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# Hyperbolic Weingarten surfaces

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*Abstract*

Weingarten surfaces which can be represented locally as solutions to second order hyperbolic partial differential equations are examined in this paper. In particular, the geometry of the families of curves corresponding to characteristics on these surfaces is investigated and the relationships of these curves with other curves on the surface such as asymptotic lines and lines of curvature are explored. It is shown that singularities in the lines of curvature, i.e. umbilic points, correspond to singularities in the families of characteristics, and that lines of curvature are non-characteristic curves. If there is a linear relation between the Gaussian and mean curvatures and real characteristics exist, then the characteristics form a Tchebychef net on the corresponding Weingarten surface.

## 1. *Introduction*

A Weingarten (or W-) surface is a surface on which there exists a relationship between the principal curvatures. Surfaces of constant Gaussian/mean curvature and surfaces of revolution are familiar examples. The systematic study of surfaces on which there exists such a relationship was initiated by Weingarten (*c.* 1862) and for the next thirty some years these surfaces were investigated by Beltrami, Darboux and Lie among others. This early research focused primarily on problems of a local nature. A complete account of their work can be found in Darboux[**6**]; abridged accounts can be found in Eisenhart[**7**] and Forsyth[**8**]. More recently, attention has turned to global properties of Weingarten surfaces, particularly closed ones. Chern[**3**, **4**], Hartman and Wintner[**9**], H. Hopf[**10**, **11**] and Voss[**17**] among others have investigated properties of these surfaces (see also Palais and Terng[**14**] and Švec[**16**]. Most of the investigations (both local and global) concerned surfaces embedded in Euclidean three space; the notion of a Weingarten surface, however, has been abstracted and extended [e.g. cf. T. Milnor, **13**; Švec **15**].

In this paper, a class of Weingarten surfaces characterized by the property of being described (at least locally) by second order nonlinear hyperbolic partial differential equations is investigated. The emphasis is on the geometry of the two families of curves corresponding to characteristics. The classification of the W-surface based on the underlying partial differential equations is discussed in the remainder of this section. The approach follows that given by Hopf[**10**]. The characteristics for a W-surface are defined in the next section, and their geometry is explored in Section 3, particularly relationships with other families of curves such as asymptotic lines and lines of curvature. As will be shown, the lines of curvature dictate the behaviour of

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the characteristics: they are non-characteristic and their singularities (i.e. umbilic points) induce singularities in the characteristics. In the last section the conditions under which the characteristics form a Tchebychef net are derived.

Weingarten surfaces are characterized by the existence of some relation $W(\kappa_1, \kappa_2) = 0$ between the principal curvatures $\kappa_1$ and $\kappa_2$ or, equivalently, some relation $U(K, J) = 0$ between the Gaussian curvature $K = \kappa_1 \kappa_2$, and the mean curvature $J = \frac{1}{2}(\kappa_1 + \kappa_2)$. The existence of such a relation can be expressed by a Jacobian condition

$$\frac{\partial(\kappa_1, \kappa_2)}{\partial(u^1, u^2)} = \frac{\partial(K, J)}{\partial(u^1, u^2)} = 0,$$

where the $u^k$ are curvilinear coordinates on the surface (assuming the appropriate smoothness conditions). The above condition leads to a third order partial differential equation, which though useful for verifying the existence of such a relation, is not so useful for examining geometrical properties. The Weingarten relation provides a second order partial differential equation, which, in general, is fully nonlinear (i.e. not quasi-linear).

For definiteness, the convention $\kappa_1 \geqslant \kappa_2$ is used i.e.

$$\kappa_1 = J + \{J^2 - K\}^{\frac{1}{2}}, \quad \kappa_2 = J - \{J^2 - K\}^{\frac{1}{2}}.$$

This convention is, to a degree, arbitrary, but it does highlight umbilic points which loom large in the geometry of Weingarten surfaces. It will be assumed throughout the remainder of this paper (though reiterated in the theorems) that the Weingarten function $W(\kappa_1, \kappa_2)$ is smooth with respect to the principal curvatures. This implies that $U(K, J)$ is a smooth function of $K$ and $J$ except perhaps at umbilic points. Under the additional assumption that

$$W_{\kappa_1}\left(= \frac{\partial W}{\partial \kappa_1}\right) \quad \text{and} \quad W_{\kappa_2}\left(= \frac{\partial W}{\partial \kappa_2}\right)$$

do not vanish simultaneously, the Weingarten relation $W(\kappa_1, \kappa_2) = 0$ defines some curve in the $\kappa_1 \kappa_2$-plane lying below or on the 'umbilic line' $\kappa_1 = \kappa_2$. As will be seen, the sign of the derivative $d\kappa_2/d\kappa_1$ determines if the associated partial differential equation is elliptic, parabolic, or hyperbolic.

Let

$$\Sigma = \{\mathbf{r}(\mathbf{u}) = (x(\mathbf{u}), y(\mathbf{u}), z(\mathbf{u})) : \mathbf{u} = (u^1, u^2) \in S \subseteq \mathscr{R}^2\}$$

describe (some portion of) a Weingarten surface. Here $\mathbf{r}$ is the position vector and $S$ is some region in $\mathscr{R}^2$. The parametrization of $\Sigma$ is assumed regular, i.e. for all $\mathbf{u} \in S$, $\mathbf{r}$ is a smooth function of $\mathbf{u}$ and

$$|\mathbf{r}_1 \wedge \mathbf{r}_2| \neq 0,$$

where

$$\mathbf{r}_j = \frac{\partial \mathbf{r}}{\partial u^j}.$$

The components of the first fundamental form in $\Sigma$ are $g_{jk} = \mathbf{r}_j . \mathbf{r}_k$, and for

$$\mathbf{n} = \frac{\mathbf{r}_1 \wedge \mathbf{r}_2}{|\mathbf{r}_1 \wedge \mathbf{r}_2|},$$

$$\mathbf{r}_{jk} = \frac{\partial^2 \mathbf{r}}{\partial u^j \, \partial u^k},$$

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the components of the second fundamental form are $L_{jk} = \mathbf{n}.\mathbf{r}_{jk}$. The Gaussian and mean curvatures are thus given by

$$K = \frac{L_{11}L_{22} - L_{12}^2}{H^2},$$

$$J = \frac{g_{22}L_{11} - 2g_{12}L_{12} + g_{11}L_{22}}{2H^2},$$

where $$H^2 = g_{11}g_{22} - g_{12}^2.$$

The relation $W(\kappa_1, \kappa_2) = U(K, J) = 0$ can thus be regarded as a tensor equation, say $F(g_{jk}, L_{jk}) = 0$; the latter yields a nonlinear equation involving the components of $\mathbf{r}_{jk}$.

Suppose now that $\Sigma$ can be represented in the 'graphical' form

$$\mathbf{r}(\mathbf{u}) = (u^1, u^2, \phi(u^1, u^2)).$$

The Weingarten relation then defines some second order nonlinear partial differential equation for $\phi$,

$$F(p, q, r, s, t) = 0,$$

where $$p = \frac{\partial\phi}{\partial u^1}, \quad q = \frac{\partial\phi}{\partial u^2}, \quad r = \frac{\partial^2\phi}{\partial u^{1^2}}, \quad s = \frac{\partial^2\phi}{\partial u^1 \partial u^2}, \quad t = \frac{\partial^2\phi}{\partial u^{2^2}}.$$

A central question concerns whether the partial differential equation above, is elliptic, hyperbolic etc. and this entails determining the sign of the discriminant

$$\Delta = F_r F_t - \tfrac{1}{4}F_s^2,$$

where $$F_r = \frac{\partial F}{\partial r} \quad \text{etc.}$$

The hyperbolicity (ellipticity) of a partial differential equation reflects a geometrical property of the solution surface, namely the existence of real (imaginary) families of curves, characteristics, whereon there may be second order derivative discontinuities. This is pursued in the next section. It suffices here to note that this observation motivates the search for a 'geometrical' means of determining the sign of $\Delta$ for a given Weingarten relation.

Hopf[**10**] has shown that the sign of $\Delta$ is the same as that of $W_{\kappa_1} W_{\kappa_2}$. A slight generalization of this result, which does not depend on the graphical representation, is given by the following lemma:

LEMMA 1. *Let $W(\kappa_1, \kappa_2) = 0$ be a Weingarten relation, where $W$ is a smooth function of the principal curvatures, such that for the equivalent relation $U(K, J) = 0$, $U$ is a smooth function of $K$ and $J$, and let $F(g_{jk}, L_{jk}) = 0$ be the associated tensor equation. Then*

$$\operatorname{sgn}(F_{11}F_{22} - \tfrac{1}{4}F_{12}^2) = \operatorname{sgn}(W_{\kappa_1} W_{\kappa_2}) \tag{1·1}$$

*where* $$F_{jk} \equiv \frac{\partial F}{\partial L_{jk}}.$$

*Proof.* By hypothesis the tensor equation $F(g_{jk}, L_{jk}) = 0$ corresponds to a Weingarten relation $W(\kappa_1, \kappa_2) = 0$ or, in terms of the Gaussian and mean curvatures

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$U(K, J) = 0$. The functional occurrence of the tensor components is thus restricted to combinations corresponding to principal curvatures. Let

$$U_K = \frac{\partial U}{\partial K}, \quad U_J = \frac{\partial U}{\partial J}.$$

Now
$$F_{11} = U_K \frac{\partial K}{\partial L_{11}} + U_J \frac{\partial J}{\partial L_{11}} = U_K \frac{L_{22}}{H^2} + \frac{g_{22}}{2H^2} U_J, \tag{1·2a}$$

and similarly
$$F_{12} = \frac{-2L_{12}}{H^2} U_K - \frac{g_{12}}{H^2} U_J, \tag{1·2b}$$

$$F_{22} = \frac{L_{11}}{H^2} U_K + \frac{g_{11}}{2H^2} U_J; \tag{1·2c}$$

consequently,

$$\begin{aligned} F_{11}F_{22} - \tfrac{1}{4}F_{12}^2 &= \frac{1}{H^4}\left\{\left(L_{11}U_K + \frac{g_{11}}{2}U_J\right)\left(L_{22}U_K + \frac{g_{22}}{2}U_J\right) - \tfrac{1}{4}(2L_{12}U_K + g_{12}U_J)^2\right\} \\ &= \frac{1}{H^2}\{U_K^2 \kappa_1 \kappa_2 + \tfrac{1}{2}U_K U_J(\kappa_1 + \kappa_2) + \tfrac{1}{4}U_J^2\} \\ &= \frac{1}{H^2}\{(U_K \kappa_2 + \tfrac{1}{2}U_J)(U_K \kappa_1 + \tfrac{1}{2}U_J)\} \\ &= \frac{1}{H^2} W_{\kappa_1} W_{\kappa_2}, \end{aligned}$$

which implies equation (1·1). ∎

This lemma, as noted, does not rely on the existence (globally) of a graphical (or any other single chart) representation for the surface, and the 'hyperbolicity' (or 'ellipticity' etc.) of the tensor equation is independent of any particular parametrization. Locally, of course, any portion of the surface can be represented in a graphical form and the sign of $W_{\kappa_1} W_{\kappa_2}$ determines the nature of the corresponding partial differential equation. This motivates.

*Definition* 1. Let $\Sigma$ be a Weingarten surface with a Weingarten relation $W(\kappa_1, \kappa_2) = 0$, where $W$ is a smooth function of the principal curvatures. The surface $\Sigma$ is said to be *elliptic* if $W_{\kappa_1} W_{\kappa_2} > 0$, *parabolic* if $W_{\kappa_1} W_{\kappa_2} = 0$, and *hyperbolic* if $W_{\kappa_1} W_{\kappa_2} < 0$.

The above definition classifies certain types of Weingarten surfaces based on the nature of the underlying partial differential equations. As with partial differential equations, there are Weingarten surfaces which do not fit under any of the above categories ($W_{\kappa_1} W_{\kappa_2}$ in general can have different signs in different regions of $\Sigma$); however, many surfaces of interest do fit into these categories.

*Example* 1. *Surfaces of constant mean curvature*

The Weingarten relation for a surface of constant mean curvature is $W(\kappa_1, \kappa_2) = \kappa_1 + \kappa_2 - \Lambda = 0$, where $\Lambda$ is some constant. Since at all points on the surface $W_{\kappa_1} W_{\kappa_2} = 1$, surfaces of constant mean curvature are elliptic.

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*Example* 2. *Surfaces of constant Gaussian curvature*

The Weingarten relation for surfaces of constant $K$ is $\kappa_1 \kappa_2 - \Lambda = 0$, where $\Lambda$ is some constant. Now $W_{\kappa_1} W_{\kappa_2} = \Lambda$; thus, the surface is elliptic if $\Lambda > 0$ and hyperbolic of $\Lambda < 0$. Evidently developable surfaces are parabolic.

*Example* 3. *Ellipsoid surface of revolution*

Consider the surface of revolution defined by

$$\mathbf{r}(\mathbf{u}) = (a \cos u^1 \cos u^2, a \cos u^1 \sin u^2, b \sin u^1),$$

where $a, b > 0$ and $a \neq b$. This surface corresponds to a surface of revolution generated by rotating an ellipse about one of its axis. Calculations indicate that

$$\kappa_1 = \frac{b}{a(a^2 + (b^2 - a^2)\cos^2 u^1)^{\frac{1}{2}}}, \quad \kappa_2 = \frac{ab}{(a^2 + (b^2 - a^2)\cos^2 u^1)^{\frac{3}{2}}},$$

so that the Weingarten relation is

$$W(\kappa_1, \kappa_2) = \frac{a^4}{b^2}\kappa_1^3 - \kappa_2 = 0.$$

Now

$$W_{\kappa_1} W_{\kappa_2} = \frac{-a^4}{b^2}\kappa_1^2,$$

and since $\kappa_1$ is nowhere zero, the surface is hyperbolic.

*Example* 4

A somewhat more exotic example of a Weingarten relation arises from a problem considered originally by Weingarten in connection with determining surfaces which have lines of curvature whose spherical image corresponds to geodesic ellipses or hyperbolae. The unusual surface is described further by Darboux [6] and Eisenhart [7]. The Weingarten relation arising from this problem is

$$W(\kappa_1, \kappa_2) = \sin\left\{2\left(\frac{1}{\kappa_1} + \frac{1}{\kappa_2}\right)\right\} - 2\left(\frac{1}{\kappa_1} - \frac{1}{\kappa_2}\right) = 0,$$

and calculations show that

$$W_{\kappa_1} W_{\kappa_2} = \frac{-4\sin^2\left(2\left(\frac{1}{\kappa_1} + \frac{1}{\kappa_2}\right)\right)}{\kappa_1^2 \kappa_2^2};$$

consequently, the surface is hyperbolic.

Following Hopf[**10**], the sign of $W_{\kappa_1} W_{\kappa_2}$ can also be conveniently interpreted in terms of the curve produced by the Weingarten relation. It has been assumed that $W_{\kappa_1}$ and $W_{\kappa_2}$ do not vanish simultaneously, and since

$$W_{\kappa_1} d\kappa_1 + W_{\kappa_2} d\kappa_2 = 0,$$

it is clear that either $d\kappa_2/d\kappa_1$ or $d\kappa_1/d\kappa_2$ is finite. If the surface is elliptic or hyperbolic then both are finite and their sign is opposite to that of $W_{\kappa_1} W_{\kappa_2}$. If the surface is parabolic then $d\kappa_2/d\kappa_1$ is either zero or infinite. The nature of the underlying partial

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differential equations is thus dependent on the monotonicity of the Weingarten relation. In summary,

$$0 < \frac{d\kappa_2}{d\kappa_1} < \infty \Rightarrow \Sigma \text{ is hyperbolic,}$$

$$\frac{d\kappa_2}{d\kappa_1} = 0 \quad \text{or} \quad \frac{d\kappa_1}{d\kappa_2} = 0 \Rightarrow \Sigma \text{ is parabolic,}$$

$$-\infty < \frac{d\kappa_2}{d\kappa_1} < 0 \Rightarrow \Sigma \text{ is elliptic.}$$

Much consideration has been afforded to elliptic Weingarten surfaces, particularly minimal surfaces. Aside from minimal surfaces, elliptic Weingarten surfaces have been studied by Chern [**3**, **4**], Hartman and Wintner [**9**] and Hopf [**10**] among others in connection with characterizing (under various assumptions) the sphere as the only closed Weingarten surface of genus zero. The general Dirichlet problem has been considered by Caffarelli *et al.* [**1**, **2**] and Korevaar [**12**]. In contrast, there is a paucity of work concerning the hyperbolic problem. The remainder of this paper is concerned with hyperbolic Weingarten surfaces and the geometrical rôle of characteristics for these surfaces.

## 2. *Characteristics for hyperbolic Weingarten surfaces*

A generalization of a characteristic is a curve on a surface along which given the surface unit normal the components of the second fundamental form are indeterminate. In the graphical parametrization, $\mathbf{r}(\mathbf{u}) = (u^1, u^2, \phi(\mathbf{u}))$, the $L_{jk}$ components correspond to the second derivatives of $\phi$, and in this respect it is in agreement with the usual definition. As the partial differential equations of interest here are the ones which arise from Weingarten relations, attention is focused on equations of the type

$$F(g_{jk}, L_{jk}) = 0. \tag{2·1}$$

The initial data for equation (2·1) analogous to that for the Cauchy problem consists of a curve

$$\Gamma \equiv \{\mathbf{r}(\mathbf{u}(\sigma)) = (x(\mathbf{u}(\sigma)), y(\mathbf{u}(\sigma)), z(\mathbf{u}(\sigma))) : \sigma \in \eta \subseteq \mathscr{R}\}$$

on which the unit normal $\mathbf{n}(\mathbf{u}(\sigma))$ is specified. In a more compressed form, the initial data will be denoted by $\mathbf{r}_0(\sigma) = \mathbf{r}(\mathbf{u}(\sigma)), x_0(\sigma) = x(\mathbf{u}(\sigma))$ etc. Here, $\eta$ is some interval on the real line, and $\mathbf{r}_0(\sigma)$ is assumed to be a smooth function of $\sigma$ such that $|\mathrm{d}/\mathrm{d}\sigma\, \mathbf{r}(\sigma)| \neq 0$ for all $\sigma \in \eta$.

Given these initial data, a moving trihedron can be constructed on $\Gamma$. Since $\Gamma$ must lie on an integral surface $\Sigma$ of equation (2·1), a unit tangent to $\Gamma$, $\mathbf{t}_0(\sigma)$, must lie in the plane tangent to $\Sigma$ at $\mathbf{r}(\sigma)$. As $\mathbf{n}_0(\sigma)$ is given, the vector $\mathbf{n}_0(\sigma) \wedge \mathbf{t}_0(\sigma)$ is known on $\Gamma$ and must also lie in the tangent plane. Let $\mathbf{r}_{0_1}(\sigma) = \mathbf{t}_0(\sigma), \mathbf{r}_{0_2}(\sigma) = \mathbf{n}_0(\sigma) \wedge \mathbf{t}_0(\sigma)$. The vectors $\mathbf{r}_{0_1}$ and $\mathbf{r}_{0_2}$ are known on $\Gamma$, and this means that the components of the metric tensor are also known on $\Gamma$; like the Cauchy problem, however, the initial data and relation (2·1) may not be sufficient to determine the $L_{jk}$. Now

$$\mathbf{n}_0(\sigma) . \mathbf{r}'_{01}(\sigma) = \mathbf{n}_0 . (\mathbf{r}_{0_{11}} u'^1 + \mathbf{r}_{0_{12}} u'^2) = a_1(\sigma),$$
$$\mathbf{n}_0(\sigma) . \mathbf{r}'_{02}(\sigma) = \mathbf{n}_0 . (\mathbf{r}_{0_{12}} u'^1 + \mathbf{r}_{0_{22}} u'^2) = a_2(\sigma),$$

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where $'$ denotes $d/d\sigma$, $\mathbf{r}_{0_{jk}} = \partial^2\mathbf{r}(\mathbf{u})/\partial u^j\,\partial u^k|_{\mathbf{u}=\mathbf{u}(\sigma)}$, and the $a_k(\sigma)$ are known functions; thus,

$$L_{0_{11}}u'^1 + L_{0_{12}}u'^2 = a_1(\sigma),$$
$$L_{0_{12}}u'^1 + L_{0_{22}}u'^2 = a_2(\sigma),$$

and these equations together with equation (2·1) provide three equations for the three unknown tensor components $L_{0_{11}}, L_{0_{12}}, L_{0_{22}}$. Evidently the degenerate case corresponding to characteristic initial data is when these equations are functionally dependent, i.e. when

$$F_{11}(u'^2)^2 - F_{12}u'^1u'^2 + F_{22}(u'^1)^2 = 0. \tag{2·2}$$

Let $$\Xi = \{\xi(\tau) = (x(\mathbf{u}(\tau)), y(\mathbf{u}(\tau)), z(\mathbf{u}(\tau))) : \tau \in \gamma \subseteq \mathscr{R}\},$$

where $\gamma$ is some interval, define a smooth curve on an integral surface $\Sigma$ to (2·1). From the above arguments it is clear that the components of the second fundamental form on $\Sigma$ are indeterminate along curves satisfying the differential equation

$$F_{11}(\dot{u}^2)^2 - F_{12}\dot{u}^1\dot{u}^2 + F_{22}(\dot{u}^1)^2 = 0, \tag{2·3}$$

where $\cdot$ denotes $d/d\tau$. Curves satisfying this differential equation will be called characteristics. As functional dependence is not changed by coordinate transformations, it is clear that these curves are invariant with respect to non-singular curvilinear coordinate transformations.

Given a Weingarten relation $W(\kappa_1, \kappa_2) = 0$, it was shown that the sign of the discriminant of (2·3) is invariant; consequently, a hyperbolic Weingarten surface has two real families of characteristics. Assuming there are no umbilic points, the Weingarten relation $U(K, J) = 0$ is a smooth function of $K$ and $J$, and differential equation (2·3) for the characteristics can be written in terms of the Weingarten relation using equations (1·2$a$)–(1·2$c$), for $F_{11}, F_{12}, F_{22}$. The characteristic equation (2·3) can thus be written

$$\frac{U_K}{H^2}(L_{11}(\dot{u}^1)^2 + 2L_{12}\dot{u}^1\dot{u}^2 + L_{22}(\dot{u}^2)^2) + \frac{U_J}{2H^2}(g_{11}(\dot{u}^1)^2 + 2g_{12}\dot{u}^1\dot{u}^2 + g_{22}(\dot{u}^2)^2) = 0,$$

i.e. $$U_K\kappa_n + \tfrac{1}{2}U_J = 0, \tag{2·4}$$

where $\kappa_n$ denotes the normal curvature. In terms of the principal curvatures, the characteristic equation implies

$$\kappa_n = \frac{\kappa_1 + \kappa_2\, d\kappa_1/d\kappa_2}{1 + d\kappa_1/d\kappa_2}. \tag{2·5}$$

Examining equation (2·4) it is clear that there will be problems at umbilic points, where $\kappa_n = \kappa_1 = \kappa_2$: equation (2·4) is either satisfied for all directions in the tangent plane or not satisfied for any. Characteristics are thus especially sensitive to umbilic points.

Equation (2·4) provides an ordinary differential equation for the characteristics in terms of a relation between the parameter space variables $u^1, u^2$, namely

$$(U_K L_{mn} + \tfrac{1}{2}U_J g_{mn})\frac{du^m}{d\tau}\frac{du^n}{d\tau} = 0 \tag{2·6}$$

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(summation over the repeated indices). The above differential equation defines some curve $(u^1(\tau), u^2(\tau))$ in the parameter space analogous to the 'characteristic projections'. Once the differential equation (2·6) is solved for $(u^1(\tau), u^2(\tau)) = \mathbf{u}(\tau)$, the characteristic itself is given by $\mathbf{r}(\mathbf{u}(\tau))$. The choice $\tau = u^1$ or $u^2$ simplifies the nonlinear first order differential equation, and if $u^k = \text{const.}$ corresponds to the lines of curvature on the surface, the differential equations have the particularly simple form,

$$\left(\frac{du^2}{du^1}\right)^2 = \frac{d\kappa_1}{d\kappa_2}\frac{g_{11}}{g_{22}}. \tag{2·7}$$

A 'geometrical' formulation of equation (2·7) can be achieved using a certain parametrization which arises naturally from the Mainardi–Codazzi equations. If the curves $u^k = \text{const.}$ correspond to lines of curvature, it can be shown (cf. Darboux [6]; Forsyth [8]) that there exists a parametrization such that

$$g_{11} = \exp\left\{2\int \frac{d\kappa_1}{\kappa_2 - \kappa_1}\right\}, \tag{2·8a}$$

$$g_{22} = \exp\left\{2\int \frac{d\kappa_2}{\kappa_1 - \kappa_2}\right\}; \tag{2·8b}$$

consequently,

$$\frac{g_{11}}{g_{22}} = g_{11}^2(\kappa_1 - \kappa_2)^2,$$

and differential equation (2·6) becomes

$$\left(\frac{du^2}{du^1}\right)^2 = \frac{d\kappa_1}{d\kappa_2}(\kappa_1 - \kappa_2)^2 \exp\left\{4\int \frac{d\kappa_1}{\kappa_2 - \kappa_1}\right\}. \tag{2·9}$$

The Picard–Lindelöf existence and uniqueness theorem for ordinary differential equations can be applied to equation (2·6) to establish the existence of precisely two characteristics through any given non-umbilic point. Specifically, let $\mathbf{p}$ be a non-umbilic point on $\Sigma$. The smoothness of $\Sigma$ implies that there exists a neighbourhood $N_p \subseteq \Sigma$ containing $\mathbf{p}$ such that $N_p$ is free from umbilic points. The portion $N_p$ of the surface $\Sigma$ can thus be parametrized using its lines of curvature. Let $\mathbf{r}(\mathbf{u}), \mathbf{u} \in S_p$ be such a parametrization. The characteristics at $\mathbf{p}$ are defined by the first order ordinary differential equations

$$\frac{du^2}{du^1} = \pm f(\mathbf{u}), \tag{2·10}$$

where

$$f(\mathbf{u})^2 = \frac{d\kappa_1}{d\kappa_2}\frac{g_{11}}{g_{22}},$$

subject to the initial condition

$$u^2(u_0^1) = u_0^2,$$

where $\mathbf{r}(u_0^1, u_0^2) = \mathbf{p}$. The surface $N_p$ is smooth as is also the Weingarten relation, and this and the hyperbolicity condition $d\kappa_1/d\kappa_2 > 0$ ensure that $f$ is a smooth function of $\mathbf{u}$ in $S_p$. The function $f$ is a fortiori continuous with respect to $u^1$ and Lipschitz continuous with respect to $u^2$ for $\mathbf{u} \in S_p$. Once the sign affixed to $f$ is specified, the Picard–Lindelöf theorem (cf. Coddington and Levinson [5]) implies that there

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exists a unique solution to the differential equation, satisfying the given initial-value, defined in some neighbourhood of $\mathbf{p}$. Consequently, at each non-umbilic point $p \in \Sigma$ there are precisely two characteristics which are defined at least locally.

The local solutions guaranteed by the Picard–Lindelöf theorem can be continued using standard continuation arguments (cf. Coddington and Levinson[5], section 2·1). If $f$ is a smooth function of $\mathbf{u}$ for $\mathbf{u} \in S$, where $S$ is some region and $S_p \subseteq S$, then the solution to differential equation (2·10) is defined for an interval $u_0^1 \leqslant u^1 < b$. Here $b$ may be infinite. If $b < \infty$, then either $u^2(u^1)$ approaches the boundary of $S$ as $u^1 \to b$ or $u^2(u^1) \to \infty$ as $u^1 \to b$. Let $\Psi \subseteq \Sigma$ be some bounded surface such that for $\bar{\Psi} \equiv \Psi \cup \partial\Psi$, where $\partial\Psi$ denotes the boundary of $\Psi$, $\bar{\Psi} \subseteq \Sigma$, $\bar{\Psi} \cap \partial\Psi = \varnothing$, and $\bar{\psi}$ is free from umbilic points. Then there exists a parametrization of $\bar{\Psi}$ using the lines of curvature such that the parameter space $S$, is bounded. Now the boundary of $S$ maps to $\partial\Psi$, and this means that if $p \in \Psi$, the characteristics through $p$ can be continued until they approach $\partial\Psi$.

Let $\Sigma_u \equiv \{p \in \Sigma : p \text{ is umbilic}\}$ and $\Sigma_0 \equiv \{p \in \Sigma : p \notin \Sigma_u\}$. The smoothness of $\Sigma$ implies that $\Sigma_0$ is open, and from the above discussion it is clear that characteristics on $\Sigma_0$ can be continued arbitrarily close to the boundary of $\Sigma_0$ by a suitable choice of $\Psi$. The only possibilities where a characteristic cannot be continued are thus when it either approaches the boundary of $\Sigma$ or an umbilic point. The following theorem summarizes the conclusions above.

THEOREM 1. *Let $\Sigma$ be a smooth hyperbolic W-surface satisfying a smooth Weingarten relation. At any non-umbilic point $p \in \Sigma$, there exist precisely two characteristics. These characteristics can be continued until they either approach the boundary of $\Sigma$ or an umbilic point.*

*Example* 5

Consider the ellipsoid of example 3. It was shown that this surface is a smooth hyperbolic W-surface with a Weingarten relation $W(\kappa_1, \kappa_2) = a^4/b^2\kappa_1^3 - \kappa_2 = 0$. The parametrization of the ellipsoid is along its lines of curvature and thus a differential equation for the characteristics is

$$\left(\frac{du^1}{du^2}\right)^2 = \frac{d\kappa_2}{d\kappa_1}\frac{g_{22}}{g_{11}} = \frac{3a^4\kappa_1^2 a^2 \cos^2 u^1}{b^2(a^2 + (b^2 - a^2)\cos^2 u^1)}$$

$$= \frac{3a^4 \cos^2 u^1}{a^2 + (b^2 - a^2)\cos^2 u^1},$$

i.e.
$$u^2(u^1) = \pm\left\{\frac{1}{\sqrt{3}} \ln\left|\frac{1 + \tan\frac{1}{2}u^1}{1 - \tan\frac{1}{2}u^1}\right| + \frac{b^2 - a^2}{\sqrt{3a^2}} \sin u^1\right\} + \text{const.}$$

As the above expression for $u^2$ illustrates, the characteristics on the ellipsoid are defined everywhere except at the umbilic points where $u^1 = \pm\frac{1}{2}\pi$. The set $S$ here is unbounded ($u^2 \to \pm\infty$ as $u^1 \to \pm\frac{1}{2}\pi$).

It is interesting to note that the existence of lines of curvature on $\Sigma$ ensures that the characteristics can be continued. Where the lines of curvature are defined so are the characteristics: the relationship between these families of curves is a close one.

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### 3. *Characteristics, asymptotic lines, and lines of curvature*

Asymptotic lines, it is recalled, are the curves on a surface for which the normal curvature $\kappa_n$ vanishes. Evidently, surfaces with negative Gaussian curvature have two such families of curves; whereas, surfaces with positive Gaussian curvature do not have any (real) asymptotic lines. For the case $K < 0$, the question arises: under what conditions do the characteristics on a hyperbolic W-surface coincide with its asymptotic lines? As characteristics are defined in terms of the normal curvature, the question can be readily answered:

THEOREM 2. *If the characteristics on a smooth hyperbolic W-surface $\Sigma$ coincide with the asymptotic lines on $\Sigma$, then $\Sigma$ is a surface of constant negative Gaussian curvature.*

*Proof.* The above theorem can be established directly from equation (2·4). Suppose $\Sigma$ is a smooth hyperbolic Weingarten surface and that the characteristics on $\Sigma$ coincide with the asymptotic lines on $\Sigma$. As $K < 0$, there are no umbilic points on $\Sigma$; the normal curvature along a characteristic is given by equation (2·5). If, in addition, the curve is an asymptotic line, then the Weingarten relation must satisfy

$$\kappa_1 + \kappa_2 \frac{d\kappa_1}{d\kappa_2} = 0,$$

which implies that $\Sigma$ must be a surface of constant Gaussian curvature. As $K < 0$ the constant must be negative. ∎

Whereas the asymptotic lines coincide with the characteristics on surfaces of constant negative Gaussian curvature, lines of curvature, can never coincide with the characteristics.

THEOREM 3. *Let $\Sigma$ be a smooth hyperbolic W-surface. The lines of curvature on $\Sigma$ are non-characteristic curves. At each non-umbilic point of $\Sigma$, the lines of curvature bisect the angles between the characteristics.*

*Proof.* Let $\phi_1$, denote the angle between the characteristic defined by

$$\frac{du^2}{du^1} = \left\{\frac{d\kappa_1}{d\kappa_2}\frac{g_{11}}{g_{22}}\right\}^{\frac{1}{2}}$$

and the line of curvature corresponding to $u^2 = \text{const}$. Then

$$\tan\phi_1 = \sqrt{\frac{d\kappa_1}{d\kappa_2}}. \tag{3·1a}$$

Similarly if $\phi_2$ denotes the angle between the other characteristic and the same line of curvature, then

$$\tan\phi_2 = -\sqrt{\frac{d\kappa_1}{d\kappa_2}}. \tag{3·1b}$$

The angle $\alpha$ between the two characteristics is given by

$$\cos\alpha = \frac{1 - d\kappa_1/d\kappa_2}{1 + d\kappa_1/d\kappa_2}. \tag{3·2}$$

Since $0 < d\kappa_1/d\kappa_2 < \infty$, equation (3·1$a$) (and (3·1$b$)) implies that characteristics can

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never be tangent to lines of curvature. Indeed, even if $d\kappa_1/d\kappa_2$ were allowed to vanish or become infinite, the 'characteristics' would not be orthogonal by virtue of equation (3·2). ∎

The relationship between lines of curvature and characteristics is intimate: the above theorem essentially means that the behaviour of the characteristics is dictated by that of the lines of curvature, e.g. a singularity in the vector field defining the characteristics on a smooth surface can occur only at umbilic points, and the Poincaré index of such a singularity is the same as that for the corresponding umbilic point. If the surface is free from umbilic points then the characteristics can be used to define a non-singular net on the surface. A well-known example of this is the Tchebychef net, which has a metric of the form

$$ds^2 = (du^1)^2 + 2g_{12}\,du^1\,du^2 + (du^2)^2. \tag{3·3}$$

The asymptotic lines on a surface of constant negative Gaussian curvature correspond to a Tchebychef net and, as indicated above, for these W-surfaces, the asymptotic lines correspond to characteristics. Other W-surfaces with this property are explored in the next section.

Although characteristics on the general smooth hyperbolic Weingarten surface may not correspond to a Tchebychef net, it is none the less of interest to note that a conformal relationship exists between a Tchebychef net and characteristics.

THEOREM 4. *Let $\Sigma$ be a smooth hyperbolic Weingarten surface free from umbilic points. Then there exists a parametrization of $\Sigma$ using the characteristics such that the metric has the form*

$$(ds)^2 = f(\kappa_1,\kappa_2)\left\{(du^1)^2 + 2\frac{(1-d\kappa_1/d\kappa_2)}{(1+d\kappa_1/d\kappa_2)}du^1\,du^2 + (du^2)^2\right\}, \tag{3·4}$$

*where*

$$f(\kappa_1,\kappa_2) = \left(1+\frac{d\kappa_1}{d\kappa_2}\right)\exp\left(2\int\frac{d\kappa_1}{\kappa_2-\kappa_1}\right). \tag{3·5}$$

*Proof.* The surface $\Sigma$ is free from umbilic points and therefore can be parametrized using the lines of curvature. For $k = 1, 2$, let the curves $\hat{u}^k = \text{const.}$ on $\Sigma$ correspond to the lines of curvature and the curves $u^k = \text{const.}$ on $\Sigma$ correspond to the characteristics. As noted earlier there exists a parametrization of $\Sigma$ using the lines of curvature such that

$$\hat{g}_{11} = \exp\left(2\int\frac{d\kappa_1}{\kappa_2-\kappa_1}\right),$$

$$\hat{g}_{22} = \exp\left(2\int\frac{d\kappa_2}{\kappa_1-\kappa_2}\right),$$

(cf. equation 2·8). Using this parametrization and equation (2·7), the metric tensor component $g_{11}$ using the characteristics is

$$\begin{aligned} g_{11} &= \hat{g}_{11}\left(\frac{\partial\hat{u}^1}{\partial u^1}\right)^2 + \hat{g}_{22}\left(\frac{\partial\hat{u}^2}{\partial u^1}\right)^2 \\ &= \hat{g}_{11}\left(1-\frac{d\kappa_1}{d\kappa_2}\right)\left(\frac{\partial\hat{u}^1}{\partial u^1}\right)^2 \\ &= f(\kappa_1,\kappa_2)\left(\frac{\partial\hat{u}^1}{\partial u^1}\right)^2. \end{aligned}$$

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Similarly,

$$g_{12} = \frac{\partial \hat{u}^1}{\partial \hat{u}^1}\frac{\partial \hat{u}^1}{\partial u^2}\hat{g}_{11}\left(1-\frac{d\kappa_1}{d\kappa_2}\right)$$

$$= \frac{\partial \hat{u}^1}{\partial u^1}\frac{\partial \hat{u}^1}{\partial u^2} f(\kappa_1, \kappa_2)\frac{(1-d\kappa_1/d\kappa_2)}{(1+d\kappa_1/d\kappa_2)},$$

$$g_{22} = f(\kappa_1, \kappa_2)\left(\frac{\partial \hat{u}^1}{\partial u^2}\right)^2.$$

Consider the transformation from $\hat{\mathbf{u}}$ to $\mathbf{u}$ defined by

$$\hat{u}^1 = u^1 - u^2 \tag{3·6a}$$

$$\frac{\partial \hat{u}^2}{\partial u^1} = \frac{\partial \hat{u}^2}{\partial u^2} = G(\kappa_2), \tag{3·6b}$$

where

$$G(\kappa_2) = \sqrt{\frac{d\kappa_1}{d\kappa_2}}(\kappa_1 - \kappa_2)\, e^{2\int \frac{d\kappa_1}{\kappa_2 - \kappa_1}}.$$

Since $G(\kappa_2) > 0$, this transformation is clearly non-singular. Now

$$d\hat{u}^1 = du^1 - du^2,$$

$$d\hat{u}^2 = G(\kappa_2)(du^1 + du^2),$$

so that for the curve $u^1 = \text{const.}$ on $\Sigma$

$$\frac{d\hat{u}^2}{d\hat{u}^1} = G(\kappa_2),$$

and for the curve $u^2 = \text{const.}$

$$\frac{d\hat{u}^2}{d\hat{u}^1} = -G(\kappa_2);$$

thus, equation (2·9) indicates that the curves $u^k = \text{const.}$ defined by this transformation are characteristics. The new parameters are given by

$$u^1 = \frac{1}{2}\left\{\int \frac{d\hat{u}^1}{G(\kappa_2)} + \hat{u}^2\right\}, \tag{3·7a}$$

$$u^2 = \frac{1}{2}\left\{\int \frac{d\hat{u}^1}{G(\kappa_2)} - \hat{u}^2\right\}, \tag{3·7b}$$

and transformation (3·6) indicates that

$$g_{11} = g_{22} = f(\kappa_1, \kappa_2), \quad g_{12} = f(\kappa_1, \kappa_2)\frac{(1-d\kappa_1/d\kappa_2)}{(1+d\kappa_1/d\kappa_2)}. \quad \blacksquare \tag{3·8}$$

A remarkable property of Weingarten surfaces first established by Lie is that the lines of curvature on these surfaces can be obtained by quadratures (cf. Forsyth [**8**]). Transformation (3·7) indicates that this is also true for the characteristics.

COROLLARY 5. *Let $\Sigma$ be as defined in Theorem* 4. *Then the characteristics on $\Sigma$ can be obtained by quadratures.*

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Another feature of this transformation is that the components of the second fundamental form also have a particularly symmetric form, namely

$$L_{11} = L_{22} = h(\kappa_1, \kappa_2), \tag{3·9a}$$

$$L_{12} = \frac{h(\kappa_1, \kappa_2)(\kappa_1 - \kappa_2 d\kappa_1/d\kappa_2)}{\kappa_1 + \kappa_2 d\kappa_1/d\kappa_2}, \tag{3·9b}$$

where $$h(\kappa_1, \kappa_2) = \left(\kappa_1 + \kappa_2 \frac{d\kappa_1}{d\kappa_2}\right) \exp\left(2 \int \frac{d\kappa_1}{\kappa_2 - \kappa_1}\right).$$

## 4. *Characteristics and Tchebychef nets*

A natural question in view of theorem 4 is whether there are hyperbolic W-surfaces other than those of constant $K$ for which the characteristics correspond to a Tchebychef net. Evidently, if $f(\kappa_1, \kappa_2)$ in equation (3·5) is a constant, then a simple rescaling of the curvilinear coordinates will produce a Tchebychef net. In fact, the characteristics cannot correspond to a Tchebychef net unless $f(\kappa_1, \kappa_2)$ is constant.

LEMMA 2. *Let $\Sigma$ be a smooth hyperbolic W-surface free from umbilic points with a smooth Weingarten relation $\kappa_1 = g(\kappa_2)$ and suppose there exists a regular parametrization of $\Sigma$ using the characteristics such that these curves form a Tchebychef net on $\Sigma$. Then $f(\kappa_1, \kappa_2)$* must be a constant.

*Proof.* Suppose $\mathbf{r}(\hat{\mathbf{u}}), \hat{\mathbf{u}} = (\hat{u}^1, \hat{u}^2)$ is a parametrization of $\Sigma$ using the characteristics and that under this parametrization the metric has the form

$$ds^2 = (d\hat{u}^1)^2 + 2Q(\hat{\mathbf{u}})\, d\hat{u}^1\, d\hat{u}^2 + (d\hat{u}^2)^2.$$

The surface $\Sigma$ satisfies the conditions of theorem 4 and thus there exists a parametrization $\mathbf{r}(\mathbf{u})$ of $\Sigma$ using the characteristics which yields a metric of the form (3·4). There must thus exist a non-singular transformation $\mathbf{u} = \mathbf{u}(\hat{\mathbf{u}})$ between the two coordinate systems which satisfies

$$\mathbf{UAU}^T = \frac{1}{f(\kappa_1, \kappa_2)}, \tag{4·1a}$$

$$\mathbf{VAV}^T = \frac{1}{f(\kappa_1, \kappa_2)}, \tag{4·1b}$$

where $$\mathbf{U} = \left(\frac{\partial u^1}{\partial \hat{u}^1}, \frac{\partial u^2}{\partial \hat{u}^1}\right), \quad \mathbf{V} = \left(\frac{\partial u^1}{\partial \hat{u}^2}, \frac{\partial u^2}{\partial \hat{u}^2}\right),$$

and, for $$F(\kappa_1, \kappa_2) = \frac{1 - d\kappa_1/d\kappa_2}{1 + d\kappa_1/d\kappa_2},$$

the matrix $\mathbf{A}$ is $$\mathbf{A} = \begin{pmatrix} 1 & F(\kappa_1, \kappa_2) \\ F(\kappa_1, \kappa_2) & 1 \end{pmatrix}.$$

Since the parametrization is assumed to use the characteristics, the relation

$$\mathbf{UAV}^T = \frac{F(\kappa_1, \kappa_2)}{f(\kappa_1, \kappa_2)} \tag{4·1c}$$

must also be satisfied.

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The components of the second fundamental form in the $\hat{u}$ coordinates are

$$\hat{L}_{11} = h(\kappa_1, \kappa_2)\,\mathbf{UBU}^T, \tag{4·2a}$$

$$\hat{L}_{12} = h(\kappa_1, \kappa_2)\,\mathbf{UBV}^T, \tag{4·2b}$$

$$\hat{L}_{22} = h(\kappa_1, \kappa_2)\,\mathbf{VBV}^T, \tag{4·2c}$$

where
$$\mathbf{B} = \begin{pmatrix} 1 & M(\kappa_1, \kappa_2) \\ M(\kappa_1, \kappa_2) & 1 \end{pmatrix},$$

and
$$M(\kappa_1, \kappa_2) = \frac{\kappa_1 - \kappa_2 d\kappa_1/d\kappa_2}{\kappa_1 + \kappa_2 d\kappa_1/d\kappa_2}.$$

Now the parametrization $\mathbf{r}(\hat{\mathbf{u}})$ is assumed to refer to the characteristics; consequently, the normal curvature of the parameter curves in both systems is the same. Therefore,

$$\kappa_n = \frac{L_{11}}{g_{11}} = \frac{h(\kappa_1, \kappa_2)}{f(\kappa_1, \kappa_2)} = \hat{L}_{11} = \hat{L}_{22}, \tag{4·3}$$

and this equation along with equations (4·1$a$) and (4·2$a$) imply that either

$$F(\kappa_1, \kappa_2) = M(\kappa_1, \kappa_2) \tag{4·4}$$

or
$$\frac{\partial u^1}{\partial \hat{u}^1}\frac{\partial u^2}{\partial \hat{u}^1} = 0. \tag{4·5a}$$

Similarly equations (4·1$b$), (4·2$b$) and (4·3) imply that either (4·4) is satisfied or

$$\frac{\partial u^1}{\partial \hat{u}^2}\frac{\partial u^2}{\partial \hat{u}^2} = 0. \tag{4·5b}$$

If equation (4·4) is satisfied, then, since $d\kappa_1/d\kappa_2 > 0$, the Weingarten relation must satisfy the differential equation

$$\frac{d\kappa_1}{d\kappa_2} = 1 - \frac{\kappa_1}{\kappa_2} \tag{4·6}$$

(i.e. $K = \kappa_2^2/2 + \text{const.}$), and the components of the first and second fundamental forms are

$$\hat{g}_{11} = \hat{g}_{22} = 1, \quad g_{12} = F(\kappa_1, \kappa_2), \quad \hat{L}_{11} = \hat{L}_{22} = \frac{h(\kappa_1, \kappa_2)}{f(\kappa_1, \kappa_2)}, \quad \hat{L}_{12} = h(\kappa_1, \kappa_2)\frac{F(\kappa_1, \kappa_2)}{f(\kappa_1, \kappa_2)}.$$

The Gaussian curvature is thus

$$K = \kappa_1 \kappa_2 = \frac{\hat{L}_{11}\hat{L}_{22} - \hat{L}_{12}^2}{\hat{g}_{11}\hat{g}_{22} - \hat{g}_{12}^2} = \frac{h^2(\kappa_1, \kappa_2)}{f^2(\kappa_1, \kappa_2)}$$

$$= \frac{\kappa_2^3}{2\kappa_2 - \kappa_1},$$

the last equality following from equation (4·6). The above equation, however, implies that

$$(\kappa_2 - \kappa_1)^2 = 0,$$

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which contradicts the hypothesis that $\Sigma$ is free from umbilic points. The transformation must therefore satisfy (4·5), and since the transformation is non-singular, this means $f$ must be a constant. ∎

The above lemma indicates that if a W-surface has characteristics which correspond to a Tchebychef net, then for some positive constant $\Lambda$

$$\left(1+\frac{d\kappa_1}{d\kappa_2}\right)\exp\left(2\int\frac{d\kappa_1}{\kappa_2-\kappa_1}\right)=\Lambda. \tag{4·7}$$

This equation is a nonlinear integro-differential equation for the Weingarten relation. Under the assumption the Weingarten relation is smooth (and in particular that $d^2\kappa_1/d\kappa_2^2$ exists), equation (4·7) can be converted into the second-order nonlinear ordinary differential equation

$$(\kappa_2-\kappa_1)\frac{d^2\kappa_1}{d\kappa_2^2}+2\frac{d\kappa_1}{d\kappa_2}\left(1+\frac{d\kappa_1}{d\kappa_2}\right)=0. \tag{4·8}$$

The hyperbolicity requirement and equation (4·7) itself preclude solutions to this differential equation such as $\kappa_1 = \text{const.}$ and $\kappa_1 = -\kappa_2$. One viable family of solutions to (4·7) is already known, viz. surfaces of constant $K$; this family can be generalized to Weingarten relations which are linear in the Gaussian and mean curvatures. Specifically, differential equation (4·8) can be transformed to

$$\frac{d^2\Delta}{dJ^2}=\frac{1-(d\Delta/dJ)^2}{\Delta},$$

where $\Delta=\frac{1}{2}(\kappa_1-\kappa_2)$ denotes the 'skew curvature'. Using the transformation $\mu=\Delta\, d\Delta/dJ$, this differential equation simplifies to

$$\frac{d\mu}{dJ}=1;$$

thus, the corresponding Weingarten relation is

$$W(\kappa_1,\kappa_2)=U(K,J)=K+2c_1J+c_2=0, \tag{4·9}$$

where $c_1, c_2$ are integration constants. For the Weingarten relation (4·9)

$$W_{\kappa_1}W_{\kappa_2}=\kappa_1\kappa_2+c_1(\kappa_1+\kappa_2)+c_1^2=c_1^2-c_2,$$

and the hyperbolicity requirement will be satisfied if

$$c_2>c_1^2. \tag{4·10}$$

Substituting the Weingarten relation (4·9) into equation (4·7) yields

$$f(\kappa_1,\kappa_2)=\frac{1}{c_2-c_1^2},$$

so that for $\Lambda=1, c_2=c_1^2+1$. Inequality (4·10) it is also noted precludes the possibility of umbilic points. The above conclusions are summarized by the following theorem:

THEOREM 6. *Let $\Sigma\subseteq\mathscr{R}^3$ be some smooth surface satisfying the Weingarten relation*

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(4·9) *subject to inequality* (4·10), *then* $\Sigma$ *is a hyperbolic W-surface and there exists a transformation such that the characteristics on* $\Sigma$ *form a Tchebychef net.*

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