Euclidean constructions with a right-angle tool

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In [1], the following problem appears on page 205.

12. *A right-angle tool can be used to draw a line through two points and erect a perpendicular to a given line at a given point. Use this tool to drop a perpendicular from a given point to a given line.*

No solution is given but it is not too difficult to determine the steps needed to do this construction. Euclidean constructions are classically done with a straight edge and compasses, but this problem raises the following question. What Euclidean constructions can be done with only a right-angle tool which also functions as a straight edge? The short answer is, not many, but if a slight adjustment is made as to how the tool can be used, many other Euclidean constructions are possible.

More formally, we can think of the right-angle tool (RAT) as a flat, rigid piece of material with one right-angle corner. The sides can be thought of as unmarked of infinite length and the tool can be moved freely around the plane. We may assume from the definition of a RAT that it can also serve as a straight edge. In the building trade, this tool in its finite form is known as a framing or carpenter's square.

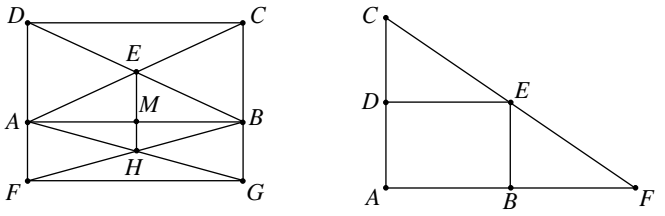
For points X and Y in a plane, $\overline{XY}$ represents the line segment joining the points and XY is the distance between them. We let $\angle ABC$ represent both the angle and its measure and $\angle D = \angle E$ meaning equal measures. The justifications for the following results have mostly been left out due to the simplicity of the arguments.

If L is any line with distinct points A and B, we shall first show, using the RAT, how to construct a segment that will be a rational multiple of this distance. We first construct a rectangle with one side $\overline{AB}$, its diagonals, and their intersection point. At point A, draw the perpendicular to $\overline{AB}$ and choose point D on it. Draw $\overline{DB}$. Construct the perpendiculars at D to $\overline{AD}$ and at B to $\overline{AB}$ and locate the intersection point at C. Draw $\overline{AC}$ and label the intersection with $\overline{DB}$ as E (Figure 1(a)).

To find the midpoint of $\overline{AB}$, construct rectangles $ABCD$ and $ABGF$ with diagonal intersections at E and H (Figure 1(a)). Draw $\overline{EH}$ and label the intersection of $\overline{EH}$ with $\overline{AB}$ as M. M will be the midpoint of $\overline{AB}$.

To double the length of $\overline{AB}$, construct the perpendicular at A to $\overline{AB}$ and choose point C on it (Figure 1(b)). Find the midpoint D, construct rectangle $ADEB$, draw $\overline{CE}$ and extend it to intersect the extended line $\overleftrightarrow{AB}$ at F. Thus $CD = DA = EB$, $\overline{CA} \,//\, \overline{EB}$, and $\overline{DE} \,//\, \overline{AF}$. These imply that $\triangle CDE \cong \triangle EBF$ and consequently that $AB = DE = BF$. Applying this process to $\overline{BF}$ yields another segment of length AB and so, iteratively, any point X can be found on this line such that $AX = mAB$ where m is a positive integer.

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(a) (b)

FIGURE 1

To divide $\overline{AB}$ into three equal parts, draw the perpendicular at A to $\overline{AB}$ and choose point D on it (Figure 2). Extend $\overline{AD}$ to find points C and E so that $AD = DC = CE$. Draw $\overline{EB}$. Construct perpendiculars at D and C to $\overline{AE}$ and extend them to intersect $\overline{EB}$ at points G and F. Construct perpendiculars at F to $\overline{CF}$ and at G to $\overline{DG}$ and extend them to intersect $\overline{AB}$ at points H and I. H and I divide $\overline{AB}$ into three equal parts.

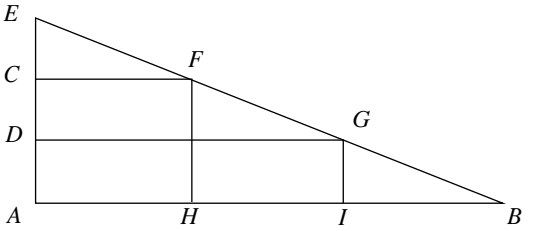


FIGURE 2

To divide $\overline{AB}$ into n equal parts, follow the same idea except now extend $\overline{AD}$ n times.

A rational number multiple m/n of $\overline{AB}$ is found by constructing a segment of length mAB and then dividing that into n segments.

We now solve the original problem: Given $\overline{AB}$ and a point P not on $\overleftrightarrow{AB}$, draw the line through P perpendicular to $\overline{AB}$. We can assume that $\overline{AB}$ is not perpendicular to $\overline{AP}$ (Figure 3).

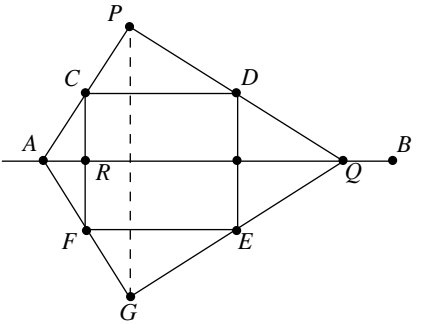


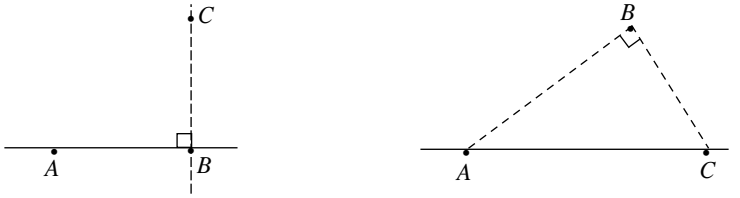
FIGURE 3

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Draw $\overline{AP}$ and use the RAT to construct the perpendicular to $\overline{AP}$ at P and assume it intersects $\overline{AB}$ at Q. Find midpoints C and D of $\overline{AP}$ and $\overline{PQ}$ respectively and draw $\overline{CD}$. Construct perpendiculars to $\overline{CD}$ at points C and D, and once the intersections with $\overline{AB}$ are found, double those lengths to obtain points F and E. Draw $\overline{FE}$. Draw $\overline{AF}$ and $\overline{QE}$ and extend the segments to intersect at point G. Draw $\overline{PG}$. Now $\triangle ACR \cong \triangle AFR$, $CDEF$ is a rectangle, and $\triangle PCD \cong \triangle GFE$ which imply that C and F are midpoints of $\overline{AP}$ and $\overline{AG}$. The Midpoint Theorem ensures that $\overline{FC} \,//\, \overline{PG}$ and since $\overline{CF}$ is constructed to be perpendicular to $\overline{AB}$, then $\overline{PG} \perp \overline{AB}$.

One consequence of this is that given a line L and a point P not on L, the RAT can construct a line K through P that is parallel to L. Furthermore, any distance AB on L can then be reproduced on the line K. There is, however, a shortcoming of the RAT, namely, that it cannot reproduce a given distance on a line that is not parallel to L. This is not too surprising since this is a critical property of compasses. We are left then with no way to bisect angles or to construct equilateral triangles and squares.

Alternatives to using straightedge and compasses to do geometric constructions have arisen over the years and an interesting collection of them can be found in [2]. To expand our use of the RAT, we simply add another way that it can be employed.



(a) (b)

FIGURE 4

Using the initial definition of the RAT, we start with points A and B and the line joining them (Figure 4(a)) and then placing an edge of the RAT on $\overline{AB}$ with corner at B, we mark point C and draw a unique segment $\overline{BC}$. We now let the RAT also do the following. Beginning with two points A and C and the line joining them (Figure 4(b)), we adjust the RAT so that its two edges pass through A and C and then mark point B at the right-angle corner. In particular, B can be located on a specific line if desired. This new definition will be referred to as the M-RAT. This addition to the definition is hinting at the idea of compasses since $\overline{AC}$ can be thought of as a diameter with point B being on the circle. With this new capability, we now show that AB can be reproduced along an arbitrary line L (see Figure 5(a)). The construction relies on the fact that the midpoint of the hypotenuse of a right triangle is equidistant from the vertices.

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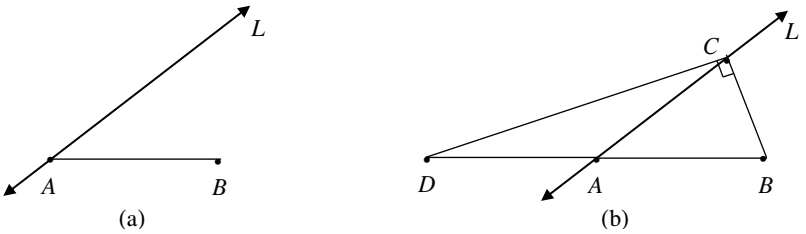


FIGURE 5

Double $\overline{AB}$ to obtain point D giving $AD = AB$ (Figure 5(b)). With point D on one side of the RAT and point B on the other, move the tool so that the corner lies on L. Mark that point C. Now $AB = AC = AD$ and so AB has been reproduced along L. This also implies that $\angle ADC = \angle ACD$ and since $\angle CAB$ is an exterior angle for ΔACD, then $\angle D = \frac{1}{2}\angle CAB$. Thus, this technique can also be used to find angle bisectors.

The question now is, can the M-RAT do everything that a straightedge and compasses (SC) can do? For a long period of time, there were certain constructions like the trisection of an angle, squaring a circle, and duplicating a cube that defied solution using only a SC. In the early nineteenth century, when the SC constructions were given an algebraic representation, it became clear exactly what SC could and could not do. We give the key results [3].

We begin with a line and a unit length on that line. If a segment of length x can be constructed using a SC, then x is called a constructible number. To put this in the context of the Euclidean plane, the point (x, y) is constructible if both x and y are constructible.

Theorem: A number is constructible if, and only if, a segment of that length can be constructed from rational number lengths in a finite number of steps using only the operations of addition, subtraction, multiplication, division, and taking square roots.

Given the unit segment on the x-axis from (0, 0) to (1, 0), the RAT (and hence the M-RAT) can construct all rational numbers on it and since it can also reproduce distances on parallel lines, then additions and subtractions of constructible numbers can be done (Figure 6).

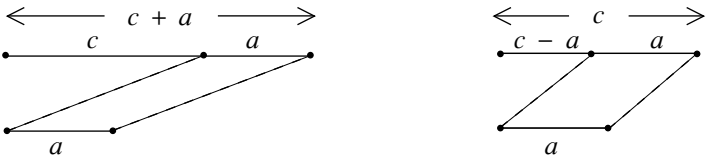


FIGURE 6

Since the RAT cannot reproduce distances in arbitrary directions, one needs the M-RAT to construct the product and quotient of constructible

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numbers. Construct segments with distances 1, a, and b, draw $\overline{AB}$, and construct $\overline{CD}$ so that $\overline{AB} \,//\, \overline{CD}$ (Figure 7).

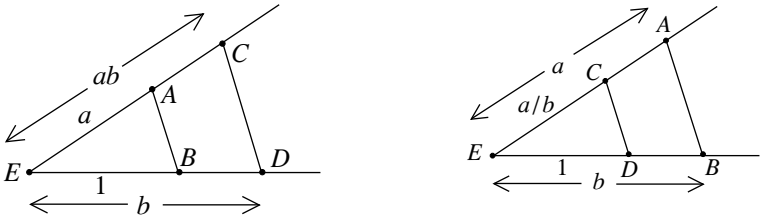


FIGURE 7

In both cases, ΔAEB is similar to ΔCED. Thus, with the M-RAT, we can do the usual arithmetic operations on the constructible numbers. The following shows how to construct square roots (Figure 8). Let r be a constructible number.

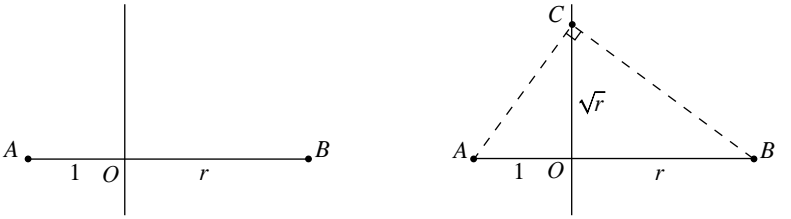


FIGURE 8

Start with $\overline{AB}$ where $AO = 1$ and $OB = r$. Construct the perpendicular at O and use the M-RAT to find point C so that $\angle ACB$ is right. Then, similar triangles give $OC^2 = AO \cdot OB$ and so $OC = \sqrt{r}$.

Thus the numbers constructible using the M-RAT are the same as the numbers constructible using SC. There is nothing in the definition of the M-RAT that would suggest that it could do more than SC. On the other hand, there is one obvious shortcoming, namely, that the M-RAT cannot draw a circle. It can find points determining a circle but that is the extent of it.

References

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