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No elliptic limits for quadratic maps

CARSTEN LUNDE PETERSEN
IMFUFA, Roskilde University, Postbox 260, DK-4000 Roskilde, Denmark
(e-mail: lunde@.ruc.dk)

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Abstract. We establish bounds for the multipliers of those periodic orbits of $R_\mu(z) = z(z+\mu)/(1+\overline{\mu}z)$, which have a Poincaré rotation number p/q. The bounds are given in terms of p/q and the (logarithmic) hororadius of μ to $e^{2\pi ip/q}$. The principal tool is a new construction denoted a 'star' of an immediate attracting basin. The bounds are used to prove properties of the space of Möbius conjugacy classes of quadratic rational maps. These properties are related to the mating and non-mating conjecture for quadratic polynomials [**Ta**]. Moreover they are also reminiscent of Chuckrows theorem on the non-existence of elliptic limits of loxodromic elements in quasiconformal deformations of Kleinian groups. We bear this analogy further by proving an analog of Chuckrows theorem for deformations of certain holomorphic maps.

0. *Introduction*

Let R be a rational map. The Fatou set $F_R \subset \overline{\mathbb{C}}$ for R is the maximal subset on which the family of iterates $\{R^n\}_{n\geq 0}$ is locally a normal family in the sense of Montel. The Fatou set is trivially open. The compact complement $J_R = \overline{\mathbb{C}} - F_R$ is called the Julia set for R. We use the term Fatou domain to denote a connected component of F_R. A point z is periodic for R with (exact) period (k minimal) k, if $R^k(z) = z$. For a k-periodic point z the complex number $\lambda = D_z R^k$ is called the multiplier/eigenvalue of z or of the orbit of z. The periodic point z is called attracting if $|\lambda| < 1$, repelling if $|\lambda| > 1$ and indifferent if $|\lambda| = 1$. Furthermore z is called rationally indifferent or parabolic, if λ is a root of unity. Repelling and parabolic periodic points are contained in the Julia set, whereas attracting periodic points are contained in the Fatou set.

Let $\mu \in \mathbb{D}$ and define $R_\mu(z) = z(z+\mu)/(1+\overline{\mu}z)$. The points 0 and ∞ are fixed points for R_μ with multipliers μ and $\overline{\mu}$ respectively. Moreover the unit circle $\mathbb{S}^1$ contains all the repelling periodic orbits and equals the Julia set. A q-periodic R_μ-orbit in $\mathbb{S}^1$ is said to have *(Poincaré) rotation number* p/q, $(p,q) = 1$, if the restriction of R_μ to this q-periodic orbit coincides with an orientation preserving circle homeomorphism with rotation number p/q. Each rational number p/q, $(p,q) = 1$ is the rotation number for precisely one q-periodic orbit for R_μ (see [**Go**]). This orbit will be denoted the p/q orbit for R_μ.

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For p/q a rotation number we shall always assume $(p, q) = 1$. The number p is however only unique modulo q. For instance $-p/q$ is equivalent to $(q - p)/q$.

Given $M \in \mathbb{C}$ with $\Re(M) \neq 0$ and p/q with $(p, q) = 1$ we define the number

$$r(M, p/q) = \frac{|M - (p/q)2\pi i|}{2 \cdot \sin \eta}$$

where η is the angle between $M - (p/q)2\pi i$ and $2\pi i$. In terms of polar coordinates the number $r = r(M, p/q)$ is the radius of the circle $C(\pm r + 2\pi i p/q, r)$ through M, tangent to the imaginary axis at $(p/q)2\pi i$ [here the $\pm$ indicates the sign of $\Re(M)$].

Let $\mu \in \mathbb{C}^* - \mathbb{S}^1$, so that $\Re(\log \mu) \neq 0$. Moreover let a rotation number p/q be given. Then there is at least one and at most two logarithms M of μ such that $|M - (p/q)2\pi i|$ is minimal among all choices of logarithm of μ. We define two radii:

$$r(\mu, p/q) = r(M, p/q) \quad \text{and} \quad \frac{1}{\widehat{r}(\mu, p/q)} = \sum_{n \in \mathbb{Z}} \frac{1}{r(M, (p + nq)/q)}.$$

We remark that these radii satisfy $r(\mu, p/q) < \widehat{r}(\mu, p/q)$, depend only on μ and $p/q \bmod \mathbb{Z}$ and are thus independent of the integer part of p/q.

THEOREM A. *Let $\mu \in \mathbb{D}^*$ and let $\rho > 1$ be the multiplier of the p/q orbit for R_μ. Then:*

$$0 < \log \rho < q^2 \cdot \widehat{r}(\mu, p/q) < q^2 \cdot r(\mu, p/q).$$

Let f be holomorphic in an open neighbourhood $U \subseteq \mathbb{C}$ of α with $f(\alpha) = \alpha$. Suppose $\gamma_0, \ldots, \gamma_{q-1}$ are q arcs in U starting at α and otherwise disjoint, labelled in their counter clockwise direction around α. The q arcs are said to be p/q rotated, or to form a p/q cycle under f, iff $(p, q) = 1$ and for all $j = 0, \ldots, q - 1$, $f(\gamma_j) \subseteq (\supseteq)\gamma_{(j+p) \bmod q}$.

Let R be a rational map with α a repelling or parabolic k-periodic point. We say that α has *(combinatorial) rotation number p/q*, if there exists a simply connected Fatou domain Λ and a kq-periodic access $(\alpha, [\gamma])$ with $\gamma \in \Lambda$ such that, the q curves $\gamma, R^k(\gamma), \ldots, R^{k(q-1)}(\gamma)$ form a p/q-cycle under R^k and define q distinct accesses to α.

Let

$$R_{\lambda,\mu}(z) = z\frac{z + \lambda}{1 + \mu z}, \quad \lambda \neq \mu^{-1} \in \mathbb{C}.$$

The maps $R_{\lambda,\mu}$ have fixed points at 0 and ∞ with multipliers λ and μ respectively. The following theorem and Corollary 1 was conjectured by Milnor.

THEOREM B. *Suppose the Julia set $J_{\lambda,\mu}$ of the rational map $R_{\lambda,\mu}$ is connected for some pair of parameters $\mu \in \mathbb{D}^*$ and $|\lambda| > 1$. Then the repelling fixed point* 0 *has a well defined rotation number p/q and its multiplier λ verifies*

$$r(\lambda, p/q) < \widehat{r}(\mu, -p/q) < r(\mu, -p/q).$$

Let $P_c(z) = z^2 + c$, $c \in \mathbb{C}$. The connectedness locus of this the quadratic polynomial family is the Mandelbrot set $M = \{c \in \mathbb{C} \mid J_c \text{ is connected}\}$. Let $Q_\lambda(z) = \lambda z + z^2$, $\lambda \in \mathbb{C}$. The two polynomials Q_λ and P_c with $c = \pi(\lambda) = (\lambda/2) - (\lambda^2/4)$ are affine conjugate and thus equivalent holomorphic dynamical systems. The connectedness locus

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M_0 of the family $\{Q_\lambda\}_{\lambda\in\mathbb{C}}$ is thus a double cover of M. Let $\theta \in \mathbb{T} := \mathbb{R}/\mathbb{Z}$, and define the θ limb $M_{0,\theta}$ of M_0 as follows. If θ is irrational set $M_{0,\theta} = \{e^{i2\pi\theta}\}$ and if $\theta = p/q$ is rational, $(p, q) = 1$ let $M_{0,p/q}$ be the set of λ for which the fixed point 0 for Q_λ has rational rotation number p/q. The Douady landing theorem for polynomials (see [**Pe,** Corollary B.2]) implies that for any $\lambda \in M_0 \setminus (\overline{\mathbb{D}} \setminus \exp(i2\pi\mathbb{Q}))$ the fixed point 0 of Q_λ has a rotation number. Thus $M_0 = \mathbb{D} \cup \cup_{\theta\in\mathbb{T}} M_{0,\theta}$. Moreover, Douady and Hubbard have shown that the set M_0 is connected and that its limbs are connected and mutually disjoint.

The family of maps $R_{\lambda,\mu}$ defined above parametrizes the space $\mathcal{M}_2$ of Möbius conjugacy classes of rational maps of degree 2 except for the single class of maps with a single but triple-fixed point (see [**Mi**]). For $\mu \in \mathbb{D}$ fixed we define the connectedness locus $M_\mu := \{\lambda \mid J_{\lambda,\mu}$ is connected$\}$. By the theory of polynomial like mappings, (see [**DH**]), there exists a dynamically defined quasi-conformal homeomorphism, between open neighbourhoods of M_μ and M_0, mapping M_μ onto M_0. The restriction $\Pi_\mu : M_\mu \to M_0$, of the above homeomorphism, is unique on the boundary of M_μ and is unique in the interior of M_μ, if we require in addition that Π_μ is conformal there. In particular Π_μ is the identity on $\overline{\mathbb{D}}$. Here dynamically defined means that $R_{\lambda,\mu}$ and $Q_{\Pi_\mu(\lambda)}$ are conjugate on neighbourhoods of their filled Julia sets, by quasiconformal homeomorphisms, which are conformal on the interior of the filled Julia sets. We define the limbs $\{M_{\mu,\theta}\}_{\theta\in\mathbb{T}}$ of M_μ analogously to the limbs of M_0. Thus, Π_μ preserves limbs for all $\mu \in \mathbb{D}$. We obtain an immediate corollary of Theorem B.

COROLLARY 1. *Let $r > 0$ and $p/q \in \mathbb{Q}/\mathbb{Z}$ be given. Then for all $\mu \in \mathbb{D}^*$ with $r(\mu, p/q) \le r$ the p/q-limb of M_μ satisfies $M_{\mu,p/q} \subset \exp(\mathbb{D}(-r + i2\pi p/q, r)) \cup \{e^{2\pi i p/q}\}$.*

One may ask if Corollary 1 just exhibits some peculiarity of the normalization made by choosing the family $R_{\lambda,\mu}$. For $p/q = 0/1$ the answer is 'yes', by our normalization we are inhibiting the two fixed points 0 and ∞ in merging. However for $p/q \neq 0/1$ the answer is 'no', there are no elliptic limits in $\mathcal{M}_2$ as the following Corollary 2 shows. Furthermore at the end of this paper we shall formulate and prove a general version of Corollary 2 independent of the rational maps of degree two context. For a direct proof of Corollary 2 we need a relation, which has been proved by Milnor (see [**Mi**]). Let κ denote the multiplier of the third fixed point for $R_{\lambda,\mu}$, then $\lambda\mu\kappa - (\lambda + \mu + \kappa) + 2 = 0$ or equivalently, since $\lambda\mu \neq 1$

$$\kappa = \frac{2 - \lambda - \mu}{1 - \lambda\mu}. \tag{1}$$

We shall use the following terminology (a longer, but more precise term would be strictly less than second order tangential). A sequence $\{\mu_n\}_{n\ge1} \subset \mathbb{D}$ is said to *converge non-quadratically* to $\mu_0 = e^{i2\pi p/q}$, $(p, q) = 1$, if $r(\mu_n, p/q) \underset{n\to\infty}{\longrightarrow} 0$.

COROLLARY 2. *Let $p/q \neq 0/1$ and let $\lambda(\mu) \in M_{\mu,p/q}$. If $\mu \in \mathbb{D}^*$ tends to $e^{-(p/q)2\pi i}$ non-quadratically, then $\kappa(\mu) \to \infty$, in particular $R_{\lambda(\mu),\mu}$ leaves any compact subset of the space of rational maps of degree two.*

A hyperbolic component of the space $\mathcal{M}_2$ is a maximal connected open subset on which both the critical points of the corresponding maps are attracted to attracting periodic orbits.

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COROLLARY 3. [**Mi**] *Any hyperbolic component of $\mathcal{M}_2$ containing a polynomial is unbounded.*

For $\mu \in \mathbb{D}$ and $\lambda \in M_\mu$ we can interpret the map $R_{\lambda,\mu}$ as the mating of the two quadratic polynomials $Q_{\Pi_\mu(\lambda)}$ and Q_μ. More precisely any polynomial-like restriction of $R_{\lambda,\mu}$ to a neighbourhood of the filled Julia set has internal class $Q_{\Pi_\mu(\lambda)}$ and external class R_μ.

The only if part of Tan Lei's mating conjecture states that for $p/q \neq 0/1$, one cannot mate a quadratic polynomial of $M_{0,p/q}$ with a polynomial of the conjugate limb $M_{0,-p/q}$. Our Theorem B and its corollaries shows that, if we keep $Q_{\Pi_\mu(\lambda)}$ in $M_{0,p/q}$ and let μ tend non-quadratically to the root of the conjugate limb $M_{0,-p/q}$, then the mating $R_{\lambda,\mu}$ of Q_μ and $Q_{\Pi_\mu(\lambda)}$ tends to infinity in the space of quadratic rational maps.

The geometry hidden in Corollary 2 is that the maps $R_{\lambda,\mu}$ converge locally uniformly (away from $z = -e^{(p/q)2\pi i}$) to the elliptic Möbius transformation $z \mapsto ze^{(p/q)2\pi i}$, when $\mu \in \mathbb{D}$ tends to $e^{-(p/q)2\pi i}$ non-quadratically. That is, at the limit, the map $R_{\lambda,\mu}$ has left the space of rational maps of degree 2 and has become a rational map of degree 1 (for a more general discussion of this point see [**Mi,** pp. 8–10]). This situation is reminiscent of the conclusion in Chuckrows theorem on quasiconformal deformations of Kleinian groups (see [**Ma,** p. 97]). If a loxodromic element of some Kleinian group converges to an elliptic element under some deformation of the group, then it is not possible to normalize the deformed groups in such a way as to retain the whole group in the algebraic limit. Either at least the loxodromic element (and the elliptic) is lost or the rest of the group is lost and only the elliptic element remains. With the normalization we have chosen for the maps $R_{\lambda,\mu}$ we end up with only the elliptic map and we loose the rest of the dynamics, when we try to force 0 and ∞ to become 'elliptic'. At the end of this paper we isolate the above phenomena and prove that the dynamically trivial 'elliptic' limit is caused by the geometry of the immediate attracted basin of ∞ and not by the fact that the map is rational (of degree two).

1. *Proof of Theorem B and its corollaries*

The proof of Theorem B is based on Theorem A and the PLY-inequality (see below). Let us recall the PLY-inequality and some of its prerequisites. The *conjugate multiplier* corresponding to $(\alpha, [\gamma])$ a periodic access ($[\gamma]$ to α) in a simply connected periodic Fatou domain Λ, is defined as follows. Let $\phi : \Lambda \to \mathbb{D}$ be a Riemann map and let $R_\Lambda := \phi \circ R^n \circ \phi^{-1}$, where n is the period of Λ. The periodic access $(\alpha, [\gamma])$ corresponds under ϕ to a periodic point $\alpha' \in \mathbb{S}^1$ for R_Λ. The *conjugate multiplier* corresponding to $(\alpha, [\gamma])$ is the multiplier of α'. This number is real, greater than or equal to 1 and does not depend on the various choices made.

THEOREM. (The PLY-inequality) *Suppose α is a repelling periodic point with multiplier λ and rotation number p/q for a rational map R. Let $\rho_1, \ldots, \rho_m$ be conjugate multipliers corresponding to distinct orbits of periodic accesses to α. Then:*

$$r(\lambda, p/q) \leq \frac{1}{q^2 \cdot \sum_{j=1}^{m} 1/\log \rho_j}.$$

For a proof see, for example, [**Pe**].

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Proof of Theorem B. Suppose the Julia set $J_{\lambda,\mu}$ of the rational map $R_{\lambda,\mu}$ is connected for some pair of parameters $\mu \in \mathbb{D}^*$ and $|\lambda| > 1$. Then the attracting basin Λ of ∞ is connected and simply connected. Let $\phi : \Lambda \to \mathbb{D}$ be a Riemann map, which maps ∞ to 0. Composing ϕ with a rotation if necessary, we can suppose the conjugate map equals R_μ. The Douady Landing theorem (see [**Pe,** Theorem B]) implies that the repelling fixed point 0 has a rational rotation number p/q. Then the $-p/q$ orbit for R_μ corresponds under ϕ to a (unique) p/q-cycle of accesses to 0 from Λ. The multiplier ρ of the $-p/q$ orbit is the conjugate multiplier of the corresponding p/q-cycle of accesses to 0 from Λ and the PLY-inequality yields:

$$r(\lambda, p/q) \leq \frac{\log \rho}{q^2}.$$

For $\mu \neq 0$ we obtain Theorem B by combining the above inequality with Theorem A for rotation number $-p/q$. □

2. *An asymptotic formula*

The inequality in Theorem A is not best possible. Indeed if we let μ tend to 0, then the bound in the inequality tends to infinity. Fortunately we have quite good estimates from other sources, when μ is close to 0, viz.

$$\rho \leq \left(1 + \frac{1 + |\mu|}{1 - |\mu|}\right)^q,$$

where ρ is the multiplier of any q-periodic orbit for R_μ in $\mathbb{S}^1$ (see [**Pe,** p. 20]). However the inequality in Theorem A is asymptotically best possible to first order (in $r = r(\mu, p/q)$), when μ tends to $e^{i2\pi p/q}$ non-quadratically. To see this we shall use the holomorphic index of rational maps (see [**Mi**]). Let R be any rational map of degree at least two and without parabolic fixed points, then

$$1 = \sum_{z, R(z)=z} \frac{1}{1 - D_z R}. \tag{2}$$

The inequality in Theorem A for the eigenvalue ρ of the R_μ-fixed point in $\mathbb{S}^1$ can be improved to an exact value.

LEMMA 2.1. [**Mi**] *For μ in the horocycle $C(1 - r, r) \setminus \{1\}$ of $\mathbb{D}$ the multiplier ρ of the R_μ-fixed point in $\mathbb{S}^1$ is given by*

$$\rho = \frac{1}{1 - r}.$$

Proof. For $\mu \in C(1 - r, r) \setminus \{1\}$ we get from (2)

$$\frac{1}{\rho - 1} = \frac{1}{1 - \mu} + \frac{1}{1 - \overline{\mu}} - 1 = \frac{2\Re(1 - \mu)}{|1 - \mu|^2} - 1 = \frac{1}{r} - 1.$$ □

We could proceed using (2) and the expression for ρ in Lemma 2.1 to obtain the exact value of the multiplier of the period two orbit for R_μ. We shall leave this computation to the

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reader. However, for period three and higher, one cannot use this method to obtain exact formulae, because there is more than one orbit of any given period bigger than two. We are thus left with estimates. To prove the above statement about the asymptotic effectiveness of the bound in Theorem A we shall hence use (2) for R_μ^q. It yields

$$1 = \sum_{z, R^q(z)=z} \frac{1}{1 - D_z R^q}. \tag{3}$$

Let $x_0, \ldots, x_{q-1}$ be any q-periodic orbit with multiplier ρ. Then we get from (3)

$$q\frac{1}{\rho - 1} = -1 + \frac{1}{1-\mu^q} + \frac{1}{1-\overline{\mu}^q} + \sum_{\substack{R^q(z)=z \\ z \neq x_j, 0, \infty}} \frac{1}{1 - D_z R^q}.$$

The terms of the last sum are all negative thus for $\mu^q \in \mathbb{D} \setminus \mathbb{D}(1 - r', r')$ we have

$$q\frac{1}{\rho - 1} < -1 + \frac{2\Re(1-\mu^q)}{|1-\mu^q|^2} \leq \frac{1}{r'} - 1 \quad \text{and} \quad 1 + q\frac{r'}{1-r'} < \rho. \tag{4}$$

For $r > 0$ let $0 < r'(r) = r' = (1 - \exp(-2qr))/2 < 1/2$. Given $\mu \in \mathbb{D}^*$ with $r = r(\mu, p/q) < \pi/q$, we claim (and prove below) that $\mu^q \in \mathbb{D} \setminus \mathbb{D}(1 - r', r')$. Let ρ denote the multiplier of the unique p/q orbit for R_μ. Combining this claim with (4) and Theorem A we obtain after a few elementary manipulations the following sandwich for ρ:

$$1 + q\tanh(qr) < \rho < \exp(q^2 r). \tag{5}$$

The two bounds agree to first order terms in r, which means that the upper bound is best possible to first order in r.

To complete the proof of (5) we shall show that $\mu^q \in \mathbb{D} \setminus \mathbb{D}(1 - r', r')$. Recall that $r(\mu, p/q) = r$ implies that there exists logarithm M of μ with $|M - (-r + i2\pi p/q)| = r$. But then $\mu^q = \exp(qM) = \exp(qM - p2\pi i)$ and $|qM - (-qr + p2\pi i)| = qr$. Hence, μ^q belongs to the $\mathbb{D}^*$-horocycle, which is tangent to $\mathbb{S}^1$ at 1 and which passes through $\exp(-2qr)$. As exp is injective on $\mathbb{D}(-qr, qr)$ for $r \leq \pi/q$ the claim is a consequence of Lemma 2.2 below.

LEMMA 2.2. *For any $r > 0$ the $\mathbb{D}^*$ horodisk $\exp(\mathbb{D}(-r, r))$ contains the $\mathbb{D}$ horodisk $\mathbb{D}(1 - r', r')$, where*

$$0 < r'(r) = r' = \frac{1 - \exp(-2qr)}{2} < \frac{1}{2}.$$

Proof. The horodisks are obtanied in the limit $t \to 1$ of the hyperbolic disks with center t and radius equal to the distance from t to $1 - 2r'$. Hence, it suffices to prove that for any $1 - 2r' < t < 1$

$$D_{\mathbb{D}}(t, d_{\mathbb{D}}(t, 1 - 2r')) \subseteq D_{\mathbb{D}^*}(t, d_{\mathbb{D}^*}(t, 1 - 2r')).$$

Let $\tau : \mathbb{D}^* \to \mathbb{R}_+$ denote the coefficient function of the infinitessimal hyperbolic metric on $\mathbb{D}^*$ relative to the ditto on $\mathbb{D}$. Then clearly $\tau(z) = \tau(|z|)$ and an elementary calculation shows that τ is monotonically decreasing from ∞ to 1 on the interval $]0, 1[$. Let $1 - 2r' < t < 1$ and let $z \in \mathbb{D}$ be arbitrary with $d_{\mathbb{D}}(t, z) = d_{\mathbb{D}}(t, 1 - 2r') = l_t$. We

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need to prove that $d_{\mathbb{D}^*}(t,z) \le d_{\mathbb{D}^*}(t,1-2r')$. Let $\gamma : [0,l_t] \to \mathbb{D}$ be the parametrization by $\mathbb{D}$-length of the $\mathbb{D}$-geodesic from t to z. And let similarly $\gamma_0 : [0,l_t] \to \mathbb{D}$ parametrize the geodesic segment $[t,1-2r']$ by $\mathbb{D}$-length. As the later is also a $\mathbb{D}^*$ geodesic segment we obtain

$$d_{\mathbb{D}^*}(t,z) \le l_{\mathbb{D}^*}(\gamma) = \int_0^{l_t} \tau(\gamma(t))\,dt \le \int_0^{l_t} \tau(\gamma_0(t))\,dt = l_{\mathbb{D}^*}(\gamma_0) = d_{\mathbb{D}^*}(t,1-2r'). \quad \square$$

3. *A 'star' for an immediate attracting basin and proof of Theorem A*

Let $U \subset \overline{\mathbb{C}}$ be a domain and let $f : U \to \overline{\mathbb{C}}$ be holomorphic. Suppose $z_0 \in U$ is an attracting fixed point for f. Let $\Lambda(z_0) = \{z \mid f^n(z) \to z_0 \text{ as } n \to \infty\}$ denote the attracting basin for z_0 and let $\Lambda_0(z_0)$ denote the immediate attracting basin, the connected component of $\Lambda(z_0)$ containing z_0.

Definition 3.1. The immediate basin $\Lambda = \Lambda_0(z_0)$ is called a degree d (immediate) basin if the restriction $f : \Lambda \to \Lambda$ is a proper map of degree d. For $d = 2$ we shall also use the term quadratic basin for Λ. (Note that $d \ge 2$, when f is non-trivial, i.e. if f does not coincide with a Möbius transformation.)

To simplify notation we shall write '$f : U \to \overline{\mathbb{C}}$ has a degree d basin Λ around the fixed point z_0 with multiplier μ', for the above. To condense even more we simply write a sixtuple $(f, U, z_0, \mu, \Lambda, d)$. If z_0 is periodic with exact period n we obtain similar definitions by replacing f with f^n in the above discussion.

Given $(f, U, z_0, \mu, \Lambda, d)$ we shall define certain subsets denoted *stars* of Λ. We define these stars in the general case, though we in this paper shall study them only in the case of a simply connected quadratic basin. Studying these stars for R_μ with $\Lambda = \mathbb{D}$ leads to a proof of Theorem A and studying them for general simply connected quadratic basins yields an analog of Chuckrows theorem for holomorphic maps with such a basin.

Let $(f, U, z_0, \mu, \Lambda, d)$ be given and let $\psi : \Lambda \to \mathbb{C}$ be a linearizing coordinate for f on Λ, i.e. ψ is holomorphic and locally injective at z_0, with $\psi(z_0) = 0$ and $\psi \circ f = \mu \cdot \psi$. Let $c_1, \ldots, c_k$ denote the critical points for f in Λ, not in the backwards orbit of z_0. (Note that $k \le d-1$ if Λ is simply connected, by the Hurewiz formula.) Then the set $\mathrm{Crit}'_{\psi,\Lambda} = \{\mu^{-n}\psi(c_i) \mid n \ge 0, 1 \le i \le k\}$ consists of the non-zero critical values of the (infinite degreed) branched covering ψ. For $1 \le i \le k$ let ζ_i denote a logarithm of $\psi(c_i)$ and let G denote the half grid $G := \{n \cdot M + m \cdot 2\pi i + \zeta_i \mid n, m \in \mathbb{Z}, n \le 0, 1 \le i \le k\}$. So that $z \in G$ if and only if $\exp(z) \in \mathrm{Crit}'_{\psi,\Lambda}$.

For the following discussion we fix a logarithm M of μ and a rotation number p/q. We consider the straight lines through G parallel to $q \cdot M - p \cdot 2\pi i$. Their projection to $\mathbb{C}^*$ by exp are $k' \cdot q$ disjoint logarithmic spirals (possibly straight lines), which are grouped into k' orbits under multiplication by μ, each orbit performing a p/q rotation around 0 and a $-p/q$ rotation around ∞, because the exponential map semi-conjugates translation by M to multiplication by μ. Moreover the spirals contain all the non-zero critical values of ψ. Here $1 \le k' \le k$ takes into account, the possibility that $\{\mu^{-n}\psi(c_i)\}_{n\ge 0}$ and $\{\mu^{-n}\psi(c_{i'})\}_{n\ge 0}$ may happen to lie on the same q spirals. The complement of the spirals in $\mathbb{C}^*$ has $k' \cdot q$ connected components $\widehat{U}_0, \ldots, \widehat{U}_{k'q-1}$ say labelled in their counter clockwise order with $\psi(c_1)$ on the common boundary of $\widehat{U}_0$ and $\widehat{U}_{k'q-1}$. Fix, for each $0 \le j < k'q$,

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a connected component $\widetilde{U}_j$ of $\exp^{-1}(\widehat{U}_j)$. Then $\widetilde{U}_j$ is a straight strip invariant under the translation $z \mapsto z + qM - p2\pi i$. Denote by $\widetilde{\gamma}_j$ the central straight line in $\widetilde{U}_j$, invariant under the above translation. Moreover, denote by $\widehat{\gamma}_j \subset \widehat{U}_j$ the projection $\exp(\widetilde{\gamma}_j)$, so that $\widehat{\gamma}_j$ is a logarithmic spiral parallel to the boundaries of $\widehat{U}_j$. Also the arcs $\widehat{\gamma}_j$ form k' p/q-cycles for multiplication by μ and so do the 'strips' $\widehat{U}_j$, each of which is p/q rotated around 0 under multiplication by μ, i.e. $\mu \cdot \widehat{U}_j = \widehat{U}_{(j+k'p) \bmod q}$ for each j.

Let $U_j \subset \Lambda$ be the unique connected component of $\psi^{-1}(\widehat{U}_j)$, whose closure contains 0. Then the restriction $\psi : U_j \to \widehat{U}_j$ is biholomorphic, because the restriction

$$\psi : \Lambda \setminus \psi^{-1}(\{0\} \cup \mathrm{Crit}'_{\psi,\Lambda}) \to \mathbb{C}^* \setminus \mathrm{Crit}'_{\psi,\Lambda}$$

is a covering map and $\widehat{U}_j$ is a simply connected subset of the range. Furthermore let $\gamma_j = \psi^{-1}(\widehat{\gamma}_j) \cap U_j$ so that ψ maps γ_j homeomorphically unto $\widehat{\gamma}_j$. Then the 'strips' U_j and the arcs γ_j each form k' orbits under f, and each orbit performs a p/q rotation around z_0 under f, because ψ semi-conjugates f to multiplication by μ.

Definition 3.2. The $(M, p/q)$*-star* $\Sigma_{M,p/q,\Lambda,f}$ of Λ for f is the open subset of Λ given by

$$\Sigma_{M,p/q,\Lambda,f} = \left(\overset{\circ}{\bigcup_{j=0}^{k'q-1}} \overline{U}_j \right).$$

For $0 \leq j < k'q$ the set U_j is called the *jth strip*, the set $\overline{U}_j - \Lambda$ is the *jth tip* or tip of U_j and the arc γ_j is the *jth wire* of the star $\Sigma_{M,p/q,\Lambda,f}$ or the wire of U_j.

When the map f and the basin Λ is obvious from the context we shall write $\Sigma_{M,p/q}$ for the $(M, p/q)$-star of Λ for f.

Note that the star $\Sigma_{M,p/q,\Lambda,f}$ is an open neighbourhood of z_0 and that f maps the star univalently into itself. In general one should not expect the tips of the wires to be one point sets. With mild extra assumptions, they form however a periodic orbit, with period dividing q.

LEMMA 3.3. *Given $(f, U, z_0, \mu, \Lambda, d)$, suppose that f^q is defined and holomorphic in a neighbourhood of a point x in the ith tip of $\Sigma_{M,p/q}$. Then $f^q(x) = x$, the ith tip equals x and if f is continuous on neighbourhoods of the orbit of x, then this periodic orbit form the tips of the orbit of U_i. Moreover, for any neighbourhood ω of x, there exists $R > 0$ such that $z \in \omega$, whenever $z \in U_i$ and $|\psi(z)| > R$.*

Proof. The proof is adapted from a classical argument introduced by Sullivan, Douady and Hubbard. Let $W \subset \Lambda$ be a neighbourhood relative to Λ of $\partial\Lambda$, without critical points for f and bounded by a Jordan preimage $\psi^{-1}(C(0, r))$ for some sufficiently big r. Moreover, let $W' \subset W = f^{-1}(W) \cap W$. Then the restriction $f : W' \to W$ is a degree d covering and hence expanding with respect to the hyperbolic metric on W. Let f^{-q} denote the branch of the inverse, which maps U_i biholomorphically onto itself. Let F denote the closure of a fundamental domain for the action of f^q on U_i bounded by $\partial W'$ and $f^{-q}(\partial W')$. The W-hyperbolic distance between z and $f^{-q}(z)$ is uniformly bounded for $z \in F$ and F has finite W-hyperbolic diameter, because F is relatively compact in W. Hyberbolic balls of a given diameter have spherical diameters, which tends to 0, when the balls approach the

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boundary. Hence, x and any other accumulation point of U_i (or equivalently of γ_i) not in U_i, but where f^q is defined, is a fixed point of f^q. But as f^q is holomorphic in a neighbourhood of x, it is an isolated fixed point. Hence, the entire tip, being a continuum, equals x and given any neighbourhood ω of x, the outer end of U_i is contained in it. By continuity the tips of the strips in the orbit of U_i equals the orbit of x. □

When $d = 2$ and Λ is simply connected in the above lemma, then $1 = k' = k$ and the orbit of x is the unique q-periodic orbit with rotation number p/q relative to Λ.

LEMMA 3.4. *Given $(f, U, z_0, \mu, \Lambda, d)$, suppose the tips of the q strips $\{U_{i+jpk'}\}_{j=0}^{q-1}$ are formed by a periodic orbit with exact period n. Then either $n = q$ or $n = 1$.*

Proof. Let $0 \le j_0 \le j_1 \le \cdots \le j_{q-1} < qk'$ denote the indices of the q strips above, ordered increasingly, so that $f(U_{j_l}) = U_{j_{(l+p) \bmod q}}$. To avoid double indexing we shall write U_l for U_{j_l} and γ_l for γ_{j_l}, etc. Let $x_0, \ldots, x_{q-1}$ denote the periodic orbit forming the tips of the above cycle of strips and write $q = n \cdot m$, where n is the exact period of the x_j. Then for each $j = 0, \ldots, n-1$ the m wires (arcs) $\gamma_j, \gamma_{j+n}, \ldots, \gamma_{j+(m-1)n}$ all land on $x_j = x_{j+n} = \cdots = x_{j+(m-1)n}$. Suppose $n < q$ so that $m > 1$ and let X_i, $0 \le i < m$ be the connected component of $\overline{\mathbb{C}} - \cup_{l=0}^{m-1} \overline{\gamma_{ln}}$ bounded by $\overline{\gamma_{in} \cup \gamma_{(i+1)n}}$. If $n > 1$, then the set $\cup_{l=0}^{m-1} \gamma_{1+ln} \cup \{x_1\}$ is a connected subset of $\cup_{i=0}^{m-1} X_i$. And moreover $\gamma_{1+ln} \subset X_l$ for $0 \le l < m$. Thus, a contradiction. This proves the lemma. □

Proof of Theorem A. Let $\mu \in \mathbb{D}^*$, $M \in \log \mu$ and a rotation number p/q be given. The tips of the $(M, p/q)$-star $\Sigma_{M,p/q}$ of $\mathbb{D}$ for R_μ are formed by a q-periodic orbit $x_0, \ldots, x_{q-1}$, because R_μ is holomorphic in a neighbourhood of $\overline{\mathbb{D}}$. The exact period n of this orbit equals 1 or q by Lemma 3.4. Moreover $n = q$ because any periodic orbit for R_μ in $\mathbb{S}^1$ has combinatorial rotation number $0/1$. Hence $x_0, \ldots, x_{q-1}$ is the p/q orbit for R_μ. Let $\rho > 1$ denote the multiplier of this orbit.

Recall that $\log \circ \psi : U_j \to \widetilde{U}_j$ is biholomorphic and conjugates dynamics for each j. Thus the quotient U_j/R_μ^q is isomorphic to the cylinder $\widetilde{U}_j/(z \mapsto z + qM - p2\pi i)$. This cylinder has the modulus $\operatorname{mod}(U_j/R_\mu^q) = \pi/(q^2 \cdot r(M, p/q))$. On the other hand, if ω is a sufficiently small disk around x_j, then $C = (\omega \cap \mathbb{D})/R_\mu^q$ is a cylinder of modulus $\pi/\log(\rho)$. As proved above, one end of U_j is entirely contained in ω, thus $C_M = U_j/R_\mu^q$ naturally embeds non-null homotopically and properly into C.

Fix a point say x_0 in the p/q-orbit for R_μ and the cylinder $C = (\omega \cap \mathbb{D})/R_\mu^q$. As above let $C_M = U_0/R_\mu^q$ denote the non-(homotopically) trivial subcylinder of C corresponding to the strip U_0. Choosing another logarithm $M' = M + n2\pi i$, $n \in \mathbb{Z}$ of μ we obtain similarly a non-trivial subcylinder $C_{M'} = U_j'/R_\mu^q \subseteq C$ corresponding to U_j' for some $0 \le j < q$. We claim that $C_M \cap C_{M'} = \emptyset$. To see this note that both cylinders C_M and $C_{M'}$ are naturally foliated by analytic Jordan curves corresponding to the straight lines in $\mathbb{C}$ parallel to $qM - p2\pi i$ and $qM' - p2\pi i = qM - (p - qn)2\pi i$ respectively. These straight lines intersect at constant angle say η. Hence, if the two cylinders intersect, then the corresponding foliations intersect transversely at constant angle η. But then the leaves of the two foliations cannot all be Jordan curves, because both C_M and $C_{M'}$ are subcylinders of the cylinder C. Hence, $C_M \cap C_{M'} = \emptyset$. As $M \ne M'$ were arbitrary logarithms of μ we

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obtain from the Grötzsch-inequality for annuli that

$$\sum_{n\in\mathbb{Z}} \frac{\pi}{q^2 \cdot r(M + n \cdot i2\pi,\, p/q)} < \frac{\pi}{\log \rho}$$

and thus

$$\log \rho < q^2 \cdot \widehat{r}(\mu,\, p/q) < q^2 \cdot r(\mu,\, p/q). \qquad \square$$

4. *Further results for simply connected quadratic basins*

Let $\epsilon(z) = 1/|z|$ denote the coefficient function of the complete Euclidean metric on $\mathbb{C}^*$ and let $d_\epsilon(\cdot,\cdot)$ denote the corresponding distance function. The metric given by ϵ is equivalent to the standard Euclidean metric on $\mathbb{C}$. Moreover the exponential map $(\mathbb{C}, 1) \xrightarrow{\exp} (\mathbb{C}^*, \epsilon)$ is a local isometry, where the constant function 1 indicates the standard Euclidean metric on $\mathbb{C}$. For $U \subset \overline{\mathbb{C}}$ a domain with universal covering isomorphic to $\mathbb{D}$, let λ_U denote the coefficient function of the hyperbolic metric on U and let d_U denote the corresponding distance function. One of the basic results on hyperbolic metrics is that for any simply connected domain $U \subset \mathbb{C}^*$ and any $z \in U$: $\epsilon(z) \le 2 \cdot \lambda_U(z)$.

Let the sixtuple $(f, U, z_0, \mu, \Lambda, 2)$ with Λ simply connected and a star $\Sigma_{M,p/q,\Lambda,f}$ be given. Using that the dynamics of f^q on each U_j are conjugate to the translation on a certain strip $\widetilde{U}_j$, one calculates that for all $z \in U_j$: $d_{\lambda_j}(z, f^q(z)) \le q^2 \cdot r(M, p/q)$, with equality precisely on the wire γ_j, where λ_j denotes the hyperbolic metric on U_j.

PROPOSITION 4.1. *Suppose the map $f : U \to \overline{\mathbb{C}}$ has a simply connected quadratic basin $\Lambda \subset \mathbb{C}$ around the fixed point 0 with multiplier $\mu \in \mathbb{D}^*$. For any star $\Sigma_{M,p/q}$ the function $f^q(z)/z$ has a unique, holomorphic logarithm $h = h_{M,p/q}$ defined on the star with $h(0) = qM - p2\pi i$. Moreover for any $0 \le j < q$ and any $z \in \gamma_j$*

$$|h(z)| \le 2 \cdot q^2 \cdot r(M, p/q) \quad \text{and} \quad d_\epsilon(f^q(z), z) \le 2 \cdot q^2 \cdot r(M, p/q). \tag{1}$$

Proof. The univalent function f^q on the simply connected domain $\Sigma_{M,p/q} \subset \mathbb{C}$ maps zero to zero. Thus the function $f^q(z)/z$ has a holomorphic logarithm h on the star $\Sigma_{M,p/q}$. As $\exp(qM - i2\pi\, p/q) = \mu^q = (f^q)'(0)$, we may specify such a logarithm uniquely by requiring that $h(0) = qM - i2\pi\, p/q$. This proves existence and uniqueness of h.

For each $0 \le j < q$ let V_j denote a connected component of $\exp^{-1}(U_j)$ and let $\widetilde{\lambda}_j$ denote the coefficient function of the hyperbolic metric on V_j. Then $1 \le 2\widetilde{\lambda}_j(z)$ for all $z \in V_j$ by the remark preceding this proposition and the fact that exp is a local isometry for the hyperbolic metrics of V_j and U_j, and for the Euclidean metrics on $\mathbb{C}$ and $\mathbb{C}^*$. Let $F : V_j \to V_j$ denote the conjugate of f^q by exp, i.e. $\exp \circ F = f^q \circ \exp$ on V_j. Moreover, let $\kappa_j = \exp^{-1}(\gamma_j) \cap V_j$. Then $d_{\widetilde{\lambda}_j}(z, F(z)) = q^2 \cdot r(M, p/q)$ for all $z \in \kappa_j$ as exp conjugates dynamics. Let $z \in \kappa_j$ be arbitrary and parametrize κ_j by $\widetilde{\lambda}_j$-length from z, so that $\kappa_j(0) = z$ and $\widetilde{\lambda}_j(\kappa_j(s)) \cdot |\kappa_j'(s)| = 1$ for all s, where s is the parameter. Then

$$|F(z) - z| \le \int_0^{q^2 \cdot r(M,p/q)} |\kappa_j'(s)|\, ds$$

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$$\leq \int_0^{q^2 \cdot r(M,p/q)} 2 \cdot \widetilde{\lambda}_j(\kappa_j(s)) \cdot |\kappa_j'(s)|\, ds$$
$$= 2 \int_0^{q^2 \cdot r(M,p/q)} ds = 2 \cdot q^2 \cdot r(M, p/q).$$

On the other hand $h(e^z) = F(z) - z$ for $z \in V_j$, because $\exp(F(z) - z) = f^q(e^z)/e^z$ and $F(z) - z \underset{\Re(z)\to-\infty}{\longrightarrow} qM - p2\pi i$. Thus $|h(z)| \leq 2 \cdot q^2 \cdot r(M, p/q)$ for all $z \in \kappa_j$.

This proves the first inequality of (1). The second inequality follows from the first and the fact that h is a logarithm of $f^q(z)/z$, because $\exp : \mathbb{C} \to \mathbb{C}^*$ is a local Euclidean isometry. □

When we speak of a continuous family of continuous/holomorphic maps or a family of continuous/holomorphic maps depending continuously on some parameter, we shall always infer continuity in the compact open topology on continuous/holomorphic maps, i.e. with respect to the topology induced by uniform convergence on compact sets. Moreover, a family f_μ of holomorphic maps with $\mu \in U$, U an open subset of $\mathbb{C}$ is said to depend holomorphically on μ, if the map $(z, \mu) \mapsto f_\mu(z)$ is holomorphic as a function of the two complex variables z and μ.

Definition 4.2. Suppose $f : U \to \overline{\mathbb{C}}$ is a holomorphic map, where $U \subset \overline{\mathbb{C}}$ is an open subset containing a f-fixed point $\alpha \in \mathbb{C}$ and $f'(\alpha) = \exp(i2\pi p/q)$ for some rotation number p/q. We shall say that α is an elliptic fixed point, if $f^q(z) \equiv z$ on a neighbourhood of α and parabolic otherwise. For a k-periodic α the terms elliptic and parabolic are defined analogously. Note that rational maps of degree $d \geq 2$ cannot have an elliptic periodic orbit.

Let $d_\sigma : \overline{\mathbb{C}} \times \overline{\mathbb{C}} \to [0, \pi]$ denote the spherical distance on $\overline{\mathbb{C}}$. For $\Sigma_{M,p/q,\Lambda,f}$ a star around $z_0 \in \Lambda$ as above let the spherical star radius $r_\sigma(\Sigma_{M,p/q,\Lambda,f})$ be given by

$$r_\sigma(\Sigma_{M,p/q,\Lambda,f}) = \sup\{d_\sigma(z_0, z) \mid z \in \overline{\gamma_j}, 0 \leq j < k'q\}$$

where γ_j denotes the jth wire of the star $\Sigma_{M,p/q,\Lambda,f}$.

The following Proposition 4.3 and its Corollaries 4.7 and 4.8 are the analogues of Chuckrows theorem for holomorphic maps with a simply connected proper quadratic basin.

PROPOSITION 4.3. *Let $\{(f_n, U_n, 0, \mu_n, \Lambda_n, 2)\}_{n\geq 1}$ be any sequence of holomorphic maps each with a simply connected quadratic basin Λ_n around the attracting fixed point 0 with multiplier $\mu_n \in \mathbb{D}^*$ and satisfying for some rotation number p/q:*

(1) *$r(\mu_n, p/q) \to 0$ as $n \to \infty$.*

(2) *There exists $\epsilon > 0$ a lower bound on the spherical radii $r_\sigma(\Sigma_{M_n,p/q,\Lambda_n,f_n}) > \epsilon$, where M_n denotes the logarithm of μ_n closest to $i2\pi p/q$.*

If there exists a (holomorphic) map f_0 such that $f_n \underset{n\to\infty}{\rightrightarrows} f_0$ uniformly on a neighbourhood of 0. Then f_0 has an elliptic fixed point at 0.

Proof. Suppose $f_n \underset{n\to\infty}{\rightrightarrows} f_0$ on some disk $\mathbb{D}_r$ of radius r around 0. Then f_0 is holomorphic on $\mathbb{D}_r$ and $f_0'(0) = \mu_0 = \exp(i2\pi p/q) \neq 0$. Thus decreasing r if necessary, we shall suppose f_0 is injective on $\mathbb{D}_r$. Then all but finitely many f_n are also injective on any fixed compactly contained sub disk, say $\mathbb{D}_{r/2}$. It follows that $\overline{\mathbb{C}} \setminus \Lambda_n \not\subseteq \mathbb{D}_{r/2}$ for all but finitely

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many n. This is because any sufficiently thin annulus in Λ_n along $\partial\Lambda_n$ is double covered by an annulus inside itself.

For each $n \geq 1$ let $z_n \in \overline{\mathbb{C}} - \Lambda_n$ be a point closest to infinity. Passing to a subsequence, if necessary we can suppose $z_n \to z_\infty \in \overline{\mathbb{C}}$. By the argument above $z_n \notin \mathbb{D}_{r/2}$ for all but finitely many n and thus $z_\infty \notin \mathbb{D}_{r/2}$.

Define $H_y(z) = (y \cdot z)/(y - z)$ for $y \in \mathbb{C}^*$ and $H_\infty(z) = z$, so that H_y depends holomorphically on $y \in \overline{\mathbb{C}} - \{0\}$. We can suppose that $z_n \equiv \infty$, conjugating each f_n by H_{z_n} and f_0 by H_{z_∞}, if necessary, and possibly replacing ϵ in (2) by some smaller value. In particular we shall suppose that each $\Lambda_n \subset \mathbb{C}$ so that Proposition 4.1 applies.

Let $\gamma_{j,n}$ denote the jth wire of the star $\Sigma_{M_n, p/q, \Lambda_n, f_n}$ and let $Y_n = \cup_{0 \leq j < q} \overline{\gamma_{j,n}} \subset \overline{\mathbb{C}}$. Then Y_n is compact, connected and contains 0. We can suppose $Y_n \to Y_0 \in \mathrm{Comp}(\overline{\mathbb{C}})$ as $n \to \infty$, extracting a subsequence if necessary, where $Y_0 \subset \overline{\mathbb{C}}$ is some compact and connected set containing 0. This is because $\mathrm{Comp}(\overline{\mathbb{C}})$, the space of compact subsets of $\overline{\mathbb{C}}$, is a compact space, when equipped with the Hausdorff metric. Moreover, the spherical radius $\sup\{d_\sigma(0, z) \mid z \in Y_0\}$ is at least ϵ, as this holds for any of the Y_n.

We can suppose that f_n^q converges uniformly to f_0^q on $\mathbb{D}_r$, decreasing r if necessary. Clearly $f_0(0) = f_0^q(0) = 0$. Let $x_0 \in Y_0 \cap \mathbb{D}_r - \{0\}$ be arbitrary and choose a sequence $\{x_n \in Y_n - \{0\}\}_{n \geq 1}$ with $x_n \to x_0$ as $n \to \infty$. Then there exists n_0 such that $x_n \in \mathbb{D}_r$ for $n \geq n_0$ and for all such n we have

$$\begin{aligned} d_\epsilon(f_0^q(x_0), x_0) \leq\ & d_\epsilon(f_0^q(x_0), f_0^q(x_n)) + d_\epsilon(f_0^q(x_n), f_n^q(x_n)) \\ & + d_\epsilon(f_n^q(x_n), x_n) + d_\epsilon(x_n, x_0) \end{aligned}$$

the right-hand side converges to 0 as $n \to \infty$, because f_0^q is continuous, $f_n^q \rightrightarrows f_0^q$ locally uniformly on the neighbourhood $\mathbb{D}_r$ of x_0 and

$$d_\epsilon(f_n^q(x_n), x_n) \leq 2 \cdot q^2 \cdot r(\mu_n, p/q) \underset{n \to \infty}{\longrightarrow} 0$$

by (1). Thus $f_0^q(x_0) = x_0$ and hence $f_0^q(z) = z$ for all $z \in Y_0 \cap \mathbb{D}_r$. Finally $f_0^q(z) \equiv z$ on $\mathbb{D}_r$, by analytic continuation and the fact that $Y_0 \cap \mathbb{D}_r$ is a continuum. □

LEMMA 4.4. *Let $f_n : U_n \to \overline{\mathbb{C}}$, $n \geq 1$ be a sequence of holomorphic maps with $f_n(0) = 0$, $f_n'(0) = \mu_n \underset{n \to \infty}{\to} \mu_0 \neq 1$. If $f_n \rightrightarrows f_0$ uniformly on a neighbourhood of 0, then there exist $r > 0$ and $n_0 \geq 1$ such that 0 is the unique fixed point of f_n in the disk $\mathbb{D}_r$ of radius r around 0.*

The proof is an elementary application of Rouches theorem and is left to the reader.

Definition 4.5. Let $f : U \to \overline{\mathbb{C}}$ be a holomorphic map with a simply connected quadratic basin Λ around a fixed point z_0 with multiplier $\mu \in \mathbb{D}^*$. Given a rotation number $p/q \neq 0/1$ we shall say that Λ is a p/q-loxodromic quadratic basin (for α), iff $\alpha \in \partial\Lambda$ is a repelling or parabolic, f-fixed point, with combinatorial rotation number p/q relative to Λ.

Example 4.6. Given a rotation number $p/q \neq 0/1$. We shall give some examples of continuous families of maps with a p/q-loxodromic quadratic basin. Let $\mathbb{D} \ni \mu \mapsto \lambda(\mu) \in M_{\mu, p/q}$ be any continuous function. Each map of the family $\mathbb{D}^* \ni \mu \mapsto R_{\lambda(\mu),\mu} \in \mathcal{M}_2$

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has a p/q-loxodromic quadratic basin. A particular case of this is when $\Pi_\mu(\lambda(\mu)) \equiv \lambda_0 \in M_{0,p/q}$. The later is however also a special case of the following situation: suppose $f_{M_0} : U \to \overline{\mathbb{C}}$ has a p/q-loxodromic quadratic basin Λ around a fixed point say at 0 with multiplier $\mu = \exp(M_0)$, where $\Re(M_0) < 0$. Using the idea of wringing and stretching the complex structure (see, for example, [**BH**]) on the grand orbit of Λ we obtain a continuous family of holomorphic maps $F_M : U_M \to \overline{\mathbb{C}}$, $M \in \mathbb{H}_- = \{z \mid \Re(z) < 0\}$ each with a p/q-loxodromic quadratic basin Λ_M around the attracting fixed point 0 with multiplier $\mu_M = \exp(M)$.

COROLLARY 4.7. *Let $p/q \neq 0/1$ be a rotation number. Suppose*

$$\{(f_n, U_n, 0, \mu_n, \Lambda_n, 2)\}_{n\geq 1}$$

is a sequence of holomorphic maps each with a simply connected p/q-loxodromic quadratic basin Λ_n for α_n. If $r(\mu_n, p/q) \to 0$, (that is the multipliers $\{\mu_n\}_{n\geq 1} \in \mathbb{D}^*$ converge non-quadratically to $\mu_0 = e^{i2\pi p/q}$) and $f_n \rightrightarrows f_0$ locally uniformly in a neighbourhood of 0, for some (holomorphic) map f_0. Then the map f_0 has an elliptic fixed point at 0.

Proof. The limiting map f_0 is automatically holomorphic in a neighbourhood of 0, by the Weierstrass theorem on uniform convergence of holomorphic maps. Moreover $f_0(0) = 0$ and $f_0'(0) = \mu_0$. According to Lemma 4.4 the sequence of f_n-fixed points α_n stay away from a neighbourhood of 0. However, the tips of the star $\Sigma_{M_n, p/q, \Lambda_n, f_n}$ all equal α_n, because Λ_n is p/q-loxodromic for α_n. Thus, the requirements of Proposition 4.3 are satisfied and $f_0^q(z) \equiv z$ on a neighbourhood of 0. □

Deformations of maps with a parabolic fixed point are much studied and the local dynamics of such deformations is to some extent understood: let $\mu_0 = e^{i2\pi p/q}$, $(p, q) = 1$ and $\epsilon > 0$ be given. Suppose $f_\mu : \mathbb{D} \to \mathbb{C}$ is a continuous family of holomorphic maps such that 0 is a fixed point for f_μ with multiplier μ for $\mu \in \mathbb{D}(\mu_0, \epsilon)$ and that 0 is a parabolic fixed point for f_{μ_0}. Then there exists $n \geq 1$ with the property that given $\delta > 0$ there exists $\epsilon_0 > 0$ such that f_μ has n q-periodic orbits, counted with multiplicity, in the disk $\mathbb{D}(0, \delta)$ for $\mu \in \mathbb{D}(\mu_0, \epsilon_0)$. Generally speaking f_μ has some number $n \geq 1$ of q-periodic orbits converging to the fixed point 0, when μ converges to μ_0. But which orbits converge to the fixed point?

COROLLARY 4.8. *Let $\mu_0 = e^{i2\pi p/q}$ and suppose f_μ, $\mu \in \{z \in \mathbb{D} \mid |\mu - \mu_0| < \epsilon\} \cup \{\mu_0\}$, $\epsilon > 0$ is a continuous family of holomorphic maps such that:*

(1) *The map f_{μ_0} has a parabolic fixed point at 0, with multiplier μ_0.*
(2) *For $\mu \in \mathbb{D}$ the map f_μ has a simply connected quadratic basin Λ_μ around the fixed point 0 with multiplier μ.*

Then

(a) *There exists $r_0 > 0$ such that for $\mu \in \mathbb{D}$ with $r(\mu, p/q) \leq r_0$ the boundary of Λ_μ contains an exact q-periodic orbit $z_0(\mu), \ldots, z_{q-1}(\mu) \in U$ with rotation number p/q relative to Λ.*
(b) *The q-periodic orbit $z_0(\mu), \ldots, z_{q-1}(\mu)$ converge to 0 uniformly as μ converge to μ_0 non-quadratically in $\mathbb{D}$.*

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Proof. Define $\Xi = \{\mu \in \mathbb{D} \mid |\mu - \mu_0| < \epsilon\}$ and $\Xi_0 = \Xi \cup \{\mu_0\}$. Let $\delta > 0$ be such that $f_{\mu_0}^q$ is defined and holomorphic on $\mathbb{D}(2\delta)$. Decreasing ϵ if necessary we can suppose f_μ^q is defined and holomorphic on $\mathbb{D}(\delta)$, as $f_\mu \rightrightarrows f_{\mu_0}$, when $\mu \to \mu_0$. For $\mu \in \Xi$ let M_μ denote the logarithm of μ closest to $i2\pi\, p/q$. Moreover let $s(\mu) = r_\sigma(\Sigma_{M_\mu, p/q, \Lambda_\mu, f_\mu})$.

CLAIM. *When* $r(\mu, p/q) \to 0$, *then* $s(\mu) \to 0$.

Proof of the claim. If the claim is false there exists $\eta > 0$ and a sequence $\{\mu_n\}_{n\geq 1} \subset \Xi$ with $r(\mu, p/q) \to 0$ as $n \to \infty$ such that $s(\mu_n) \geq \eta$ for all $n \geq 1$. But then f_{μ_0} has an elliptic fixed point at 0 by Proposition 4.3, this contradicts the assumption (1). □

We conclude from the claim that there exists $r_0 > 0$ such that the union of the closures of the q wires of the star $\Sigma_{M_\mu, p/q, \Lambda_\mu, f_\mu}$ is contained in $\mathbb{D}(\delta)$ for all $\mu \in \Xi$ with $r(\mu, p/q) \leq r_0$. In particular the tips of those wires are contained in $\mathbb{D}(\delta)$, where f^q is defined and holomorphic. Thus, those tips are formed by a q-periodic orbit $z_0(\mu), \ldots, z_{q-1}(\mu) \in \partial\Lambda_\mu$ for $r(\mu, p/q) \leq r_0$. Decreasing r_0, if necessary, we can suppose that f_μ is not p/q loxodromic, for $\mu \in \Xi$ with $r(\mu, p/q) \leq r_0$, by Corollary 4.7. Thus, the exact period of the orbit $z_0(\mu), \ldots, z_{q-1}(\mu)$ is q. Finally applying the claim once more also proves (b). This completes the proof. □

The conclusions of the above corollary are not surprising. For quadratic polynomials, the conclusions follow from the combinatorial description of the Mandelbrot set and the underlying combinatorial description of the Julia sets of quadratic polynomials obtained by Douady and Hubbard. Thus, if the maps f_μ have restrictions to neighbourhoods of Λ_μ as quadratic polynomial like maps, then the conclusion can also be obtained by the theory of polynomial like families [**DH**]. However, the corollary is a stronger statement, because the hypotheses on f_μ outside Λ_μ is limited to the bare minimum that f_μ converge to f_{μ_0} locally uniformly on a neighbourhood of 0. Moreover the proof is based directly on the dynamics, and not on the structure of some parameter space. As an illustration the following is an example, where the above Corollary 4.8 applies but where the argument of polynomial like maps does not work.

Example 4.9. Given a rotation number p/q, consider a one parameter family of cubic polynomials $P_\mu(z) = \mu z + b(\mu)z^2 + z^3$, $\mu \in \overline{\mathbb{D}}$ with:

(1) for each $\mu \in \mathbb{D}$ the polynomial P_μ has a simply connected quadratic basin Λ_μ around the fixed point 0 with multiplier μ;

(2) the q-periodic P_μ-orbit $z_0(\mu), \ldots, z_{q-1}(\mu)$ with rotation number p/q relative to Λ_μ is parabolic with multiplier 1.

The above Corollary 4.8 implies that the orbit $z_0(\mu), \ldots, z_{q-1}(\mu)$ converges to 0, when μ converges to $\mu_0 = \exp(i2\pi\, p/q)$ non-quadratically. In particular the cubic polynomial P_{μ_0} satisfies $P_{\mu_0}^q(z) = z(1 + az^{2q} + \mathcal{O}(z^{2q+1}))$ in a neighbourhood of 0. But P_μ has no restriction as a quadratic polynomial like map in a neighbourhood of Λ_μ, because of the parabolic orbit $z_0(\mu), \ldots, z_{q-1}(\mu) \in \partial\Lambda_\mu$. Thus the polynomial like argument does not yield the desired conclusion.

Though the stars are defined for general proper basins, this paper has only made use of them in the case of simply connected quadratic basins Λ. These are particularly simple,

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because precisely one periodic orbit on the boundary of Λ can have rotation number p/q relative to Λ. However, for $d \geq 3$, there are more than one q-periodic orbit with Poincaré rotation number p/q for a(ny) degree d covering of the circle by itself. Thus, one also needs a discussion on which q-periodic orbit(s) can form a p/q rotation set for a given degree d Blaschke model. The notion of rotation set for a degree d covering is defined in [**Go**]. There it is shown that there can be at most $d-1$ (q-periodic) orbits in a p/q rotation set and the number of q-periodic orbits with rotation number p/q is calculated as a function of d. Unfortunately the question 'which q-periodic orbit(s) can form a p/q rotation set for $z \mapsto z^d$?' is not touched upon. Solving this question and combining the answer with the 'stars' defined above, should yield much insight into the structure of, for example, the hyperbolic components, corresponding to degree 3 proper basins, of the connectedness locus for degree three polynomials.

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