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On the convergence to equilibrium states for certain non-hyperbolic systems

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Abstract. We study the convergence to equilibrium states for certain non-hyperbolic piecewise invertible systems. The multi-dimensional maps we shall consider do not satisfy Renyi's condition (uniformly bounded distortion for any iterates) and do not necessarily satisfy the Markov property. The failure of both conditions may cause singularities of densities of the invariant measures, even if they are finite, and causes a crucial difficulty in applying the standard technique of the Perron–Frobenius operator. Typical examples of maps we consider admit indifferent periodic orbits and arise in many contexts. For the convergence of iterates of the Perron–Frobenius operator, we study continuity of the invariant density.

1. *Introduction*

In this paper, we study the convergence to equilibrium states of iterates of the Perron–Frobenius operator for multi-dimensional non-invertible maps which are not hyperbolic and have discontinuities. In the category of (piecewise) smooth dynamical systems, non-hyperbolic phenomena may be caused by either critical points of the derivatives or indifferent periodic points. It is known that such maps may still admit invariant measures μ which are absolutely continuous with respect to some reference measure λ under certain conditions ([**2**, **8**, **9**, **18**, **19**, **24**, **25**]) and μ is characterized by a variational principle [**32**]. However, the problem of the convergence to the density of the equilibrium states μ absolutely continuous with respect to λ still remains open. We shall be interested in this problem and will solve it for certain non-hyperbolic systems which are piecewise C^1-invertible maps on subsets of $\mathbf{R}^d$ with indifferent periodic points satisfying the FRS condition (see the definition in §2). Our approach has the following advantage: this method admits singularities of the invariant densities for which the usual approach for hyperbolic systems does not apply. In [**31**], a relation between singularities of the invariant density and indifferent periodic orbits was discussed.

Our results easily extend to more general systems, e.g. *fibred systems* for which λ is not necessarily Lebesgue measure (see §2 and [**20**]). The theory of fibred systems

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first appeared in [**19**] and it was adapted to the analysis of piecewise C^1-invertible maps admitting non-hyperbolic periodic orbits, with a slight generalization in [**18**, **19**] (see [**8**, **30**–**36**]). In addition a theory of fibred systems satisfying the Markov property (see §2) was satisfactorily developed in [**2**] to study the dynamics on Julia sets of parabolic rational maps.

The paper is organized as follows. In §2 we collect together definitions and previous results. In §3 we state our results for piecewise C^1-invertible maps on subsets of $\mathbf{R}^d$ satisfying the FRS condition, i.e. the (piecewise) continuity of the density of μ with respect to λ (Theorem 3.1), and a characterization of $d\mu/d\lambda$ by a limit of the powers of the Perron–Frobenius operator for a certain class of functions (Theorem 3.2). In §4 we apply our results to non-hyperbolic examples admitting indifferent periodic points for which reference measures are the Lebesgue measure. Proofs of the result in §3 are postponed until §5.

2. *Preliminaries*

In this section, we first explain the classes of dynamical systems to which we can apply our results. Let $(X, \mathcal{F}, \lambda)$ be a non-atomic probability space. Suppose that $T : X \to X$ is a non-singular transformation with respect to λ ($\lambda T^{-1} \sim \lambda$) and that there exists a countable measurable partition $Q = \{X_a\}$ of X, generating $\mathcal{F}$ satisfying local invertibility, i.e. for every $X_a \in Q$, $\lambda(X_a) > 0$, and T is non-singular, invertible on X_a. Such a system (T, X, Q) is called a *fibred system* in Schweiger's sense. We denote $X_{a_1} \cap T^{-1}X_{a_2} \cap \cdots \cap T^{-(n-1)}X_{a_n}$ by $X_{a_1 \ldots a_n}$ if the set has positive λ measure and call $X_{a_1 \ldots a_n}$ a *cylinder* of rank n. If $TX_b = \bigcup_{X_a \in Q, \lambda(X_a \cap TX_b) > 0} X_a \pmod{\lambda}$ $\forall X_b \in Q$, we say that T satisfies the *Markov property* with respect to Q. Even if the system (T, X, Q) does not satisfy this property, the next condition leads us to a useful countable state symbolic dynamics, similar to sofic systems [**32**].

Finite range structure (FRS). Put $\mathcal{U} = \{T^n X_{a_1 \ldots a_n} : \forall X_{a_1 \ldots a_n}, \forall n > 0\}$. Then $\mathcal{U}$ consists of only finitely many subsets of X with positive λ measure.

Typical examples satisfying the FRS condition are number theoretical transformations. In most cases, those transformations satisfy the following:

1. T is defined on a bounded subset X of an Euclidian space $\mathbf{R}^d$.
2. $Q = \{X_a\}_{a \in I}$ is a generating countable partition of X, consisting of measurable connected subsets with piecewise smooth boundaries.
3. For each X_a, $T|_{X_a} : X_a \to TX_a$ is a C^1-diffeomorphism, so that the reference measure λ is the Lebesgue measure.

In this category, if the FRS condition is satisfied, we call $(T, X, Q, \mathcal{U})$ a *piecewise invertible system with finite range structure*. Let (T, X, Q) be a fibred system. Then for each cylinder $X_{a_1 \ldots a_n}$ we can define

$$\Delta(a_1 \ldots a_n) = \left. \frac{d\lambda T^n}{d\lambda} \right|_{X_{a_1 \ldots a_n}} \qquad (>0 \text{ a.e.})$$

and we denote

$$C(a_1 \ldots a_n) = \frac{\operatorname{ess\,sup}_{x \in X_{a_1 \ldots a_n}} \Delta(a_1 \ldots a_n)(x)}{\operatorname{ess\,inf}_{x \in X_{a_1 \ldots a_n}} \Delta(a_1 \ldots a_n)(x)}.$$

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For a given $C \geq 1$ define $\mathcal{R}(C.T)$ to be the set of cylinders satisfying $C(a_1 \dots a_n) < C$. Measure theoretical results in the general theory of multi-dimensional dynamical systems (hyperbolic systems like Axiom-A diffeomorphisms, expanding endomorphisms of compact manifolds, piecewise expanding maps of compact metric spaces) rely heavily on, not only the existence of a generating Markov partition, but also the following condition whether it is invertible or non-invertible, which implies a uniform bound for distortion under iteration of T ([**3**–**5**, **11**, **12**, **14**, **22**, **27**, **28**, **29**]).

Renyi's condition. There exists a constant $C \geq 1$ such that all cylinders belong to $\mathcal{R}(C.T)$.

Even if both these conditions (the Markov condition and Renyi's condition) fail, we can still establish measure theoretical results which are true in the general category of metric dynamical systems. For example, the following theorem which was proved in [**8**, **30**] is the fundamental result for developing ergodic theory of non-hyperbolic systems in the category of piecewise C^1-invertible maps, where the reference measure is the Lebesgue one.

THEOREM 2.1. (The existence of an ergodic invariant measure equivalent to Lebesgue measure.) *Let $(T, X, Q, \mathcal{U})$ be a piecewise invertible system with FRS. Assume that there exists a constant $C > 1$ such that $\mathcal{R}(C.T) \neq \emptyset$ and satisfying the following conditions:*

(C1) $X_{b_1 \dots b_k a_1 \dots a_n} \in \mathcal{R}(C.T)$ *if* $X_{a_1 \dots a_n} \in \mathcal{R}(C.T)$. *(The local Renyi condition.)*

(C2) *Each* $U \in \mathcal{U}$ *contains a cylinder* $X_{a_1 \dots a_s}$ *satisfying* $X_{a_s} \in \mathcal{R}(C.T)$ *and* $T^s X_{a_1 \dots a_s} = X$. *(The transitivity condition.)*

(C3) *Define* $\mathcal{D}_n = \{X_{a_1 \dots a_n} : X_{a_1 \dots a_i} \notin \mathcal{R}(C.T)$ *for* $1 \leq i \leq n\}$ *and put* $D_n = \bigcup_{X_{a_1 \dots a_n} \in \mathcal{D}_n} X_{a_1 \dots a_n}$. *(For* $n = 0$ *we put* $D_0 = X$.*) Then* $\sum_{n=0}^{\infty} \lambda(D_n) < \infty$, *where* λ *is the normalized Lebesgue measure of* X.

Under the above conditions, $(T, X, Q, \mathcal{U})$ admits a finite ergodic invariant measure μ equivalent to λ, and with respect to μ, T is exact.

Remark A. If we replace condition (C3) by the weaker condition

(C3*) $\lim_{n \to \infty} \lambda(D_n) = 0$,

then we still have an exact invariant measure equivalent to λ. This measure need not be finite (although it will be σ-finite) (see [**8**] and [**32**]).

Further statistical results are contained in [**30**–**36**]. In particular, the invariant measure μ is characterized by some variational principle. The paper [**32**] contains a result from which the next theorem follows immediately.

THEOREM 2.2. (Variational principle.) *The T-invariant measure μ in Theorem 2.1 gives an equilibrium state $\mu_0 \equiv \mu/\mu(X)$ for $-\log|\det DT(x)| \in L^1(\mu)$ in the sense of Walters* [**27**] *when $H_\mu(Q) \equiv -\sum_{a \in I} \mu(X_a) \log \mu(X_a) < \infty$. The Rohlin formula $h_{\mu_0}(T) = \int_X \log|\det DT(x)| \, d\mu_0(x)$ is valid and we have the following variational principle, $\forall m$ a T-invariant ergodic probability measure defined on the Borel σ-field $\mathcal{F}$:*

$$h_{\mu_0}(T) - \int_X \log|\det DT(x)| \, d\mu_0(x) \geq \int_X \{I_m(\mathcal{F}|T^{-1}\mathcal{F}) - \log|\det DT(x)|\} \, dm(x),$$

where $I_m(\mathcal{F}|T^{-1}\mathcal{F})$ denotes the conditional information of $\mathcal{F}$ with respect to $T^{-1}\mathcal{F}$.

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Remark B. In [**31**], we have already obtained sufficient conditions for the validity of $\log|\det DT| \in L^1(\mu)$, $H_\mu(Q) < \infty$, and so of the Rohlin formula.

Theorem 2.1 still remains true for fibred systems (T, X, Q) defined on some general non-atomic probability space $(X, \mathcal{F}, \lambda)$ satisfying the FRS condition. Indeed, in order to prove Theorem 2.1, we applied Schweiger's method for Bernoulli fibred systems in the same way as it was done in [**18**, **19**], with a slight generalization to the sofic (FRS) fibred systems. Fibred systems satisfying the Markov condition have also been studied in [**2**] with an application to parabolic rational maps on the Riemann sphere, which admit a non-atomic h-conformal measure on their Julia set where h is the Hausdorff dimension. Assuming Schweiger's property (which corresponds to the local Renyi condition) and the FRS condition on Markov fibred systems, the same results as Theorem 2.1 were established in [**2**]. To extend Theorem 2.2 to fibred systems, we need to assume X is a metric space (see §3).

In the rest of this section, we recall some definitions and previous facts, which will be used later. Let $(X, \mathcal{F}, \lambda)$ be the normalized Lebesgue space and let $(T, X, Q, \mathcal{U})$ be a piecewise invertible system with FRS satisfying Theorem 2.1. $\mathcal{L}^{(n)}$ denotes the family of all cylinders of rank n with respect to T and $\mathcal{L} = \bigcup_{n=1}^{\infty} \mathcal{L}^{(n)}$. For $n \geq 1$, put

$$\mathcal{B}_n = \{X_{a_1 \dots a_n} \in \mathcal{L}^{(n)} : X_{a_1 \dots a_{n-1}} \in \mathcal{D}_{n-1}, X_{a_1 \dots a_n} \in \mathcal{R}(C.T)\}$$

and denote $B_n = \bigcup_{X_{a_1 \dots a_n} \in \mathcal{B}_n} X_{a_1 \dots a_n}$. It follows from (C3*) that $\bigcup_{n=1}^{\infty} B_n = X \,(\lambda \bmod 0)$ (see [**18**–**20**]). Define a map $T_R : \bigcup_{n=1}^{\infty} B_n \to X$ by $T_R x = T^j x$ for $x \in B_j$. We call the T_R a *jump transformation* over $\mathcal{R}(C.T)$. By restricting T_R on $X_R = X \setminus \bigcup_{m=0}^{\infty} T_R{}^{-m}(\bigcap_{n \geq 0} D_n)$, we have a transformation of X_R, and as $X_R = X \,(\lambda \bmod 0)$ we use the same notation T_R for this restriction on X_R. Define $I_R = \bigcup_{n=1}^{\infty} \{(a_1 \dots a_n) \in I^n : X_{a_1 \dots a_n} \in \mathcal{B}_n\}$ and $Q^R = \{X_\alpha\}_{\alpha \in I_R}$. $\mathcal{L}_R^{(n)}$ and $\mathcal{L}_R$ are also defined as above, with respect to T_R. Put $\mathcal{U}^R = \{T_R^n X_{\alpha_1 \dots \alpha_n} : n > 0, X_{\alpha_1 \dots \alpha_n} \in \mathcal{L}_R^{(n)}\}$. Then $(T_R, X_R, Q^R = \{X_\alpha\}_{\alpha \in I_R}, \mathcal{U}^R)$ is a piecewise invertible system with FRS satisfying Renyi's condition for $C \geq 1$, so T_R admits a finite ergodic invariant measure ν equivalent to λ with density bounded away from zero and infinity ($G^{-1} \leq d\nu/d\lambda \leq G$). The σ-finite T-invariant measure μ is given by using this T_R-invariant measure ν as follows (cf. [**8**]):

$$\mu(E) = \sum_{n=0}^{\infty} \nu(T^{-n} E \cap D_n) \quad (\forall E \in \mathcal{F}). \tag{2.1}$$

From the above formula, we can also find the formula of the invariant density of μ:

$$\frac{d\mu}{d\lambda}(x) = \sum_{n=0}^{\infty} \sum_{X_{d(n)} \in \mathcal{D}_n} |\det D\psi_{d(n)}(x)| \frac{d\nu}{d\lambda}(\psi_{d(n)}(x)) I_{T^n X_{d(n)}}(x) \tag{2.2}$$

for a.e. $x \in X$ (where $d(n)$ stands for $(d_1 \dots d_n)$ and $\psi_{d_1 \dots d_n}$ denotes the inverse map of $T^n|_{X_{d_1 \dots d_n}}$, $(T^n|_{X_{d_1 \dots d_n}})^{-1} : T^n X_{d_1 \dots d_n} \to X_{d_1 \dots d_n}$).

Remark C. As we have already mentioned, the above definitions and the facts are valid for fibred systems. We can easily see that no difficulty appears in the arguments used in [**8**] and [**30**–**32**] even if we replace the Jacobians

$$|\det D(T^n|_{X_{a_1 \dots a_n}})|, \quad |\det D\psi_{d_1 \dots d_n}|$$

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by $\Delta(a_1 \dots a_n)$, and the measurable function $\omega(d_1 \dots d_n) : X \to \mathbf{R}$ satisfying $\lambda(X_{d_1 \dots d_n} \cap T^{-n}E) = \int_E \omega(d_1 \dots d_n)(x)\, d\lambda(x)$ respectively.

3. *Main results*

In this section, we suppose that all the assumptions of Theorem 2.1 are satisfied. For establishing the convergence theorem, we shall first prove the (piecewise) continuity of the density of μ with respect to λ assuming the following two conditions.

(E1) There is a sequence $\{E_k\}_{k>0}$ with $0 < E_k < \infty$ such that

$$|1 - |\det D\psi_{a(n)}(x)/\det D\psi_{a(n)}(y)|| \le E_k \|x - y\|,$$

$\forall x, y \in D_k^c \cap T^n X_{a(n)}, \forall X_{a(n)} \in \mathcal{L}^{(n)}$ and $\forall n > 0$.

(E2) There is a sequence $\{F_k\}_{k>0}$ with $0 < F_k < \infty$ such that

$$\|\psi_{a(n)}(x) - \psi_{a(n)}(y)\| \le F_k \|x - y\|,$$

$\forall x, y \in D_k^c \cap T^n X_{a(n)}, \forall X_{a(n)} \in \mathcal{L}^{(n)}$ and $\forall n > 0$.

For a function $f : X(\subset \mathbf{R}^d) \to \mathbf{R}$, $(\partial f/\partial x)^j (j \in \{1, 2, \dots, d\})$ denotes the jth component of the Jacobian matrix of f. If T is piecewise twice differentiable, the next condition is sufficient for the validity of (E1).

(E0) There is a sequence $\{E_k\}_{k>0}$ with $0 < E_k < \infty$ such that

$$\frac{\sup_{\theta \in D_k^c} \max_{1 \le j \le d} |\{\partial/\partial x(\det D\psi_{a(n)}(\theta))\}^j|}{|\det D\psi_{a(n)}(y)|} \le E_k,$$

$\forall y \in D_k^c \cap T^n X_{a(n)}$ and $\forall X_{a(n)} \in \mathcal{L}^{(n)}$ for all $n > 0$.

For a sequence $1 \le j_1 < \cdots < j_r \le d$, let $p_{j_1 \dots j_r} : \mathbf{R}^d \to \mathbf{R}^r$ be a projection defined by $p_{j_1 \dots j_r}(x^1, \dots, x^d) = (x^{j_1}, \dots, x^{j_r})$.

Remark D. We will see that for all examples in §4 the sequences $\{E_k\}$ and $\{F_k\}$ can be chosen as follows: $\exists E, F$ with $0 < E, F < \infty$ and $\exists l > 0, l' > 0$ such that

$$E_k = E\{\mathrm{diam}(p_{j_1 \dots j_l}(D_k))\}^{-l}, \quad F_k = F\{\mathrm{diam}(p_{j_1 \dots j_l}(D_k))\}^{-l'},$$

where $j_i (1 \le i \le l)$ belongs to $\{1, 2, \dots, d\} \backslash \{j \in \{1, 2, \dots, d\} : \{\partial/\partial x \det DT(x)\}^j \equiv 0\}$. So E_k, F_k diverge as k goes to infinity.

Let $\mathcal{V}$ be the disjoint partition generated by $\mathcal{U}$ and let $\mathcal{V}_R \equiv \mathcal{V} \cap X_R$.

THEOREM 3.1. (Continuity of the invariant density.) *Suppose that all assumptions of Theorem 2.1 and* (E1), (E2) *are satisfied. If $d\nu/d\lambda$ is Lipshitz continuous on each atom of the partition $\mathcal{V}_R$, then $d\mu/d\lambda$ is piecewise continuous on each atom of the partition: $\mathcal{V}_R \cap \cup_{i=1}^k B_i$, $\forall k > 0$.*

Under Renyi's condition, the following three conditions guarantee Lipshitz continuity of the invariant density (see [**12**], [**18**] and [**21**]).

(WSN1) $\exists E_0$ with $0 < E_0 < \infty$ such that

$$\|\psi^R_{\alpha_1 \dots \alpha_n}(x) - \psi^R_{\alpha_1 \dots \alpha_n}(y)\| \le E_0 \|x - y\|$$

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$\forall x, y \in T_R^n(X_{\alpha_1\dots\alpha_n}), \forall X_{\alpha_1\dots\alpha_n} \in \mathcal{L}_R^{(n)}$, and $\forall n > 0$. (Here $\psi^R_{\alpha_1\dots\alpha_n} = (T_R^n|_{X_{\alpha_1\dots\alpha_n}})^{-1}$.)

(WSN2) $\exists F_0$ with $0 < F_0 < \infty$ such that

$$|| \det D\psi^R_{\alpha_1\dots\alpha_n}(x)| - |\det D\psi^R_{\alpha_1\dots\alpha_n}(y)|| \le F_0\lambda(X_{\alpha_1\dots\alpha_n})\|x-y\|$$

$\forall x, y \in V_R (V_R \in \mathcal{V}_R)$, $\forall X_{\alpha_1\dots\alpha_n} \in \mathcal{L}_R^{(n)}$ and $\forall n > 0$.

(WSN3) $\sup_{X_{\alpha_1\dots\alpha_n}\in\mathcal{L}_R^{(n)}} \operatorname{diam} X_{\alpha_1\dots\alpha_n} \to 0$ as $n \to \infty$.

As we have already mentioned in §2, the jump transformation T_R satisfies Renyi's condition. When T_R is piecewise twice differentiable we have the next condition for the validity of (WSN1).

(WSN0) $\exists E_0$ with $0 < E_0 < \infty$ such that $|\partial(\psi^R_{\alpha_1\dots\alpha_n})^j/\partial x_i| \le E_0$ uniformly in $X_{\alpha_1\dots\alpha_n} \in \mathcal{L}_R^{(n)}$, $i, j \in \{1, 2, \dots d\}$ and $\forall n > 0$.

Remark E. We can easily check the various Lipshitz conditions, (E0, 1, 2), (WSN0, 1, 2) for many number theoretical transformations (cf. [**12**], [**21**], [**28**], [**35**], [**36**]). In fact, we will see in §4 that (E0, 1, 2) for $(T, X, Q, \mathcal{U})$ and (WSN0, 1, 2) for $(T_R, X_R, Q^R, \mathcal{U}^R)$ are verified for all examples. These conditions can be weakened by replacing $\|x-y\|$ by $\|x-y\|^\epsilon$ for a fixed $\epsilon > 0$.

We shall use the next result when proving Theorem 3.1.

PROPOSITION 3.1. *Suppose that all assumptions of Theorem 2.1 are satisfied. Let* $\Gamma(\mathcal{D}) = \cup_{U\in\mathcal{U}(\mathcal{D})}\partial U \setminus \cup_{a\in I}\partial X_a$, *where* $\mathcal{U}(\mathcal{D}) = \{T^n X_{d(n)} : X_{d(n)} \in \mathcal{D}_n, n > 0\}$, *and let* $\mathcal{V}(\mathcal{D})$ *denote the disjoint partition generated by* $\mathcal{U}(\mathcal{D})$. *If* $x \in X_{\mathcal{R}} \cap \Gamma(\mathcal{D})^c$, *then there exists a positive finite number* $K(x)$ *such that*

$$\Psi(x) \equiv \sum_{n=0}^{\infty}\sum_{X_{d(n)}\in\mathcal{D}_n} |\det D\psi_{d(n)}(x)| 1_{T^n X_{d(n)}}(x) \le K(x) < \infty.$$

The proof of the convergence theorem will come via Theorem 3.1 and three further lemmas. Define a generalized Perron–Frobenius–Ruelle–Kuzmin operator for a function ϕ, $\mathcal{L}_\phi$, by

$$\mathcal{L}_\phi f(x) = \sum_{y\in T^{-1}x} f(y)e^{\phi(y)} = \sum_{a\in I} f(\psi_a x)e^{\phi(\psi_a x)}1_{TX_a}(x) \quad (\forall f \in L^1(\lambda), \forall x \in X).$$

(Cf. [**15**], [**27**].) Inductively, we can define $\mathcal{L}_\phi^n$ by

$$\mathcal{L}_\phi^n f(x) = \sum_{y\in T^{-n}x} f(y)e^{\phi(y)} = \sum_{a(n)\in I^n} f(\psi_{a(n)}x)e^{\phi(\psi_{a(n)}x)}1_{T^n X_{a(n)}}(x).$$

As mentioned in §2, we have already obtained the invariant density which is a fixed point of $\mathcal{L}_\phi$ with $\phi(x) = -\log|\det DT(x)|$ and $\mu/\mu(X)$ is the unique equilibrium state for the ϕ when $-\phi \in L^1(\mu)$ [**32**]. We denote $h(x) \equiv (d\mu/d\lambda)(x)$, and define $g(x) \equiv (h(x)/hT(x))|\det DT(x)|^{-1}$. The operator $\mathcal{L}_{\log g}$ is given by

$$\begin{aligned}\mathcal{L}_{\log g} f(x) &= \sum_{y\in T^{-1}x} \frac{h(y)}{h(x)} f(y)|\det DT(y)|^{-1} \\ &= \sum_{a\in I} \frac{h(\psi_a x)}{h(x)} f(\psi_a x)|\det D\psi_a(x)| 1_{TX_a}(x) \quad (\forall f \in L^1(\lambda)).\end{aligned}$$

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We denote $-\log|\det DT(x)|$ by $\phi(x)$. We can easily see the following.

LEMMA 3.1.
(1) *$\mathcal{L}_\phi$, $\mathcal{L}_{\log g}$ are positive linear operators of $L^1(\lambda)$, $L^1(\mu)$ respectively.*
(2) *g is defined on X_R.*
(3) $\sum_{y\in T^{-1}x} g(y) = 1$.
(4) $\mathcal{L}^*_{\log g}\mu = \mu$, *where $\mathcal{L}^*_{\log g}$ is the dual of $\mathcal{L}_{\log g}$, i.e. $\mathcal{L}^*_{\log g}\mu(f) = \mu(\mathcal{L}_{\log g} f)$ (here $\mu(f)$ denotes $\int_X f\, d\mu$).*

The following properties of the invariant density h are readily verified for various number theoretical transformations admitting indifferent periodic points. These conditions are important in establishing the convergence theorem.
(E3a) There exists a sequence $\{\Delta(k)\}_{k>0}$ such that $\sup_k \Delta(k) < \infty$ and

$$\sup_{x,x'\in B_k} |1 - h(x)/h(x')| \le \Delta(k).$$

(E3b) There exist $0 < H_*, H^* < \infty$ such that

$$H_* \le \frac{\inf_{x\in B_{k+1}} h(x)}{\sup_{x'\in B_k} h(x')} \le H^* \quad (\forall k > 0).$$

(E4) For each $l > 0$,

$$w_l = \sup_{X_{b(l)}\in\mathcal{B}_l} \sum_{n=0}^{\infty} \sum_{X_{d(n)}\in\mathcal{D}_n} \inf_{x\in T^n X_{d(n)}\cap X_{b(l)}} |\det D\psi_{d(n)}(x)| < \infty.$$

Remark F. For each $m > 0$, put $W_m = \max\{w_l : 1 \le l \le m\}$. Assuming Theorem 2.1 with (C3*) the following inequality was established in [**32**]:

$$h(x) \equiv \frac{d\mu}{d\lambda}(x) \le C^3 G W_m < \infty \quad (\forall x \in \cup_{i=1}^m B_i).$$

Further, it follows from Theorem 3.2 in [**31**] that

$$h(x) \ge (C^2G)^{-1}\frac{\min_{U\in\mathcal{U}}\lambda(U)}{\max_{U\in\mathcal{U}}\lambda(U)}\Psi(x) = (C^2G)^{-1}\min_{U\in\mathcal{U}}\lambda(U)\Psi(x).$$

Condition (E4) and the fact in Remark F immediately bring us to the following result.

LEMMA 3.2. *Suppose that all assumptions of Theorem 2.1 and* (E4) *are satisfied. Then there exists a sequence $\{\Psi^*(k)\}_{k>0}$ with $0 < \Psi^*(k) < \infty$ such that $\sup_{x\in D_k^c}\Psi(x) \le \Psi^*(k) < \infty$.*

For all examples in §4, we can observe the property $\sup_k \Psi^*(k) \to \infty$ as $k \to \infty$.
Let $V \in \mathcal{V}$ and define for $x, x' \in V$,

$$C_g(x,x') = \sup_{n\ge 1} \sup_{\substack{y\in T^{-n}x,\\ y'\in T^{-n}x'}} \prod_{i=0}^{n-1} \frac{g(T^i y)}{g(T^i y')} (\ge 1).$$

LEMMA 3.3. *Suppose that all assumptions of Theorem 3.1 are satisfied. Assume further that* (E3a), (E3b) *and* (E4) *hold. Then there is a sequence $\{C_g(k)\}_{k>0}$ with $C_g(k) \ge 1$ such that $C_g(x,x') \le C_g(k)$, $\forall x, x' \in D_k^c \cap \operatorname{int} V (V \in \mathcal{V})$, and $C_g(x,x') \to 1$ as $\|x - x'\| \to 0$.*

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Remark G. The above property corresponds to condition (iii) in [**27**]. Let $x, x' \in V \in \mathcal{V}$ and $y \in T^{-n}x$, $y' \in T^{-n}x'$. As

$$\prod_{i=0}^{n-1} \frac{g(T^i y)}{g(T^i y')} = \frac{h(y)}{h(y')} \frac{h(T^n y')}{h(T^n y)} \left| \frac{\det DT^n(y')}{\det DT^n(y)} \right|,$$

we can write $C_g(x, x')$ in the following form:

$$C_g(x, x') = \sup_{n \geq 1} \sup_{X_{a(n)} \in \mathcal{L}: T^n X_{a_1 \dots a_n} \supseteq V} \frac{h(\psi_{a(n)}x)}{h(\psi_{a(n)}x')} \frac{h(x')}{h(x)} \frac{|\det D\psi_{a(n)}(x)|}{|\det D\psi_{a(n)}(x')|}.$$

If Renyi's condition is valid, the invariant density $h \equiv d\mu/d\lambda$ is bounded away from zero and infinity so that Lemma 3.3 is true without any further assumptions on $h(x)$ like (E3) and (E4).

Combining Theorem 3.1 (Proposition 3.1) and Lemmas 3.1–3.3, we now come to the convergence theorem.

THEOREM 3.2. *Suppose that all assumptions of Theorem 2.1 are satisfied. Assume further that conditions* (E1, 2, 4), (E3a), (E3b), *and* (WSN1–3) *hold. Let* $h(x)^{-1} \in C(\overline{X})$, *where* $\overline{X}$ *is the closure of* X *and* $C(\overline{X})$ *denotes the set of continuous functions of* $\overline{X}$. *Then for* $f \in C(\overline{X})$

$$\mathcal{L}_\phi^n f \Longrightarrow \frac{h}{\mu(X)} \left(\int_X f \, d\lambda \right) \quad (n \to \infty)$$

uniformly on compact subsets of $\bigcup_{i=1}^{\infty} B_i \cap (\partial\mathcal{V})^c$.

In particular, for $f \equiv 1$ we have the following.

COROLLARY 3.1.

$$\mathcal{L}_\phi^n 1 \Longrightarrow \frac{h}{\mu(X)} \quad (n \to \infty)$$

uniformly on compact subsets of $\bigcup_{i=1}^{\infty} B_i \cap (\partial\mathcal{V})^c$.

Let $C(Q), C(\mathcal{V})$ denote the set of all piecewise uniformly continuous functions on X with respect to $Q, \mathcal{V}$ respectively.

COROLLARY 3.2. *We can relax the condition* $h^{-1} \in C(\overline{X})$ *in Theorem 3.2 as* $h^{-1} \in C(Q)$. *In particular, if* $\mathcal{V} \subseteq Q$ *(i.e.* Q *is the Markov partition), the condition* $h^{-1} \in C(\mathcal{V})$ *implies that* $h^{-1} \in C(Q)$.

It follows from Theorem 2.1 that h^{-1} is bounded from above and is integrable. However, in general, we do not know any further properties of h^{-1}. For all examples in §4, we know asymptotic formulas of h so that we can explicitly observe that h^{-1} belongs either to $C(\overline{X})$ or $C(\mathcal{V})$.

Remark H. $\mathcal{P}(\phi) \equiv 0$, where $\mathcal{P}(\phi) = \log \lambda(\mathcal{L}_\phi 1)$.

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In fact, it is easy to see that

$$\begin{aligned}\exp \mathcal{P}(\phi) &= \lambda(\mathcal{L}_\phi 1) = \int_X \sum_{a\in I} |\det D\psi_a(x)| 1_{TX_a}\, d\lambda(x)\\ &= \sum_{a\in I} \lambda(X_a) = \lambda(X) = 1.\end{aligned}$$

In Remark H we used the definition of pressure for a non-singular transformation (T, λ) given in [**27**] (see Lemma 2.1). On the other hand, Rohlin's formula implies that the left-hand side of the inequality in Theorem 2.2 (which is the usual definition of the pressure) is zero. Further properties on pressure can be found in [**33**].

Our main results are established in the case when μ is finite. Theorem 3.1 remains true in the case when μ is infinite, without any change of proof. On the other hand, the proof of Theorem 3.2 relies heavily on the finiteness of μ. Now we ask the following question.

Question. When $\mu(X) = \infty$, can we find an appropriate sequence $\{a_n\}_{n>0}$ such that

$$a_n \mathcal{L}_\phi^n f \Longrightarrow h\left(\int_X f\, d\lambda\right) \quad (n \to \infty)$$

uniformly on compact subsets of $\bigcup_{i=1}^{\infty} B_i \cap (\partial\mathcal{V})^c$?

We mention a result by Thaler [**26**], related to this problem for one-dimensional intermittent Bernoulli maps preserving absolutely continuous infinite invariant measures.

THEOREM 3.3. (One-dimensional case.) *Assume that* $T : [0, 1] \to [0, 1]$ *satisfies* (1)–(4) *in* [**26**]. *Then there exists* $\{a_n\}_{n>0}$ *such that for all Riemann integrable functions* f *on* $[0, 1]$,

$$1/a_n \sum_{k=0}^{n-1} \mathcal{L}_\phi^n f \Longrightarrow h\left(\int_X f\, d\lambda\right) \quad (n \to \infty)$$

uniformly on compact subsets of $\bigcup_{i=1}^{\infty} B_i$.

We now discuss some generalizations of our results. As we have mentioned in §1 and 2, Theorem 2.1 can be extended to a fibred system (T, X, Q) satisfying the FRS condition, by defining $R(C.T)$ in terms of $d(\lambda T)/d\lambda$. (Here $(X, \mathcal{F}, \lambda)$ is a non-atomic probability space.) Assume further that X is a bounded metric space, $\mathcal{F}$ is the σ-algebra of Borel sets of X and $\exists\overline{X}$ a compact metric space such that $\overline{X} \supseteq X$ and $X_0 \equiv \bigcup_{a\in I} \text{int}\, X_a$ is an open dense subset of $\overline{X}$. Now we suppose that for each $X_a \in Q$, $T|_{X_a} : X_a \to TX_a$ is a homeomorphism so that $T : X_0 \to X$ is continuous, then Theorem 2.2 still remains true. To establish Theorems 3.1 and 3.2 under the above conditions we shall change slightly the assumptions (E1), (WSN2) and (E4) on the Jacobians, i.e. as mentioned in Remark C we need to replace these conditions by new assumptions (E1)*, (WSN2)* and (E4)* on the measurable functions $\omega(a_1 \ldots a_n)$ with respect to T and $\omega_R(\alpha_1 \ldots \alpha_n)$ with respect to T_R. Then the result of Theorem 3.2 remains true under these new conditions and (E2), (WSN1, 3) and (E3).

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4. *Examples and applications*

In this section, we shall show some examples to which we can apply the results of §3.

Example 1. (A one-parameter family of maps on the interval $[0, 1]$*.)* Let $X = [0, 1]$ and for α with $0 < \alpha < 1$ define

$$f_\alpha(x) = \begin{cases} \dfrac{x}{(1-x^\alpha)^{1/\alpha}} & \text{on } X_0 = [0, (1/2)^{1/\alpha}) \\ \dfrac{x}{(1/2)^{1/\alpha} - 1} + \dfrac{1}{1-(1/2)^{1/\alpha}} & \text{on } X_1 = [(1/2)^{1/\alpha}, 1]. \end{cases}$$

The map f_α has an indifferent fixed point 0. Since $f_\alpha''(0) = \infty$, the map does not belong to the class of one-dimensional maps with infinite invariant measures which Thaler studied (see [**24**] and [**25**]). In [**30**], Theorem 2.1 was applied to f_α so that we have a finite ergodic invariant measure μ equivalent to λ which is exact. We can take $2^{1+1/\alpha}$ for f_α as the local Renyi constant C, so that we see the validity of (C3) as follows. Let $X_{\underbrace{0\ldots0}_{n}} = [0, d_n)$ and $X_{\underbrace{0\ldots01}_{n}} = [d_n, d_{n-1})$. Then, for $C = 2^{1+1/\alpha}$ we have $\mathcal{D}_n = \{X_{\underbrace{0\ldots0}_{n}}\}$ and $\mathcal{B}_n = \{X_{\underbrace{0\ldots01}_{n}}\}$ and so

$$\lambda(D_n) = \lambda(X_{\underbrace{0\ldots0}_{n}}) = 1/(n+1)^{1/\alpha} \quad \text{(Figure 1).}$$

(See [**30**] and [**31**]). By Theorem 3.1 in [**31**] we know that the indifferent fixed point 0

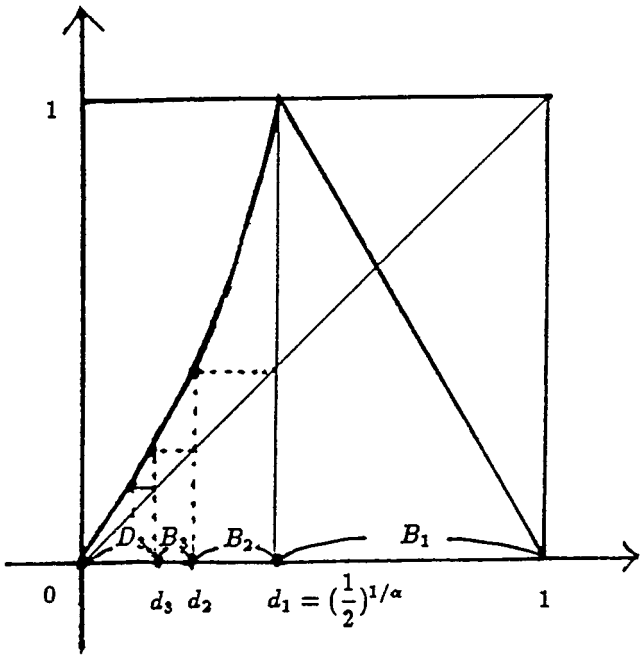


FIGURE 1. $D_n = [0, d_n)$, $B_n = [d_n, d_{n-1})$, $d_n = (1/(n+1))^{1/\alpha}$.

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is a singular point of the invariant density $h = d\mu/d\lambda$ and that

$$h(x) \sim \text{constant} \times x^{-\alpha} \quad (x \to 0)$$

i.e. the density should be of order $x^{-\alpha}$ near the fixed point (cf. [**24**]). To see this, it suffices to observe the fact:

$$\text{constant} \times h(x) \sim \sum_{n=0}^{\infty} |\psi'_{\underbrace{0\ldots0}_{n}}(x)| = \sum_{n=0}^{\infty} \frac{1}{(1+nx^{\alpha})^{1+1/\alpha}}$$

(Theorem 3.2 in [**31**]) and the following fact for $\alpha = 1$ which was proved in [**24**]:

$$\sum_{n=0}^{\infty} |\psi'_{\underbrace{0\ldots0}_{n}}(x)| = \sum_{n=0}^{\infty} \frac{1}{(1+nx)^2} \sim \text{constant} \times x^{-1}.$$

We shall check all conditions of Theorem 3.2. Observe that $\psi_{\underbrace{0\ldots0}_{l}}(x) = x/(1+lx^{\alpha})^{1/\alpha}$ then it is easy to see that $\forall x, \theta \in B_k$,

$$\frac{|\psi''_{0\ldots0}(\theta)|}{|\psi'_{\underbrace{0\ldots0}_{l}}(x')|} \le E_k,$$

where $E_k = E_0(\text{diam}\, D_k)^{-1} = E_0(k+1)^{1/\alpha}$ and $E_0 = 2(\alpha+1)$. Thus

$$\left|1 - \frac{|\psi'_{0\ldots0}(x)|}{|\psi'_{\underbrace{0\ldots0}_{l}}(x')|}\right| \le E_k|x-x'|$$

for $x, x' \in B_k$. Furthermore, there exists a constant $C'(\alpha)$ depending on α with $0 < C'(\alpha) < \infty$ such that

$$\left|1 - \left|\frac{\psi'_{a_1\ldots a_n 1}(x)}{\psi'_{a_1\ldots a_n 1}(x')}\right|\right| \le C'(\alpha)|x-x'| \quad (\forall X_{a_1\ldots a_n} \in \mathcal{L}).$$

To see this, the following observations ([**30**] and [**31**]) are important.

(1) ψ'_0 is a positive and monotone increasing function on $[0,1]$.

(2) ψ''_0/ψ'_0 is a negative and monotone increasing function on $[0,1]$.

(3) $C(\underbrace{0\ldots01}_{n}) \le \psi'_0(d_n)/\psi'_0(1) \le C(\alpha)$, where $C(\alpha) = 2^{1+1/\alpha}$, the local Renyi constant.

(4) $\exists n_0 \equiv n_0(\alpha) < \infty$ s.t. for $n \ge n_0(\alpha)$, $|\psi'_1| < \inf_{x \in X_{\underbrace{0\ldots01}_{n}}} |\psi'_0(x)|$.

(5) $X_{a_1\ldots a_n} \in \mathcal{L}$ with $a_k = 1$ $(\exists k \in \{1,2,\ldots,n-1\})$ satisfies

$$d_{n-1}\left(= \sup_{x \in X_{\underbrace{0\ldots01}_{n}}} \{x\}\right) \le \inf_{x \in X_{a_1\ldots a_n}} \{x\}.$$

(6)

$$\left|\log \frac{|\psi'_{a_1\ldots a_n}(x)|}{|\psi'_{a_1\ldots a_n}(x')|}\right| \le \sum_{j=1}^{n-1} \inf_{X_{\underbrace{0\ldots01}_{n-j}}} \frac{|\psi''_0|}{\psi'_0}\left(\prod_{h=1}^{n-j-1} \inf_{\xi \in X_{\underbrace{0\ldots01}_{n-j-h}}} |\psi'_0(\xi)|\right)(|\psi'_1|/\psi'_0(1))^{n_0(\alpha)}|x-x'|.$$

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Using the above facts, we have $C'(\alpha) = (|\psi_1'|/\psi_0'(1))^{n_0(\alpha)}|\psi_1'|^{-1}\log C(\alpha)$. Finally, we come to the following inequalities which show the validity of (E1): for $(a_1 \dots a_n) = (a_1 \dots a_{n-l}\underbrace{10\dots0}_{l})$ and for $x, x' \in D_k^c$,

$$\begin{aligned}\frac{|\psi'_{a_1\dots a_n}(x')|}{|\psi'_{a_1\dots a_n}(x)|} &= \frac{|\psi'_{a_1\dots a_{n-l}1}(\psi_{0\dots0}x')|}{|\psi'_{a_1\dots a_{n-l}1}(\psi_{0\dots0}x)|}\frac{|\psi'_{0\dots0}(x')|}{|\psi'_{\underbrace{0\dots0}_{l-1}}(x)|}\\ &\le (1 + C'(\alpha)|\psi_{0\dots0}x' - \psi_{0\dots0}x|)(1 + E_k|x - x'|)\\ &\le 1 + (C'(\alpha) + E_k)|x - x'| + O(|x - x'|^2).\end{aligned}$$

Similarly we have the lower bound of $|\psi'_{a_1\dots a_n}(x')|/|\psi'_{a_1\dots a_n}(x)| - 1$. Further, the fact

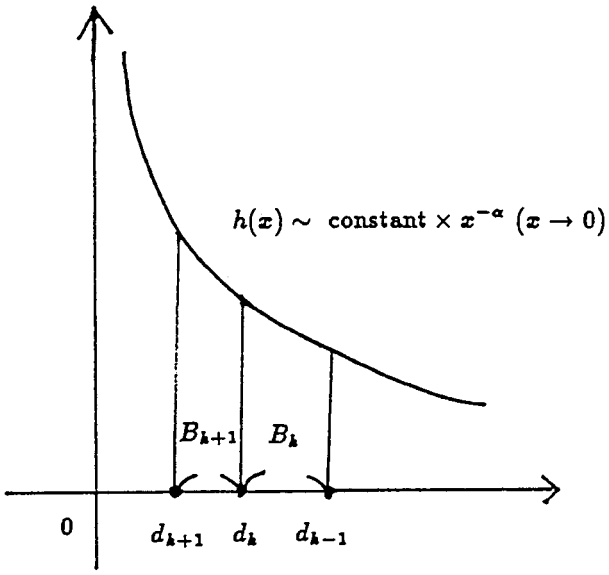


FIGURE 2.

that $\sup_{\theta\in[0,1]}\sup_n|\psi'_{a_1\dots a_n}(\theta)| \le 1$ allows us to verify (E2) and (WSN1). (WSN3) was verified in [**30**]. To check (E3a), it is suffices to take $\Delta(k) = k^{-1}$. Since $\inf_{x\in B_{k+1}} h(x)/\sup_{x'\in B_k} h(x') = 1$ (Figure 2), we have $H^* = H_* = 1$ in (E3b). Note that $X_{a_1\dots a_n} \in \mathcal{L}_R$ implies $a_n = 1$. Then we see that

$$|\psi'_{a_1\dots a_n}(x) - \psi'_{a_1\dots a_n}(x')| \le \lambda(X_{a_1\dots a_{n-1}1})E_1|\psi_1'||x - x'|,$$

$\forall x, x' \in [0, 1]$ and $\forall X_{a_1\dots a_n} \in \mathcal{L}_R$. Thus (WSN2) is satisfied. (E4) was verified in [**30**] as follows:

$$W_m \le (m + 1)^{1+1/\alpha}\sum_{n=0}^{\infty} 1/(1 + n)^{1+1/\alpha} < \infty.$$

Thus all conditions of Theorem 3.2 are verified.

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Example 2. (A skew product two-dimensional map which is related to Diophantine approximation in the inhomogeneous linear class.) Let $X = \{(x_1, x_2) \in \mathbf{R}^2 : 0 \le x_2 \le 1, -x_2 \le x_1 \le -x_2 + 1\}$. Define T on X by

$$T(x_1, x_2) = (1/x_1 - [(1 - x_2)/x_1] + [-(x_2/x_1)], -[-(x_2/x_1)] - (x_2/x_1)).$$

The invariant density of the finite invariant measure of T is given by

$$h(x_1, x_2) = \frac{1}{(2 \log 2)(1 - {x_1}^2)}$$

([**30**] and [**31**]). So the singular points of the density $h(x_1, x_2)$ are $(1, 0)$ and $(-1, 1)$. These points are periodic points with period 2 which are indifferent. Let

$$a(x_1, x_2) = [(1 - x_2)/x_1] - [-(x_2/x_1)], b(x_1, x_2) = -[-(x_2/x_1)],$$
$$a_n(x_1, x_2) = a(T^{n-1}(x_1, x_2)),$$
$$b_n(x_1, x_2) = b(T^{n-1}(x_1, x_2)).$$

Let $I = \{(a, b) \in \mathbf{Z}^2 : a > b > 0 \text{ or } a < b < 0\}$. For each $(a, b) \in I$, define

$$X_{(a,b)} = \{(x_1, x_2) \in X : a(x_1, x_2) = a, b(x_1, x_2) = b\}.$$

Then $Q = \{X_{(a,b)}\}_{(a,b)\in I}$ is a countable partition of X. On a cylinder $X_{(a,b)}$, $T(x_1, x_2) = ((1/x_1) - a, -(x_2/x_1) + b)$, and for any $X_{a_1 \ldots a_n} \in \mathcal{L}$, $T^n X_{a_1 \ldots a_n} = X$, that is T is a Bernoulli map. Let us define inductively $q_n = a_n q_{n-1} + q_{n-2}$ $(q_0 = 1, q_{-1} = 0)$. Then we have

$$|\det D\psi_{(a_1,b_1)\ldots(a_n,b_n)}(x_1, x_2)| = \frac{1}{|q_n|^3 |1 + (q_{n-1}/q_n)x_1|^3},$$

and we remark that

$$q_{n-1}/q_n = \cfrac{1}{a_n + \cfrac{1}{a_{n-1} + \cdots \cfrac{1}{a_1}}} \in [-1, 1].$$

We can choose 3^3 as the local Renyi constant (see [**31**]), and we can define $\mathcal{D}_k, \mathcal{B}_k$ for each $k > 0$ as follows:

$$\mathcal{D}_k = \{X_{\underbrace{(2,1)(-2,-1)(2,1)(-2,-1)\ldots}_{k}}, X_{\underbrace{(-2,-1)(2,1)(-2,-1)(2,1)\ldots}_{k}}\}$$

$$\mathcal{B}_k = \{X_{(a_1,b_1)\ldots(a_k,b_k)} : X_{(a_1,b_1)\ldots(a_{k-1},b_{k-1})} \in \mathcal{D}_{k-1}, a_k \neq \pm 2 \text{ or } a_{k-1}a_k > 0 \text{ if } |a_k| = 2\}$$

(Figure 3). Observe that $\operatorname{diam} p_1(D_{k+1}) = 2/(2k + 3)$ (Figure 4) and

$$1 - \sup\{|x| : x \in p_1(D_k^c)\} \ge \operatorname{diam} p_1(D_{k+1}).$$

Then we see that

$$\left|1 - \frac{|\det D\psi_{(a_1,b_1)\ldots(a_n,b_n)}(x_1, x_2)|}{|\det D\psi_{(a_1,b_1)\ldots(a_n,b_n)}(x_1', x_2')|}\right| \le \frac{3\|(x_1, x_2) - (x_1', x_2')\|}{(\operatorname{diam} p_1(D_{k+1}))^3}.$$

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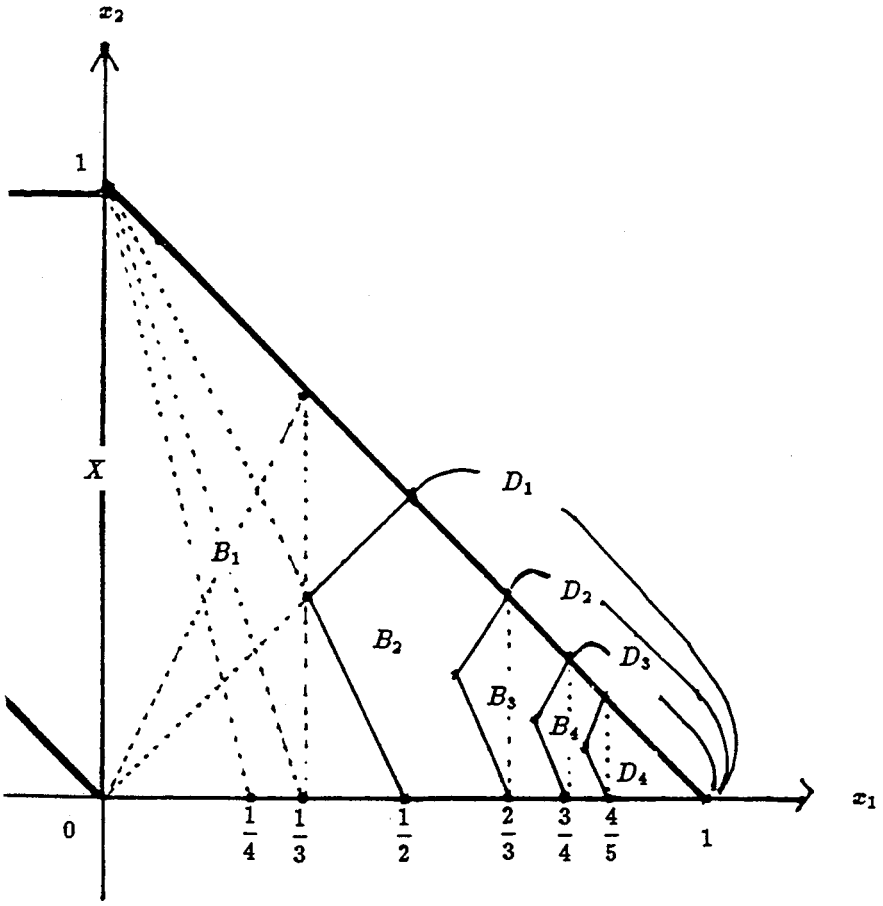


FIGURE 3.

Hence, for (E1) we can take $E_k = 3(\text{diam } p_1(D_{k+1}))^{-3}$. Similarly, the validity of (E2) can be readily verified as follows. First observe that

$$\begin{aligned} & p_1\psi_{(a_1,b_1)\dots(a_n,b_n)}(x_1, x_2) - p_1\psi_{(a_1,b_1)\dots(a_n,b_n)}(x_1', x_2') \\ = \quad & \frac{x_1' - x_1}{q_n^2(1 + (q_{n-1}/q_n)x_1)(1 + (q_{n-1}/q_n)x_1')}. \end{aligned}$$

Then we see

$$\begin{aligned} \|p_1\psi_{(a_1,b_1)\dots(a_n,b_n)}(x_1, x_2) - p_1\psi_{(a_1,b_1)\dots(a_n,b_n)}(x_1', x_2')\| \quad & \leq \quad \frac{|x_1 - x_1'|}{|q_n|^2(1 - |x_1|)(1 - |x_1'|)} \\ & \leq \quad \frac{|x_1 - x_1'|}{(\text{diam } p_1(D_{k+1}))^2}. \end{aligned}$$

Thus we can take $F_k = (\text{diam } p_1(D_{k+1}))^{-2} = O(k^2)$. Since we know the invariant

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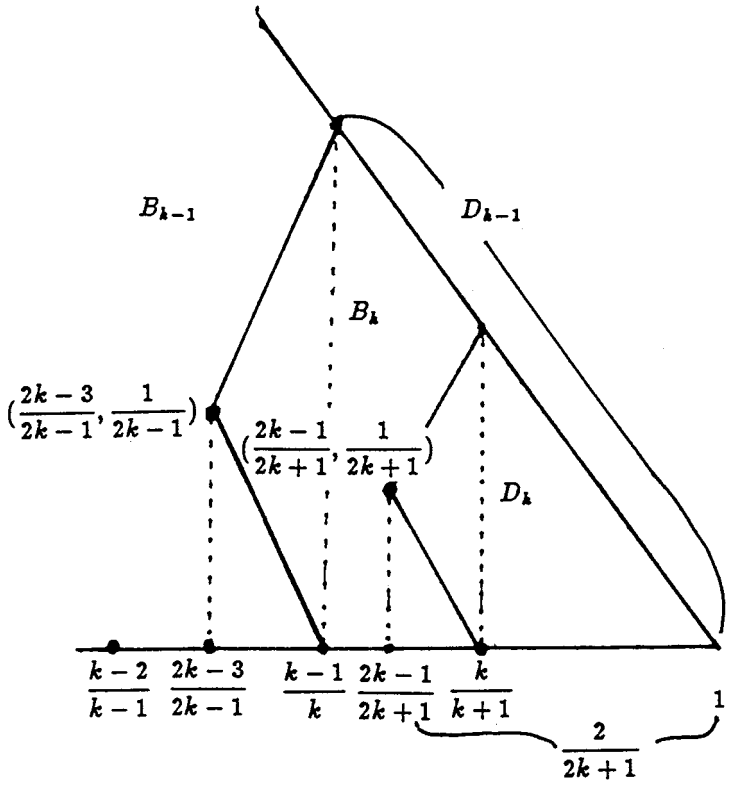


FIGURE 4.

density $h(x_1, x_2)$ explicitly, it is easy to verify (E3a). In fact, we have

$$\sup_{(x_1,x_2),(x_1',x_2')\in B_k} \frac{h(x_1, x_2)}{h(x_1', x_2')} = \sup_{x_1,x_1'\in p_1(B_k)} \frac{1/(1-x_1^2)}{1/(1-x_1'^2)} = \frac{1-(2k-3)^2/(2k-1)^2}{1-k^2/(k+1)^2}.$$

Hence we have $\Delta(k) = O(k^{-1})$. Since

$$\frac{\inf_{(x_1,x_2)\in B_{k+1}} h(x_1, x_2)}{\sup_{(x_1,x_2)\in B_k} h(x_1, x_2)} = \frac{1-k^2/(k+1)^2}{1-(2k-1)^2/(2k+1)^2} \to 1/2 \quad (k \to \infty),$$

(E3b) is also valid. Note that $X_{(a_1,b_1)\ldots(a_n,b_n)} \in \mathcal{R}(C.T)$ implies $|a_n| > 3$ and so $|q_n|/|q_{n-1}| > |a_n| - 1 \geq 2$. Combining the above facts, we immediately obtain (WSN2). A similar observation allows us to verify (WSN1) explicitly. The validity of (E4) was shown in [**30**] as follows:

$$W_m \leq \sum_{n=1}^{\infty} O(n^{-3}) \left(1 - \sup_{x_1\in p_1(D_m^c)} |x_1|\right)^{-3} \quad (\forall m > 0).$$

(WSN3) was also verified in [**30**]. Thus all assumptions of Theorem 3.2 are satisfied.

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Example 3. (A real two-dimensional map which is related to a complex continued fraction expansion.) Let $\alpha = 1+i$. Then we denote $X = \{z = x_1\alpha + x_2\overline{\alpha} : -(1/2) \le x_1, x_2 \le 1/2\}$ and define $T : X \to X$ by $Tz = 1/z - [1/z]_1$, where $[z]_1$ denotes $[x_1+1/2]\alpha + [x_2+1/2]\overline{\alpha}$ for a complex number $z = x_1\alpha + x_2\overline{\alpha}$. Let the index set I be $I = \{n\alpha + m\overline{\alpha} : m, n \in \mathbf{Z}\}\backslash\{0\}$. For each $n\alpha + m\overline{\alpha} \in I$, we define

$$X_{n\alpha+m\overline{\alpha}} = \{z \in X : [1/z]_1 = n\alpha + m\overline{\alpha}\}.$$

Then we have a countable partition $Q = \{X_a\}_{a\in I}$ of X. The map T induces a continued fraction like expansion of $z \in X$:

$$z = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots \cfrac{1}{a_n + \cdots}}}},$$

where each a_i is contained in I. The family $\mathcal{U}$ consists of five subsets of X and T has indifferent periodic points ± 1, $\pm i$, and the invariant density which was obtained by Tanaka [**23**] has singularities at these points and no others. On the other hand, T satisfies all assumptions of Theorem 2.1 and so admits a finite ergodic invariant measure μ equivalent to λ which is exact (see [**30**]). The weak Bernoulli property and further dynamical properties are discussed in [**30**] and [**31**]. We can choose $C = 5^4$ as the local Renyi constant and for each $n > 0$, $\mathcal{D}_n$ consists of cylinders of rank n which touch the indifferent periodic points. Put

$$p_{-1} = \alpha, \quad p_0 = 0, \quad p_n = a_n p_{n-1} + p_{n-2} \quad (n \ge 1)$$
$$q_{-1} = 0, \quad q_0 = \alpha, \quad q_n = a_n q_{n-1} + q_{n-2} \quad (n \ge 1).$$

Then

$$\psi_{a_1\dots a_n}(z) = \frac{p_n + zp_{n-1}}{q_n + zq_{n-1}}.$$

The following properties are needed to verify the assumptions of Theorem 3.2.

(1) $\left|\dfrac{q_{n-1}}{q_n}\right| \le 1$, $(\forall n > 0)$.

(2) $q_n p_{n-1} - q_{n-1}p_n = 2i(-1)^n$.

(3) $|\det D\psi_{a_1\dots a_n}(z)| = |q_n + zq_{n-1}|^{-4}$.

(4) For each $k > 0$, $\rho_k = \sup\{|z| : z \in D_k^c\} = 1 - O(k^{-1})$ (Figure 5).

(5) For $X_{a_1\dots a_n} \in \mathcal{R}(C.T)$, $|q_{n-1}/q_n| < 2/3$.

(Cf. [**23**] and [**30**].) To verify (E1), we use the inequality

$$\frac{|\det D\psi_{a_1\dots a_n}(z)|}{|\det D\psi_{a_1\dots a_n}(z')|} \le \left(1 + \frac{|q_{n-1}/q_n||z - z'|}{|1 + (q_{n-1}/q_n)z|}\right)^4,$$

and facts (1), (3) and (4). We then have $E_k = O(k^4)$. Furthermore, from fact (2), (E2) is easily verified by taking k^2 as F_k. The formula for the invariant density $h \equiv d\mu/d\lambda$ allows us to verify (E3a) and (E3b) explicitly. In fact, we can see that on neighbourhoods of indifferent periodic points $\pm i$, ± 1,

$$h(z)\left(\equiv \frac{d\mu}{d\lambda}(z)\right) \sim \frac{\text{constant}}{(1-|z|^2)^2} \quad (|z| \to 1).$$

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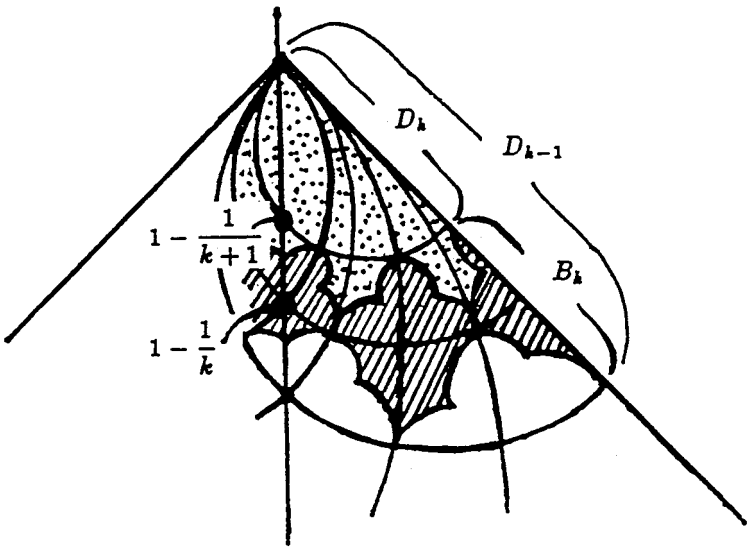


FIGURE 5.

So (E3a) and (E3b) are easily verified. For (WSN1) and (WSN2), (5) allows us to have

$$E_0 = 3^4, \quad F_0 = 2CE_0(1+\operatorname{diam} X)^4\left(\min_{U\in\mathcal{U}}\lambda(U)\right)^{-1}.$$

The validity of (E4) and the validity of (WSN3) were shown in [**30**]. Thus all assumptions of Theorem 3.2 are verified.

5. *Proofs*

Proof of Proposition 3.1. Let $x \in X_R \cap \Gamma(D)^c$. Then it follows from the assumption that there is a positive integer $l > 0$ such that $X_{b(l)}(x) \in \mathcal{R}(C.T)$ and $X_{b(l)} \subset V$ for some $V \in \mathcal{V}(\mathcal{D})$ (see [**31**] and [**32**]). Here $X_{b(l)}(x)$ is the unique cylinder of rank l containing x. Some interesting properties are discussed in [**34**] including the fact that we can choose $C^2\sum_{n=1}^{l}\lambda(D_n)/\lambda(X_{b(l)})$ as $K(x)$. We give an outline of the proof here. First observe that for $y \in T^n X_{d(n)} \cap X_{b(l)}$,

$$\begin{aligned}|\det D\psi_{d(n)b(l)}(T^l y)| &= |\det D\psi_{d(n)}(y)||\det D\psi_{b(l)}(T^l y)| \\ &\le C\inf_{y'\in T^n X_{d(n)}\cap X_{b(l)}}|\det D\psi_{d(n)b(l)}(T^l y')| \\ &\le C^2\inf_{y'\in T^n X_{d(n)}\cap X_{b(l)}}|\det D\psi_{d(n)}(y')|\inf_{y'\in T^l X_{b(l)}}|\det D\psi_{b(l)}(y')|.\end{aligned}$$

Hence we have

$$|\det D\psi_{d(n)}(y)| \le C^2\inf_{y'\in T^n X_{d(n)}\cap X_{b(l)}}|\det D\psi_{d(n)}(y')|.$$

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As $X_{b(l)} \subset V(\exists V \in \mathcal{V}(\mathcal{D}))$ implies that $T^n X_{d(n)} \cap X_{b(l)} = X_{b(l)}$, we can easily see that

$$\begin{aligned}\Psi(x) &\le C^2 \sum_{n=0}^{\infty} \sum_{X_{d(n)} \in \mathcal{D}_n} \inf_{y \in X_{b(l)}(x)} |\det D\psi_{d(n)}(y)| \\ &\le C^2 \sum_{n=1}^{l} \lambda(D_n)/\lambda(X_{b(l)}(x)) < \infty\end{aligned}$$

because of the following facts:

$$\lambda(X_{d(n)b(l)})/\lambda(T^n X_{d(n)} \cap X_{b(l)}) \ge \inf_{y \in T^n X_{d(n)} \cap X_{b(l)}} |\det D\psi_{d(n)}(y)|,$$
$$X_{d(n)b(l)} \subset D_n \cap D_{n+l}^c.$$

Thus we have the desired result.

Proof of Theorem 3.1. First we claim that for $X_{d(n)} \in \mathcal{L}$, $|\det D\psi_{d(n)}(x)|$ is continuous on $T^n X_{d(n)}$. Let $x \in \operatorname{int} V_R^j$ for some $V_R^j \in \mathcal{V}_R$. Then $\exists V_R^{j'} \in \mathcal{V}_R$ such that $\psi_{d(n)} x \in \psi_{d(n)} V_R^j \subset V_R^{j'}$. In fact, if not, $\operatorname{int}(\psi_{d(n)} V_R^j) \cap (\cup_{V_R \in \mathcal{V}_R} \partial V_R) \neq \emptyset$ and so

$$T^n|_{X_{d(n)}}(\operatorname{int}(\psi_{d(n)} V_R^j) \cap (\cup_{V_R \in \mathcal{V}_R} \partial V_R)) \neq \emptyset.$$

It follows from the local invertibility and 3 of Theorem 10.1 in [**32**] that

$$\begin{aligned}T^n|_{X_{d(n)}}(\operatorname{int}(\psi_{d(n)} V_R^j) \cap (\cup_{V_R \in \mathcal{V}_R} \partial V_R)) &\subset \operatorname{int}(V_R^j) \cap T^n(\cup_{V_R \in \mathcal{V}_R} \partial V_R) \\ &\subset \operatorname{int}(V_R^j) \cap (\cup_{V_R \in \mathcal{V}_R} \partial V_R).\end{aligned}$$

Comparing these, we have a contradiction.

Thus, Lipshitz continuity of $d\nu/d\lambda$ on $V_R^{j'} \in \mathcal{V}_R$ leads us to Lipshitz continuity of $d\nu/d\lambda$ on $\psi_{d(n)} V_R^j$. Let $x, y \in V_R^j \cap (\bigcup_{i=1}^k B_i) (= V_R^j \cap D_k^c)$. Then

$$\begin{aligned}&\left|\frac{d\mu}{d\lambda}(x) - \frac{d\mu}{d\lambda}(y)\right| \\ &\le \sum_{n=0}^{\infty} \sum_{X_{d(n)} \in \mathcal{D}_n} ||\det D\psi_{d(n)}(x)| - |\det D\psi_{d(n)}(y)|| 1_{T^n X_{d(n)}}(x) \frac{d\nu}{d\lambda}(\psi_{d(n)}(x)) \\ &\quad + \sum_{n=0}^{\infty} \sum_{X_{d(n)} \in \mathcal{D}_n} |\det D\psi_{d(n)}(x)| \left|\frac{d\nu}{d\lambda}(\psi_{d(n)}(x)) - \frac{d\nu}{d\lambda}(\psi_{d(n)}(y))\right| 1_{T^n X_{d(n)}}(x) \\ &\le G \sum_{n=0}^{\infty} \sum_{X_{d(n)} \in \mathcal{D}_n} |\det D\psi_{d(n)}(x)| \left|1 - \left|\frac{\det D\psi_{d(n)}(y)}{\det D\psi_{d(n)}(x)}\right|\right| 1_{T^n X_{d(n)}}(x) \\ &\quad + L \sum_{n=0}^{\infty} \sum_{X_{d(n)} \in \mathcal{D}_n} |\det D\psi_{d(n)}(x)| \|\psi_{d(n)}(x) - \psi_{d(n)}(y)\| 1_{T^n X_{d(n)}}(x),\end{aligned}$$

where L denotes the Lipshitz constant for $d\nu/d\lambda$.

Using (E1, 2) and Proposition 3.1, we have

$$\begin{aligned}\left|\frac{d\mu}{d\lambda}(x) - \frac{d\mu}{d\lambda}(y)\right| &\le \|x - y\| \sum_{n=0}^{\infty} \sum_{X_{d(n)} \in \mathcal{D}_n} |\det D\psi_{d(n)}(x)| (GE_k + LF_k) \qquad (5.1) \\ &= \|x - y\| \Psi(x)(GE_k + LF_k) \le \|x - y\| K(x)(GE_k + LF_k).\end{aligned}$$

We have finished the proof.

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Proof of Lemma 3.1. (1)–(3) are immediate consequences of definitions. We can easily verify (4) as follows. For all $f \in L^1(\mu)$,

$$\begin{aligned}(\mathcal{L}^*_{\log g}\mu)(f) &= \int_X \mathcal{L}_{\log g} f(x)\, d\mu(x) = \int_X \sum_{y\in T^{-1}x} f(y)g(y)\, d\mu(x)\\ &= \sum_{a\in I}\int_X f(\psi_a x)\frac{h(\psi_a x)}{h(x)}|\det D\psi_a(x)|1_{TX_a}(x)h(x)\, d\lambda(x)\\ &= \sum_{a\in I}\int_{TX_a} f(\psi_a x)h(\psi_a x)|\det D\psi_a(x)|1_{TX_a}(x)\, d\lambda(x)\\ &= \sum_{a\in I}\int_{X_a} f(x)h(x)\, d\lambda(x) = \int_X f(x)\, d\mu(x).\end{aligned}$$

Proof of Lemma 3.3. First we show that for each $k > 0$, $\exists 0 < \delta_k < \infty$ such that for $x, x' \in D_k^c \cap \operatorname{int} V (V \in \mathcal{V})$,

$$\left|1 - \frac{h(x')}{h(x)}\right| \le \delta_k \|x - x'\|.$$

In fact, from (5.1) we have

$$|h(x) - h(y)| \le \|x - y\|\Psi(x)(GE_k + LF_k)$$

and, furthermore,

$$\begin{aligned}\left|1 - \frac{h(x)}{h(y)}\right| &\le \frac{\Psi(x)}{h(x)}(GE_k + LF_k)\|x - y\|\\ &\le \sup_{x\in D_k^c}\frac{\Psi(x)}{h(x)}(GE_k + LF_k)\|x - y\|.\end{aligned}$$

Since $h(x) \ge (d\nu/d\lambda)(x) \ge G^{-1}$, Lemma 3.2 allows us to choose δ_k as

$$\delta_k \equiv G\Psi^*(k)(GE_k + LF_k).$$

Next we note that for $x, x' \in \bigcup_{i=1}^k B_i \cap \operatorname{int} V (V \in \mathcal{V})$,

$$\psi_{a(n)}x,\ \psi_{a(n)}x' \in D^c_{n+k} \cap \operatorname{int} V' (\exists V' \in \mathcal{V}).$$

(Recall the proof of Theorem 3.1 and the local Renyi condition.) We see that in the case when $\sup_k \delta_k \equiv \delta_\infty < \infty$,

$$\begin{aligned}\sup_{X_{a(n)}\in\mathcal{L}:T^nX_{a(n)}\supseteq V}\frac{h(\psi_{a(n)}x)}{h(\psi_{a(n)}x')} &\le 1 + \delta_{n+k}\|\psi_{a(n)}x - \psi_{a(n)}x'\|\\ &\le 1 + \delta_{n+k}F_k\|x - x'\| \le 1 + \delta_\infty F_k\|x - x'\|.\end{aligned}$$

Hence we have

$$\sup_{n\ge1}\sup_{X_{a(n)}\in\mathcal{L}:T^nX_{a(n)}\supseteq V}\frac{h(\psi_{a(n)}x)}{h(\psi_{a(n)}x')} \le 1 + \delta_\infty F_k\|x - x'\| \le 1 + \delta_\infty F_k(\operatorname{diam} X).$$

So we can take $C_g(k) = (1 + \delta_\infty F_k \operatorname{diam} X)(1 + E_k \operatorname{diam} X)(1 + \delta_k \operatorname{diam} X)$, and

$$C_g(x, x') \le (1 + \delta_\infty F_k\|x - x'\|)(1 + E_k\|x - x'\|)(1 + \delta_k\|x - x'\|) \to 1$$

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as $\|x - x'\| \to 0$. Thus we have shown the claimed result.

Next we assume that $\sup_k \delta_k \equiv \delta_\infty = \infty$. (Without loss of generality we can assume that $\delta_i \to \infty$ as $i \to \infty$ monotonically.) First we shall show that there exists a sequence $\{H_\infty(k)\}_{k>0}$ with $0 < H_\infty(k) < \infty$ satisfying

$$\sup_{n\geq 1} \sup_{X_{a(n)}\in\mathcal{L}:T^n X_{a(n)}\supseteq V} \frac{h(\psi_{a(n)}x)}{h(\psi_{a(n)}x')} \leq H_\infty(k).$$

Then we can choose a sufficiently large number $n_0 \equiv n_0(k) > 0$ depending on k such that

$$1 + \delta_{k+n_0(k)} F_k \|x - x'\| > H_\infty(k) \qquad (*)$$

because of the hypothesis $\delta_i \to \infty$ as $i \to \infty$. This implies that

$$\sup_n \sup_{X_{a(n)}\in\mathcal{L}:T^n X_{a(n)}\supseteq V} \frac{h(\psi_{a(n)}x)}{h(\psi_{a(n)}x')} \leq 1 + \delta_{k+n_0(k)} F_k \|x - x'\|.$$

If $x, x' \in B_i$ $(1 \leq i \leq k)$ and $\psi_{a(n)}x, \psi_{a(n)}x' \in B_j$ $(1 \leq \exists j \leq n + k)$, then condition (E3a) allows us to write

$$\sup_{X_{a(n)}\in\mathcal{L}:T^n X_{a(n)}\supseteq V} \frac{h(\psi_{a(n)}x)}{h(\psi_{a(n)}x')} \leq H \equiv 1 + \sup_k \Delta(k).$$

If $x, x' \in \bigcup_{i=1}^k B_i$, then $\psi_{a(n)}x, \psi_{a(n)}x' (\in \bigcup_{i=1}^{k+n} B_i)$ do not necessarily belong to a single B_j $(1 \leq j \leq n + k)$. We need a further observation. If $X_{a(n)} \subseteq B_i$ for some $1 \leq i \leq n$, then $\psi_{a(n)}x, \psi_{a(n)}x' \in B_i$ and thus we have

$$\frac{h(\psi_{a(n)}x)}{h(\psi_{a(n)}x')} \leq \sup_{y,y'\in B_i} \frac{h(y)}{h(y')} \leq H.$$

If $X_{a(n)} \in \mathcal{D}_n$, then $\psi_{a(n)}x, \psi_{a(n)}x' \in \bigcup_{i=n+1}^{n+k} B_i$ because of the local Renyi condition. Thus we have

$$\frac{h(\psi_{a(n)}x)}{h(\psi_{a(n)}x')} \leq \max_{1\leq i,j\leq k} \frac{\sup_{y\in B_{n+i}} h(y)}{\inf_{y'\in B_{n+j}} h(y')}.$$

From conditions (E3a) and (E3b) we see that

$$\frac{\sup_{x\in B_{k+1}} h(x)}{\sup_{x'\in B_k} h(x')} \leq H^*(1 + \Delta(k+1)) \leq H_\infty < \infty,$$

where $H_\infty = H^*(1 + \sup_k \Delta(k))$ and putting $H_i \equiv H(H_\infty)^{i-1}$, we have the sequence $\{H_i\}_{i>0}$ satisfying

$$\frac{\sup_{x\in B_{n+i}} h(x)}{\inf_{x'\in B_{n+1}} h(x')} \leq H_i.$$

Here we simply use the equality

$$\frac{\sup_{x\in B_{n+i}} h(x)}{\inf_{x'\in B_{n+1}} h(x')} = \frac{\sup_{x\in B_{n+1}} h(x)}{\inf_{x\in B_{n+1}} h(x)} \frac{\sup_{x\in B_{n+2}} h(x)}{\sup_{x\in B_{n+1}} h(x)} \cdots \frac{\sup_{x\in B_{n+i}} h(x)}{\sup_{x\in B_{n+i-1}} h(x)}.$$

Similarly, we have a sequence $\{H'_i\}_{i>0}$ with $0 < H'_i < \infty$ s.t.

$$\frac{\sup_{x\in B_{n+1}} h(x)}{\inf_{x'\in B_{n+i}} h(x')} \leq H'_i.$$

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Combining the above observations, we have for $\psi_{a(n)}x, \psi_{a(n)}x' \in \bigcup_{i=n+1}^{n+k} B_i \cap \operatorname{int} V' (V' \in \mathcal{V})$,

$$\frac{h(\psi_{a(n)}x)}{h(\psi_{a(n)}x')} \leq \max_{1\leq i,j\leq k} \frac{\sup_{y\in B_{n+i}} h(y)}{\inf_{y'\in B_{n+j}} h(y')} \leq \max_{1\leq j<i\leq k} \{H_{i-j+1}, H'_{i-j+1}\} \equiv \overline{H}_k.$$

Since

$$\overline{\lim_{n\to\infty}} \sup_{y,y'\in\bigcup_{i=n+1}^{n+k} B_i} \frac{h(y)}{h(y')} \leq \overline{H}_k < \infty,$$

by putting $H_\infty(k) \equiv \max\{H, \overline{H}_k\}$ we have the desired sequence. Finally, we can choose a sufficiently large number $n_0 \equiv n_0(k) > 0$ satisfying (*) and $\forall n \geqq 1$,

$$\sup_{X_{a(n)}\in\mathcal{L}:T^nX_{a(n)}\supseteq V} \frac{h(\psi_{a(n)}x)}{h(\psi_{a(n)}x')} \leq 1 + \delta_{k+n_0(k)} F_k \|x - x'\|.$$

Thus for $x, x' \in \bigcup_{i=1}^k B_i \cap \operatorname{int} V (V \in \mathcal{V})$,

$$\sup_{n\geq 1} \sup_{X_{a(n)}\in\mathcal{L}:T^nX_{a(n)}\supseteq V} \frac{h(\psi_{a(n)}x)}{h(\psi_{a(n)}x')} \leq 1 + \delta(k) F_k \|x - x'\|,$$

where $\delta(k) = \max_{1\leq n\leq n_0(k)}\{\delta_{k+n_0(k)}, \delta_{n+k}\}$, and consequently

$$\begin{aligned} C_g(x,x') &\leq (1+\delta(k)F_k\|x-x'\|)(1+E_k\|x-x'\|)(1+\delta_k\|x-x'\|) \\ &\leq (1+\delta(k)F_k \operatorname{diam} X)(1+E_k \operatorname{diam} X)(1+\delta_k \operatorname{diam} X). \end{aligned}$$

Since $C_g(x,x') \geq 1$, by the definition, we have $C_g(x,x') \to 1$ as $\|x - x'\| \to 0$. In fact, $C_g(x,x') \leq 1 + Z(k)\|x - x'\|$, where

$$(1+\delta(k)F_k\|x-x'\|)(1+E_k\|x-x'\|)(1+\delta_k\|x-x'\|) = 1 + Z(k)\|x-x'\|.$$

Taking

$$C_g(k) \equiv (1+\delta(k)F_k \operatorname{diam} X)(1+E_k \operatorname{diam} X)(1+\delta_k \operatorname{diam} X),$$

we have the desired result.

Proof of Theorem 3.2. It follows from property (3) in Lemma 3.1 and Lemma 3.3 that for $x, x' \in D_k^c \cap V (V \in \mathcal{V})$ and $\forall f \in C(\overline{X})$,

$$\begin{aligned} &|\mathcal{L}^n_{\log g} f(x) - \mathcal{L}^n_{\log g} f(x')| \\ &\leq \sup_{X_{a(n)}\in\mathcal{L}:T^nX_{a(n)}\supseteq V} |f(\psi_{a(n)}x) - f(\psi_{a(n)}x')| \sum_{X_{a(n)}\in\mathcal{L}:T^nX_{a(n)}\supseteq V} \prod_{i=0}^{n-1} g(\psi_{a_{i+1}\dots a_n}x) \\ &\quad + \|f\| \left| \sum_{X_{a(n)}\in\mathcal{L}:T^nX_{a(n)}\supseteq V} \prod_{i=0}^{n-1} g(\psi_{a_{i+1}\dots a_n}x) \left| 1 - \prod_{i=0}^{n-1} \frac{g(\psi_{a_{i+1}\dots a_n}x')}{g(\psi_{a_{i+1}\dots a_n}x)} \right| \right| \\ &\leq \sup_{X_{a(n)}\in\mathcal{L}:T^nX_{a(n)}\supseteq V} |f(\psi_{a(n)}x) - f(\psi_{a(n)}x')| + \|f\|\{C_g(x,x') - 1\} \\ &\leq \sup\left\{|f(y) - f(y')| : y, y' \in \bigcup_{i=1}^{n+k} B_i, \|y - y'\| \leq F_k\|x - x'\|\right\} \\ &\quad + \|f\| Z(k)\|x - x'\|. \end{aligned}$$

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Therefore $\forall f \in C(\overline{X})$ $\{\mathcal{L}^n_{\log g} f\}_{n\geq 0}$ is equicontinuous on $\bigcup_{i=1}^k B_i \cap V$ for each $k > 0$ and each $V \in \mathcal{V}$. In addition to this we have $\|\mathcal{L}^n_{\log g} f\| \leq \|f\|$ $(\forall f \in C(\overline{X}))$ because of property (3) in Lemma 3.1. Thus the closure of $\{\mathcal{L}^n_{\log g} f\}$ in $C(\overline{X})$ is compact. Hence there is a sequence $\{n_i\}_{i>0} \to \infty (i \to \infty)$ and f^* such that

$$\mathcal{L}^{n_i}_{\log g} f \Longrightarrow f^* \quad (i \to \infty)$$

uniformly on any compact subset of $\bigcup_{i=1}^\infty B_i \cap (\partial\mathcal{V})^c$ for $f \in C(\overline{X})$. As $f(x), h^{-1}(x)$ are continuous on $\overline{X}$, $f(x)/h(x)$ is also uniformly continuous on $\overline{X}$. Hence we can apply the above fact to $f(x)/h(x)$ as follows:

$$\mathcal{L}^{n_i}_{\log g}(f/h) \Longrightarrow (f/h)^* \quad (i \to \infty) \tag{5.2}$$

uniformly on compact subsets of $\bigcup_{i=1}^\infty B_i \cap (\partial\mathcal{V})^c$. We remark that

$$\begin{aligned} \mathcal{L}^n_{\log g}(f)(x) &= \frac{\sum_{a(n)} |\det D\psi_{a(n)}(x)|(f \times h)(\psi_{a(n)}x)}{h(x)} \\ &= h(x)^{-1}\mathcal{L}^n_{-\log|\det DT(x)|}(f \times h)(x). \end{aligned}$$

Hence $h(x)\mathcal{L}^n_{\log g}(f/h)(x) = \mathcal{L}^n_{-\log|\det DT(x)|} f(x)$ and (5.2) means that

$$\mathcal{L}^{n_i}_{\phi}(f) \Longrightarrow (f/h)^* h \quad (i \to \infty) \tag{5.3}$$

uniformly on compact subsets of $\bigcup_{i=1}^\infty B_i \cap (\partial\mathcal{V})^c$, $\forall f \in C(\overline{X})$. We claim that the exactness of (T, μ) can be characterized as follows:

$$\|\mathcal{L}^n_\phi f - \mathcal{L}^n_\phi h\|_1 = \int_X |\mathcal{L}^n_\phi f - h|\, d\lambda \to 0 \quad (n \to \infty), \tag{5.4}$$

$\forall f \in L^1(\lambda)$ with $\int_X f\, d\lambda = \int_X h\, d\lambda (= \mu(X))$ (see [**10**]). Choose a function $f \in C(\overline{X})$ with $\int_X f\, d\lambda = \int_X h\, d\lambda$. Observe the following inequality:

$$\int |\mathcal{L}^{n_i}_\phi f - h|\, d\lambda \geq \int ||\mathcal{L}^{n_i}_\phi f - (f/h)^* h| - |(f/h)^* h - h||\, d\lambda.$$

It follows from (5.4) that the right-hand side must go to zero as $i \to \infty$. Then if $|(f/h)^* h - h| > 0$ on a positive measure set, we have a contradiction immediately because of (5.3). Thus $(f/h)^*$ is a.e. constant and so is $\mathcal{L}_{\log g}$-invariant. Hence

$$\mathcal{L}^n_{\log g} f/h \Longrightarrow (f/h)^* \quad (n \to \infty)$$

and it follows from (5.3) that

$$\mathcal{L}^n_\phi f \Longrightarrow (f/h)^* h \quad (n \to \infty)$$

uniformly on compact subsets of $\bigcup_{i=1}^\infty B_i \cap (\partial\mathcal{V})^c$. On the other hand, from (4) in Lemma 3.1 we have

$$\mu((f/h)^*) = (f/h)^* \mu(X) = \mu(f/h)$$

and so

$$(f/h)^* = 1/\mu(X) \int f d\lambda.$$

This completes the proof.

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