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Weak compactness in constructive spaces

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Abstract

We introduce a constructively weak variant of compactness for locales as a supplement to weak closure, a notion which has featured in the constructive formulation of some recent results in locale theory.

Introduction

Locales play a fundamental role in the constructive setting of topos theory, in particular as the true objects of study if one wishes to give generally valid formulations of results from topology. (We therefore follow [**5**] in referring to locales as *spaces*.) In this paper we shall be interested in the connection between compactness and closedness of a subspace, as expressed by the pair of statements:

(I) a closed subspace of a compact space is compact;
(II) a compact subspace of a regular space is closed.

With the standard (localic) usage of the terms involved these are elementary facts, with proofs (see e.g. [**3**]) that are naturally constructive in the required sense.

If X is an ordinary topological space and Y a subspace of X, a (typical) point of the closure of Y may be characterized in alternative ways: either negatively, as not possessing an open neighbourhood contained in the (pseudo-)complement of Y, or positively, by the requirement that all the open neighbourhoods of the point meet Y. Constructively, the second property is stronger and leads to a weaker notion of closure. Insofar as these two ideas of closure differ, each is evidently important in its own right.

What has just been said with reference to topological spaces becomes applicable to localic spaces by simply replacing 'a (typical) point' by 'the generic point'. In this context the formulation of closure in terms of complements is clearly the more basic, and indeed standard one. Attention to the weak closure has only recently been drawn by P. T. Johnstone [**4**] who introduced it (albeit in a slightly different form) in order to obtain a constructive version of the fact [**1**] that subgroups of groups in the category of locales are always closed. Associated with weak closure is an obvious notion of weakly regular space. Our purpose here is to introduce a corresponding, constructively weaker notion of compactness, such that the relationships (I) and (II) remain true if each term is replaced by its weak counterpart.

In fact, a few observations will suffice to point to the appropriate candidate for weak compactness. Recall that an ideal of opens of a space X is a down-set which contains the suprema of its empty and binary (i.e. 2-indexed) subsets). So, by

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iteration, an ideal is closed under Kuratowski-finite suprema. X is compact exactly when an ideal of opens can cover X only by having it as a member. The connection between (standard) compactness and closure depends essentially on the fact that closure commutes with Kuratowski-finite union of subspaces.

Now, like standard closure, weak closure preserves empty and binary, and therefore all Kuratowski-finite unions. However, it additionally commutes with any union indexed by a subsingleton set, that is, a set of subspaces having at most one element; in a topos this is not a trivial fact, since there is no reason for a subsingleton set to be either empty or a singleton. We shall call a set weakly finite if it can be obtained from its singletons by iterating binary and/or subsingleton unions, or equivalently, sub-Kuratowski-finite unions. Then weak closure commutes with weakly finite union. A strong ideal of opens is a down-set closed under weakly finite suprema; and X is said to be weakly compact if each of its covering strong ideals of opens contains it as a member. Notably, any subset of a weakly finite set is weakly finite, thus recapturing an aspect of 'boundedness' for which the Kuratowski notion is too restrictive.

After dealing with the preliminaries, and some needed preparation concerning weak finiteness and weak closure, we show that weak compactness of spaces behaves much as one would like, and in particular that weakly closed subspaces of a weakly compact space are weakly compact (Theorem 3·7). Finally, the converse for weakly regular spaces is treated (Theorem 4·3).

0. *Notation and preliminaries*

Sets. In dealing with our sets, which are the objects of an unspecified topos $\mathscr{E}$, we shall adopt the informal 'naive' approach of e.g. [**5**]. All arguments will be correct as long as the law of the excluded middle and the axiom of choice are avoided. Nevertheless, each construction or statement has ultimately a precise categorical meaning in a topos, which it would be useful to have some familiarity with (by consulting e.g. [**2**], 5·4).

Some of our central arguments will involve the set Ω of truth values. There are two distinguished truth values, true and false, but potentially many more. Each proposition p about sets and elements has, and so gives a name to, a unique truth value $[\![p]\!]$; conversely, any $v \in \Omega$ is the value of 'v = true'. Let $1 = \{*\}$ be some chosen one element set. The subsets of 1 are in bijective correspondence with truth value via the assignment $p \mapsto \{* \mid p\}$ to a proposition p. A subsingleton is precisely a set indexed by — and therefore isomorphic to — a subset of 1. (The subsingleton subsets of a set S are also referred to as partial elements, cf. [**7**].) A set having at least one element is said to be inhabited.

$\mathscr{P}(S)$ denotes the power set of S, into which S is embedded as the set of singletons. In $\mathscr{P}(S)$, $\mathscr{K}(S)$ denotes the smallest collection of sets containing the singletons while closed under empty and binary unions. By definition, S is Kuratowski-finite exactly when it is a member of $\mathscr{K}(S)$; it follows that $\mathscr{K}(S)$ consists of all Kuratowski-finite subsets of S. Kuratowski-finiteness is preserved under Kuratowski-finite coproducts, as well as images and binary products, but not necessarily inherited by subsets; see [**6**]. We shall use the following 'elementwise' construction of Kuratowski-finite subsets:

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LEMMA 0·1. *$\mathscr{K}(S)$ is the smallest $\mathscr{J} \subseteq \mathscr{P}(S)$ for which* (i) $\varnothing \in \mathscr{J}$ *and* (ii) $J \in \mathscr{J}$ *implies* $J \cup \{x\} \in \mathscr{J}$ *for all* $x \in S$. ▮

Lemma 0·1 has a straightforward proof, using the finite enumerability of Kuratowski-finite sets. A proof which avoids induction on the natural numbers may be found in [**8**].

Fixed points. Let $r : C \to C$ be an order-preserving operation on a complete lattice which is inflationary (i.e. $r \geqslant \mathrm{id}$ pointwise). Then for any $x \in C$,

$$\mathrm{fix}\,(r)\,(x) \equiv \bigwedge\{y \geqslant x \mid r(y) \leqslant y\}$$

is the least fixed point of r containing x. The operation fix (r) is again order-preserving and inflationary; it is, moreover, idempotent, i.e. a closure operation on C. We note that fix : $C^C \to C^C$ is itself a closure operation (of which the fixed points are exactly the closure operations on C).

LEMMA 0·2. *Let $c(r) : \mathscr{P}(C) \to \mathscr{P}(C)$ be the closure operation which assigns to $S \subseteq C$ the least $J \subseteq C$ containing S such that* (1) *$x \in J$ and $y \leqslant x$ implies $y \in J$ (i.e. J is down-closed)*, (2) *$x \in J$ implies $r(x) \in J$, and* (3) *$D \subseteq J$ directed implies $\bigvee D \in J$. Then for any directed $D \subseteq C$, $c(r)\,(D) = \downarrow \mathrm{fix}\,(r)\,(\bigvee D)$ ($\downarrow x \equiv$ the down-segment of x).*

Proof. Given directed $D \subseteq C$, let $K = \{x \in C \mid x \vee d \in c(r)\,(D) \text{ for all } d \in D\}$. Then K is clearly down-closed since $c(r)\,(D)$ is. Also, if $x \in K$, then for any $d \in D$,

$$r(x) \vee d \leqslant r(x \vee d) \in c(r)\,(D),$$

i.e. $r(x) \vee d \in c(r)\,(D)$, which means that $r(x) \in K$. Now, if $S \subseteq C$ is directed, then for any $d \in D$, $(\bigvee S) \vee d = \bigvee(S \vee d)$ while the set $S \vee d \equiv \{x \vee d \mid x \in S\}$ is directed; so

$$S \subseteq K \Rightarrow S \vee d \subseteq c(r)\,(D) \Rightarrow (\bigvee S) \vee d \in c(r)\,(D) \Rightarrow \bigvee S \in K.$$

It follows that K is $c(r)$-closed. But $D \subseteq K$, since D is directed and $c(r)\,(D)$ is down-closed. So $c(r)\,(D) \subseteq K$, i.e. for any $x \in c(r)\,(D)$ and $d \in D$, we have $x \vee d \in c(r)\,(D)$. Now let

$$J = \{y \in C \mid x \vee y \in c(r)\,(D) \text{ for all } x \in c(r)\,(D)\}.$$

As before, J is $c(r)$-closed, and $D \subseteq J$ by what we have just shown. So $c(r)\,(D) \subseteq J$, which means that $c(r)\,(D)$ is closed under binary suprema and therefore directed. By the third closure condition, $p = \bigvee c(r)\,(D) \in c(r)\,(D)$, i.e. $c(r)\,(D) = \downarrow(p)$. But then $r(p) \in c(r)\,(D)$, and so $r(p) \leqslant p$, i.e. p is a fixed point of r. Finally, observe that p is in the down-set of any fixed point q of r containing $\bigvee D$, since $\downarrow(q)$ is $c(r)$-closed, and therefore contains $(c(r)\,(D)$. ▮

Frames and spaces (cf. [**3**, **5**]). Recall that a frame is a complete lattice in which the distributive law

$$x \wedge \bigvee S = \bigvee(x \wedge S)$$

is satisfied. Frame homomorphisms are functions which preserve suprema, top $(\top)$ and binary meet. The category of spaces (or locales or formal spaces) is dual to that of frames: a space X is determined by its frame of opens $\mathscr{O}X$, and a map of spaces $f : Z \to X$ by a frame homomorphism $f^* : \mathscr{O}X \to \mathscr{O}Z$. Each $U \in \mathscr{O}X$ is realized as an open subspace ($\equiv$ regular subobject) of X, such that $f^*(U) \in \mathscr{O}Z$ is exactly the inverse image ($\cong$ pullback) of U along f.

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A set S may be regarded as a discrete space, with $\mathcal{O}S \equiv \mathcal{P}(S)$. In particular, the notation $\mathcal{O}1 \equiv \mathcal{P}(1) \simeq \Omega$ is consistent, since the discrete space 1 is indeed terminal. Pulling back along the unique map $!: X \to 1$ assigns to $v \in \Omega$ the open set $\bigvee\{X \mid v = \text{true}\}$ of X, which we shall also denote by v. X is inhabited if the map $!$ is surjective, i.e. if $!^*$ reflects the order on truth values. Since $X \leqslant [\![\mathscr{C} \text{ inhabited}]\!]$ whenever $X \leqslant \bigvee\mathscr{C}$, it follows that X is inhabited if and only if each cover of X is inhabited as a set.

The opens of a subspace $i: Y \hookrightarrow X$ are all of the form $Y \wedge U \simeq i^*(U)$ for some $U \in \mathcal{O}X$. We say $Q \in \mathcal{O}X$ is Y-saturated if $Y \wedge U \leqslant Q$ (i.e. Q covers U in Y) implies $U \leqslant Q$, for all $U \in \mathcal{O}X$. The Y-saturated opens of X are closed under intersection in $\mathcal{O}X$, and the fixed points of a local (i.e. binary meet preserving closure) operator

$$l_Y: U \mapsto \bigwedge\{Q \geqslant U \mid Q \in \mathcal{O}X,\ Y\text{-saturated}\}$$

on $\mathcal{O}X$. It follows that $Y \wedge U \leqslant V$ if and only if $U \leqslant l_Y(V)$, and the assignment $Y \wedge U \mapsto l_Y(U)$ yields a (well-defined) bijection between $\mathcal{O}Y$ and the Y-saturated opens of X.

Given any set $E \subseteq \mathcal{O}X \times \mathcal{O}X$ of pairs, a unique subspace Y of X is defined by requiring that the conditions

$$Y \wedge U \leqslant V \quad \text{whenever} \quad \langle U, V\rangle \in E$$

be satisfied universally. By definition any map $f: Z \to X$ then factors through Y if and only if $f^*(U) \leqslant f^*(V)$ for all $\langle U, V\rangle \in E$. Any Y is obtained in this way by letting

$$E = \{\langle U, V\rangle \mid U, V \in \mathcal{O}X, Y \wedge U \leqslant V\}.$$

E is stable if for $\langle U, V\rangle \in E$ and $U' \leqslant U$, some $V' \leqslant V$ can always be found such that $\langle U', V'\rangle \in E$.

LEMMA 0·3. *If Y is defined by a stable set of conditions E, then $Q \in \mathcal{O}X$ is Y-saturated if and only if $U \wedge V \leqslant Q$ implies $U \leqslant Q$ for all $\langle U, V\rangle \in E$.*

Proof. See [**5**, III·4]. ∎

An easy (relevant) example will illustrate the formalism. Given any subspace Y of X, the closure of $\bar{Y}$ of Y is defined by the stable condition(s)

$$\bar{Y} \wedge U \leqslant 0 \quad \text{whenever} \quad Y \wedge U \leqslant 0,$$

where $U \in \mathcal{O}X$ and 0 is the empty subspace (bottom of $\mathcal{O}X$). Let $\neg Y$ be the open pseudo-complement of Y:

$$\neg Y \equiv \bigvee\{U \in \mathcal{O}X \mid Y \wedge U \leqslant 0\},$$

the largest open disjoint to Y. Then by Lemma 0·3, $Q \in \mathcal{O}X$ is $\bar{Y}$-saturated exactly when $\neg Y \leqslant Q$, and the local operator associated with Y can be given explicitly:

$$l_{\bar{Y}}: U \mapsto \neg Y \vee U.$$

The subspaces of X form a complete distributive lattice (in fact the dual of a frame) in which $\bar{Y}$ is the complement of $\neg Y$: if $(\neg Y \vee \bar{Y}) \wedge U \leqslant V$ for $U, V \in \mathcal{O}X$, then $\neg Y \wedge U \leqslant V$ and $Y \wedge U \leqslant V$ giving $U \leqslant \neg Y \vee V$; so $U \leqslant (\neg Y \wedge U) \vee V \leqslant V$, or $X \wedge U \leqslant V$, which shows that $X \leqslant \neg Y \vee \bar{Y}$. $Y \mapsto \bar{Y}$ is clearly a closure operation on subspaces, and by definition Y is closed whenever $Y = \bar{Y}$. $\bar{Y}$ is completely determined by $\neg Y$, so any

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open has a closed complement $-U$. Therefore $Y \wedge U \leqslant V$ if and only if $Y \leqslant -U \vee V$ for $U, V \in \mathcal{O}X$; if the conditions E define Y, then explicitly,

$$Y = \bigwedge\{-U \vee V \mid \langle U, V\rangle \in E\}.$$

1. *Weak finiteness*

Weak finiteness incorporates 'at most one' into 'finitely enumerably many' (Kuratowski-finiteness). The definition mimics that of Kuratowski-finiteness: for a set S let $\mathscr{W}(S)$ be the smallest collection of subsets of S which contains the singletons, and is closed under subsingleton and binary unions.

Definition 1·1. S is said to be weakly finite if $S \in \mathscr{W}(S)$.

Since the empty subset of S is in $\mathscr{W}(S)$, the latter clearly contains $\mathscr{K}(S)$ as a subset: hence any Kuratowski-finite set is weakly finite. The proofs of the following facts are fairly routine, and similar to their Kuratowski-finite counterparts, for which cf. [**6**].

Proposition 1·2. *For any set S, $\mathscr{W}(S)$ consists exactly of the weakly finite subsets of S, and is closed under weakly finite unions.* ▮

Proposition 1·3. *Weak finiteness is stable under forming* (1) *weakly finite-indexed coproducts,* (2) *images, and* (3) *finite products.* ▮

Remark 1·4. In contrast to the Kuratowski-finite case, it is (unfortunately) not always true that a weakly finite set in the image of a mapping f results as the image of some weakly finite subset of the domain of f (unless f happens to be weakly finite-to-one, so that weak finiteness is preserved under pullback along f). The problem is that partial elements cannot always be lifted to partial elements, or for that matter (and this seems obvious, though we have not actually constructed an example), to weakly finite sets. Being a somewhat counter-intuitive feature of weak finiteness, it demands particular care.

In the remainder of the present section, A is an arbitrary frame.

Definition 1·5. A subset D of A is said to be strongly directed if it contains an upper bound for each of its weakly finite subsets. A down-closed, strongly directed set is called a strong ideal.

It is clear that a strongly directed subset D of A is indeed directed. The converse holds precisely when D is Ω-directed, i.e. contains upper bounds for its partial elements.

Lemma 1·6. *A subset D of A is Ω-directed if and only if $x \wedge [\![x \in D]\!] \subseteq \downarrow D$ for all $x \in A$.*

Proof. If D is Ω-directed, then since, for arbitrary $x \in A$, the subsingleton set

$$\{z \in A \mid z = \bigvee S \text{ and } x \in D\}$$

is contained in D, it follows that

$$x \wedge [\![x \in D]\!] = \bigvee\{z \in A \mid z = \bigvee S \text{ and } x \in D\} \in \downarrow D.$$

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Conversely, suppose the last equality holds for all $x \in A$, and consider $S \subseteq D$ with at most one element. If S is inhabited, S is a singleton and $\bigvee S \in D$. Hence

$$S \subseteq \{z \in A \mid z = \bigvee S \text{ and } S \text{ inhabited}\} \subseteq \{z \in A \mid z = \bigvee S \text{ and } \bigvee S \in D\},$$

i.e. $\bigvee S \leqslant \bigvee S \wedge [\![\bigvee S \in D]\!] \in {\downarrow} D$. So D contains an upper bound for S. ▮

A down-set of A is a strong ideal exactly when closed under weakly finite joins. consequently, it is possible to speak of the strong ideal generated by a subset of A.

LEMMA 1·7. *The strong ideal generated by $S \subseteq A$ is the closure of ${\downarrow} S$ under weakly finite joins.*

Proof. Let J be the closure of ${\downarrow} S$ under weakly finite joins. Let $K = \{x \in A \mid {\downarrow}(x) \subseteq J\}$. If $W \subseteq K$ is weakly finite and $x \leqslant \bigvee W$, then $x = x \wedge \bigvee W = \bigvee(x \wedge W)$ while $x \wedge W \subseteq J$ is weakly finite by Proposition 1·3(2). So $\bigvee W \in K$, i.e. K is closed under weakly finite joins. Now clearly $S \subseteq K$. So $K \subseteq K$, i.e. J is down-closed. ▮

Remark 1·8. Note that, due to (1·4), a set $S \subseteq A$ is not generally closed off under weakly finite joins by the set of suprema of weakly finite subsets of S. This will in particular mean (see below) that weak compactness cannot be defined with direct reference to weak finiteness.

COROLLARY 1·9. *For any set S, $\mathscr{W}(S)$ is the strong ideal generated in $\mathscr{P}(S)$ by the singletons.*

Thus, in striking contrast to the Kuratowski-finite case, we have

PROPOSITION 1·10. *A subset of a weakly finite set is weakly finite.* ▮

Definition 1·11. An element x of A is said to be weakly compact if for any strong ideal $J \subseteq A$, $x \leqslant \bigvee J$ only if $x \in J$.

If $D \subseteq A$ is only known to be strongly directed, and $x \leqslant \bigvee D$, it is due to (1·4) not possible to conclude that $x \in D$ from weak compactness alone, unless $x = \top$. But even in the case of the top element, some machinery is needed for the proof, which is delayed until Section 3. Meanwhile the following result is clear:

LEMMA 1·12. *The set of weakly compact elements of A is closed under weakly finite joins.* ▮

The basic example of weak compactness is of course weak finiteness:

PROPOSITION 1·13. *For a set S, $\mathscr{W}(S)$ is the set of weakly compact elements of $\mathscr{P}(S)$.*

Proof. That weakly finite subsets of S are weakly compact is immediate from (1·12) and the fact that singletons are weakly compact (trivially). Conversely, suppose $W \subseteq S$ is weakly compact. Then the set of weakly finite subsets of W is a strong ideal of $\mathscr{P}(S)$ (by Lemma 1·7), covering W (since it contains the singleton subsets of W), hence contains W as an element. ▮

2. *Weakly closed subspaces*

Let X be a space, and Y any subspace of X. The weak closure $\bar{Y}^{\mathrm{w}}$ of Y is defined by the set of conditions

$$\bar{Y}^{\mathrm{w}} \wedge U \leqslant [\![\mathscr{C} \text{ inhabited}]\!]$$

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where U ranges over open subspaces of X and $\mathscr{C}$ over covers of $Y \wedge U$. $\bar{Y}^{\mathrm{W}}$ is the locale of the generic 'adherent' point of Y, characterized by having U as an open neighbourhood only when Y meets U, i.e. when each cover of $Y \wedge U$ is inhabited. It is not difficult to see that for the 'test covers' $\mathscr{C}$ it suffices to consider subsets of the singleton $\{X\}$, so that $\bar{Y}^{\mathrm{W}}$ may alternatively be defined by the requirement (which is essentially the definition of [**4**]):

$$Y^{\mathrm{W}} \wedge U \leqslant v \quad \text{whenever} \quad Y \wedge U \leqslant v, \quad v \in \Omega. \tag{*}$$

Equivalently,

$$\bar{Y}^{\mathrm{W}} = \bigwedge \{ -U \vee v \mid Y \wedge U \leqslant v, U \in \mathcal{O}X, v \in \Omega \}.$$

The operation $Y \mapsto \bar{Y}^{\mathrm{W}}$ is order preserving, inflationary and idempotent, i.e. a closure operator in the order-theoretic sense on the set of subspaces of X (a straightforward consequence of the definition). So Y is said to be weakly closed if $\bar{Y}^{\mathrm{W}} \leqslant Y$, i.e. if Y 'contains its adherent points'.

Proposition 2·1 (cf. [**4**]). *The following subspaces are weakly closed*:
(1) *a subspace with an open complement* (*i.e. closed in the ordinary sense*);
(2) *any intersection of weakly closed subspaces*;
(3) *the inverse image of a weakly closed subspace*;
(4) *a Kuratowski-finite union of weakly closed subspaces.* ▮

As indicated, the last fact may indeed by strengthened:

Proposition 2·2. *A weakly finite union of weakly closed subspaces is weakly closed.* ▮

Proof. It remains to show that $Y \equiv \bigvee \mathscr{C}$ is weakly closed if $\mathscr{C}$ is a set of weakly closed subspaces of X with at most one element. Now clearly if $\mathscr{C}$ is inhabited, then Y is weakly closed, and $\bar{Y}^{\mathrm{W}} \leqslant Y$. So $Y \leqslant [\![\mathscr{C} \text{ inhabited}]\!] \leqslant [\![\bar{Y}^{\mathrm{W}} \leqslant Y]\!]$, giving $\bar{Y}^{\mathrm{W}} \leqslant [\![\bar{Y}^{\mathrm{W}} \leqslant Y]\!]$, i.e. $\bar{Y}^{\mathrm{W}} = \bar{Y}^{\mathrm{W}} \wedge [\![\bar{Y}^{\mathrm{W}} \leqslant Y]\!] \leqslant Y$. ▮

By Proposition 2·2, any open subspace of 1 is weakly closed. It is in fact not difficult to see that open subspaces of discrete spaces are always weakly closed [**4**]; since any subspace of a space is the intersection of joins of pairs of closed and open subspaces, using Proposition 2·1 we have

Proposition 2·3. *Any subspace of a discrete space is weakly closed.* ▮

For the ordinary closure $\bar{Y}$ of Y, extracting information from a statement of the form $\bar{Y} \wedge U \leqslant V$ in terms of the opens U and V is in principle easy: $U \leqslant \neg Y \vee V$, where $\neg Y$ is the open pseudo-complement of Y. By its nature the weak case involves more work, for which we shall mainly depend on Lemma 2·5 below.

Lemma 2·4. *$U \in \mathcal{O}X$ is $\bar{Y}^{\mathrm{W}}$-saturated exactly when for all $C \in \mathcal{O}X$,*

$$Y \wedge U \leqslant [\![C \leqslant U]\!] \text{ implies } C \leqslant U.$$

Proof. Since the defining conditions (*) of $\bar{Y}^{\mathrm{W}}$ are stable, by Lemma 0·3 $U \in \mathcal{O}X$ will be $\bar{Y}^{\mathrm{W}}$-saturated if and only if for all $C \in \mathcal{O}X$, $C \leqslant U$ whenever there exists a truth value v such that $Y \wedge C \leqslant v$ and $v \wedge C \leqslant U$, i.e. whenever $Y \wedge C \leqslant [\![C \leqslant U]\!]$. ▮

Now, for $U \in \mathcal{O}X$, let

$$\rho_Y(U) \equiv \{C \in \mathcal{O}X \mid Y \wedge C \leqslant [\![C \leqslant U]\!]\}.$$

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Next, let $w_Y : \mathscr{P}(\mathcal{O}X) \to \mathscr{P}(\mathcal{O}X)$ be the closure operator which assigns to $\mathscr{S} \subseteq \mathcal{O}X$ the least $\mathscr{J} \subseteq \mathcal{O}X$ containing $\mathscr{S}$ and satisfying

(W1) $U \in \mathscr{J}$ and $V \leqslant U \Rightarrow V \in \mathscr{J}$;

(W2) $U \in \mathscr{J}$ and $\mathscr{F} \subseteq \rho_Y(U)$ Kuratowski-finite $\Rightarrow \bigvee \mathscr{F} \in \mathscr{J}$;

(W3) $\mathscr{D} \subseteq \mathscr{J}$ directed $\Rightarrow \bigvee \mathscr{D} \in \mathscr{J}$.

LEMMA 2·5. *For any $U \in \mathcal{O}X$ and directed $\mathscr{D} \in \mathcal{O}X$,*

$$\bar{Y}^{\mathrm{W}} \wedge U \leqslant \bigvee \mathscr{D} \text{ if and only if } U \in w_Y(\mathscr{D}).$$

Proof. Lemma 2·4 effectively says that U is $\bar{Y}^{\mathrm{W}}$-saturated, i.e. fixed by $l_{\bar{Y}^{\mathrm{W}}}$, if and only if it is a fixed point of the order-preserving, inflationary operation $r : U \mapsto \bigvee \rho_Y(U)$. So $\bar{Y}^{\mathrm{W}} \wedge U \leqslant \bigvee \mathscr{D}$ if and only if $U \leqslant$ the least fixed point of r containing $\bigvee \mathscr{D}$, which by (0·2) is equivalent to U being a member of the least $\mathscr{J}$ containing $\mathscr{D}$ which satisfies (W1) and (W3), and has $\bigvee \rho_Y(U) \in \mathscr{J}$ whenever $U \in \mathscr{J}$. But in the presence of (W1) and (W3), the last condition is fulfilled exactly when $\mathscr{J}$ satisfies (W2). ∎

3. *Weakly compact spaces*

Let X be a space.

Definition 3·1. X is said to be weakly compact if it is so in its frame of opens.

As we would like, the following holds:

LEMMA 3·2. *An open subspace U of X is weakly compact if and only if U is weakly compact in $\mathcal{O}X$.*

Proof. A strong ideal in $\mathcal{O}U \simeq {\downarrow}(U)$ is clearly a strong ideal in $\mathcal{O}X$. Conversely, suppose that $\mathscr{J}$ is strong ideal in $\mathcal{O}X$. Then $U \wedge \mathscr{J} \subseteq \mathscr{J}$, so that from Definition 1·5 it follows that $U \wedge \mathscr{J}$ is a strong ideal in ${\downarrow}(U)$. ∎

Some technical work is still required before we can deduce the standard kind of consequences which issue from the definition. The following lemma serves as a general purpose tool for extracting facts from weak compactness.

LEMMA 3·3. *Suppose that X is weakly compact and $\mathscr{D}$ is a directed open cover of X. Then a proposition p is true whenever it can be shown to follow by assuming that X is covered by $\{D, [\![p]\!]\}$ for some $D \in \mathscr{D}$.*

Proof. Define a mapping $r : \mathcal{O}X \to \Omega^{\Omega}$ by letting for $T \in \mathcal{O}X$, $v \in \Omega$,

$$r(T) : v \mapsto [\![X \leqslant D \vee v \vee T \text{ for some } D \in \mathscr{D}]\!].$$

Then each $r(T)$ is an order-preserving and inflationary operation on Ω, so for each $v \in \Omega$ there is a least fixed point $\mathrm{fix}\,(T)(v)$ of $r(T)$ containing v. Since r is itself order-preserving, so is $\mathrm{fix} : \mathcal{O}X \to \Omega^{\Omega}$. Now let

$$\mathscr{J} = \{U \in \mathcal{O}X \mid \mathrm{fix}\,(U \vee T) = \mathrm{fix}\,(T)\ \forall T \in \mathcal{O}X\}.$$

Then $\mathscr{J}$ contains $\mathscr{D}$, since $r(D \vee T) = r(T)$ for any $D \in \mathscr{D}$, $\mathscr{D}$ being directed. $\mathscr{J}$ is down-closed, since $V \leqslant U$ implies $\mathrm{fix}\,(T) \leqslant \mathrm{fix}\,(V \vee T) \leqslant \mathrm{fix}\,(U \vee T)$, so that the three operations coincide if $U \in \mathscr{J}$. Also, if U, $V \in \mathscr{J}$, then since

$$\mathrm{fix}\,((U \vee V) \vee T) = \mathrm{fix}\,(U \vee (V \vee T)) = \mathrm{fix}\,(\mathrm{V} \vee T) = \mathrm{fix}\,(T)$$

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for any $T \in \mathcal{O}X$, $\mathcal{J}$ is closed under binary suprema. Suppose now that $\mathcal{U} \subseteq \mathcal{J}$ has at most one element. Fix any $T \in \mathcal{O}X$. Then, if $\mathcal{U}$ is inhabited, for any $v \in \Omega$ one has,

$$\begin{aligned} r(\bigvee \mathcal{U} \vee T)(v) &\leqslant [\![X \leqslant D \vee v \vee U \vee T \text{ for some } D \in \mathcal{D}, U \in \mathcal{J}]\!] \\ &= \bigvee \{r(U \vee T)(v) \mid \mathcal{U} \in \mathcal{J}\} \\ &\leqslant \bigvee \{\mathrm{fix}\,(U \vee T)(v) \mid U \in \mathcal{J}\} \\ &= \mathrm{fix}\,(T)(v). \end{aligned}$$

Hence $\bigvee \mathcal{U} \leqslant [\![\mathcal{U} \text{ inhabited}]\!] \leqslant \mathrm{fix}\,(T)(v)$ in $\mathcal{O}X$, so that

$$\begin{aligned} r(\bigvee \mathcal{U} \vee T)(v) &\leqslant [\![X \leqslant D \vee v \vee \mathrm{fix}\,(T)(v) \vee T \text{ for some } D \in \mathcal{D}]\!] \\ &= r(T)(\mathrm{fix}\,(T)(v)) \qquad (\text{since } v \leqslant \mathrm{fix}\,(T)(v)) \\ &= \mathrm{fix}\,(T)(v), \end{aligned}$$

since $\mathrm{fix}\,(T)(v)$ is fixed by $t(T)$. In particular,

$$r(\bigvee \mathcal{U} \vee T)(\mathrm{fix}\,(T)(v)) \leqslant \mathrm{fix}\,(T)(\mathrm{fix}\,(T)(v)) \leqslant \mathrm{fix}\,(T)(v),$$

since $\mathrm{fix}\,(T)$ is idempotent, so that $\mathrm{fix}\,(T)(v)$ is a fixed point of $r(\bigvee \mathcal{U} \vee T)$ containing v. But then $\mathrm{fix}\,(\bigvee \mathcal{U} \vee T)(v) \leqslant \mathrm{fix}\,(T)(v)$. Since v and T were chosen arbitrary, it follows that $\bigvee \mathcal{U} \in \mathcal{J}$. This shows that $\mathcal{J}$ is a strong ideal of $\mathcal{O}X$, covering X since $\mathcal{D} \subseteq \mathcal{J}$. So $X \in \mathcal{J}$ by compactness, which means (choosing $T \equiv 0$) that

$$\text{true} = \mathrm{fix}\,(X)(\text{false}) = \mathrm{fix}\,(0)(\text{false}).$$

But now, if $X \leqslant D \vee [\![p]\!]$ implies p for all $D \in \mathcal{D}$, then $[\![p]\!]$ is a fixed point of $r(0)$, so that

$$\text{true} = \mathrm{fix}\,(0)(\text{false}) \leqslant [\![p]\!]$$

(using $0 \leqslant p$), i.e. p is true. ▮

Now, as mentioned in Section 1, we have

LEMMA 3·4. *Suppose X is weakly compact and $\mathcal{D}$ is a strongly directed cover of X. Then $X \in \mathcal{D}$.*

Proof. Since $\mathcal{D}$ is directed, it is by Lemma 3·3 only necessary to show that $X \in \mathcal{D}$ if it is known that $X \leqslant D \vee [\![X \in \mathcal{D}]\!]$ for some $D \in \mathcal{D}$. But, using Lemma 1·6, this is immediate. ▮

PROPOSITION 3·5. *Weak compactness is inherited by weakly finite coproducts and images.*

Proof. The constituent spaces of a coproduct are open subspaces (cf. [5], V·2·1). So the first part is immediate from Lemma 3·2 and Lemma 1·12. Suppose now that $f: X \to Y$ is a surjective map of spaces. Then the inverse image mapping $f^*: \mathcal{O}Y \to \mathcal{O}X$ is injective. So if $\mathcal{J}$ is a strong ideal in $\mathcal{O}Y$ covering Y, its inverse image is strongly directed, and covers X. So $X \in f^*[\mathcal{J}]$ by Lemma 3·4, i.e. $Y \in \mathcal{J}$ by the injectivity of f^*. ▮

COROLLARY 3·6. *The weakly compact subspaces of a space are closed under weakly finite unions.* ▮

THEOREM 3·7. *A weakly closed subspace of a weakly compact space is weakly compact.*

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Proof. Let X be weakly compact and Y a weakly closed subspace of X. Suppose that $\mathscr{I} \subseteq \mathcal{O}Y$ is a strong ideal in $\mathcal{O}Y$ covering Y. Then the set $\mathscr{D} = \{D \in \mathcal{O}X \mid Y \wedge D \in \mathscr{I}\}$ is an (ordinary) ideal in $\mathcal{O}X$. Now let

$$\mathscr{J} = \{U \in \mathcal{O}X \mid \forall D \in \mathscr{D},\ X \leqslant U \vee D \Rightarrow Y \in \mathscr{I}\}.$$

It will be enough to show that $X \in \mathscr{J}$, since then $Y \in \mathscr{I}$ will follow from the fact that $0 \in \mathscr{D}$. But since $\bar{Y}^{\mathrm{w}} \wedge X = Y \leqslant \mathscr{I} \leqslant \bigvee \mathscr{D}$, Lemma 2·5 gives $X \in w_Y(\mathscr{D})$. Also, clearly $\mathscr{D} \subseteq \mathscr{J}$. So it is sufficient to prove that $\mathscr{J}$ satisfies the closure conditions defining w_Y. (W1) is trivially satisfied. As for (W2), suppose that $U \in \mathscr{J}$ and let

$$\mathbb{H} = \{\mathscr{H} \subseteq \rho_Y(U) \mid \bigvee \mathscr{H} \vee U \in \mathscr{J}\}.$$

Then $\varnothing \in \mathbb{H}$ by assumption. Further, if $\mathscr{H} \in \mathbb{H}$ and $C \in \rho_Y(U)$, then for any $D \in \mathscr{D}$,

$$X \leqslant \bigvee \mathscr{H} \vee C \vee U \vee D \Rightarrow Y \wedge C \leqslant Y \wedge [\![X \leqslant (\bigvee \mathscr{H} \vee U) \vee D]\!] \leqslant Y \wedge [\![Y \in \mathscr{I}]\!] \in \mathscr{I}$$
$$\Rightarrow C \in \mathscr{D} \Rightarrow C \vee D \in \mathscr{D}$$

since $\mathscr{I}$ is Ω-directed in $\mathcal{O}Y$ and $\mathscr{D}$ is an ideal in $\mathcal{O}X$. But this means that

$$X \leqslant \bigvee \mathscr{H} \vee C \vee U \vee D = (\bigvee \mathscr{H} \vee U) \vee (C \vee D) \Rightarrow Y \in \mathscr{I},$$

i.e. $\mathscr{H} \cup \{C\} \in \mathbb{H}$. So, by Lemma 0·1, $\mathbb{H}$ contains the Kuratowski-finite subsets of $\rho_Y(U)$; since $\mathscr{J}$ is down-closed, (W2) follows. Finally, if $\mathscr{K} \subseteq \mathscr{J}$ is directed and $X \leqslant \bigvee \mathscr{K} \vee D = \bigvee (\mathscr{K} \vee D)$ for some $D \in \mathscr{D}$, then $\mathscr{K} \vee D$ is an open directed cover of X and Lemma 3·3 may be applied to show that $Y \in \mathscr{I}$: if $X \leqslant K \vee D \vee [\![Y \in \mathscr{I}]\!]$ for some $K \in \mathscr{K}$, then $K \in \mathscr{J}$ and since $[\![Y \in \mathscr{I}]\!] \in \mathscr{D}$, i.e. $D \vee [\![Y \vee \mathscr{I}]\!] \in \mathscr{D}$, $Y \in \mathscr{I}$ follows. Thus $\bigvee \mathscr{K} \in \mathscr{J}$, which shows that (W3) is satisfied. ∎

COROLLARY 3·8. *All subspaces of a discrete weakly compact space are weakly compact.*

Proof. The result follows by Proposition 2·3. ∎

As is easily verified, a discrete space is weakly compact exactly when its set of points is weakly finite. Thus, Proposition 3·8 may be regarded as a strengthening of Proposition 1·10.

4. *Separation and weak compactness*

As expected, Theorem 3·7 has a converse for weakly regular spaces. Let X be a space. X is weakly regular [**4**] if each of its opens U may be covered by open subspaces V with $\bar{V}^{\mathrm{w}} \leqslant U$. Obvious examples of weakly regular spaces are (of course) regular spaces (by Proposition 2·1) and discrete spaces (by Proposition 2·3); the latter are regular only if decidable.

LEMMA 4·1. *Suppose that X is weakly regular and $Y \leqslant U$ are subspaces of X, with Y weakly compact and U open. Then $\bar{Y}^{\mathrm{w}} \leqslant U$.*

Proof. By weak regularity, $Y \leqslant \bigvee \{D \in \mathcal{O}X \mid \bar{D}^{\mathrm{w}} \leqslant U\} \equiv \bigvee D$, and $\mathscr{D}$ is directed (by Proposition 2·1). So $Y \wedge \mathscr{D}$ is a directed open cover of Y. suppose now $Y \leqslant (Y \wedge D) \vee (Y \wedge [\![\bar{Y}^{\mathrm{w}} \leqslant U]\!])$, i.e. $Y \leqslant D \vee [\![\bar{Y}^{\mathrm{w}} \leqslant U]\!]$ for some $D \in \mathscr{D}$. Then we have $\bar{Y}^{\mathrm{w}} \leqslant \bar{D}^{\mathrm{w}} \vee [\![\bar{Y}^{\mathrm{w}} \leqslant U]\!]$ as a result of Proposition 2·2, i.e.

$$\bar{Y}^{\mathrm{w}} \leqslant \bar{D}^{\mathrm{w}} \vee (\bar{Y}^{\mathrm{w}} \wedge [\![\bar{Y}^{\mathrm{w}} \leqslant U]\!]) \leqslant U.$$

Now apply Lemma 3·3. ∎

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Remark 4·2. Observe that in the proof of Lemma 4·1, the set $\mathscr{D}$ is in fact a strong ideal of $\mathscr{O}X$ by Proposition 2·2; unfortunately however, unless Y is open, this fact cannot be used to conclude that Y is covered by a single element of $\mathscr{D}$: due again to Remark 1·4, the set $Y \wedge \mathscr{D}$ need not be Ω-directed.

LEMMA 4·3. *Let Y be a weakly compact subspace of X, with X weakly regular, and suppose $Y \wedge U \leqslant V$, where U, V are open. Then $\bar{Y}^{\mathrm{w}} \wedge P \leqslant V$ (for $P \in \mathscr{O}X$) whenever there exists $Q \in \mathscr{O}X$ such that $\bar{Y}^{\mathrm{w}} \leqslant Q$ and $\bar{P}^{\mathrm{w}} \wedge Q \leqslant U$.*

Proof. Fix $P \in \mathscr{O}X$, and consider the set

$$\mathscr{J} = \{Q \in \mathscr{O}X \mid \text{for all } R \in \mathscr{O}X \text{ with } \bar{Y}^{\mathrm{w}} \wedge P \wedge R \leqslant V, \quad Y \leqslant Q \vee R \Rightarrow \bar{Y}^{\mathrm{w}} \wedge P \leqslant V\}.$$

By choosing $R \equiv 0$, the result will follow if for all $Q \in \mathscr{O}X$, one has $Q \in \mathscr{J}$ whenever $\bar{P}^{\mathrm{w}} \wedge Q \leqslant U$. By Lemma 2·5, this amounts to showing that $w_P(\{U\}) \subseteq J$, which will be the case if $U \in \mathscr{J}$ and $\mathscr{J}$ satisfies (W1)–(W3). Now, if $\bar{Y}^{\mathrm{w}} \wedge P \wedge R \leqslant V$ for $R \in \mathscr{O}X$, then

$$\begin{aligned} Y \leqslant U \vee R &\Rightarrow Y \leqslant (Y \wedge U) \vee R \leqslant V \vee R \\ &\Rightarrow \bar{Y}^{\mathrm{w}} \leqslant V \vee R \qquad \text{(by (4·1))} \\ &\Rightarrow Y^{\mathrm{w}} \wedge P \leqslant V \vee (\bar{Y}^{\mathrm{w}} \wedge P \wedge R) \leqslant V. \end{aligned}$$

So $U \in \mathscr{J}$. (W1) is trivially satisfied by $\mathscr{J}$. For (W2), suppose $Q \in \mathscr{J}$ and let

$$\mathbb{H} = \{\mathscr{H} \subseteq \rho_P(Q) \mid \bigvee \mathscr{H} \vee Q \in \mathscr{J}\}.$$

Then $\varnothing \in \mathbb{H}$ by assumption. Further, if $\mathscr{H} \in \mathbb{H}$ and $\bar{Y}^{\mathrm{w}} \wedge P \wedge R \leqslant V$ with $R \in \mathscr{O}X$, then for any $C \in \rho_P(Q)$,

$$\begin{aligned} Y \leqslant \bigvee \mathscr{H} \vee C \vee Q \vee R &\Rightarrow P \wedge C \leqslant [\![Y \leqslant \bigvee \mathscr{H} \vee Q \vee R]\!] \leqslant [\![\bar{Y}^{\mathrm{w}} \wedge P \leqslant V]\!] \\ &\Rightarrow \bar{Y}^{\mathrm{w}} \wedge P \wedge C \leqslant V \\ &\Rightarrow \bar{Y}^{\mathrm{w}} \wedge P \wedge (C \vee R) \leqslant V, \end{aligned}$$

which means that $Y \leqslant \bigvee \mathscr{H} \vee C \vee Q \vee R$ implies $\bar{Y}^{\mathrm{w}} \wedge P \leqslant V$, i.e. $\mathscr{H} \cup \{C\} \in \mathbb{H}$. It follows (by Lemma 0·1) that $Q \in \mathscr{J}$ implies $\bigvee \mathscr{F} \in \mathscr{J}$ for any Kuratowski-finite $\mathscr{F} \subseteq \rho_P(Q)$. Finally, suppose that $\mathscr{K} \subseteq \mathscr{J}$ is directed and $Y \leqslant \bigvee \mathscr{K} \vee R$, with $\bar{Y}^{\mathrm{w}} \wedge P \wedge R \leqslant V$ for some $R \in \mathscr{O}X$. Since $\bar{Y}^{\mathrm{w}} \wedge P \wedge [\![Y^{\mathrm{w}} \wedge P \leqslant V]\!] \leqslant V$, it follows for any $Q \in \mathscr{J}$ that $\bar{Y}^{\mathrm{w}} \wedge P \leqslant V$ whenever $Y \leqslant Q \vee ([\![\bar{Y}^{\mathrm{w}} \wedge W \leqslant V]\!] \vee R)$, i.e. whenever

$$Y \leqslant (Y \wedge (Q \vee R)) \vee (Y \wedge [\![\bar{Y}^{\mathrm{w}} \wedge P \leqslant V]\!]).$$

This is in particular so for $Q \in \mathscr{K}$, while $Y \wedge (\mathscr{K} \vee R)$ is an open directed cover of Y. So, by Lemma 3·3, $\bigvee \mathscr{K} \in \mathscr{J}$ and (W3) is satisfied. ∎

THEOREM 4·3. *A weakly compact subspace of a weakly regular space is weakly closed.*

Proof. Let X be regular, with Y a weakly compact subspace of X. Given arbitrary $U, V \in \mathscr{O}X$ with $Y \wedge U \leqslant V$, it has to be shown that $\bar{Y}^{\mathrm{w}} \wedge U \leqslant V$. Now, since X is weakly regular, it is sufficient to have $\bar{Y}^{\mathrm{w}} \wedge P \leqslant V$ for $P \in \mathscr{O}X$ with $\bar{P}^{\mathrm{w}} \leqslant U$. But this is immediate from Lemma 4·2. ∎

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