rho}(\gamma_j)$, but $\tilde{\psi}_j$ ($= \psi_j \circ \phi_j$) is only C^1-near to ψ_j.

Take now generators $\gamma_1, \ldots, \gamma_k$ of $\pi_1(\hat{S}_k)$ such that $\pi_1(\hat{S}_k) = \langle \gamma_1, \ldots, \gamma_k \,|\, \gamma_1 * \ldots * \gamma_k \rangle$. Then we have the relation $\tilde{\rho}(\gamma_1) \circ \ldots \circ \tilde{\rho}(\gamma_k) = id$, i.e.

$$\tilde{\psi}_1 \circ R_{\alpha_1} \circ \tilde{\psi}_1^{-1} \circ \tilde{\psi}_2 \circ R_{\alpha_2} \circ \tilde{\psi}_2^{-1} \circ \ldots \circ \tilde{\psi}_k \circ R_{\alpha_k} \circ \tilde{\psi}_k^{-1} = id$$

or

$$R_{\alpha_1} \circ \omega_1 \circ R_{\alpha_2} \circ \omega_2 \circ \ldots \circ \omega_{k-1} \circ R_{\alpha_k} = \omega_1 \circ \omega_2 \circ \ldots \circ \omega_{k-1},$$

where $\{\alpha_j\}$ are small and every ω_j ($= \tilde{\psi}_j^{-1} \circ \tilde{\psi}_{j+1}$) is C^1-near to η_j.

We have to prove that (for a C^∞-dense choice of $\eta_1, \ldots, \eta_{k-1}$) this relation implies $\alpha_j = 0$ for all $j = 1, \ldots, k$, i.e. $\tilde{\rho}(\gamma_j) = id$ for all $j = 1, \ldots, k$. From this fact the structural stability of $\mathscr{F}$ follows easily [**8**]. So the proof of Theorem 2 is completed by the following

LEMMA. *There exists a C^∞-dense subset $\Omega \subset Diff_+^\infty(\mathbf{S}^1)^{k-1}$ such that for every $(\eta_1, \ldots, \eta_{k-1}) \in \Omega$ there exists $\epsilon > 0$ such that the relation*

$$R_{\alpha_1} \circ \omega_1 \circ R_{\alpha_2} \circ \omega_2 \circ \ldots \circ \omega_{k-1} \circ R_{\alpha_k} = \omega_1 \circ \omega_2 \circ \ldots \circ \omega_{k-1},$$

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with $\|\omega_j-\eta_j\|_{C^1}<\epsilon$ *for all* $j=1,\dots,k-1$, $|\alpha_j|<\epsilon$ *for all* $j=1,\dots,k$, *implies* $\alpha_j=0$ *for all* $j=1,\dots,k$.

Proof. Let $Diff^l_+(\mathbf{S}^1)$ be given with the C^l-topology and consider the map

$$F:\mathbf{T}^k\times Diff^1_+(\mathbf{S}^1)^{k-1}\to Diff^0_+(\mathbf{S}^1)\times Diff^1_+(\mathbf{S}^1)^{k-1}$$

$$(\alpha_1,\dots,\alpha_k,\eta_1,\dots,\eta_{k-1})\mapsto(G(\alpha_1,\dots,\alpha_k,\eta_1,\dots,\eta_{k-1}),\eta_1,\dots,\eta_{k-1}),$$

where $$G(\alpha_1,\dots,\alpha_k,\eta_1,\dots,\eta_{k-1})=R_{\alpha_1}\circ\eta_1\circ R_{\alpha_2}\circ\dots\circ\eta_{k-1}\circ R_{\alpha_k}.$$

A straightforward computation shows that F is of class C^1. Moreover, the derivative of F at $(\underline{0},\underline{\eta})=(0,\dots,0,\eta_1,\dots,\eta_{k-1})$ is given by

$$T_{(\underline{0},\underline{\eta})}F:\mathbf{R}^k\times\chi^1_{\eta_1}(\mathbf{S}^1)\times\dots\times\chi^1_{\eta_{k-1}}(\mathbf{S}^1)\to\chi^0_{\eta_1\circ\dots\circ\eta_{k-1}}(\mathbf{S}^1)\times\chi^1_{\eta_1}(\mathbf{S}^1)\times\dots\times\chi^1_{\eta_{k-1}}(\mathbf{S}^1)$$

$$(c_1,\dots,c_k,v_1,\dots,v_{k-1})\mapsto(H_{\underline{\eta}}(c_1,\dots,c_k,v_1,\dots,v_{k-1}),v_1,\dots,v_{k-1}),$$

where $\chi^k_\eta(\mathbf{S}^1)$ denotes the space of C^k-vector fields along η (with the C^k-topology) and

$$\begin{aligned}H_{\underline{\eta}}(c_1,\dots,c_k,v_1,\dots,v_{k-1})&=c_1+v_1\circ\eta_2\circ\dots\circ\eta_{k-1}+\eta_1'((\eta_2\circ\dots\circ\eta_{k-1})(\cdot))\\&\cdot[c_2+v_2\circ\eta_3\circ\dots\eta_{k-1}+\eta_2'((\eta_3\circ\dots\circ\eta_{k-1})(\cdot))\\&\cdot[c_3+\dots\cdot[c_{k-1}+v_{k-1}+\eta_{k-1}'(\cdot)\cdot c_k]]\dots]].\end{aligned}$$

Setting $n_j=\eta_j\circ\eta_{j+1}\circ\dots\circ\eta_{k-1}$ for all $j=1,\dots,k-1$:

$$H_{\underline{\eta}}(c_1,\dots,c_k,v_1,\dots,v_{k-1})=n_1'(\cdot)\cdot\left[\frac{c_1+v_1\circ n_2}{n_1'}+\frac{c_2+v_2\circ n_3}{n_2'}+\dots+\frac{c_{k-1}+v_{k-1}}{n_{k-1}'}+c_k\right].$$

The linear operator $T_{(\underline{0},\underline{\eta})}F$ is injective if and only if

$$\frac{c_1}{n_1'}+\frac{c_2}{n_2'}+\dots+\frac{c_{k-1}}{n_{k-1}'}+c_k=0\Rightarrow c_j=0\quad\forall j=1,\dots,k.$$

Hence we define

$$\Omega=\{(\eta_1,\dots,\eta_{k-1})\in Diff^\infty_+(\mathbf{S}^1)^{k-1}\mid\text{the functions }\{1/n_j'\}_{j=1}^k,\text{ where }n_j=\eta_j\circ\dots\circ\eta_{k-1}$$
$$\forall j=1,\dots,k-1,n_k=id,\text{ are }\mathbf{R}\text{-linearly independent}\}.$$

The set Ω is clearly C^∞-dense in $Diff^\infty_+(\mathbf{S}^1)^{k-1}$.

Moreover, for every $(\eta_1,\dots,\eta_{k-1})$ the linear map $T_{(\underline{0},\underline{\eta})}F$ has closed image: $Im\,T_{(\underline{0},\underline{\eta})}F$ is the direct sum of the images of the two linear maps

$$A_{\underline{\eta}}:(v_1,\dots,v_{k-1})\mapsto\left(\sum_{j=1}^{k-1}\frac{n_1'}{n_j'}v_j\circ n_{j+1},v_1,\dots,v_{k-1}\right)$$

and

$$B_{\underline{\eta}}:(c_1,\dots,c_k)\mapsto\left(\sum_{j=1}^{k}\frac{n_1'}{n_j'}c_j,0,\dots,0\right).$$

But $Im\,A_{\underline{\eta}}$ is clearly closed (it is the graph of the linear continuous map

$$(v_1,\dots,v_{k-1})\mapsto\sum_{j=1}^{k-1}\frac{n_1'}{n_j'}v_j\circ n_{j+1})$$

and $Im\,B_{\underline{\eta}}$ is closed because finite dimensional. Hence $Im\,T_{(\underline{0},\underline{\eta})}F$ is also closed.

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We shall use the following result, which is stated as an exercise in [3], chapter X, section 3, problem 1*a*, but for which we are unable to give a reference with a proof.

SUBLEMMA. *Let X, Y be Banach spaces, $V \subset X$ open, $f: V \to Y$ a C^1 map. Let $\bar{x} \in V$ be a point where $T_{\bar{x}}f: X \to Y$ is injective and with closed image. Then there exists a neighbourhood U of $\bar{x}$ such that f is injective in U.*

Proof. By Banach's theorem $T_{\bar{x}}f$ is an isomorphism onto its image and so there exists $\lambda > 0$ such that $\|(T_{\bar{x}}f)(x)\| \geqslant \lambda\|x\|$ for x in X. Set $\phi(x) = f(x) - (T_{\bar{x}}f)(x - \bar{x})$ for x in V. From $T_{\bar{x}}\phi = 0$ and $\|\phi(x_1) - \phi(x_2)\| \leqslant \sup_{x \in [x_1, x_2]} \|T_x\phi\| \cdot \|x_1 - x_2\|$, there exists a neighbourhood U of $\bar{x}$ such that $\|\phi(x_1) - \phi(x_2)\| < \frac{1}{2}\lambda\|x_1 - x_2\|$ $\forall x_1, x_2 \in U$. Now

$$\|f(x_1) - f(x_2)\| \geqslant \big|\, \|(T_{\bar{x}}f)(x_1 - x_2)\| - \|\phi(x_1) - \phi(x_2)\| \,\big| \geqslant (\lambda - \tfrac{1}{2}\lambda)\, \|x_1 - x_2\|$$

for every $x_1, x_2 \in U$, and this implies the injectivity of f in U.

By the sublemma above, if $(\eta_1, \ldots, \eta_{k-1}) \in \Omega$ then F is injective in a neighbourhood of $(\underline{0}, \underline{\eta})$, i.e. there exists $\epsilon >$ s.t.

$$F(\underline{\alpha}, \underline{\omega}) = F(\underline{0}, \underline{\omega}), \quad |\alpha_j| < \epsilon, \quad \|\omega_j - \eta_j\|_{C^1} < \epsilon \quad \forall j \Rightarrow \underline{\alpha} = \underline{0}$$

or $\quad R_{\alpha_1} \circ \omega_1 \circ \ldots \circ \omega_{k-1} \circ R_{\alpha_k} = \omega_1 \circ \ldots \circ \omega_{k-1}, |\alpha_j| < \epsilon, \|\omega_j - \eta_j\|_{C^1} < \epsilon \quad \forall j \Rightarrow \underline{\alpha} = \underline{0}.$

This completes the proof.

Remark. If we fix $(\eta_1, \ldots, \eta_{k-1}) \in \Omega$ then the map

$$F(\cdot, \underline{\omega}): \mathbf{T}^k \to Diff^0_+(\mathbf{S}^1) \times Diff^1_+(\mathbf{S}^1)^{k-1}$$

is C^1 and injective near $\underline{0} \in \mathbf{T}^k$ for every $\underline{\omega}$ C^0-near to $\underline{\eta}$ (because Ω is C^0-open). Hence for every $\underline{\omega}$ C^0-near to $\underline{\eta}$ there exists $\epsilon = \epsilon(\underline{\omega})$ such that

$$R_{\alpha_1} \circ \omega_1 \circ \ldots \circ \omega_{k-1} \circ R_{\alpha_k} = \omega_1 \circ \ldots \circ \omega_{k-1}, |\alpha_j| < \epsilon \, \forall j \Rightarrow \underline{\alpha} = \underline{0}.$$

If we were able to show that ϵ is uniform in $\underline{\omega}$, then we would obtain the C^1-structural stability.

Remark. The condition '$(\eta_1, \ldots, \eta_{k-1}) \in \Omega$' is well defined, in the sense that it is invariant by the equivalence relation $\{\eta_1, \ldots, \eta_{k-1}\} \sim \{\bar{\eta}_1, \ldots, \bar{\eta}_{k-1}\}$ introduced above.

Proof of Theorem 3. We use the construction of foliations based on the Lickorish–Wallace theorem [4]: every closed orientable 3-manifold M can be obtained by gluing together an S_k-bundle M_0 over $\mathbf{S}^1$ with solid tori $N_1, \ldots, N_h$.

Every solid torus N_j is foliated as a Reeb component, with ∞-flat holonomy. The component M_0 is foliated by a hyperbolic turbulization along ∂M_0 of the S_k-fibration. Between every N_j and M_0 we insert a component C_j diffeomorphic to $\mathbf{T}^2 \times [0, 1]$ and with a foliation $\mathscr{H}_j$ given by the suspension of a representation $\rho_j: \pi_1(\mathbf{T}^2) \to Diff^\infty_+([0, 1])$. If α, β are generators of $\pi_1(\mathbf{T}^2)$ and $f_j = \rho_j(\alpha)$, $g_j = \rho_j(\beta)$, then we require:

(1) $g_j = id$,

(2) $Fix(f_j) = \{0, \frac{1}{2}, 1\}$, 0 and $\frac{1}{2}$ are hyperbolic, f_j is ∞-tangent to id at 1,

(3) $f_j|_{[0, \frac{1}{2}]}$ belongs to $\mathscr{Z}$ (see proof of Theorem 1), modulo the identification $[0, \frac{1}{2}] \simeq [0, 1]$.

We denote by D_j the subset of $C_j \simeq \mathbf{T}^2 \times [0, 1]$ corresponding to $\mathbf{T}^2 \times [0, \frac{1}{2}]$. Clearly, $\mathbf{T}^2 \times \{0\}$ is glued with M_0 and $\mathbf{T}^2 \times \{1\}$ is glued with N_j, in such a way that the result

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is a smooth proper foliation $\mathscr{F}$. We may suppose that the holonomy of every leaf ∂N_j contains a contracting element, so that by Hirsch's theorem these leaves persist after C^1-perturbations.

We claim that $\mathscr{F}$ is C^1-structurally semistable.

Let $\mathscr{G}$ be C^1-near to $\mathscr{F}$. The choice of f_j, $j = 1, \dots, h$, implies, as in Theorem 1, that $\mathscr{G}$ has a saturated set $\tilde{M}_1$, near to $M_1 = M_0 \cup_j D_j$, such that $\mathscr{G}|_{\tilde{M}_1}$ is topologically conjugate to $\mathscr{F}|_{M_1}$. In particular, there exists a saturated set $\tilde{M}_0$, near to M_0, such that $\mathscr{G}|_{\tilde{M}_0}$ is topologically conjugate to $\mathscr{F}|_{M_0}$. Moreover (for every $j = 1, \dots, h$) $\mathscr{G}$ has a Reeb component $\tilde{N}_j$ near N_j, by the remark above on the existence of contracting elements in the holonomy of ∂N_j.

We may assume $\tilde{M}_1 = M_1$, $\tilde{M}_0 = M_0$ and $\tilde{N}_j = N_j$ for all j.

Every C_j is foliated by $\mathscr{G}$ by the suspension of a representation

$$\tilde{\rho}_j : \pi_1(\mathbf{T}^2) \to Diff_+^\infty([0,1]),$$

C^1-near to ρ_j. Let $\tilde{f}_j = \tilde{\rho}_j(\alpha)$, $\tilde{g}_j = \tilde{\rho}_j(\beta)$. Then $\tilde{g}_j = id$ on $[0, \frac{1}{2}]$, and $0, \frac{1}{2}$ are hyperbolic fixed points of $\tilde{f}_j$. Every $\tilde{f}_j$ has a fixed point $a_j \in (\frac{1}{2}, 1]$ (near to 1) such that $\tilde{f}_j$ is without fixed points on $(\frac{1}{2}, a_j)$. Because $\tilde{g}_j = id$ on $[0, \frac{1}{2}]$, we must have also $\tilde{g}_j = id$ on $[\frac{1}{2}, a_j]$, $\frac{1}{2}$ being hyperbolic for $\tilde{f}$ [7]. It follows that $\mathscr{G}|_{\mathbf{T}^2 \times [0, a_j]}$ is topologically conjugate to $\mathscr{F}|_{C_j}$, for every $j = 1, \dots, h$.

From this one easily deduces that there exists a surjective continuous map $h : M \to M$ sending leaves of $\mathscr{G}$ to leaves of $\mathscr{F}$ (the set $\mathbf{T}^2 \times [a_j, 1]$ is mapped to $\mathbf{T}^2 \times \{1\}$).

Appendix

In [7, pp. 171–174], the existence is proved of an open and dense subset

$$\mathscr{W} \subset Diff_+^\infty([0,1])$$

in the C^r-topology, $r \geqslant 2$, such that every $f \in \mathscr{W}$ has trivial centralizer. We prove here a similar statement in the C^1-topology, which is sufficient to construct C^1-structurally stable actions of $\mathbf{Z} \oplus \mathbf{Z}$ on $[0,1]$ and hence C^1-structurally stable foliations on $\mathbf{T}^2 \times [0,1]$.

As in [7], it is sufficient to consider only the open subset $\mathscr{D} \subset Diff_+^\infty([0,1])$ composed by hyperbolic diffeomorphisms f such that $f(x) > x \; \forall x \in (0,1)$.

Let $\mathscr{Z}_0 \subset \mathscr{D}$ be the subset of $\mathscr{D}$ composed by diffeomorphisms f which satisfy the following condition:

there exist $h \in Diff_+^\infty([0,1])$, $a, b, z \in (0,1)$, $t \in (0, +\infty)$, $N \in \mathbf{N}$, such that

(i) $f_0 \overset{\text{def}}{=} h \circ f \circ h^{-1}$ is linear on $[0,a] \cup [b,1]$:

$$f_0(x) = \lambda x \quad \forall x \in [0,a] \quad (\lambda = f'(0) > 1)$$
$$f_0(x) = 1 - \mu(1-x) \quad \forall x \in [b,1] \quad (\mu = f'(1) < 1),$$

(ii) $\lambda^t z \in (0,a)$, $\quad f_0^N(z) \in (b,1)$,

(iii) $$\frac{1 - f_0^N(\lambda^t z)}{1 - f_0^N(z)} \neq \mu^t.$$

PROPOSITION. (1) *$\mathscr{Z}_0$ is C^1-open and C^∞-dense in $\mathscr{D}$.* (2) *For every $f \in \mathscr{Z}_0$ there exist C^1-neighbourhoods U of f and V of id in $Diff_+^\infty([0,1])$ such that $Z(\tilde{f}) \cap V = \{id\}$ for all $\tilde{f} \in U$.* (3) *$\mathscr{Z}_0$ coincides with the set $\mathscr{W}_0 = \{f \in \mathscr{D} | Z(f)$ is discrete$\}$.*

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Proof. The C^∞-density is clear. Let $\tilde f$ be C^1-near to $f\in\mathscr{Z}_0$, then there exists, by Sternberg's theorem, an $\tilde h\in Diff^\infty_+([0,1])$ C^0-near to h such that $\tilde f_0 \overset{\text{def}}{=} \tilde h\circ\tilde f\circ\tilde h^{-1}$ is linear on $[0,\tilde a]\cup[\tilde b,1]$, with $\tilde a,\tilde b$ near to a,b. Hence $\tilde f_0$ is C^0-near to f_0, and moreover $\tilde\lambda=\tilde f'(0)$, $\tilde\mu=\tilde f'(1)$ are near to λ,μ. This guarantees that, choosing $\tilde f$ sufficiently C^1-near to f, we still have

$$\tilde\lambda^t z\in(0,\tilde a),\quad \tilde f_0^N(z)\in(\tilde b,1),\quad \frac{1-\tilde f_0^N(\tilde\lambda^t z)}{1-\tilde f_0^N(z)}\neq\tilde\mu^t.$$

Hence $\tilde f\in\mathscr{Z}_0$ ($\tilde z=z$, $\tilde t=t$, $\tilde N=N$), and this proves that $\mathscr{Z}_0$ is C^1-open.

Remark that if $f\in\mathscr{Z}_0$ then there exists $\epsilon>0$ such that

$$\lambda^s z\in(0,a),\quad \frac{1-f_0^N(\lambda^s z)}{1-f_0^N(z)}\neq\mu^s\quad\text{for } s\in[t-\epsilon,t+\epsilon].$$

The same arguments as before show that this remains true for every $\tilde f$ sufficiently C^1-near to f (so that $\tilde f_0$ is sufficiently C^0-near to f_0):

$$\tilde\lambda^s z\in(0,\tilde a),\quad \frac{1-\tilde f_0^N(\tilde\lambda^s z)}{1-\tilde f_0^N(z)}\neq\tilde\mu^s\quad\text{for } s\in[t-\epsilon,t+\epsilon].$$

Let $g\in Z(\tilde f_0)$, and suppose $g'(0)=l=\tilde\lambda^r$, $r\in[t-\epsilon,t+\epsilon]$; then g must be linear on neighbourhoods of 0 and 1 [7]. More precisely one finds:

$$g(x)=lx\quad\text{for } x\in\left[0,\frac{\tilde a}{l}\right]$$

$$g(x)=1-m(1-x)\quad\text{for } x\in[\tilde b,1],$$

where $m=g'(0)$. The relation $g\circ\tilde f_0=\tilde f_0\circ g$ implies $m=\tilde\mu^r$ [7]. The property $\tilde\lambda^r z\in(0,\tilde a)$ shows that $g(z)=\tilde\lambda^r z$. Now,

$$\frac{1-\tilde f_0^N(\tilde\lambda^r z)}{1-\tilde f_0^N(z)}=\frac{1-\tilde f_0^N(g(z))}{1-\tilde f_0^N(z)}=\frac{1-g(\tilde f_0^N(z))}{1-\tilde f_0^N(z)}=m=\tilde\mu^r$$

and we have reached a contradiction. This shows that if $g\in Z(\tilde f_0)$ then $g'(0)$ cannot belong to $\{\tilde\lambda^s\mid s\in[t-\epsilon,t+\epsilon]\}$. Because $Z(\tilde f)$ is conjugate to $Z(\tilde f_0)$, the same is true for $g\in Z(\tilde f)$. So, $Z(\tilde f)$ being a group, if $g\in Z(\tilde f)$ then $g'(0)$ cannot belong to $[\tilde\lambda^{-\epsilon},1)\cup(1,\tilde\lambda^{\epsilon}]$. As $\tilde f$ varies in a C^1-neighbourhood of f, $\tilde\lambda$ varies in a neighbourhood of λ; hence there is $\delta>0$, depending only on f, such that for every $\tilde f$ sufficiently C^1-near to f and for every $g\in Z(\tilde f)$ we have $g'(0)\notin(1-\delta,1)\cup(1,1+\delta)$. But $g'(0)=1\Rightarrow g=id$, hence if $g\neq id$ we must have $\|g-id\|_{C^1}\geqslant\delta$, and this proves (2).

In particular, f in $\mathscr{Z}_0$ implies $Z(f)$ is discrete. Let $f\in\mathscr{D}\setminus\mathscr{Z}_0$, and let $h\in Diff^\infty_+([0,1])$, $a,b\in(0,1)$, be such that $f_0=h\circ f\circ h^{-1}$ is linear on $[0,a]\cup[b,1]$. Fix any $t\in(0,+\infty)$ and consider the map $g_t:[0,\lambda^{-t}a]\to[0,a]$, $x\mapsto\lambda^t a$ ($\lambda=f'(0)$). This map commutes with f_0, and hence can be extended to a diffeomorphism g_t of $[0,1)$, which commutes with f_0. Because $f_0\notin\mathscr{Z}_0$, for every $z\in(0,a)$, $N\in\mathbf{N}$ such that $\lambda^t z\in(0,a)$, $f_0^N(z)\in(b,1)$ we necessarily have

$$\frac{1-f_0^N(\lambda^t z)}{1-f_0^N(z)}=\mu^t,$$

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that is $$g_t(f_0^N(z)) = f_0^N(g_t(z)) = f_0^N(\lambda^t z) = 1-\mu^t(1-f_0^N(z)).$$

Every $w \in (b,1)$ is of the form $f_0^N(z)$, $z \in (0, \lambda^{-t}a)$; hence g_t is linear on $(b,1)$ and can be extended to $g_t \in Diff_0^\infty([0,1])$. This proves that $Z(f_0)$ (i.e. $Z(f)$) is not discrete, and $\mathscr{L}_0 = \mathscr{W}_0$.

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