13

Applications

Laplace operators and their spectral properties are powerful tools for the analysis of networks in the social and the biological sciences and in other domains. In computer science, the theory of families of expander graphs is particularly important. Eigenvalues are also a key for quantifying synchronization and other features of nonlinear dynamics.

13.1 Social Sciences

Sociology is about the structure and the dynamics of relations between individuals or groups, and therefore, it is natural for the field to invoke network concepts. In fact, sociology pioneered the analysis of empirical network data a long time before other disciplines took this up. It started with the work of Moreno and Jennings [212, 213] who explicitly proposed to analyze the formal structure of networks of interpersonal relations to find explanations for individual behavior. Harary [123–125] then introduced and investigated the structure of signed graphs, and we have made substantial use of that concept at various places of this monograph; see in particular Section 4.6. Of course, the sign of an edge is interpreted as the positive or negative nature of a relation between persons. And, when persons switch their attitudes, as formalized in Definition 4.6.4, the signs of all their relations change.

Other important developments in social network analysis, although not directly related to spectral theory, include the discovery of the small world phenomenon and the role of weak ties. The small world phenomenon means that in the modern world, any two individuals v, v' are separated only by very few intermediates; that is, in the network formed by personal contacts between people, one finds individuals $v = v_1 \sim v_2 \dots v_{k-1} \sim v_k = v'$ for a rather small value of k. That is, the diameter of the network of personal contacts is quite

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small. This was convincingly demonstrated by the experiments of Milgram in the 1960s.

The importance of weak ties was emphasized by Granovetter [115]. It means that for many purposes, like getting new information or spotting new opportunities, strong ties between your circle of close friends are often less helpful than the much weaker ties with loose acquaintances.

But there are also concepts in social network analysis that depend on eigenvalues and eigenvectors of operators. Let us describe some.

13.1.1 Eigenvector Centrality

In a network, for instance, a social one, it is natural to try to quantify the importance of various nodes in the network. A first attempt might be to call a node important if it is connected to many others, that is, if it has a high degree. A better option might be to call a node important if it is connected to other important nodes. That now is a recursive definition. It was first formalized by Bonacich [33]. We let $A = (a_{vv'})$ be the adjacency matrix of the network. (For simplicity, we consider an unweighted and undirected network here, but the construction can easily be extended to the weighted case. The directed case is more complicated.) Then, if x_v is the importance of node v, it should be proportional to $\sum_{v'} a_{vv'} x_{v'}$, that is,

$$x_v = \lambda \sum_{v'} a_{vv'} x_{v'} \tag{13.1.1}$$

for some positive λ, which then is an eigenvalue of A. Also, all the values of x_v should be positive themselves. Now it follows from the Perron–Frobenius Theorem that, when the non-negative matrix A is irreducible, the largest eigenvalue of A is simple and has an eigenvector with positive entries. Thus, such an eigenvector solves the problem, and the value x_v for a suitably normalized such eigenvector then is called the *eigenvector centrality* of the node v.

In fact, these properties can be derived in a very simple manner. First, irreducibility of A means that whenever the vertex set V is decomposed into two nonempty subsets V_1, V_2, there exists at least one pair $v_1 \in V_1, v_2 \in V_2$ with

$$a_{v_1 v_2} \neq 0. \tag{13.1.2}$$

But this simply means that the underlying graph $\Gamma = (V, E)$ is connected.

By Theorem 2.6.1, the largest eigenvalue of A can be obtained as

$$\lambda = \max_{\mathbf{x} \neq 0} \frac{\langle \mathbf{x}, A\mathbf{x} \rangle}{\langle \mathbf{x}, \mathbf{x} \rangle} \tag{13.1.3}$$

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as the adjacency matrix of an undirected graph is symmetric; here $\langle \cdot, \cdot \rangle$ is the Euclidean product in $\mathbb{R}^N$ where N is the number of nodes of Γ. When for some $\mathbf{x} \in \mathbb{R}^N$, we replace every negative entry $x_v < 0$ by $|x_v|$, we do not change the denominator in (13.1.3) and do not decrease the numerator in (13.1.3). Therefore, we may assume that all $x_v \geq 0$ for a maximizer of (13.1.3). In particular, the maximum is positive, and therefore, we obtain a solution of (13.1.1) with some $\lambda > 0$. We want to exclude that some x_v vanish. But since not all x_v can be zero, for some such $x_v = 0$, we find some v' with $a_{vv'} \neq 0$ and $x_{v'} > 0$, since A is irreducible. But since all $x_{v'} \geq 0$, then by (13.1.1), also $x_v > 0$, a contradiction. Therefore, λ has an eigenvector with positive entries. Thus, from any maximizer of (13.1.3), by changing signs of the negative entries, if any, we have produced a maximizer with only positive entries. But then, when the original had both positive and negative entries, since A is irreducible, there would be some $a_{vv'} \neq 0$ connecting some v with $x_v > 0$ and some v' with $x_{v'} < 0$. But then, the replacement would have strictly increased the value of (13.1.3), contradicting the maximizing property. This argument shows that all eigenvectors are either positive or negative everywhere. Therefore, in particular, eigenvectors for this largest eigenvalue λ cannot be orthogonal to each other, and the eigenspace for λ is one-dimensional, indeed.

Thus, a positive eigenvector for the largest eigenvalue of A provides the centrality measure.

While such centrality measures were first conceived in mathematical sociology, they have found widespread applications also in other fields. In particular, Google's page rank algorithm is based on some such centrality measure. See, for instance, [34].

13.1.2 Cohesion of Networks

Personal ties can break in a social group, and this can lead to a split into subgroups that are no longer connected with each other. Or, in order to prevent the spreading of an epidemic, contacts between different groups should be reduced. Thus, depending on the interpretation of the edges, for instance, whether they correspond to the spreading of information, opinions, or diseases, it can be desirable or undesirable to break up a network into disconnected components. In any case, this raises the question how easy or difficult that is. This issue has been formally addressed in Section 4.3, and we have described there how this is controlled by the first nonzero eigenvalue λ_2 of the Laplacian. When that eigenvalue is small, it is easy to break up the network into large disconnected

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components by cutting a small number of edges. And the nodal domains of the corresponding eigenfunction yield an approximation of those components. See Chapter 9.

13.2 Biological Networks

There is an abundance of network-type data in the biological sciences, ranging from the species to the molecular level (see, for instance, [148]). The descendence relations between species, derived from paleontological data, and the genetic relationships, derived from the comparison of gene sequences, can ideally be organized in phylogenetic trees (see, for instance, [245] for the mathematics of such trees), although in practice processes such as hybridization, incipient speciations, or horizontal gene transfer may necessitate deviations from such a simple arrangement (see [14] for a mathematical theory that can incorporate that). Also foodwebs, that is, trophic interactions between species in ecosystems, are arranged in directed graphs – who eats whom. Interactions between individuals in animal groups can also be represented by directed or undirected graphs.

At the molecular level, networks can be constructed from correlation patterns, for instance, between gene expression in cells, chemical interactions as in metabolic networks, and physical interactions, for instance, between proteins but also by the geometric proximity of molecules or their parts in cells.

For an application of spectral analysis, let us consider the example of protein–protein interaction (PPI) networks, following [17]. Such a PPI network is modeled by a graph whose nodes stand for the proteins and edges indicate interactions. We plot the spectrum of the PPI networks of different species with a smoothing kernel (Figure 13.1). More precisely, the spectral density is represented as a sum of Lorentz distributions, $\rho(\lambda) = \sum_{k=2}^{N} \frac{\gamma}{(\lambda_k - \lambda)^2 + \gamma^2}$ with width $\gamma = .08$ where $\lambda_2, \ldots, \lambda_N$ are the positive eigenvalues. In Figure 13.2, we see the prominent peak near 1, indicating a high multiplicity of the eigenvalue 1.

This should be compared with the analogous representation in Figure 13.2 of the spectrum of a simulated network with repeated node duplication and subsequent random mutations. In fact, we have seen in Section 4.1 that repeated node duplications lead to high multiplicity of the eigenvalue 1. The simulated network therefore suggests a plausible evolutionary process behind the spectral structure of PPI networks. The mechanism should work through repeated duplications of the genes coding for the proteins in question, with subsequent mutations so as to diversify their function.

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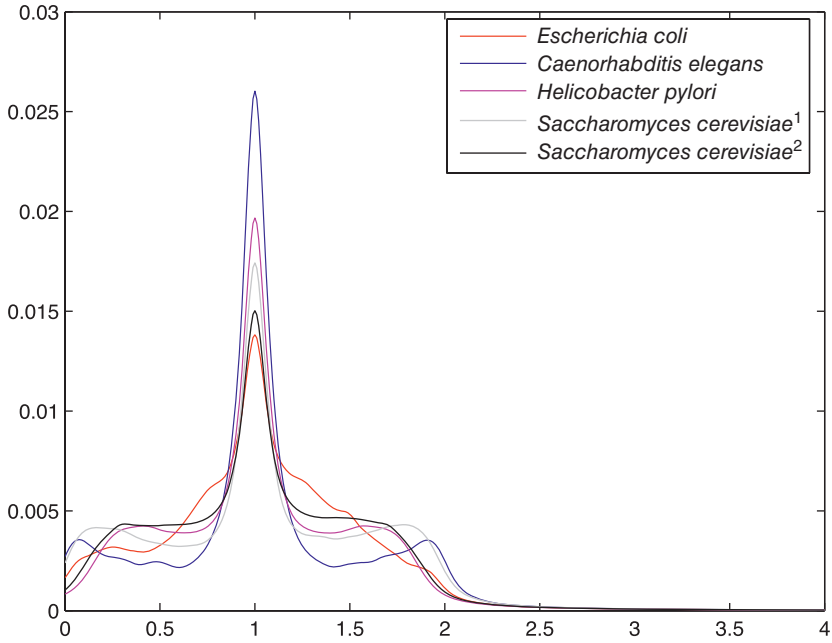


Figure 13.1 Spectra of PPI networks of different species, convolved with a Lorentz kernel.

In fact, in [18, 20], a systematic program has been proposed to classify biological and other networks according to their large-scale spectral properties. The approach of Section 10.2 may provide the appropriate mathematical structure for that program.

Another example of spectral analysis for biological networks is given in [216]. Here, networks of genetic expressions are modeled by *hypergraphs with real coefficients* (Definition 2.4.5), as follows. Each vertex v represents a cell, each edge e represents a gene, and the coefficient $\varphi(v,e)$ represents the fraction of transcripts in cell v mapping to gene e. It is assumed that the coefficients are normalized with respect to cells, so that $\sum_e \varphi(v,e) = 1$ for each v, as such cell-wise normalization is the norm in RNA-sequence analysis. Therefore, $\varphi(v,e)$ encodes the relative expression of gene e in cell v. The corresponding Laplace operator, which was introduced in [154] and generalizes the Laplacian for oriented hypergraphs in Chapter 4.9, is then defined as $L = D^{-1}\mathcal{I}\mathcal{I}^\top$, where D is the diagonal degree matrix and $\mathcal{I}$ is the incidence matrix of the hypergraph. For a data set of gene expression with N cells, the Laplacian spectrum consists of N real, non-negative eigenvalues, $\lambda_1 \leq \ldots \leq \lambda_N$, and the largest of these

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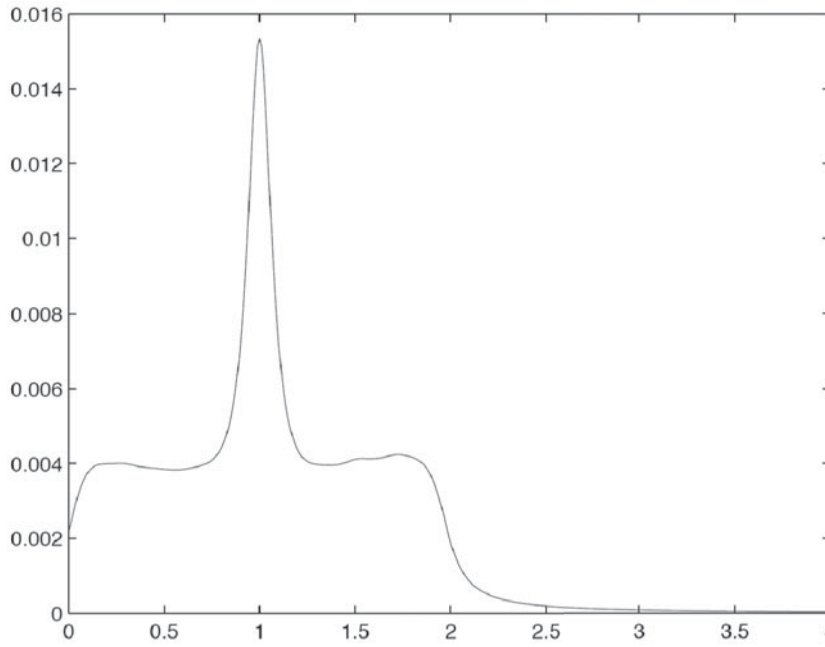


Figure 13.2 Spectrum of a model network with iterated node duplications and random mutations, again convolved with a Lorentz kernel.

eigenvalues is shown to encode *cellular redundancy*. In fact, as proved in [216], $1 \leq \lambda_N \leq N$, and λ_N is equal to 1 if and only if each edge has cardinality 1 (that is, each gene is concentrated in one single cell), while it is equal to N if and only if each gene is uniformly distributed among all cells (that is, the network has cellular redundancy). Moreover, the normalized quantity λ_N/N is chosen as a measure of cellular redundancy that is independent of the number of cells, and its accuracy is shown through an analysis of simulated and real data sets of gene expression.

13.3 Computer Science

In order to link the main applications of spectral theory in computer science with the perspective of our book, we should first clarify a mathematical issue. As we have seen in Section 4.1, there are different linear operators on the vertex functions of a graph whose spectra encode information about the underlying graph and that have been investigated in the literature in that regard. Here, we

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have mainly considered the (4.1.33), defined by

$$Lf(v) = f(v) - \frac{1}{\deg v} \sum_{v' \sim v} f(v') \tag{13.3.1}$$

for $f : V \to \mathbb{R}$ and $v \in V$. The algebraic graph Laplacian (1.2.1) is

$$Kf(v) = \deg v \, f(v) - \sum_{v' \sim v} f(v'). \tag{13.3.2}$$

Finally, the adjacency matrix operates on functions via

$$Af(v) = \sum_{v' \sim v} f(v'). \tag{13.3.3}$$

Clearly, for a general graph, the spectra of these operators are different, and there is no obvious and simple relation between them. When, however, the graph is d-regular, that is, all vertices have the same degree d,

$$\deg v = d \qquad \text{for all } v \in V, \tag{13.3.4}$$

the spectra are simply related. Whenever λ is an eigenvalue of L, then $\sigma = d\lambda$ is an eigenvalue of K, and $\mu = d - d\lambda$ is an eigenvalue of A, and conversely. Also, for a d-regular graph, the Cheeger constants of L and K are simply related to each other. In order to conform to the interpretation in the applications we have in mind, we use the notation

$$|\partial S| := |E(S, \overline{S})|, \tag{13.3.5}$$

the size of the *boundary*[1] of the vertex set $S \subset V$, where we recall that $|E(S, \overline{S})|$ is the number of edges between S and its complement. The (Pólya)–Cheeger constant for L (4.3.12), (4.3.13) is

$$h_L = \min_{S \subset V} \frac{|\partial S|}{\min(\mathrm{vol}(S), \mathrm{vol}(\overline{S}))}, \tag{13.3.6}$$

where, as we also recall,

$$\mathrm{vol}(S) = \sum_{v} \deg v. \tag{13.3.7}$$

The analogous constant for K is

$$h_K = \min_{S \subset V} \frac{|\partial S|}{\min(|S|, |\overline{S}|)}, \tag{13.3.8}$$

[1] This is the edge boundary of S. The vertex boundary would instead consider those vertices in the complement of S that are connected to some vertex in S. Since several edges from S may lead to the same vertex, the vertex boundary may have fewer elements than the edge boundary. But when the graph is d-regular, the discrepancy could be at most by a factor d. – On the other hand, as in (13.3.7), we could also count those vertices with their degrees.

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where now (not quite consistent with our earlier notation),

$$|S| = \text{number of vertices in } S. \tag{13.3.9}$$

Again, when the graph is d-regular, we have the simple relation $\mathrm{vol}(S) = d|S|$, and so

$$h_K = d\, h_L. \tag{13.3.10}$$

In particular, the Cheeger inequality of Theorem 4.3.1 then is the same for K as for L, except that (13.3.10) leads to a different scaling on the left-hand side for h_K. Thus,

$$\frac{h_K^2}{2d} \le \sigma_2 \le 2h_K, \tag{13.3.11}$$

or equivalently

$$\frac{\sigma_2}{2} \le h_K \le \sqrt{2d\sigma_2}, \tag{13.3.12}$$

where now σ_2 is the second smallest eigenvalue of K, that is, for a connected graph, its smallest nonvanishing eigenvalue. And if $\mu_1 > \mu_2 \ge \cdots \ge \mu_N$ are the eigenvalues of A ($\mu_1 = d$, $\mu_1 > \mu_2$ as we assume that the graph is connected, and $\mu_N = -d$ precisely if the graph is bipartite), the Cheeger inequality (13.3.12) becomes

$$\frac{d - \mu_2}{2} \le h_K \le \sqrt{2d(d - \mu_2)}. \tag{13.3.13}$$

Therefore, while the notion of an expander graph, which we shall discuss in this section, is usually formulated in terms of the Cheeger inequality for A, we can as well work with our preferred operator L, as long as only d-regular graphs are considered, which in fact is the setting of expander theory.

It follows from a result of Alon and Boppana [3] (see, for instance, [136] for a proof) that for a d-regular connected graph with N vertices

$$\mu_2 \ge 2\sqrt{d-1}\left(1 - O\left(\frac{1}{\log^2 N}\right)\right). \tag{13.3.14}$$

This motivates the definition of a *Ramanujan graph* as such a graph with

$$\mu_2 \le 2\sqrt{d-1}, \tag{13.3.15}$$

that is, with the largest possible spectral gap within this class. Families of Ramanujan graphs were constructed in [195, 201]; see also [61, 136].

We shall now provide the basic definition.

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Definition 13.3.1. A family $\Gamma_m = (V_m, E_m)$ of d-regular graphs with $|V_m| \to \infty$ as $m \to \infty$ is called an *expander family* if

$$h_K(\Gamma_m) \geq \varepsilon \quad \text{for all } m \text{ for some } \varepsilon > 0. \tag{13.3.16}$$

The idea is to have families of graphs with two seemingly contradictory properties, sparsity, and high connectivity. Of course, every individual connected graphs satisfies (13.3.16) for some $\varepsilon > 0$, but the point of the definition is to require this uniformly on families of graphs whose size goes to infinity.

Pinsker [229] showed with probabilistic arguments that almost all graphs satisfy some expansion property. Margulis [199] constructed the first explicit expander family. In his construction, $d = 8$ and $V_m = \mathbb{Z}_m \times \mathbb{Z}_m$. With everything understood mod m, the neighbors of the vertex (x, y) are $(x + y, y)$, $(x - y, y)$, $(x, y + x)$, $(x, y - x)$, $(x + y + 1, y)$, $(x - y + 1, y)$, $(x, y + x + 1)$, and $(x, y - x + 1)$. The explicit eigenvalue bound (13.3.15) was derived in [102].

This example, however, does not represent just some isolated discovery but emerges from a wider context opened by Kazhdan [170]. For the purposes here, it suffices to consider a discrete group G with a finite symmetric set Σ of generators.

Definition 13.3.2. The discrete group G possesses *Kazhdan's property (T)* if for some symmetric set Σ of generators, there exists some $\delta > 0$ with the property that for every unitary representation ρ of G in some Hilbert space H that does not fix any nonzero element of H, for every $v \in H, v \neq 0$, there exists some $s \in \Sigma$ with

$$\frac{\|\rho(s)v - v\|}{\|v\|} \geq \delta. \tag{13.3.17}$$

(In fact, when (13.3.17) holds for all $v \neq 0$, then Σ has to generate G.)

This may sound like a rather special property, but Kazhdan discovered that it is possessed by all lattices in simple Lie groups H of $\mathbb{R}$-rank ≥ 2, like $\mathrm{Sl}(n, \mathbb{R})$ for $n \geq 3$. A lattice is a discrete subgroup Γ for which the quotient $\Gamma \backslash (H/K)$ (where K is a maximal compact subgroup of H) has finite volume (induced by the natural Riemannian metric on H/K); an example is $\mathrm{Sl}(n, \mathbb{Z})$ for $n \geq 3$. $\mathrm{Sl}(2, \mathbb{Z})$, however, does not possess property (T).

Definition 13.3.3. Let Σ be a finite symmetric set of generators of the discrete group G. The *Cayley graph* $C(G, \Sigma)$ has G as its vertex set and connects $g, h \in G$ when there exists some $s \in \Sigma$ with $h = gs$.

We observe that the symmetry of Σ makes the Cayley graph undirected, even if G is not commutative.

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Margulis [199, 200] then found that Kazhdan's property (T) of G is equivalent to the expansion property of the Cayley graphs of its quotients by finite index normal subgroups and applied this to the construction of expanders. Examples then are the Cayley graphs of $\mathrm{Sl}(n,\mathbb{Z}/p\mathbb{Z})$ for $n \geq 3$ and primes p.

This already suggests that the theory of expander graphs has surprisingly rich and diverse connections with other areas of mathematics, such as number theory or representation theory; see, for instance, the survey in [193].

For practical purposes, since computing h_K is an **NP**-hard problem, one usually requires an eigenvalue bound, more precisely, a lower bound on the spectral gap $d - \mu_2$, as for instance in (13.3.15). By now, many other expander families are known, such as the Ramanujan graphs already mentioned. See, for instance, [61, 136] for details.

There are other variants of the expansion property; for instance, one considers a bipartite graph $\Gamma = (V,E)$, with V consisting of the two disjoint vertex classes L,R, with $|L| = n$. Every vertex in L should have d neighbors, which by bipartiteness are all in R. The set $N(S) \subset R$ of neighbors of any $S \subset L$ with $|S| \leq \frac{n_1}{2}$ should satisfy

$$|N(S)| \geq |S|. \tag{13.3.18}$$

In (13.3.18), the expansion factor is 1. But if S consists of a single vertex, $|S| = 1$, then $|N(S)| = d|S|$ by assumption, and so, for small $|S|$, one may require an expansion factor even larger than 1, and in fact even larger than $\frac{d}{2}$. [136] speaks of a *magical graph* in such a case.

We now introduce another important concept.

Definition 13.3.4. For a (directed) graph $\Gamma = (V,E)$, let $I,O \subset V$, called the input and the output set, both have n vertices. Γ is called a *super concentrator* if for every k and every $S \subset I$ and $T \subset O$ with $|S| = |T| = k$, there exist k vertex disjoint paths in Γ from S to T.

Super concentrators find many applications for the efficient design of networks. Super concentrators can be constructed from graphs with suitable expansion properties like (13.3.16) or (13.3.18), see, for instance, [61, 136, 199].

Another important application of expander graphs in computer science concerns error-correcting codes. The theoretical framework had been developed by Claude Shannon [246]. One has a number of messages, for simplicity, since we are using binary codes, 2^k for some $k \in \mathbb{N}$. To make them better distinguishable, one encodes them using $n > k$ binary symbols. Thus, every message is uniquely represented by a string of n 0s and 1s. One wants to set up the code

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in such a way that different code words differ in as many symbols as possible while not making n too large. The Hamming distance d_H between two such strings is defined as the number of positions where the two have different symbols. For example, $d_H(0101, 0011) = 2$ because the two strings differ at the second and at the third positions. When C is the set of code words, by assumption of cardinality 2^k, one wants to control the

$$\text{rate } R := \frac{\log_2 |C|}{n} = \frac{k}{n} \tag{13.3.19}$$

and the

$$\text{(normalized) distance } \delta := \frac{\min_{c \neq c' \in C} d_H(c, c')}{n} \tag{13.3.20}$$

simultaneously from below. Thus, the code words should not be too long but sufficiently different from each other. When the Hamming distance between any two code words is at least d_0, then any code word can be uniquely recovered even if transmitted with errors, as long as less than $\frac{d_0}{2}$ symbols are incorrectly transmitted. Of course, the receiver needs to know the code words, and when she receives a message, she simply looks up the code word with the smallest Hamming distance to that message.

In a line of research that started with [103] and was advanced by the discoveries in [199, 229], it was found by Sipser and Spielman [252] that the simultaneous control of the rate and the distance between code words can be achieved efficiently with the help of expander graphs, and we shall now describe this using the presentation in [136] and the concept of a magical graph described earlier.

But let us first explain the principle. The strings with entries 0 and 1 of length n are naturally considered as elements of the vector space $\mathbb{Z}_2^n$ and therefore amenable to operations from linear algebra over the field $\mathbb{Z}_2$. One can seek 2^k code words as those elements that are in the kernel of some matrix $A = (A_{ij})_{i,j=1,\dots n}$ of rank $n - k$. That matrix A then is represented as the adjacency matrix of a directed bipartite graph with one vertex class L consisting of n vertices $v_1, \dots, v_n$. The subsets S of L then represent strings by putting a 1 in the ith position if $v_i \in S$. In order to get a good code with (13.3.20) controlled, one then shows that whenever the size of S is sufficiently small, we can find some $w_j \in N(S)$, the neighborhood of S (which by bipartiteness is contained in the other vertex class R, consisting of those vertices w_j where $A_{ij} = 1$ for some i), that has a unique neighbor $v_i \in S$. But since the corresponding element $A_{ij} = 1$ and for all other k, A_{kj} is applied to 0, and the jth component of the image of the string encoded by S is 1. This implies that this string is not in the kernel of A and hence cannot be a code word.

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If the graph is magical, then a small $S \subset L$ has a set of neighbors $N(S) \subset R$ of size larger than $\frac{d}{2}|S|$. But since, by construction, every element of S has precisely d neighbors in $N(S)$, there has to exist some $w \in N(S)$ that has only one neighbor $v \in S$, which then makes the preceding argument work. If $|R| \leq \frac{3n}{4}$, then the dimension of $\ker A$ is $\geq \frac{n}{4}$, and so, we achieve a rate of 4, that is, the code words are at most four times longer than the number of bits they encode. Since the code is linear, the minimal distance between any two code words equals the minimal distance between a code word and $0 \in \ker A$. But the earlier argument shows that vectors with only a few 1s cannot be code words, and therefore, we also obtain a lower bound on the Hamming distance between code words. Thus, an efficient error-correcting code is constructed from the magical graph.

In the literature, for instance [64, 194], notions of hypergraph (in fact, simplicial complex) expanders have also been proposed, and such expanders have been shown to exist and applied to various problems in computer science. Again, these objects are required to be highly regular. A hypergraph is called r-uniform if every hyperedge connects exactly r vertices, and such a hypergraph then is d-regular, if every subset of $r-1$ vertices of such a hyperedge is contained in exactly d hyperedges. As before, one may require lower bounds for the eigenvalues of Laplace operators, now also in higher dimensions. A topological version of the expansion property was proposed by Gromov [117]. Here, a simplicial complex or a hypergraph is considered as a topological space. Gromov's condition then requires that for any continuous map from a pure r-dimensional simplicial complex (where *pure* means that every simplex is contained in some r-dimensional one) to $\mathbb{R}^{r-1}$, there exists some point p whose preimage is contained in an ε-fraction of the r-simplices. For some existence results of such families, see [100]. In the graph case, when the Cheeger inequality holds, if we consider a function f from the geometric realization of a graph to $\mathbb{R}$ and take some suitable $z \in \mathbb{R}$, the sets $\{f(v) > z\}$ and $\{f(v) < z\}$ are connected by many edges and so by the intermediate value theorem, the images of many edges have to contain z.

In any case, such higher-dimensional expanders are currently finding applications in general questions around error-correcting codes; for instance, in the quantum version, see, for example, the discussion at the end of [194].

13.4 Nonlinear Dynamics

Nonlinear dynamics is a branch of physics that overlaps with the mathematical theory of dynamical systems. One of its aims is to understand the formation of

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structures and patterns through general mechanisms of self-organization. These mechanisms include the interaction of simple units that follow some nonlinear dynamical rule in networks. The coupling between those units is typically of the diffusive type and therefore can be formalized by a Laplace operator. The spectrum of that operator and in particular its second (i.e., its first nonvanishing) and its largest eigenvalue then control properties of the network diffusion. When we have a normalized Laplace operator, the second eigenvalue should be large in order to facilitate the uniform spreading of activity through the network and to prevent the formation of local clusters that behave differently from the rest. The largest eigenvalue should often be bounded away from 2, in order to prevent period-2 oscillations of the network dynamics.

In order to illustrate these principles, let us consider a concrete setting, that of the synchronization of coupled chaotic oscillators (see [228] and the references therein). We shall follow [151] in linking this to an eigenvalue problem of the Laplacian of a connected graph Γ with vertex set V consisting of N nodes. One starts with a differentiable function $\Psi\colon \mathbb{R} \to \mathbb{R}$, often chosen to be the quadratic (logistic) map in the literature. The idea is that Ψ should generate chaotic dynamics when operating on the state of an individual node. (Here, we take a positive Lyapunov exponent of Ψ [see later] as an indication of chaotic dynamics.)

The nodes then are coupled, and one studies the system

$$u(v,n+1) = \epsilon \frac{1}{\deg v} \sum_{\substack{v' \\ v' \sim v}} \Psi(u(v',n)) + (1-\epsilon)\Psi(u(v,n)) \tag{13.4.1}$$

for $n \in \mathbb{N}$, with some initialization $u(v,0)$. We can also rewrite this as

$$\begin{aligned} u(v,n+1) &= \epsilon \left(\frac{1}{\deg v} \sum_{\substack{v' \\ v' \sim v}} \Psi(u(v',n)) - \Psi(u(v,n)) \right) \\ &\quad + \Psi(u(v,n)) = \epsilon \Delta \Psi(u(v,n)) + \Psi(u(v,n)). \end{aligned} \tag{13.4.2}$$

That is, the dynamics at the individual nodes are diffusively coupled via the (normalized) graph Laplacian Δ. Thus, while in isolation, the dynamics at the nodes might display chaotic behavior; it was a somewhat surprising finding [165] that these individual chaotic dynamics can get synchronized when diffusively coupled. The dynamics is synchronized if

$$u(v,n) = \bar{u}(n) \tag{13.4.3}$$

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for some function $\bar{u}$ that is independent of the individual nodes. This function then needs to solve the uncoupled equation

$$\bar{u}(n+1) = \Psi(\bar{u}(n)). \tag{13.4.4}$$

We then want to check the condition for linear stability of such a synchronized state. We shall use an orthonormal basis of $(u_k)_{k=1,\dots,N}$ of eigenvectors of Δ, and we recall our convention that we order the eigenvalues as $\lambda_1 = 0 < \lambda_2 \leq \dots \leq \lambda_N \leq 2$. We consider a perturbation

$$u(v,n) = \bar{u}(n) + \delta\alpha_k(n)u_k(v), \tag{13.4.5}$$

by an eigenmode u_k for small enough δ and some $k \geq 2$. Here, we only need to consider the case $k \geq 2$ because the eigenfunction u_1 for the eigenvalue $\lambda_1 = 0$ is constant and therefore does not desynchronize a state. The question of linear stability analysis then is whether $\alpha_k(n)$ goes to 0 for $n \to \infty$ if $u(v,n)$ solves (13.4.1). Inserting (13.4.5) into (13.4.2) and expanding about $\delta = 0$ yields

$$\alpha_k(n+1) = (1 - \epsilon\lambda_k)\Psi'(\bar{u}(n))\alpha_k(n), \tag{13.4.6}$$

where Ψ' denotes the derivative of Ψ. A sufficient local stability condition is

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{\alpha_k(n)}{\alpha_k(0)} = \lim_{n\to\infty} \frac{1}{n} \log \prod_{\nu=0}^{n-1} \frac{\alpha_k(\nu+1)}{\alpha_k(\nu)} < 0\,. \tag{13.4.7}$$

Here, this is

$$\log|1 - \epsilon\lambda_k| + \lim_{n\to\infty} \frac{1}{n} \sum_{\nu=0}^{n-1} \log|\Psi'(\bar{u}(\nu))| < 0\,, \tag{13.4.8}$$

with the Lyapunov exponent

$$\mu_0 = \lim_{n\to\infty} \frac{1}{n} \sum_{\nu=0}^{n-1} \log|\Psi'(\bar{u}(n))|$$

of Ψ. Therefore, the stability condition (13.4.8) is

$$|e^{\mu_0}(1 - \epsilon\lambda_k)| < 1. \tag{13.4.9}$$

The important point is that we may have temporal instability of the uncoupled dynamics, as witnessed by a positive Lyapunov exponent, that is,

$$\mu_0 > 0, \tag{13.4.10}$$

but still, (13.4.9) may hold for all $k \geq 2$, that is, the synchronized state may be linearly stable. We thus need to identify the conditions for 13.4.9 to hold for all

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$k \geq 2$. This happens if

$$\frac{1 - e^{-\mu_0}}{\lambda_2} < \epsilon < \frac{1 + e^{-\mu_0}}{\lambda_N}. \tag{13.4.11}$$

In order to satisfy (13.4.11), we need

$$\frac{\lambda_N}{\lambda_2} < \frac{e^{\mu_0} + 1}{e^{\mu_0} - 1}. \tag{13.4.12}$$

While this condition is rather tight (and in fact, synchronization can be observed in numerical simulations for a much wider range of parameters), the important point is that it can still hold in situations where $\mu_0 > 1$, that is, where Ψ generates chaotic dynamics. And it is easier to satisfy the larger λ_2 is, that is, recalling the discussion in Section 4.3, the more difficult it is to cut the network into independent pieces.

While the phenomenon that chaotic dynamics can synchronize was a surprising discovery, a synchronized dynamic is still a somewhat boring situation – everybody does the same. But we have recently found [31, 218] that when we diffusively couple chaotic dynamics on hypergraphs via the hypergraph Laplacian (2.5.39), much richer dynamical patterns can emerge.

Another surprising phenomenon [9] is that transmission delays, that is, replacing (13.4.1) by

$$u(v, n+1) = \epsilon \frac{1}{\deg v} \sum_{\substack{v' \\ v' \sim v}} \Psi(u(v', n - n_0)) + (1 - \epsilon)\Psi(u(v, n)) \tag{13.4.13}$$

for some $n_0 > 0$ can facilitate synchronization. This is surprising because each node only is influenced by past states of its neighbors but still coordinates its dynamic with their current states. It remains to study, for instance, what will happen on hypergraphs when delays are introduced.

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