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DYNAMIC BINOMIALS WITH AN APPLICATION TO GENDER BIAS ANALYSIS

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Abstract

We analyze the dynamics of a stochastic process driven by binomial random variables, where the probability of success depends on the past realization. We study the limit behavior when the group size is fixed but the number of iterations increases. It will become apparent that the so-called policy function and its fixed point play an outstanding role. Some applications to a statistical analysis of gender bias are also briefly discussed.

Keywords: Dynamic binomials; policy function; fixed point; stationary limit

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1. Introduction

To motivate our work, consider a committee of size z that needs to be renewed at each time instant n. The choice of the new members is up to the actual committee. Due to differences, e.g. in gender, one may wonder if the selection process incorporates some gender bias in that members favor candidates from their own group. To formalize things, denote the two groups with G_{B} and G_{U}, where B and U stand for 'biased' and 'unbiased'. Let Z_{n-1}, at time $n-1$, be the number of members from group G_{U} so that its fraction equals $\xi_{n-1} = Z_{n-1}/z$. This term may be relevant for the next choice in that committee members may choose candidates from G_{U} independently with a success probability

$$p_n = f(\xi_{n-1}). \tag{1}$$

The 'policy function' f describes how the current state of the committee enters into the decision-making process of the next round. For example, if we let, for $0 \le v, \delta \le 1$,

$$p_n = v\xi_{n-1} + v\delta(1-\xi_{n-1}) \equiv f(\xi_{n-1}), \tag{2}$$

then δ may be interpreted as the bias parameter and v as the proportion of unbiased people. Actually, when $\xi_{n-1} = 1$, irrespective of δ, there is no gender bias and a new candidate is selected from G_{U} with probability v and from G_{B} with probability $1-v$. On the other hand, when $\xi_{n-1} = 0$, p_n decreases to $v\delta$. In general, p_n is a convex combination of the two extremes. When $\delta = 1$ there is no gender bias. As δ gets smaller, however, the chance to select a member from G_{U} decreases with ξ_{n-1}.

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To complete our probabilistic modeling we assume that, under (1), $Z_1, Z_2, \ldots$ is a Markov sequence with initial value $Z_0 = z_0$ in $S = \{0, 1, \ldots, z\}$ and transition probabilities

$$p_{ij} = \mathbb{P}(Z_n = j \mid Z_{n-1} = i) = \binom{z}{j} f^j\left(\frac{i}{z}\right)\left(1 - f^{z-j}\left(\frac{i}{z}\right)\right), \qquad i, j \in S,$$

where $f^j(x) \equiv (f(x))^j$ denotes the jth power of the function f. Hence, conditionally on $Z_{n-1} = i$, Z_n is binomial with parameters z and $f(i/z)$. Provided that

$$0 < f\left(\frac{i}{z}\right) < 1 \quad \text{for all } i \in S, \tag{3}$$

the chain is positive recurrent and aperiodic and admits a unique stationary distribution $\boldsymbol{\pi} = (\pi_0, \pi_1, \ldots, \pi_z)$, i.e. there exists a unique probability vector $\boldsymbol{\pi}$ such that

$$\boldsymbol{\pi} = \boldsymbol{\pi}\boldsymbol{P}, \tag{4}$$

where $\boldsymbol{P} = (p_{ij})$ is the associated transition matrix from above. See, e.g. Grimmett and Stirzaker (2001). The vector $\boldsymbol{\pi}$ constitutes the unconditional distribution of Z_n on the long run. Though, in theory, $\boldsymbol{\pi}$ may be determined by solving (4), in practice this is impossible since a flexible modeling of f and, hence, $\boldsymbol{P}$ includes some unknown parameters. Therefore, a purely algebraic argument as to solving (4) is inappropriate to provide us with some insight into the equilibrium distribution of the Z-chain. Parallel questions have been investigated for the Poisson case in which the intensity λ was assumed to satisfy some autoregressive recursion, possibly perturbed. See Fokianos and Tjøstheim (2012), Fokianos *et al.* (2009), and Doukhan *et al.* (2012). Compared with their case, our state space is finite so that the fundamental properties of the chain (such as irreducibility, aperiodicity, or ergodicity) are easily obtained. A fixed group size z is required in the introductory example, where it is unlikely that z changes in a random way. Also, our parameters p_n are supposed to stay in the unit interval so that perturbation may be inappropriate. Moreover, rather than being Poissonian, our chain could be matched in distribution by a normal.

Binomial distributions in a dynamic context have also been investigated by other authors. One proposal led to Markov binomial distributions, where the probability of success at stage n depends on the result at stage $n - 1$, i.e. on X_{n-1}. Omey *et al.* (2008) then investigated the distribution of $S_n = X_1 + \cdots + X_n$ and derived conditions under which the limit distribution is of Poissonian type. Dekking and Kong (2011) is a recent work where the authors study the shape of these general Markov binomial distributions. In Drezner and Farnum (1993) the probabilities $\mathbb{P}(S_n = x)$ depend on S_{n-1}, and the main focus is on the total number of successes, as n increases. See also Kowalski (2000).

As we have seen in our introductory comments, the situation discussed in this paper is different since the group size z is kept fixed and we follow the chain from one round to another, with the success probabilities p_n being random and controlled through the policy function f.

The paper is organized as follows. In Section 2 we briefly study the $z = 1$ case, i.e. when the binomials reduce to Bernoullis. It turns out that we can fully specify the equilibrium. In Section 3 we discuss the general binomial case.

Rather than looking at (4), we study the associated Laplace transforms, both conditionally and unconditionally. We shall see that the unconditional equilibrium Laplace satisfies a certain integral equation. This will be useful in order to study moments. In particular, the first moment can be characterized through the solution of a fixed point equation.

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In Section 4 we present a discussion of higher moments under the equilibrium distribution. In Section 5 we investigate normal approximations when z is moderately large. In Section 6 we report on some numerical approximations. Proofs are postponed to Section 7.

2. The Bernoulli case

In this section we focus on the $z = 1$ case. Hence, $\xi_n = Z_n$ is an irreducible aperiodic Markov chain with state space $S = \{0, 1\}$ and conditional success probabilities

$$p_n \equiv \mathbb{E}[\xi_n \mid \xi_{n-1}] = f(\xi_{n-1}),$$

where f maps S into the open interval $(0, 1)$ and ξ_0 is an arbitrary constant. We then have the following theorem.

Theorem 1. *The variables ξ_n converge in distribution to a variable ξ_∞, which is again Bernoulli with parameter p_∞, say. Here,*

$$p_\infty = \lim_{n\to\infty} \mathbb{E}[p_n] \tag{5}$$

solves the fixed point equation

$$p_\infty = f(0)(1 - p_\infty) + f(1)p_\infty. \tag{6}$$

Example 1. For the model (2), the fixed point p_∞ equals $p_\infty = \nu\delta/(1 - \nu(1 - \delta))$. Note that p_∞ is also a fixed point of the associated function f given in (2).

Remark 1. As we shall see from the proof of Theorem 1, the fixed point p_∞ not only serves as a limit in (5) but also, at the same time, makes the operator

$$\xi_n \to \mathbb{E}[\xi_n - p_\infty \mid \mathcal{F}_{n-1}]$$

a contraction.

The last comment is meant to indicate that fixed points and a contraction argument will play an important role also in the general binomial case.

3. The general binomial case

We assume throughout that (3) holds. Then the Markov chain Z_n has a unique stationary limit distribution $\boldsymbol{\pi}$ satisfying (4). Denote with

$$\psi_n(u) = \mathbb{E}[e^{uZ_n}], \qquad n \geq 1,$$

the Laplace transform of Z_n. Then, as $n \to \infty$, ψ_n converges to $\psi_\infty(u) = \mathbb{E}[e^{uZ_\infty}]$, where Z_∞ has distribution $\boldsymbol{\pi}$.

Theorem 2. *The Laplace transform ψ_∞ satisfies the integral equation*

$$\psi_\infty(u) = \mathbb{E}[e^{uZ_\infty}] = \mathbb{E}\left[e^u f\left(\frac{Z_\infty}{z}\right) + 1 - f\left(\frac{Z_\infty}{z}\right)\right]^z. \tag{7}$$

The right-hand side of (7) equals the Laplace transform of a random variable which conditionally on $Z_\infty = k$ is binomial with parameters z and $p = f(k/z)$. Hence, the distribution of Z_∞ is a mixture of binomials. By taking derivatives on both sides of (7) and then setting $u = 0$, we obtain the following corollary.

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Corollary 1. *Under (3), the expectation under equilibrium satisfies*

$$\mathbb{E}[Z_\infty] = z\mathbb{E}\left[f\left(\frac{Z_\infty}{z}\right)\right]. \tag{8}$$

We have seen in Theorem 1 that, for $z = 1$, the targets, in the limit, can be characterized through solutions of certain fixed point equations. For the mean this also holds when $z > 1$. For the sake of simplicity, assume that the policy function f is affine, see (2):

$$f(p) = \nu p + \nu\delta(1 - p). \tag{9}$$

Theorem 3. *Under (9), the expectation of Z_∞ equals zp_∞, where p_∞ is the fixed point of f, i.e.*

$$p_\infty = \frac{\nu\delta}{1 - \nu(1 - \delta)}.$$

4. Higher moments under equilibrium

As is known from elementary probability theory, higher moments of a variable Z admitting a Laplace transform $\psi(u) = \mathbb{E}[\mathrm{e}^{uZ}]$ may be easily obtained from ψ through the equation

$$\mathbb{E}[Z^k] = \psi^{(k)}(0),$$

where $\psi^{(k)}$ denotes the kth derivative of ψ. In the context of this paper, the variable $Z = Z_\infty$ is the limit in distribution of a sequence Z_n which are binomial only conditionally so that an explicit representation of ψ is not available. Rather, in (7) we see that under equilibrium the associated ψ satisfies some integral equation. On the other hand, using Theorem 3 (and its proof), we shall show that for the first moment the recursive dynamic behavior of the Z-sequence together with a contraction argument will yield a simple representation of the mean. For higher moments, finding a proper contraction operator is far from being obvious. Therefore, we propose a recursion which traces the kth moment back to the previous ones.

Set

$$g(u) = \mathrm{e}^u f\left(\frac{Z_\infty}{z}\right) + 1 - f\left(\frac{Z_\infty}{z}\right).$$

Then

$$g'(u) = g''(u) = \cdots = \mathrm{e}^u f\left(\frac{Z_\infty}{z}\right) \qquad g(0) = 1, \qquad g'(0) = g''(0) = \cdots = f\left(\frac{Z_\infty}{z}\right).$$

Conclude that $\psi'_\infty(u) = z\mathbb{E}[g^{z-1}(u)g'(u)]$, whence

$$\mathbb{E}[Z_\infty] = z\mathbb{E}[g^{z-1}(0)g'(0)] = z\mathbb{E}\left[f\left(\frac{Z_\infty}{z}\right)\right].$$

This once again yields (8). For the second moment, we obtain

$$\begin{aligned}\psi''_\infty(u) &= \mathbb{E}[Z_\infty^2 \mathrm{e}^{uZ_\infty}] \\ &= z(z-1)\mathbb{E}[g^{z-2}(u)(g'(u))^2] + z\mathbb{E}[g^{z-1}(u)g''(u)] \\ &= z(z-1)\mathbb{E}[g^{z-2}(u)(g'(u))^2] + \psi'_\infty(u).\end{aligned}$$

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For $u = 0$, this yields

$$\mathbb{E}[Z_\infty^2] = \mathbb{E}[Z_\infty] + z(z-1)\mathbb{E}\left[f^2\left(\frac{Z_\infty}{z}\right)\right]. \tag{10}$$

Now, if, as in (9), f is affine, $f(p) = ap + b$, we are able to solve (10) for $\mathbb{E}[Z_\infty^2]$ from the following equation:

$$\left[1 - \frac{z-1}{z}a^2\right]\mathbb{E}[Z_\infty^2] = [1 + 2ab(z-1)]\mathbb{E}[Z_\infty] + z(z-1)b^2.$$

Note that for $z = 1$, we obtain $\mathbb{E}[Z_\infty^2] = \mathbb{E}[Z_\infty]$. This is trivial since in view of Theorem 1, Z_∞ is Bernoulli.

For higher moments and a general z, we could proceed as before. Some standard computations yield

$$\mathbb{E}[Z_\infty^k] = \psi^{(k)}(0) = \sum_{j=1}^{k-1} a_{j,k}\mathbb{E}[Z_\infty^j] + z(z-1)\cdots(z-k+1)\mathbb{E}\left[f^k\left(\frac{Z_\infty}{z}\right)\right],$$

where for $k \geq 2$ the coefficients $a_{j,k}$ are defined in a recursive way:

$$a_{j,k+1} = \begin{cases} k + a_{k-1,k} & \text{for } j = k, \\ a_{j-1,k} - k a_{j,k} & \text{for } 1 < j < k, \\ -k a_{1,k} & \text{for } j = 1, \end{cases}$$

with $a_{1,2} = 1$. When f is a constant, we arrive at recursive formulae for standard binomials.

5. Consistency and normal approximations

So far we have kept z fixed. As a consequence the Markov chain Z_n admits, under (3), a nondegenerate limit Z_∞ (in distribution). Moreover, the mean of Z_∞/z equals the fixed point p_∞ of the policy function f. In statistical applications it is also common to study binomials when z increases. As one of the major consequences, the law of large numbers asserts that when p is fixed and $z \to \infty$, then $\xi_n = \xi_n(z) \to p$ with probability 1. Moreover, the central limit theorem yields an asymptotic normality of $\sqrt{z}(\xi_n(z) - p)$.

The purpose of this section is to study the distributional behavior of (conditional) binomials in a dynamic framework, i.e. under (1), when z is moderately large. It is likely that in such a situation the limit distribution (depending on z) increasingly concentrates at one point. We already know the candidate, the fixed point of f.

The following result provides an L^2-bound for the deviation between ξ_n and p_∞ which holds uniformly in z and n.

Theorem 4. *Assume that the policy function f satisfies the following two conditions:*

(i) $0 < f(p) < 1$ *for all* $0 \leq p \leq 1$;

(ii) f *is Lipschitz continuous:*

$$|f(p_1) - f(p_2)| \leq c|p_1 - p_2| \quad \textit{for some } 0 \leq c < 1.$$

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Then, when the group size equals z, we have

$$\mathbb{E}[\xi_n - p_\infty]^2 \leq \frac{c_1}{z} + c_2 c^{2(n-1)},$$

where c_1 and c_2 are two constants depending on c and the starting point ξ_1, respectively.

Remark 2. Theorem 4(i) is the continuous-time analogue of (3). Together with the continuity postulated in Theorem 4(ii), we have, for some constants δ_1 and δ_2,

$$0 < \delta_1 \leq f(p) \leq \delta_2 < 1 \quad \text{for all } 0 \leq p \leq 1.$$

Together with Theorem 4(i), Theorem 4(ii) guarantees that f admits a unique fixed point p_∞ in the open interval (0,1).

Remark 3. In the classical situation of elementary probability theory, when f is a constant so that $p_n = p_0$, the constant c equals 0 so that (10) reduces to the well-known bound $0(z^{-1})$.

Corollary 2. *When $n \to \infty$ and $z \to \infty$, we have*

$$\xi_n \to p_\infty \quad \textit{and} \quad p_n \to p_\infty \quad \textit{in probability},$$

where p_∞ is the unique fixed point of f.

Proof. The first assertion follows from (10) and the Chebychev inequality. The convergence of $p_n = f(\xi_{n-1})$ to p_∞ is an immediate consequence of the first, the continuity of f, and the fact that p_∞ is a fixed point of f. □

Our final result concerns the asymptotic normality of ξ_n, as the group size z increases.

Theorem 5. *Under the assumptions of Theorem 4, we have*

$$\sqrt{z}(\xi_n - p_n) \xrightarrow{\mathrm{D}} \mathcal{N}(0, p_\infty(1 - p_\infty)) \quad \textit{as } n \to \infty,\ z \to \infty,$$

where '$\xrightarrow{\mathrm{D}}$' denotes convergence in distribution.

6. Numerical approximations

In this section we provide some numerical comparison between the equilibrium distribution of Z_∞. For $p_n = f(\xi_{n-1})$, we chose (2) with $\nu = 0.5$ and δ varying.

In Table 1, 'true' denotes the probability $\mathbb{P}(Z_\infty/\sqrt{z} \leq x)$ while 'normal' is denoted by

$$\Phi\left(\frac{x - \sqrt{z}p_\infty}{\sqrt{p_\infty(1 - p_\infty)}}\right),$$

where Φ is the cumulative distribution function of a standard normal.

Another question that arises is whether the normal approximation is not only good for Z_∞ but also for Z_n with n finite. This covers a practical situation and is needed when we are performing statistical analysis. In this respect, Table 2 exhibits equally good approximations already for sample size $z = 5$ and iteration level $n = 3$. Here, 'true' means $\mathbb{P}(Z_n/\sqrt{z} \leq x)$, while normal means the same as before.

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TABLE 1.

$z = 5.0$	$\delta = 0.2$		$\delta = 0.4$	
	True	Normal	True	Normal
$x = 0.0$	0.429	0.344	0.199	0.178
$x = 0.2$	0.794	0.788	0.561	0.528
$x = 0.4$	0.950	0.977	0.845	0.855
$x = 0.6$	0.992	0.999	0.968	0.979
$x = 0.8$	0.999	0.999	0.997	0.998
$z = 10.0$	$\delta = 0.2$		$\delta = 0.4$	
	True	Normal	True	Normal
$x = 0.1$	0.500	0.443	0.185	0.171
$x = 0.2$	0.764	0.760	0.429	0.401
$x = 0.3$	0.913	0.940	0.682	0.673
$x = 0.5$	0.994	0.999	0.954	0.967
$x = 0.7$	0.999	1.000	0.998	0.999
$z = 20.0$	$\delta = 0.2$		$\delta = 0.4$	
	True	Normal	True	Normal
$x = 0.1$	0.350	0.308	0.054	0.055
$x = 0.2$	0.756	0.758	0.293	0.273
$x = 0.3$	0.948	0.971	0.657	0.651
$x = 0.5$	0.999	0.999	0.984	0.991
$x = 0.7$	1.000	1.000	0.999	0.999

TABLE 2.

$z = 5.0$	$\delta = 0.2$		$\delta = 0.4$	
	True	Normal	True	Normal
$x = 0.0$	0.385	0.344	0.192	0.178
$x = 0.2$	0.753	0.788	0.549	0.528
$x = 0.4$	0.933	0.977	0.837	0.855
$x = 0.6$	0.988	0.999	0.965	0.979
$x = 0.8$	0.999	0.999	0.996	0.998
$z = 10.0$	$\delta = 0.2$		$\delta = 0.4$	
	True	Normal	True	Normal
$x = 0.1$	0.434	0.443	0.176	0.171
$x = 0.2$	0.706	0.760	0.414	0.401
$x = 0.3$	0.879	0.940	0.667	0.673
$x = 0.5$	0.989	0.999	0.950	0.967
$x = 0.7$	0.999	1.000	0.997	0.999

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7. Proofs

Proof of Theorem 1. Let $\mathcal{F}_n = \sigma(\xi_1, \ldots, \xi_n)$ be the σ-field generated by the first n ξs, and set $\bar{\xi}_n = \xi_n - p_\infty$. Using the fact that p_∞ satisfies (6), we obtain $\mathbb{E}[\bar{\xi}_n \mid \mathcal{F}_{n-1}] = a\bar{\xi}_{n-1}$ with $a = f(1) - f(0)$. By recursion, $\mathbb{E}[\bar{\xi}_n] = a^{n-1}\mathbb{E}[\bar{\xi}_1]$. Since $|a| < 1$, we may conclude that $\lim_{n\to\infty} \mathbb{E}[\xi_n] = \lim_{n\to\infty} \mathbb{E}[p_n] = p_\infty$. To determine the limit distribution, we consider the Laplace transform of ξ_n. For any real u,

$$\mathbb{E}[\mathrm{e}^{u\xi_n} \mid \mathcal{F}_{n-1}] = \mathrm{e}^u p_n + 1 - p_n$$

so that

$$\mathbb{E}[\mathrm{e}^{u\xi_n}] = \mathrm{e}^u \mathbb{E}[p_n] + 1 - \mathbb{E}[p_n] = \mathrm{e}^u \mathbb{E}[\xi_n] + 1 - \mathbb{E}[\xi_n] \to \mathrm{e}^u p_\infty + 1 - p_\infty,$$

which is the Laplace transform of a Bernoulli variable with success probability p_∞. This proves the distributional convergence of ξ_n to ξ_∞. □

Proof of Theorem 2. For any real u, we have

$$\begin{aligned}
\psi_\infty(u) &= \lim_{n\to\infty} \mathbb{E}[\mathrm{e}^{uZ_n}] \\
&= \lim_{n\to\infty} \mathbb{E}[\mathbb{E}[\mathrm{e}^{uZ_n} \mid \mathcal{F}_{n-1}]] \\
&= \lim_{n\to\infty} \mathbb{E}[\mathrm{e}^u f(\xi_{n-1}) + 1 - f(\xi_{n-1})]^z \\
&= \mathbb{E}\left[\mathrm{e}^u f\left(\frac{Z_\infty}{z}\right) + 1 - f\left(\frac{Z_\infty}{z}\right)\right]^z.
\end{aligned}$$

This concludes the proof. □

Proof of Theorem 3. Since

$$\mathbb{E}[Z_\infty] = \lim_{n\to\infty} \mathbb{E}[Z_n] = \lim_{n\to\infty} \mathbb{E}[\mathbb{E}[Z_n \mid \mathcal{F}_{n-1}]] = z \lim_{n\to\infty} \mathbb{E}[f(\xi_{n-1})].$$

To determine the last limit, note that

$$\begin{aligned}
\mathbb{E}[f(\xi_n) - p_\infty] &= \mathbb{E}[f(\xi_n) - f(p_\infty)] \\
&= \nu(1-\delta)\mathbb{E}[(\xi_n - p_\infty)] \\
&= \nu(1-\delta)\mathbb{E}[\mathbb{E}[(\xi_n - p_\infty) \mid \mathcal{F}_{n-1}]] \\
&= \nu(1-\delta)\mathbb{E}[f(\xi_{n-1}) - p_\infty].
\end{aligned}$$

Since $0 \le \nu(1-\delta) < 1$, the expectation operator along the sequence $f(\xi_n) - p_\infty$ is a contraction. This concludes the proof. □

Proof of Theorem 4. For group size z, we have

$$\mathbb{E}[[\xi_n - p_\infty]^2 \mid \mathcal{F}_{n-1}] = \frac{p_n(1-p_n)}{z} + (p_n - p_\infty)^2.$$

Since $(p_n - p_\infty)^2 = (f(\xi_{n-1}) - f(p_\infty))^2 \le c^2(\xi_{n-1} - p_\infty)^2$ and $p_n(1-p_n) \le \frac{1}{4}$, we obtain

$$\mathbb{E}[\xi_n - p_\infty]^2 \le \frac{1}{4z} + c^2\mathbb{E}[\xi_{n-1} - p_\infty]^2.$$

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By iteration, we obtain

$$\mathbb{E}[\xi_n - p_\infty]^2 \le \frac{1}{4z}(1 + c^2 + \cdots + c^{2(n-2)}) + c^{2(n-1)}\mathbb{E}[\xi_1 - p_\infty]^2$$
$$\le \frac{1}{4(1-c^2)z} + c^{2(n-1)}\mathbb{E}[\xi_1 - p_\infty]^2.$$

This concludes the proof. □

Proof of Theorem 5. This time we use the Fourier transform in order to guarantee that all (transformed) variables are bounded. Now, since ξ_n is conditionally binomial,

$$\mathbb{E}[\exp(\mathrm{i}u\sqrt{z}(\xi_n - p_n)) \mid \mathcal{F}_{n-1}] = \left(1 - \frac{u^2}{2z}p_n(1-p_n) + o\left(\frac{1}{z}\right)\right)^z$$
$$\xrightarrow{\mathbb{P}} \exp\left(-\frac{u^2}{2}p_\infty(1-p_\infty)\right),$$

by Theorem 4, where '$\xrightarrow{\mathbb{P}}$' denotes convergence in probability. We conclude that, by the Lebesgue dominated convergence theorem,

$$\mathbb{E}[\exp(\mathrm{i}u\sqrt{z}(\xi_n - p_n))] \to \exp\left(-\frac{u^2}{2}p_\infty(1-p_\infty)\right).$$

This completes the proof. □

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