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Local analytic integrability for nilpotent centers

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Abstract. Let $X(x, y)$ and $Y(x, y)$ be real analytic functions without constant and linear terms defined in a neighborhood of the origin. Assume that the analytic differential system $\dot{x} = y + X(x, y)$, $\dot{y} = Y(x, y)$ has a nilpotent center at the origin. The first integrals, formal or analytic, will be real except if we say explicitly the converse. We prove the following.

(a) If $X = yf(x, y^2)$ and $Y = g(x, y^2)$, then the system has a local analytic first integral of the form $H = y^2 + F(x, y)$, where F starts with terms of order higher than two.

(b) If the system has a formal first integral, then it has a formal first integral of the form $H = y^2 + F(x, y)$, where F starts with terms of order higher than two. In particular, if the system has a local analytic first integral defined at the origin, then it has a local analytic first integral of the form $H = y^2 + F(x, y)$, where F starts with terms of order higher than two.

(c) As an application we characterize the nilpotent centers for the differential systems $\dot{x} = y + P_3(x, y)$, $\dot{y} = Q_3(x, y)$, which have a local analytic first integral, where P_3 and Q_3 are homogeneous polynomials of degree three.

1. *Introduction and statement of the main results*

One of the more classical problems in the qualitative theory of planar real analytic differential systems is to characterize the local phase portrait at an isolated singular point. This problem has been solved except if the singular point is a center or a focus,

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see [**1, 2, 6**]. The problem in distinguishing between a center or a focus is called the *center–focus problem*. Once we have a center, another problem is to know whether or not there exists a local analytic first integral defined in its neighborhood. It is known that any analytic differential system having a center has a local C^∞ first integral defined in its neighborhood, see [**14**]. It is also known that any analytic differential system having a center at the point p has a local analytic first integral defined in a punctured neighborhood of p, see [**8**]; but such an analytic first integral does not need to be defined at the center point.

If a real analytic system has a center at the origin, then after a linear change of variables and a rescaling of the time variable, it can be written in one of the following three forms:

$$\begin{aligned} \dot{x} &= -y + X(x, y), \\ \dot{y} &= x + Y(x, y), \end{aligned} \tag{1}$$

called a *non-degenerate center*;

$$\begin{aligned} \dot{x} &= y + X(x, y), \\ \dot{y} &= Y(x, y), \end{aligned} \tag{2}$$

called a *nilpotent center*;

$$\begin{aligned} \dot{x} &= X(x, y), \\ \dot{y} &= Y(x, y), \end{aligned} \tag{3}$$

called a *degenerate center*, where $X(x, y)$ and $Y(x, y)$ are real analytic functions without constant and linear terms, defined in a neighborhood of the origin.

The characterization of non-degenerate centers is due to Poincaré [**13**] and Liapunov [**9**], see also Chazy [**5**] and Moussu [**11**].

NON-DEGENERATE CENTER THEOREM. *System (1) has a center at the origin if and only if there exists a local analytic first integral of the form $H = x^2 + y^2 + F(x, y)$ defined in a neighborhood of the origin, where F starts with terms of order higher than two.*

Using this theorem Poincaré and Liapunov [**9, 13**] constructed an algorithm which solves in principle the center problem for non-degenerate centers.

We say that a non-zero formal power series $H = \sum_{i,j=0}^{\infty} a_{ij} x^i y^j$ is a *formal first integral* of system (2) if it formally satisfies

$$\frac{\partial H}{\partial x}(y + X(x, y)) + \frac{\partial H}{\partial y} Y(x, y) \equiv 0. \tag{4}$$

In this work we study the existence of real (formal or analytic) first integrals defined in a neighborhood of a nilpotent center. Our main result is as follows.

THEOREM 1. *Assume that the real analytic differential system (2) has a nilpotent center at the origin.*

(a) *If $X = yf(x, y^2)$ and $Y = g(x, y^2)$, then the system has a local analytic first integral of the form $H = y^2 + F(x, y)$, where F starts with terms of order higher than two.*

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(b) *If the system has a formal first integral, then it has a formal first integral of the form* $H = y^2 + F(x, y)$*, where* F *starts with terms of order higher than two. In particular, if the system has a local analytic first integral defined at the origin, then it has a local analytic first integral of the form* $H = y^2 + F(x, y)$*, where* F *starts with terms of order higher than two.*

(c) *As an application we characterize the nilpotent centers for the differential systems* $\dot{x} = y + P_3(x, y)$*,* $\dot{y} = Q_3(x, y)$*, which have a local analytic first integral, where* P_3 *and* Q_3 *are homogeneous polynomials of degree three.*

Statements (a)–(c) of Theorem 1 will be proved in §2–4, respectively.

We note that statement (b) provides a tool for detecting whether or not a given analytic nilpotent center has a local analytic first integral. For more details see Proposition 7 and the proof of statement (c).

Analytic systems having a nilpotent singular point at the origin were studied by Andreev [**1**] in order to obtain their local phase portraits. However, Andreev's results do not distinguish between a focus and a center. Takens [**16**] provided a normal form for nilpotent centers or foci, we shall use it here. Moussu [**10**] found the C^∞ normal form for analytic nilpotent centers. Berthier and Moussu in [**4**] studied the reversibility of the nilpotent centers. Teixeira and Yang [**17**] analysed the relationship between reversibility and the center–focus problem for systems (1) and (2), written in a convenient normal form.

In Proposition 7 we provide, probably, the easiest example of an analytic nilpotent center without a local analytic first integral. We note that systems satisfying the assumptions of statement (a) of Theorem 1 are time reversible because they are invariant under the change of variables $(u, v, t) \to (u, -v, -t)$. So, from the first two statements of Theorem 1 and Proposition 7, it follows that there does not exist an analytic change of variables which transforms the analytic nilpotent center of Proposition 7 into another analytic nilpotent center of the form (2) which is time reversible under the change of variables $(u, v, t) \to (u, -v, -t)$.

For degenerate analytic centers it is known that in general they have no local analytic first integrals defined in its neighborhood. Nemytskii and Stepanov in [**12**, p. 122] give a real polynomial differential system which has a degenerate center, but the system has neither a local analytic first integral in its neighborhood, nor a formal one. In [**11**] Moussu gives another example of a real polynomial differential system having a degenerate center for which a local analytic first integral does not exist. There are partial results about the center–focus problem for degenerate centers, see for instance [**7**].

2. *Proof of statement (a)*

We assume that we have a system (2) of the form

$$\begin{aligned} \dot{x} &= y + yf(x, y^2), \\ \dot{y} &= g(x, y^2), \end{aligned} \tag{5}$$

where f and g start at least with terms of order one and two, respectively, and that system (5) has a nilpotent center at the origin.

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We make the change of variables $x = u$, $y = \sqrt{u^2 + v^2}$ and $d\tau = (\sqrt{u^2 + v^2}/v)\, dt$; or equivalently $u = x$, $v = \sqrt{y^2 - x^2}$ and $d\tau = (y/\sqrt{y^2 - x^2})\, dt$, defined between the sector $\Sigma = \{(x, y) \mid y^2 - x^2 > 0 \text{ and } y > 0\}$ and the half-plane $\{v > 0\}$. In the new variables (u, v, τ), system (5) goes over to

$$\begin{aligned} \dot{u} &= v + vf(u, u^2 + v^2), \\ \dot{v} &= -u + g(u, u^2 + v^2) - uf(u, u^2 + v^2). \end{aligned} \tag{6}$$

We note that system (6) is defined in the whole $\mathbb{R}^2$.

System (6) has a linear center at the origin. It is time reversible because it is invariant under the change of variables $(u, v, t) \to (u, -v, -t)$. Therefore, system (6) has a non-degenerate center at the origin. Hence, by the Nondegenerate Center Theorem system (6) has an analytic first integral of the form $H(u, v) = u^2 + v^2 + F(u, v)$ defined in a neighborhood of the origin, where $F(u, v)$ starts with terms of order higher than two.

Due to the invariance of system (6) under the transformation $(u, v, t) \to (u, -v, -t)$, if $(u(t), v(t))$ is a solution, then $(u(-t), -v(-t))$ also is a solution. Therefore, the phase portrait of system (6) is symmetric with respect to the u-axis. Moreover, since the u-axis is not formed by orbits, every orbit which has a point on the u-axis is symmetric, i.e. it is invariant under the time-reversible symmetry. Consequently, the first integral $H(u, v)$ must be invariant under the symmetry $(u, v) \to (u, -v)$. That is, H must be of the form $H(u, v) = u^2 + v^2 + I(u, v^2)$, where $I(u, v^2)$ starts with terms of order higher than two. We remark that H depends on v through v^2.

Going back through the change of variables we find that, in a neighborhood U_1 of the origin, the analytic function

$$\overline{H}(x, y) = H\big(x, \sqrt{y^2 - x^2}\big) = y^2 + I(x, y^2 - x^2) \tag{7}$$

is a first integral of system (5) on the sector $\Sigma \cap U_1$. Of course, $\overline{H}$ is analytic in its variables because it is a composition of analytic functions.

Since the origin of system (5) is a center, let U_2 be a neighborhood of the origin invariant by the flow of the system and such that $U_2 \subset U_1$. For every periodic orbit γ contained in U_2 there is a point $(x_0, y_0) \in \Sigma \cap U_2$ such that locally γ can be written as $y = F(x)$, where F is analytic. The arc $\gamma \cap \Sigma$ must coincide with an arc of the analytic curve $\overline{H}(x, y) = h$ for some $h \in \mathbb{R}$, because $\overline{H}$ is a first integral of system (5) in $\Sigma \cap U_2$ and it is analytic in U_2. Then, in a neighborhood of the point (x_0, y_0) the curve $\overline{H}(x, y) = h$ can be written as $y = G(x)$, where G is analytic. Since F and G are analytic functions of one variable which coincide in a small neighborhood of the point (x_0, y_0), the analytic curves γ and $\overline{H}(x, y) = h$ coincide in U_2. In short, $\overline{H}$ is an analytic first integral of system (5) in U_2. Consequently, statement (a) of Theorem 1 is proved. □

3. *Formal differential equations*

In this section we summarize some definitions and results about formal differential equations and their solutions that we shall use later on. For more details and proofs about these results see Seidenberg [**15**].

We consider the field $\mathbb{K}$ (either $\mathbb{R}$ or $\mathbb{C}$). We denote by $\mathbb{K}[[x, y]]$ the ring of formal power series, that is the ring with elements $F(x, y) = \sum_{i,j=0}^{\infty} f_{ij} x^i y^j$, with $f_{ij} \in \mathbb{K}$, with

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the usual multiplicative operation. A *unit* is an invertible element of this ring. It is well known that the ring $\mathbb{K}[[x, y]]$ is factorial.

We say that a power series $F(x, y) = \sum_{i,j=0}^{\infty} f_{ij} x^i y^j$ is *constant* if all its coefficients are identically null except, maybe, f_{00}. If all the coefficients are zero, we denote the power series by $\mathbf{0}$.

Let $F(x, y)$ be an irreducible non-unit of $\mathbb{K}[[x, y]] \setminus \mathbf{0}$. An *analytic branch centered at* $(0, 0)$ is the equivalence class of F under the equivalence $F \sim G$ if $F = \text{unit} \cdot G$. We note that here the adjective *analytic* does not mean the convergence of the power series. On the other hand $F(0, 0) = 0$ because $F(x, y)$ is a non-unit.

Given a representative of an analytic branch $F(x, y)$ centered at the origin, there exist power series $x(t) = \sum_{i=1}^{\infty} x_i t^i$ and $y(t) = \sum_{i=1}^{\infty} y_i t^i$, with $x_i, y_i \in \mathbb{K}$, not both $\mathbf{0}$, such that $F(x(t), y(t)) = 0$. Such a pair is called a *branch expansion* of the analytic branch. Note that $x(0) = 0$ and $y(0) = 0$. If $x(t), y(t)$ are not both power series in a power series $\tau(t)$ of order greater than 1, then the branch expansion is called *primitive*.

Given a branch expansion, one can deduce from it a primitive one, and we will usually tacitly confine ourselves to such primitive branch expansions. Given a primitive branch expansion $x(t), y(t)$ of F, we can get others by applying substitutions of the form $t \to ct + \cdots, c \neq 0$; and in this way we get all the primitive branch expansions of F.

A *branch* is defined as the equivalence class of (x, y), where $(x(t), y(t))$ is a primitive pair in $\mathbb{K}[[t]]$, under the equivalence $(x, y) \sim (x', y')$ if each goes over into the other under a substitution of the form $t \to ct + \cdots, c \neq 0$. Given a representative of the branch $x(t), y(t)$, there is an irreducible non-unit $F(x, y)$ in $\mathbb{K}[[x, y]] \setminus \mathbf{0}$, uniquely determined up to a unit factor, such that $F(x(t), y(t)) = 0$. $F(x, y) = 0$ is called the *equation of the branch*.

LEMMA 2. *Consider a branch $x(t), y(t)$ and its equation $F(x, y) = 0$. If $G(x, y) \in \mathbb{K}[[x, y]] \setminus \mathbf{0}$ satisfies $G(x(t), y(t)) = 0$, then there exists $K(x, y) \in \mathbb{K}[[x, y]] \setminus \mathbf{0}$ such that $G(x, y) = K(x, y)F(x, y)$.*

Proof. It follows easily from the fact that $F(x, y)$ is irreducible and the ring $\mathbb{K}[[x, y]]$ is factorial. □

Consider the formal differential equation

$$P(x, y)\, dy - Q(x, y)\, dx = 0, \tag{8}$$

where $P(x, y)$ and $Q(x, y)$ are formal power series, that is,

$$P(x, y) = \sum_{i,j=0}^{\infty} p_{ij} x^i y^j, \quad Q(x, y) = \sum_{i,j=0}^{\infty} q_{ij} x^i y^j,$$

with $p_{ij}, q_{ij} \in \mathbb{K}$.

For a formal power series $F(x, y) = \sum_{i,j=0}^{\infty} f_{ij} x^i y^j$ we define $\partial F(x, y)/\partial x$ as the formal power series $\sum_{i=1, j=0}^{\infty} i f_{ij} x^{i-1} y^j$. Analogously, we define $\partial F(x, y)/\partial y$.

By a *solution* of the formal differential equation (8) we mean an analytic branch centered at the origin which has a representative satisfying equation (8). More explicitly, one has a

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pair $x = \sum_{i=1}^{\infty} x_i t^i$, $y = \sum_{i=1}^{\infty} y_i t^i$, not both $x, y = \mathbf{0}$, which satisfies the equation

$$P(x(t), y(t))\frac{dy}{dt}(t) - Q(x(t), y(t))\frac{dx}{dt}(t) = O(F), \tag{9}$$

where $O(F)$ means that this expression vanishes with F.

Let $F(x, y) = 0$ be the equation of the branch of the solution $(x(t), y(t))$. Then,

$$\frac{\partial F}{\partial x}(x(t), y(t))\frac{dx}{dt}(t) + \frac{\partial F}{\partial y}(x(t), y(t))\frac{dy}{dt}(t) = \mathbf{0}.$$

Therefore, using (9) we get

$$P(x(t), y(t))\frac{\partial F}{\partial x}(x(t), y(t)) + Q(x(t), y(t))\frac{\partial F}{\partial y}(x(t), y(t)) = O(F).$$

Consequently,

$$P(x, y)\frac{\partial F}{\partial x} + Q(x, y)\frac{\partial F}{\partial y}$$

vanishes at $x = x(t)$ and $y = y(t)$, and by Lemma 2 we have

$$P(x, y)\frac{\partial F}{\partial x} + Q(x, y)\frac{\partial F}{\partial y} = K(x, y)F(x, y), \tag{10}$$

for some $K \in K[[x, y]]$. Conversely, every irreducible $F \in \mathbb{K}[[x, y]] \setminus \mathbf{0}$ satisfying (10) for some $K \in K[[x, y]]$ yields a solution of equation (8).

We define a *(formal) first integral* for a formal differential equation (8) as a non-constant power series $H(x, y) = \sum_{i,j=0}^{\infty} h_{ij} x^i y^j$, with $h_{ij} \in K$ satisfying

$$\frac{\partial H(x, y)}{\partial x} P(x, y) + \frac{\partial H(x, y)}{\partial y} Q(x, y) = 0. \tag{11}$$

Let $p \in \mathbb{K}^2$ be a point. Without loss of generality we can assume that p is the origin. The next result, proved in [**15**], assures the existence of a solution.

THEOREM 3. *Every formal differential equation (8) has a solution at the origin.*

By Theorem 3 it is clear that there exists at least one analytic branch $(x_p(t), y_p(t))$, such that

$$P(x_p(t), y_p(t))\frac{dy_p}{dt}(t) - Q(x_p(t), y_p(t))\frac{dx_p}{dt}(t) = \mathbf{0},$$

and $x_p(0) = 0$, $y_p(0) = 0$.

If $H(x, y)$ is a formal first integral for system (8), then there exists a constant $k \in \mathbb{K}$ such that $H(x_p(t), y_p(t)) = H(x_p(0), y_p(0)) = H(0, 0) = h_{00} = k$. This result is proved formally using definition (11) and the formal expansions of H, P, Q, x and y. Therefore, $H - h_{00}$ is an element of $\mathbb{K}[[x, y]] \setminus \mathbf{0}$ (which may be reducible). Since the equation of the branch $F(x, y)$ is uniquely determined up to a unit factor, we have that

$$H(x, y) - h_{00} = F(x, y)S(x, y), \tag{12}$$

with $S \in \mathbb{K}[[x, y]] \setminus \mathbf{0}$.

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In particular, we note that if there exist an infinite number of analytic branches passing through the origin, then a formal first integral does not exist, because the equation for every branch would need to satisfy a relation (12).

If FG is a solution and F, G are non-units without common factor, then F and G, as well as their irreducible factors, are also solutions. In particular, if there exist a formal first integral $H(x, y)$, then $H(x, y) - k$ is a solution for all $k \in \mathbb{K}$. Assume that $F(x, y)$ is the equation of the branch $(x_p(t), y_p(t))$, and we have (12). Therefore, all the non-unit irreducible factors of $S(x, y)$ are also solutions.

Let

$$X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$$

be a formal vector field associated to the formal differential equation (8). Therefore, if an irreducible non-unit $F \in \mathbb{K}[[x, y]] \setminus \mathbf{0}$ is a solution of equation (8), then, taking into account (10), it satisfies $XF = KF$, for some $K \in \mathbb{K}[[x, y]]$.

A branch $x(t) = \sum_{i=1}^{\infty} x_i t^i$ and $y(t) = \sum_{i=1}^{\infty} y_i t^i$, with $x_i, y_i \in \mathbb{K}$, centered at $(0, 0)$, is called *linear* if x_1 or y_1 is not zero. We shall use the following result from [**15**].

LEMMA 4. *Consider the formal differential system $\dot{x} = \lambda x + P(x, y)$, $\dot{y} = \mu y + Q(x, y)$, where P and Q start with terms of order higher than one and with $\lambda\mu \neq 0$. If λ, μ are rationally independent or $\lambda/\mu < 0$, then there are exactly two solutions, a linear branch with horizontal tangent and a linear branch with vertical tangent.*

4. *Proof of statement (b)*

Now we start the preparation of the proof of the first part of statement (b).

Takens [**16**] has proved that real system (2) can be reduced, via a formal change of coordinates $x = u + \cdots$, $y = v + \cdots$, to a real system of the form

$$\dot{x} = y + f(x), \quad \dot{y} = g(x), \tag{13}$$

where f and g are formal power series starting with terms of order higher than one. Assume that in (13) neither f nor g are identically zero. Then, they can be put in the form

$$f(x) = x^m(a_0 + a_1x + \cdots), \quad g(x) = x^k(b_0 + b_1x + \cdots). \tag{14}$$

Since we are interested in systems (13) having a center at the origin, they must satisfy the monodromy conditions. So, from [**3**], functions f and g must satisfy one of the following two sets of conditions:

(i) $b_0 < 0, k = 2s - 1, m > s \geq 2$;

(ii) $b_0 < 0, k = 2s - 1, m = s \geq 2, ma_0^2 + 4b_0 < 0$.

Therefore, we can assume that

$$f(x) = x^s(a_0 + a_1x + \cdots), \quad g(x) = x^{2s-1}(b_0 + b_1x + \cdots), \tag{15}$$

where $b_0 < 0$ and if $a_0 \neq 0$ then $sa_0^2 + 4b_0 < 0$. Conditions (i) and (ii) (which correspond to the cases $a_0 = 0$ and $a_0 \neq 0$, respectively) exclude the possibility of $g(x) \equiv 0$, because in this case $H(x, y) = y$ is a first integral of the system and there is no center at the origin. We can also assume that $f(x) \not\equiv 0$, because otherwise system (13) is Hamiltonian with a

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Hamiltonian of the form $H = y^2 + F(x)$, where F starts with terms of degree higher than two. Hence, in order to prove the first part of statement (b) of Theorem 1, it is sufficient to prove the next result.

PROPOSITION 5. *Assume that the real system*

$$\dot{x} = y + x^s(a_0 + a_1x + \cdots), \quad \dot{y} = x^{2s-1}(b_0 + b_1x + \cdots), \tag{16}$$

with $b_0 < 0$ and if $a_0 \neq 0$ then $sa_0^2 + 4b_0 < 0$, has a formal first integral. Then there is a formal first integral of the form $H = y^2 + F(x, y)$, where $F(x, y)$ starts with terms of order higher than two.

The following lemma provides the form of the lower terms of a formal first integral for system (2).

LEMMA 6. *If system (2) has a formal first integral H, then it is of the form $H = y^k + F(x, y)$, where F starts with terms of order higher than k, with $k \in \mathbb{N}$.*

Proof. We can write $H = H_k + H_{k+1} + \cdots$, where $k \in \mathbb{N}$, $H_k \neq 0$ and H_j is a homogeneous polynomial of degree j for $j = k, k+1, \ldots$. In order that H be a formal first integral it must satisfy equation (4). If we choose the lower terms of this equality, i.e. the terms of degree k, we get $(\partial H_k/\partial x)y = 0$. Consequently, $\partial H_k/\partial x = 0$, and $H_k = y^k$, except perhaps for a multiplicative constant which can be removed. □

Proof of Proposition 5. We consider the following blow-up $(x, z) = (x, y/x^s)$. Using this blow-up, and after the time reparametrization $d\tau = x^{s-1}\,dt$, we have that system (16) is transformed into the system

$$\dot{x} = x(z + a_0 + O(x)), \quad \dot{z} = b_0 - sz^2 - sa_0z + O(x), \tag{17}$$

where the dot now indicates a derivative with respect to the new time τ, and $O(x)$ denotes a formal series which can be written in the form $xJ(x, y)$. The two corresponding singular points in the blow-up straight line $x = 0$ are $(0, p)$ and $(0, q)$ with $p = -a_0/2 + \sqrt{\Delta}/(2\sqrt{s})$ and $q = -a_0/2 - \sqrt{\Delta}/(2\sqrt{s})$, where $\Delta = sa_0^2 + 4b_0$. The quotient of the eigenvalues at the singular point $(0, p)$ is

$$-\frac{1}{2s}\left(1 + \frac{a_0\sqrt{s}}{\sqrt{\Delta}}\right).$$

We note that this quotient is complex if $a_0 \neq 0$ and negative if $a_0 = 0$. Therefore, by Lemma 4, there are exactly two formal solutions, a linear branch with vertical tangent, which is $x = 0$, and a linear branch with horizontal tangent. For the second point we obtain a similar result. Finally, taking into account the fact that the solution $x = 0$ is introduced by the blow-up, going back through the change of variables we obtain just two solutions passing through the origin for system (16). We remark that the two linear solutions obtained for system (17) are also linear solutions for system (16), and that written in the variables (x, y) they are of the form $y + f(x, y) = 0$, where f starts with terms of order higher than one.

By assumption and from Lemma 6 we know that system (16) has a formal first integral of the form $H = y^k + F(x, y)$ with $k \in \mathbb{N}$. We also know that system (16) has exactly two

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linear solutions passing through the origin that we can write as $y + f_i(x, y)$, for $i = 1, 2$ with f_i starting with terms of order higher than one. We claim that these two solutions are complex formal power series. Now we shall prove the claim. Let $y + f(x, y)$ be one of these two solutions. Then, it must satisfy

$$(y + x^s(a_0 + a_1x + \cdots))\frac{\partial}{\partial x}(y + f(x, y)) + x^{2s-1}(b_0 + b_1x + \cdots)\frac{\partial}{\partial y}(y + f(x, y)) = K(x, y)(y + f(x, y)), \tag{18}$$

where

$$K(x, y) = \sum_{i+j=0}^{\infty} k_{ij}x^i y^j, \quad f(x, y) = \sum_{i+j=2}^{\infty} f_{ij}x^i y^j.$$

Now we shall prove that

$$b_0 + sa_0f_{s0} - sf_{s0}^2 = 0. \tag{19}$$

We start computing the coefficients of the monomials with lower degree in the equality (18). Thus, the coefficient of x in (18) is k_{00}, so $k_{00} = 0$. Then, the coefficient of xy is $2f_{20} - k_{10} = 0$. Therefore, since the coefficient of x^3 is $-f_{20}k_{10}$, it follows that $f_{20} = k_{10} = 0$. Again, the coefficient of x^2y is $3f_{30} - k_{20}$ and the coefficient of x^5 is $-f_{30}k_{20}$, so $f_{30} = k_{20} = 0$. This follows in a similar way until we reach the monomial $x^{s-1}y$ whose coefficient is $sf_{s0} - k_{s-1,0}$. Now the coefficient of x^{2s-1} is $b_0 + sa_0f_{s0} - f_{s0}k_{s-1,0}$. Hence, equation (19) follows.

When we compute f_{s0} from equation (19), we get that f_{s0} is complex because its discriminant is $s(sa_0^2 + 4b_0) < 0$. Therefore, the claim is proved.

Taking into account the results of Section 3, since H is a formal first integral and $y + f_i(x, y)$, for $i = 1, 2$, are the unique solutions passing through the origin, we can write

$$H = (y + f_1(x, y))(y + f_2(x, y))f_3(x, y), \tag{20}$$

where f_3 is a formal power series in x and y with non-zero constant term. Since the system (2) is real, and $y + f_i(x, y)$ are complex formal power series, we get that $y + f_1(x, y)$ and $y + f_2(x, y)$ are conjugated. Hence, from (20) we obtain that $H = y^2 + F(x, y)$. So, the constant term of $f_3(x, y)$ must be one. Therefore, the proposition follows. □

Now we shall prove the second part of statement (b). Assume that system (2) has a local analytic first integral H_1 defined at the origin. Then, through the formal change of variables which writes system (2) into system (16), the analytic first integral H_1 provides a formal first integral $\overline{H}_1$ of system (16). The proof of Proposition 5 says that system (16) has a formal first integral $\overline{H}_2 = y^2 + \overline{F}(x, y)$, where $\overline{F}$ starts with terms of order higher than two, and $\overline{H}_1 = \overline{H}_2^l$ for some positive integer l. Going back through the formal change of variables from system (16) to system (2), and taking into account the fact that such a formal change starts with the identity, we get for system (2) an analytic first integral $H_1 = (y^2 + F(x, y))^l$, where F starts with terms of order higher than two. So, $y^2 + F(x, y)$ is an analytic first integral of system (2), and the second part of statement (b) is proved.

Now we provide the example of an analytic nilpotent center without a local analytic first integral.

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PROPOSITION 7. *The vector field* $(y + x^2, -x^3)$ *has a nilpotent center at the origin, but there is neither a local analytic first integral defined at the origin, nor a formal one.*

Proof. By Andreev's Theorem [**1**] this system has either a focus or a center at the origin. Since this system is invariant under the transformation $(x, y, t) \to (-x, y, -t)$, it has a nilpotent center at the origin. By statement (b), if it has a local analytic first integral at the origin, it must have a local analytic first integral of the form $H = y^2 + F(x, y)$ with F starting with terms of degree higher than two.

Now we take $F = \sum_{i+j=3}^{\infty} f_{ij} x^i y^j$ and we impose that H is a first integral, i.e. that H satisfies

$$\frac{\partial H}{\partial x}(y + x^2) - \frac{\partial H}{\partial y} x^3 \equiv 0. \tag{21}$$

Tending the coefficients of the monomials of degree two to zero in the previous expression, we find that $f_{30} = f_{21} = f_{12} = 0$. Doing the same for the coefficients of the monomials of degree four, we obtain that $f_{13} = f_{22} = f_{31} = 0$ and $f_{40} = 1/2$. Then, in expression (21) the monomial x^5 has coefficient two, and consequently H cannot be a first integral. So, the proposition is proved. □

5. *Proof of statement (c)*

In this section we consider *nilpotent homogeneous systems*, i.e. polynomial differential systems of the form

$$\begin{aligned} \dot{x} &= y + X_m(x, y) = y + \sum_{k=0}^{m} a_{m-k,k} x^{m-k} y^k, \\ \dot{y} &= Y_m(x, y) = \sum_{k=0}^{m} b_{m-k,k} x^{m-k} y^k, \end{aligned} \tag{22}$$

where $a_{m-k,k}$ and $b_{m-k,k}$ are real numbers. In system (22) we assume that b_{m0} is different from zero because if it is zero the system has $y = 0$ as an invariant straight line and in this case system (22) cannot have a center at the origin. On the other hand, as $b_{m0} \neq 0$ we can make the change of variables $x = u + ky$, with $k = a_{m0}/b_{m0}$, and therefore we can assume that $a_{m0} = 0$ without loss of generality.

PROPOSITION 8. *System (22) with* $b_{m0} \neq 0$ *and* $a_{m0} = 0$ *has a center or a focus at the origin if and only if* m *is odd and* $b_{m0} < 0$.

Proof. Taking into account the fact that $a_{m0} = 0$, we obtain that $y = 0$ is a solution of the equation $y + X_m(x, y) = 0$ in a neighborhood of the origin. Then $Y_m(x, 0) = b_{m0} x^m$, and the divergence of the system satisfies $(\partial X_m/\partial x + \partial Y_m/\partial y)(x, 0) = b_{m-1,1} x^{m-1}$. Applying Andreev's Theorem [**1**], in order to have a center or a focus at the origin m must be odd and $b_{m0} < 0$. □

Since we want to study the centers of system (22) with $m = 3$, rescaling the variables x and y (if necessary) we can take $b_{30} = -1$ and $a_{30} = 0$. Therefore, in order to prove statement (d) of Theorem 1, in what follows we consider cubic systems of the form

$$\begin{aligned} \dot{x} &= y + a_{21} x^2 y + a_{12} x y^2 + a_{03} y^3, \\ \dot{y} &= -x^3 + b_{21} x^2 y + b_{12} x y^2 + b_{03} y^3. \end{aligned} \tag{23}$$

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PROPOSITION 9. *System (23) has a nilpotent center at the origin having a local analytic first integral if and only if one of the following two sets of conditions holds:*

(i) $b_{21} = a_{12} + 3b_{03} = a_{21} + b_{12} = 0$;

(ii) $b_{21} = a_{12} = b_{03} = 0$.

Proof. First we shall see that conditions (i) and (ii) are sufficient. So, we assume that one of both conditions is satisfied and we must show that system (23) has a local analytic first integral. Hence, by Proposition 8, the origin will be a center.

Suppose that condition (i) holds. Then, system (23) is a Hamiltonian system with Hamiltonian

$$H = \tfrac{1}{2}(y^2 + a_{21}x^2y^2 + \tfrac{2}{3}a_{12}xy^3 + \tfrac{1}{2}(x^4 + a_{03}y^4)).$$

In particular, H is a local analytic first integral defined at the origin.

Suppose that (ii) holds. Then, system (23) satisfies statement (a) of Theorem 1. Therefore it has a local analytic first integral defined in a neighborhood of the origin.

Now we shall prove that conditions (i) and (ii) are necessary. That is, we assume that the origin of system (23) is a center having a local analytic first integral. Then, by statement (b) of Theorem 1 there exists an analytic first integral of the form $H = y^2 + F(x, y)$ defined in a neighborhood of the origin, where F starts with terms of order higher than two. So, H satisfies

$$\frac{\partial H}{\partial x}(y + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3) + \frac{\partial H}{\partial y}(-x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3) \equiv 0. \quad (24)$$

We expand F up to terms of order 10 in x and y. We only need to consider the monomials $f_{ij}x^iy^j$ such that $i + j = 4, 6, 8, 10$, because system (23) is invariant under the change of variables $(x, y) \to (-x, -y)$.

Substituting F into (24) and letting all the coefficients of the polynomial obtained from expression (24) up to terms of order 10 vanish, we get a system of equations where the unknowns are the coefficients f_{ij} of F for $i + j = 4, 6, 8, 10$. We solve these equations recursively as follows. First we solve the equations which are obtained when the coefficients of the monomials of degree four vanish. Such equations can be solved recursively obtaining the values of f_{ij} for $i + j = 4$.

We start to solve the equations which are obtained when the coefficients of the monomials of degree six vanish. Then, the equation obtained from the coefficient of x^6 is $b_{21} = 0$. Consequently, this coefficient of system (24) must be zero in order for the first integral H to exist. The other equations obtained from degree six allow us to determine the coefficients f_{ij} with $i + j = 6$.

Now, we start to solve the equations which are obtained when the coefficients of the monomials of degree eight vanish. Therefore, the equation obtained from the coefficient of x^8 is $a_{12} + 3b_{03} = 0$. So, such a relation must be satisfied between the coefficients of system (24) due to the existence of the first integral H. The other equations obtained from degree eight allow us to determine coefficients f_{ij} with $i + j = 8$.

Finally, the equation obtained from the coefficient of x^{10} is $b_{03}(a_{21} + b_{12}) = 0$. Hence, if $b_{03} = 0$ we obtain condition (ii), and if $a_{21} + b_{12} = 0$ condition (i) follows. In short, both conditions are necessary.

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