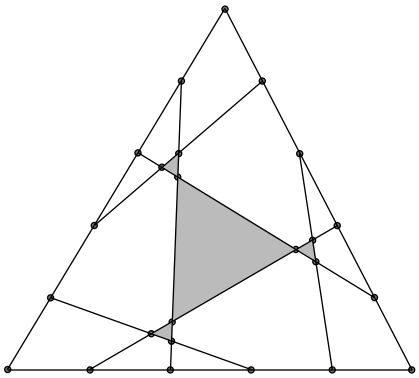
Problem Corner

Solutions are invited to the following problems. They should be addressed to **Nick Lord** at **Tonbridge School, Tonbridge, Kent TN9 1JP** (e-mail: **njl@tonbridge-school.org**) and should arrive not later than 10 March 2022.

Proposals for problems are equally welcome. They should also be sent to Nick Lord at the above address and should be accompanied by solutions and any relevant background information.

105.I (Arsalan Wares)

The figure shows a triangle with four points on each side that split the sides into five equal parts. Pairs of these points are joined by six line segments as shown, forming the four small shaded triangles. Exactly what proportion of the original triangle is shaded?



105.J (Nicolo Sartori di Borgoricco)

For positive integers $k \geqslant 2$, show that the sum of all the multiples of 3 or 5 between 1 and 10^k inclusive is 233… 34166… 68, where there are $k - 2$ digits in each block of 3's and 6's.

105.K (Isaac Sofair)

Accounts of the famous "problème des ménages" split into those giving a summation for the result and those giving a recurrence relation. Prove that these are equivalent by showing algebraically that

$$A_n = \sum_{j=0}^{n} (-1)^j \frac{2n}{2n-j} \binom{2n-j}{j} (n-j)!$$

satisfies the recurrence relation

$$(n-1)A_{n+1} = (n^2-1)A_n + (n+1)A_{n-1} + 4(-1)^n, \qquad n > 2,$$

with $A_2 = 0$ and $A_3 = 1$.

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105.L (Geoffrey Brown)

For positive integers n, evaluate:

(a) $\lim_{b\to\infty} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{b^{n+k+1}}{n+k+1}$,

(b) $\lim_{b\to\infty} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{b^{n+k+\frac{1}{2}}}{n+k+\frac{1}{2}}$.

Solutions and comments on **105.A**, **105.B**, **105.C**, **105.D** (March 2021).

105.A (Stan Dolan)

The three face angles at one vertex of a tetrahedron are right angles. Find, with proof, all such tetrahedra for which each face has integer area and the insphere has unit diameter.

Answer: There is just one such tetrahedron, with perpendicular edge lengths 2, 2, 4 giving face areas of 2, 4, 4, 6.

The solution which follows is that of the proposer, Stan Dolan. M. G. Elliott and Chris Starr's arguments were similar, but employed different search strategies at the end.

Without loss of generality, we can suppose that the areas of the faces are $A \leqslant B \leqslant C < D$. Then

$$A^2 + B^2 + C^2 = D^2, \tag{1}$$

and

$$A + B + C + D = 2\sqrt{2ABC}. \tag{2}$$

Formula (1) is known as de Gua's theorem and may be quickly proved by using a vector product to find D. Formula (2) comes from calculating the volume of the tetrahedron in two ways: directly giving $\frac{1}{6}\sqrt{8ABC}$, and splitting into three sub-tetrahedra sharing a vertex at the incentre, giving $\frac{1}{3} \times \frac{1}{2}(A + B + C + D)$.

Let the rational number $\sqrt{\frac{2AB}{C}}$ be $\frac{u}{v}$, where u and v are coprime positive integers. Then, from (1) and (2),

$$A^2 + B^2 + C^2 = \left[2\sqrt{2ABC} - (A + B + C)\right]^2$$

which gives

$$8ABC - 4(A + B + C)\sqrt{2ABC} + 2(AB + BC + CA) = 0$$

so that

$$2AB - 4(A + B + C)\sqrt{2ABC} + 2C(4AB + A + B) = 0$$

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and

$$u^2 - 4(A + B + C)uv + 2(4AB + A + B)v^2 = 0. \qquad (3)$$

Thus v is a factor of u^2, so $v = 1$ and, from (2), $\dfrac{A + B + C + D}{C} = 2u$, an even integer.

From (1), $D^2 \leqslant 3C^2$, so

$$2 < \frac{A + B + C + D}{C} \leqslant \frac{3C + \sqrt{3}C}{C} = 3 + \sqrt{3}.$$

The only possibility is $u = 2$ so that $\sqrt{\dfrac{2AB}{C}} = 2$ and $2C = AB$. Substituting for C, u, v in (3) gives $4AB - 6(A + B) + 4 = 0$ or $(2A - 3)(2B - 3) = 5$. Therefore $A = 2$, $B = 4$, $C = 4$, $D = 6$ and these face areas are realised by perpendicular edges of lengths 2, 2, 4.

Correct solutions were received from: M. G. Elliott, M. J. Emson, J. A. Mundie, C. Starr, L. Wimmer and the proposer Stan Dolan.

105.B (Michael Fox)

In the unsymmetrical plane quadrilateral $ABCD$, which can be convex, concave or crossed, the edges AB and CD are not perpendicular. Prove that the nine-point circles of triangles BCD, ACD, ABD and ABC meet at a unique point N. Prove also that if some or all of the four vertices are replaced by their reflections in N, then each new nine-point circle passes through N and has the same radius as the circle it replaces.

The four carefully argued solutions received used contrasting methods. M. G. Elliott used purely Euclidean methods, but needed to consider several possible configurations. Didier Pinchon and the proposer, Michael Fox, used the fact that every rectangular hyperbola passing through the vertices of a triangle also passes through the orthocentre and its centre lies on the nine-point circle. Stan Dolan, whose compact solution follows, used complex numbers to subsume the number of potential configurations and, together with Didier Pinchon, also placed the problem in a more general context. An *orthocentric system* of points $\{A, B, C, D\}$ is one where each point is the orthocentre of the other three, [1]: the assumptions in **105.B** mean that $ABCD$ is *not* orthocentric.

The first part of the problem is dealt with by:

Let A, B, C, D be four distinct points in the plane with no three of the points collinear. Then either the nine-point circles of the triangles BCD, ACD, ABD and ABC meet at a unique point N or $\{A, B, C, D\}$ is an orthocentric system.

Consider the complex plane with origin O the centroid of A, B, C, D. Let u, v, w be complex numbers representing the midpoints of AB, BC and CA, respectively. Then $-u, -v, -w$ represent the midpoints of CD, AD and BD, respectively.

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First suppose that two of the nine-point circles intersect at precisely two points. Without loss of generality we can suppose these are the circles through u, v, w and through u, $-v$, $-w$ and that they intersect at u and n.

Since u, v, w, n are concyclic, $\dfrac{(u-v)(w-n)}{(u-n)(w-v)}$ is real.

Since u, n, $-v$, $-w$ are concyclic, $\dfrac{(u-n)(-v+w)}{(u+w)(-v-n)}$ is real.

Multiplying, $\dfrac{(u-v)(w-n)}{(u+w)(-v-n)} = \dfrac{((-v)-(-u))(w-n)}{((-v)-n)(w-(-u))}$ is real and therefore $-v$, $-u$, w, n are concyclic. Similarly, $-w$, $-u$, v, n are concyclic and therefore n represents the unique common point N of the circles.

Now suppose that two of the nine-point circles intersect at more than two points and are therefore identical. The above argument proves that all four circles are identical and therefore $\{A, B, C, D\}$ is an orthocentric system.

The final possibility is that each pair of nine-point circles intersect at precisely one point. However, an alteration as small as we like in the position of A, say, would produce a unique common point N but then n would be as close as we like to each of $\pm u$, $\pm v$, $\pm w$. We would then have the contradiction $u = v = w$.

The second part of the problem follows by applying (repeatedly, if necessary) the result:

Let P be any point on the nine-point circle, Γ, of triangle ABC and let A' be the reflection of A in P. Then the nine-point circle of triangle $A'BC$ is the reflection of Γ in the chord joining P and the midpoint of BC,

With P as origin, let A, B and C have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, respectively. The points $\frac{1}{2}(\mathbf{a}+\mathbf{b})$ and $\frac{1}{2}(\mathbf{c}+\mathbf{a})$ are on Γ and so the reflected circle contains the reflections of these points in $\frac{1}{4}(\mathbf{b}+\mathbf{c})$, namely $\frac{1}{2}(\mathbf{c}-\mathbf{a})$ and $\frac{1}{2}(\mathbf{b}-\mathbf{a})$. Since A' has position vector $-\mathbf{a}$, the reflected circle is the nine-point circle of triangle $A'BC$, as required,

Reference

1. G. Leversha, *The geometry of the triangle*, UKMT (2013) pp. 34-35.

Correct solutions were received from: S. Dolan, M. G. Elliott, D. Pinchon and the proposer Michael Fox.

105.C (Dorin Marghidanu)

For $0 < x < \frac{1}{2}\pi$, prove that

$$\left(\sin^2 x\right)^{\sin^2 x}\left(\cos^2 x\right)^{\cos^2 x} + \left(\sin^2 x\right)^{\cos^2 x}\left(\cos^2 x\right)^{\sin^2 x} \leqslant 1.$$

There was a wide variety of methods used by solvers of this popular problem. The most common was to apply the weighted arithmetic-geometric mean inequality, $a^{\lambda}b^{1-\lambda} \leqslant \lambda a + (1-\lambda)b$ for $a, b > 0, 0 \leqslant \lambda \leqslant 1$.

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This gives

$$(\sin^2 x)^{\sin^2 x}(\cos^2 x)^{\cos^2 x} \leqslant \sin^4 x + \cos^4 x$$

and

$$(\sin^2 x)^{\cos^2 x}(\cos^2 x)^{\sin^2 x} \leqslant \cos^2 x \sin^2 x + \sin^2 x \cos^2 x$$

so that

$$(\sin^2 x)^{\sin^2 x}(\cos^2 x)^{\cos^2 x} + (\sin^2 x)^{\cos^2 x}(\cos^2 x)^{\sin^2 x} \leqslant (\cos^2 x + \sin^2 x)^2 = 1,$$

with equality if, and only if, $\sin^2 x = \cos^2 x$, i.e. when $x = \frac{\pi}{4}$.

As Henry Ricardo noted, we may also apply *Heinz's inequality*, $\sqrt{ab} \leqslant \frac{1}{2}(a^\lambda b^{1-\lambda} + a^{1-\lambda} b^\lambda) \leqslant \frac{1}{2}(a + b)$, where the (relevant) right-hand side may be proved as above (and the left-hand side is immediate from the arithmetic-geometric mean inequality).

Rob Summerson invoked Hölder's inequality,

$$\sum a_i b_i \leqslant \left(\sum a_i^p\right)^{1/p} \left(\sum b_i^q\right)^{1/q}, \qquad a_i, b_i, p, q > 0, \frac{1}{p} + \frac{1}{q} = 1$$

with the choices $a_1 = (\sin^2 x)^{\sin^2 x}$, $a_2 = (\cos^2 x)^{\sin^2 x}$, $b_1 = (\cos^2 x)^{\cos^2 x}$, $b_2 = (\sin^2 x)^{\cos^2 x}$, $p = \operatorname{cosec}^2 x$, $q = \sec^2 x$. This gives

$$(\sin^2 x)^{\sin^2 x}(\cos^2 x)^{\cos^2 x} + (\cos^2 x)^{\sin^2 x}(\sin^2 x)^{\cos^2 x} \leqslant (\sin^2 x + \cos^2 x)^{\sin^2 x}(\cos^2 x + \sin^2 x)^{\cos^2 x} = 1.$$

A succinct direct argument was given by Stan Dolan, M. G. Elliott, Phil Kitchenside, Didier Pinchon and Samuele Riccarelli. We have

$$\left[(\sin^2 x)^{\sin^2 x} - (\cos^2 x)^{\sin^2 x}\right]\left[(\sin^2 x)^{\cos^2 x} - (\cos^2 x)^{\cos^2 x}\right] \geqslant 0$$

since both square brackets are negative for $0 < x < \frac{\pi}{4}$ and positive for $\frac{\pi}{4} < x < \frac{\pi}{2}$. This expands to give

$$\sin^2 x + \cos^2 x - (\sin^2 x)^{\sin^2 x}(\cos^2 x)^{\cos^2 x} - (\cos^2 x)^{\sin^2 x}(\sin^2 x)^{\cos^2 x} \geqslant 0$$

so that

$$(\sin^2 x)^{\sin^2 x}(\cos^2 x)^{\cos^2 x} + (\sin^2 x)^{\cos^2 x}(\cos^2 x)^{\sin^2 x} \leqslant \sin^2 x + \cos^2 x = 1.$$

Finally, Peter Johnson gave a two-variable extension of **105.C** for $0 < x, y < \frac{\pi}{2}$,

$$(\sin^2 x)^{\sin^2 y}(\cos^2 x)^{\cos^2 y} + (\sin^2 x)^{\cos^2 y}(\cos^2 x)^{\sin^2 y} \leqslant 1.$$

This may be shown by adapting any of the arguments given above.

Correct solutions were received from: S. Bhattacharjee, M. V. Channakeshava, N. Curwen, P. De, S. Dolan, M. G. Elliott, P. F. Johnson, P. Kitchenside, D. Pinchon, A. Plaza, H. Ricardo (2 solutions), S. Riccarelli, T. P. Sharma, C. Starr, R. Summerson, L. Wimmer and the proposer Dorin Marghidanu.

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105.D (Isaac Sofair)

Prove that $\int_0^{\pi/3} \tan^{-1}\sqrt{\sec\theta + 1}\, d\theta = \dfrac{5\pi^2}{48}$.

The solution to this challenging integral below collates ideas from those submitted by M. V. Channakeshava, Seán Stewart and Rob Summerson.

Let $I = \int_0^{\pi/3} \tan^{-1}\sqrt{\sec\theta + 1}\, d\theta$.

Substituting $\sec\theta = x^2 + 1$ with

$$2x\frac{dx}{d\theta} = \sec\theta\tan\theta = \left(x^2 + 1\right)\sqrt{x^4 + 2x^2}$$

gives

$$I = 2\int_0^1 \frac{\tan^{-1}\sqrt{x^2 + 2}}{\left(x^2 + 1\right)\sqrt{x^2 + 2}}\, dx,$$

so it suffices to show that $I = \int_0^1 \frac{\tan^{-1}\sqrt{x^2 + 2}}{\left(x^2 + 1\right)\sqrt{x^2 + 2}}\, dx = \frac{5\pi^2}{96}$. This integral is known as *Ahmed's integral* in the literature, [1, 2, 3]. Using

$$\tan^{-1}\sqrt{x^2 + 2} = \frac{\pi}{2} - \tan^{-1}\frac{1}{\sqrt{x^2 + 2}}$$

and setting

$$J = \int_0^1 \frac{\tan^{-1}\frac{1}{\sqrt{x^2+2}}}{\left(x^2 + 1\right)\sqrt{x^2 + 2}}\, dx,$$

we have

$$\begin{aligned}\int_0^1 \frac{\tan^{-1}\sqrt{x^2+2}}{(x^2+1)\sqrt{x^2+2}}\, dx &= \frac{\pi}{2}\int_0^1 \frac{1}{(x^2+1)\sqrt{x^2+2}}\, dx - \int_0^1 \frac{\tan^{-1}\frac{1}{\sqrt{x^2+2}}}{(x^2+1)\sqrt{x^2+2}}\, dx \\ &= \frac{\pi}{2}\left[\tan^{-1}\frac{x}{\sqrt{x^2+2}}\right]_0^1 - J \\ &= \frac{\pi^2}{12} - J.\end{aligned}$$

It thus suffices to show that $J = \dfrac{\pi^2}{32}$.

To see this, let $f(u) = \int_0^1 \frac{\tan^{-1}\frac{u}{\sqrt{x^2+2}}}{\left(x^2 + 1\right)\sqrt{x^2 + 2}}\, dx$ so that $J = f(1)$.

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Differentiating under the integral sign,

$$\begin{aligned} f'(u) &= \int_0^1 \frac{1}{x^2+1}\cdot\frac{1}{x^2+u^2+2}\,dx \\ &= \int_0^1 \left(\frac{1}{x^2+1}-\frac{1}{x^2+u^2+2}\right)\frac{1}{u^2+1}\,dx \\ &= \frac{\pi}{4}\cdot\frac{1}{u^2+1}-\frac{1}{u^2+1}\left[\frac{1}{\sqrt{u^2+2}}\tan^{-1}\frac{x}{\sqrt{u^2+2}}\right]_0^1 \\ &= \frac{\pi}{4}\cdot\frac{1}{u^2+1}-\frac{\tan^{-1}\frac{1}{\sqrt{u^2+2}}}{(u^2+1)\sqrt{u^2+2}}. \end{aligned}$$

Integrating from $u = 0$ to $u = 1$ then gives $J = f(1) = \dfrac{\pi^2}{16} - J$, so $J = \dfrac{\pi^2}{32}$ as required.

The proposer, Isaac Sofair, came across the integral in **105.D** in the equivalent form $\int_0^{\pi/3} \cot^{-1}\sqrt{\sec\theta+1}\,d\theta = \dfrac{\pi^2}{16}$ while evaluating the triple integral $T = \int_0^1\int_0^1\int_0^1 \left(1+u^2+v^2+w^2\right)^{-2} du\,dv\,dw$ which was the subject of [4]. In Cartesian coordinates, the result $T = \frac{1}{2}\int_0^{\pi/3} \cot^{-1}\sqrt{\sec\theta+1}\,d\theta$ was obtained, while spherical polar coordinates gave

$$T = \frac{\pi^2}{16} - \frac{1}{2}\int_0^{\pi/3} \cot^{-1}\sqrt{\sec\theta+1}\,d\theta.$$

Equating the two results then gave

$$T = \frac{1}{2}\int_0^{\pi/3} \cot^{-1}\sqrt{\sec\theta+1}\,d\theta = \frac{\pi^2}{32}.$$

References

1. Z. Ahmed, Problem 10884, *Amer. Math. Monthly* **108** (2001) p. 556.
2. Z. Ahmed, Ahmed's integral: the maiden solution, *Math. Spectrum* **48** (2015/2016) pp. 11-12.
3. P. J. Nahin, *Inside interesting integrals*, Springer (2015) pp. 190-194.
4. M. Hajja & P. Walker, Problem 10771, *Amer. Math. Monthly* **106** (1999) p. 963.

Correct solutions were received from: M. V. Channakeshava, S. Stewart, R. Summerson and the proposer Isaac Sofair.

N.J.L.

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