Student Problems

Students up to the age of 19 are invited to send solutions to either or both of the following problems to Beth Woollacott, 20 Rectory Close, Sutton Bonington, Loughborough, Nottinghamshire, LE12 5PJ.

Two prizes will be awarded – a first prize of £25, and a second prize of £20 – to the senders of the most impressive solutions for either problem. It is not necessary to submit solutions to both. Entries should arrive by 20th September 2021 and solutions will be published in the November 2021 edition.

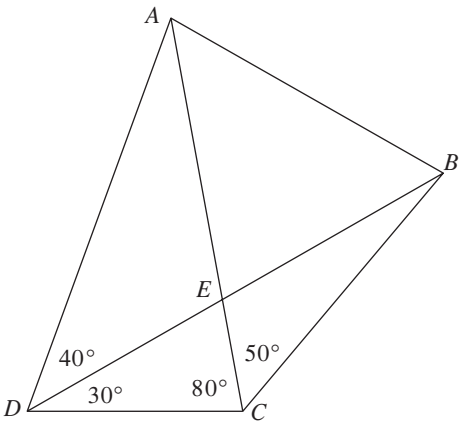
The Mathematical Association and the *Gazette* comply fully with the provisions of the 2018 GDPR legislation. Submissions **must** be accompanied by the SPC permission form which is available on the MA website

https://www.m-a.org.uk/the-mathematical-gazette

Note that if permission is not given, a pupil ***may still participate and will be eligible for a prize in the same way as others****.*

Problem 2021.3 **(Geoffrey Strickland)**

The diagram shows a quadrilateral $ABCD$ with diagonals AC, BD intersecting at E. $\angle ADB = 40°$, $\angle ACB = 50°$, $\angle ACD = 80°$ and $\angle BDC = 30°$.



Find the angles of triangle AEB.

Problem 2021.4 **(Paul Stephenson)**

Let n and k be integers greater than 1 with k a power of 2. Prove that $n^{2k} + n^k + 1$ has at least $2k$ different factors.

Note from the Editor

See *www.m-a.org.uk/the-mathematical-gazette* for the SPC questions.

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Solutions to 2021.1 and 2021.2

Both problems were solved by Christopher Mattocks (Bishops Wordsworth's School, Salisbury) and Anujan Sribavananthan (William Perkin Church of England High School, North Greenford). Problem 2021.1 was also solved by a student from St Peter's School, West Bengal, India.

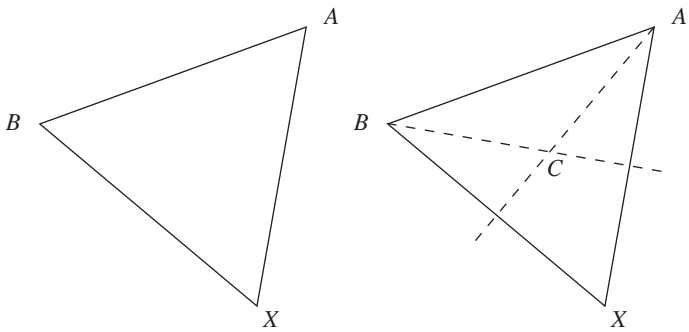
Problem 2021.1 (**Stan Dolan**)

Each vertex of a triangle is reflected in its opposite side. For what triangles will at least two of the new points be equal?

Solution

Both Christopher and Anujan gave very thorough proofs for this problem; however, I give a more direct solution here for brevity (provided by Stan Dolan).

Let A, B and C be the vertices of the triangle. If X is the image of A in BC then $XB = AB$. If X is also the image of B in AC then, similarly $XA = AB$. Therefore $\triangle ABX$ is an equilateral triangle.

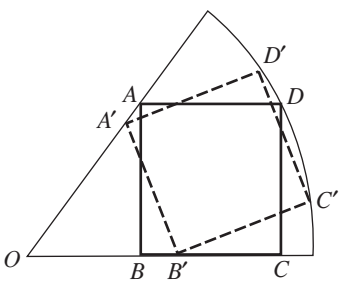


BC bisects AX and AC bisects BX. Therefore C is on the medians of ABX through A and B. Then C is the centroid of the triangle ABX and so $\triangle ABC$ is a $(30°,\ 30°,\ 120°)$ triangle.

Problem 2021.2 (**Paul Stephenson**)

The diagram shows an acute-angled sector of a circle. The sector circumscribes a unit square $ABCD$ which shares a side with the sector and circumscribes a unit square $A'B'C'D'$ which shares an axis of symmetry with the sector.

Find the angle of the sector.



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Solution

This is Anujan's very thorough proof:

Let $\angle AOB = \angle B'OA' = 2\theta$ and r be the radius of the sector. As $A'B'C'D'$ shares an axis of symmetry with the sector:

$$\angle OA'B' = \angle OB'A' = 90° - \theta$$

$$OB' = OA' = \frac{1}{2 \sin \theta}.$$

Applying the cosine rule to $\Delta OB'C'$:

$$\begin{aligned} r^2 &= OB'^2 + B'C'^2 - 2OB'.B'C' \cos \angle OB'C' \\ &= \left(\frac{1}{2 \sin \theta}\right)^2 + 1^2 - 2\left(\frac{1}{2 \sin \theta}\right)(1) \cos(180° - \theta) \\ &= \frac{1}{4 \sin^2 \theta} + 1 + \frac{\cos \theta}{\sin \theta}. \end{aligned} \tag{1}$$

Now, since $\sin 2\theta = \frac{BA}{OA} = \frac{1}{OA}$ we get $OA = \frac{1}{\sin 2\theta}$ and applying the cosine rule to ΔOAD:

$$\begin{aligned} r^2 &= OA^2 + AD^2 - 2OA.AD \cos \angle OAD \\ &= \left(\frac{1}{\sin 2\theta}\right)^2 + 1^2 - 2\left(\frac{1}{\sin 2\theta}\right)(1) \cos(180° - 2\theta) \\ &= \frac{1}{\sin^2 2\theta} + 1 + \frac{2 \cos 2\theta}{\sin 2\theta}. \end{aligned} \tag{2}$$

Then combining (1) and (2):

$$\frac{1 + 4 \sin \theta \cos \theta}{4 \sin^2 \theta} = \frac{1 + 2 \sin 2\theta \cos 2\theta}{\sin^2 2\theta}.$$

So using trigonometric identities we get $2 \sin 2\theta = 1$. Thus $2\theta = 30°$ and so the angle of the sector is 30°.

Prizewinners

I have awarded Anujan the first prize of £25 and Christopher the second prize of £20 – both showed very impressive mathematical skill.

10.1017/mag.2021.87 BETH WOOLLACOTT

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A Farewell to Stan

It is time to say thanks to Stan Dolan for all his work on Student Problem Corner over many years. He took over from Tim Cross early in 2010 and has produced and processed many issues since then. In that time, he will have dealt with dozens of young mathematicians and teachers. His efforts are greatly appreciated. We hope he will continue to write for the *Gazette* as well as being a referee and a participant in Problem Corner. We feel sure that his SPC work will be continued in the safe hands of his successor.

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