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Homogeneous line-segment processes

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Summary. A number of formulae of an integral geometric character are presented for the most general planar, homogeneous line-segment process, one in which a given segment length and orientation may depend upon (*a*) the point process of segment mid-points and (*b*) the lengths and orientations of other segments. The sense in which these formulae have a probabilistic/statistical interpretation is made precise. For the general process, two interpretations are given; one requires the theory of Palm distributions whilst the other depends upon ergodic results. When additional structure for the process is assumed, the integral geometric formulae lead to interesting, non-intuitive sampling formulae.

1. *Introduction.* Let us consider a homogeneous (stationary and isotropic) line-segment process. A line-segment process (LSP) is a point process on the set Z of all finite length line-segments. For convenience, we note that each $z \in Z$ can be parametrized by four parameters. There are a number of possible choices for these four parameters, but we will choose (x, y, l, t) where (x, y) is the mid-point of z, l its length, and t its orientation measured against some fixed axis. Thus we consider henceforth that $Z = R^2 \times (0, \infty) \times [0, \pi)$ and that an LSP is effectively this associated point process (APP) on Z, assumed to be *orderly* (no multiple points).

In Parker and Cowan (9) a fairly general type of stationary, isotropic LSP was studied and a variety of applications for LSP reviewed. In addition to these applications, which included models for the structure of paper and textiles, for random suspensions of molecules in solution or gel, and for the data derived in some cloud chamber experiments, line-segment processes are a natural model for describing the evolution in time of any point process. For example, the two-dimensional point process of plants in an area of land should more properly be viewed as an LSP in three dimensions with the lengths of the line-segments representing the lifetimes. In modern concrete technology there is an example of LSP. When objects having unusual shape and thin sections are moulded, short carpenters' nails are mixed with the concrete for reinforcing. Finally we note that an LSP can provide a close approximation to the more general process of rectifiable curves.

In the paper of Parker and Cowan, a 'mean stationary' point process of mid-points having intensity τ in R^2 was given. Then to each mid-point taken sequentially in the order of proximity to the origin a random orientation and length was assigned. The assignment was such that all lengths and orientations were mutually independent and independent of the mid-point process. Also assigned lengths had distribution function

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F whilst orientations were uniformly distributed on $[0, \pi)$. It was shown that if $K(U, \omega)$ were the number of points of the APP in $U \in \mathscr{B}(Z)$, the Borel sets of Z, then

$$EK(U) = \iiiint_{(x,y,l,t) \in U} (\tau/\pi)\, dx\, dy\, dF(l)\, dt \tag{1}$$

and this representation was used as the basis of a number of calculations.

In this paper we extend these earlier results to a quite general homogeneous LSP and provide some additional results. Such an aim is worthwhile; for in many LSP the above-mentioned independence assumptions are inapplicable. For example, the line-segments may be components of more elaborate geometric bodies such as polygons or random graphs (in the sense of Santaló (10)). Indeed the segments may be so arranged as to form a mosaic or other pattern (see Ambartsumian (1), (2) and Cowan (5)). In fact, if an LSP is such that segments cross each other then it may be useful to consider the *dual* LSP comprising the *segments of segments* created by these crossings.

In extending these results, we generate some fresh considerations concerning the *interpretation* of our new formulae. For the general setting, we find that our results, which are essentially 'integral geometric', have a non-intuitive probabilistic interpretation. When the mid-point process is assumed to be Poisson, the additional structure leads to interesting sampling interpretations. In the general setting we develop the so-called *Palm methods* of sampling. We also present some theory for the case when the LSP is assumed ergodic. This theory allows us further *large sample* probabilistic interpretations of the integral geometric formulae.

2. *The integral geometric implications of homogeneity.* We commence with only the homogeneity assumptions of stationarity and isotropy. This means that $EK(.)$ must be a measure on $(Z, \mathscr{B}(Z))$, *invariant* under the group of translations and rotations of the plane. We now show that this invariance implies that

$$EK(U) = \iiiint_{(x,y,l,t) \in U} (\tau/\pi)\, dx\, dy\, dG(l)\, dt \tag{2}$$

for $U \in \mathscr{B}(Z)$ and some function G. To see this, we prove a lemma concerning spaces, such as our Z, which can be represented as a product $C \times D$ with C being a locally compact group and D being a topological space. In our context C will be $R^2 \times [0, \pi)$ with the group multiplication being defined by $(x, y, t)(x', y', t') = (x + x', y + y', [t + t'] \bmod \pi)$, whilst D will be $(0, \infty)$.

Let $\mathscr{S}$ be the class of functions mapping $C \to C$ defined for each $a \in C$ by $f_a(c) = a^{-1}c$; $c \in C$. These functions can also be defined as a map $Z \to Z$ by the statement

$$f_a(c, d) = (a^{-1}c, d); \quad a, c \in C, d \in D.$$

Suppose there is a measure η on $(Z, \mathscr{B}(Z))$ which is invariant under the action of the functions in $\mathscr{S}$; that is for $U \in \mathscr{B}(Z)$, $a \in C$, $\eta(U) = \eta(f_a U)$. We shall show that η is a product measure on $(Z, \mathscr{B}(Z))$, the product of an arbitrary measure on $(D, \mathscr{B}(D))$ and a very special measure on $(C, \mathscr{B}(C))$. Here $\mathscr{B}(.)$ denotes the Borel sets of the said space.

LEMMA 1. *The measure η is a product measure defined on product sets $U_1 \times U_2$ by $\eta(U_1 \times U_2) = \rho(U_1)\lambda(U_2)$; $U_1 \in \mathscr{B}(C)$, $U_2 \in \mathscr{B}(D)$. Here λ is an arbitrary measure on $(D, \mathscr{B}(D))$ and ρ is a Haar measure on $(C, \mathscr{B}(C))$.*

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Proof. It is well-known (Halmos(7)) that there exists a non-zero measure, ρ say, on $(C, \mathscr{B}(C))$ that is invariant under the action of the underlying group; that is

$$\rho(a^{-1}U_1) = \rho(U_1) \quad \text{for} \quad a \in C,\ U_1 \in \mathscr{B}(C).$$

Also it is known that all non-zero measures which have this invariance property must be multiples of ρ. These invariant measures are called Haar measures.

Now $\eta(U) = \eta(f_a U)$ for $U \in \mathscr{B}(Z)$. In particular if U is a product set $U_1 \times U_2$, then $\eta(U_1 \times U_2) = \eta(f_a(U_1 \times U_2)) = \eta(f_a U_1 \times U_2) = \eta(a^{-1}U_1 \times U_2)$. So $\eta(. \times U_2)$ is a Haar measure on $(C, \mathscr{B}(C))$ for every $U_2 \in \mathscr{B}(D)$. Thus $\eta(. \times U_2)$ is a multiple of ρ, say $\rho(.)\,\lambda(U_2)$. Furthermore, it is easy to verify that the set function λ is a measure on $(D, \mathscr{B}(D))$, so we have that for product sets $U_1 \times U_2$, $\eta(U_1 \times U_2) = \rho(U_1)\,\lambda(U_2)$ thus defining η for all $U \in \mathscr{B}(Z)$. This proves the lemma.

To prove (2), it is sufficient to note that Lebesgue measure and multiples of it are invariant on the locally compact group $C = R^2 \times [0, \pi)$; so these are the Haar measures. $EK(.)$ is therefore a product measure generated by an arbitrary measure $dG(.)$ on $(0, \infty)$ and a multiple, τ/π say, of Lebesgue measure on $R^2 \times [0, \pi)$.

The integral representations given in formulae (1) and (2) are the same except that $dG(l)\,dt$ cannot be interpreted in the same way as $dF(l)\,dt$. For an $A \subset R^2$ with Lebesgue measure $m(A)$, $\tau G(l)\,m(A)\,t/\pi$ can usually be interpreted only as the *expected number* of line segments having mid-points within A, lengths $\leqslant l$ and orientations $\leqslant t$. For the case where the various independence assumptions are made then $F(l)$ and t/π have the additional interpretation as length and orientation distribution functions for 'suitably sampled' line-segments. For example, if one chooses the line-segment which has mid-point closest to some fixed point then in the case studied in Parker and Cowan, $F(l)$ and t/π equal the *probabilities* that this line has length $\leqslant l$ and orientation $\leqslant t$. Also, the integral $\int_0^\infty u\,dF(u)$, which we denote by μ, is the expected length of this chosen segment. In the general case, the same interpretation does not apply for $G(x)$ or for the integral $\int_0^\infty u\,dG(u)$ (denoted by ν). In Section 4 we shall see that G and ν have interpretations as a length distribution and mean length in two senses: a *Palm distribution* sense and an *ergodic* sense.

Now in the general case, the calculation of $EK(.)$ for appropriate subsets of Z yields a number of mean-value formulae. Let A be a Borel subset of R^2 and define $M(A)$ = the number of mid-points within A, $X(A)$ = the total length of the $M(A)$ segments that have mid-point in A, $N(A)$ = the number of segments that intersect with A, $Y(A)$ = the total length of the $N(A)$ segments that intersect with A, $S(A)$ = the total length *within* A of these $N(A)$ segments, and $T(A)$ = the number of segments wholly contained in A. Then

$$EM(A) = \tau m(A), \tag{3}$$

$$EX(A) = \tau\nu m(A), \tag{4}$$

$$ES(A) = \tau\nu m(A), \tag{5}$$

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whilst for a convex set A having perimeter $\mathscr{P}(A)$,

$$EN(A) = \tau m(A) + \tau\nu\mathscr{P}(A)/\pi, \tag{6}$$

$$EY(A) = \tau\nu m(A) + (\tau/\pi)\,\mathscr{P}(A)\int_0^\infty l^2\,dG(l) \tag{7}$$

and

$$ET(B_r) = \tau r^2\int_0^{2r}[2\cos^{-1}(u/2r) - (u/2r^2)\,(4r^2-u^2)^{\frac{1}{2}}]\,dG(u) \tag{8}$$

where B_r is the ball of radius r. For the case considered in Parker and Cowan, these expressions contain F and μ instead of G and ν. In passing, we note that (3), (4) and (5) are valid without isotropy, whilst for A a ball, (6) and (7) are also valid without isotropy. We also note that, in the general case, equations (5) and (6) yield moment estimators of τ and ν in terms of the observable variables N and S:

$$\tilde{\nu} = \pi Sm/(Nm\pi - S\mathscr{P}) \quad \text{and} \quad \tilde{\tau} = (Nm\pi - S\mathscr{P})/(\pi m^2).$$

In Section 4 it is shown that these estimates are consistent if the process is ergodic.

Further detail is possible. For example, let $N_{l,t}(A)$ be the number of segments intersecting with A which have length $\leqslant l$ and orientation $\leqslant t$. Also let $N^*_{l,t}(A)$ be the number of segments having orientation $\leqslant t$ and intersecting with A, whose length *within* A is $\leqslant l$. Appropriate integrations show that

$$EN_{l,t}(A) = (\tau/\pi)\,[tG(l)\,m(A) + \int_0^l u\,dG(u).\int_0^t \alpha(y)\,dy] \quad \text{(convex } A\text{)}, \tag{9}$$

$$EN^*_{l,t}(B_r) = tEN(B_r)/\pi \quad \text{if} \quad l > 2r \tag{10}$$

$$= (t\tau/\pi)\Big[\pi r^2 + 2r\nu - 2r^2\{1-G(l)\}\cos^{-1}(l/2r) - \tfrac{1}{2}(4r^2-l^2)^{\frac{1}{2}}\Big(2\nu - 2\int_0^l u\,dG(u) - 3l\{1-G(l)\}\Big)\Big] \quad \text{if} \quad l \leqslant 2r.$$

Here $\alpha(y)$, which is shown in Figure 1 of Parker and Cowan (9), is the 'caliper diameter' of A in direction y. For special cases of A, $\alpha(y)$ can be simple. For example, if A is itself a line-segment of length b, orientation ϕ then $\alpha(y) = b\sin|y-\phi|$ and

$$\left.\begin{aligned} EN_{l,t}(A) &= (\tau b/\pi)\,[\cos(\phi - t) - \cos\phi]\int_0^l u\,dG(u) \qquad \text{if} \quad \phi > t \\ &= (\tau b/\pi)\,[2 - \cos\phi - \cos(t-\phi)]\int_0^l u\,dG(u) \quad \text{if} \quad \phi \leqslant t. \end{aligned}\right\} \tag{11}$$

In fact, if the above reference line-segment A is *directed* and $N'_{l,t}(A)$ is defined as the number of LSP segments which intersect A, have length $\leqslant l$ and orientation, *relative to* A, $\leqslant t$, then (11) implies that

$$EN'_{l,t}(A) = (\tau b/\pi)\,[1-\cos t]\int_0^l x\,dG(x). \tag{12}$$

This leads to an interesting formula relating to the interaction of the LSP with any fixed rectifiable curve. Let C be a (directed) rectifiable curve of length q. Define $Q(C)$ as the number of intersection points of the LSP with C, and $Q_{l,t}(C)$ as the number of these intersection points which arise from segments of length $\leqslant l$ and orientation,

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relative to C, $\leqslant t$. Since a rectifiable curve is (by definition) the limit of some sequence of 'line-segment chains', we have from (12) that

$$EQ_{l,t}(C) = (\tau q/\pi)[1-\cos t]\int_0^l u\,dG(u); \tag{13}$$

$$EQ(C) = 2\tau qv/\pi. \tag{14}$$

Formula (14) follows from either (13) or (6).

3. *Coleman's process.* Coleman (3), (4) has considered an LSP where the mid-points are a *Poisson* process and the lengths and orientations assigned independently as in Parker and Cowan. For such a process it is possible, with the aid of a trivial lemma, to interpret formulae (3)–(14) in another way (that is, the versions which use F and μ).

LEMMA 2. *Suppose that U white balls and V black balls are placed in an urn. Let U and V be independent Poisson distributed random variates. Suppose that, conditional upon the event $\mathscr{F} = \{U+V \geqslant 1\}$, a ball is randomly drawn from the urn. (Here and in applications of the lemma we mean uniformly random draws.) If $\mathscr{E}$ is the event that the drawn ball is white, then*

$$P\{\mathscr{E}|\mathscr{F}\} = E(U)/E(U+V).$$

In the line-segment context, let U be the number of segments with mid-points in A whose length and orientation are respectively $\leqslant l$ and $\leqslant t$. Let V be defined as $M(A)-U$. In Coleman's process U and V are independent Poisson variates with means $\tau tF(l)\,m(A)/\pi$ and $[1-F(l)\,t/\pi]\,\tau m(A)$ respectively. Thus the lemma states that conditional upon $M(A) \geqslant 1$, if we randomly select a mid-point within A and observe the length and orientation of the associated segment, then $tF(l)/\pi$ is the joint probability that it will have length $\leqslant l$ and orientation $\leqslant t$. Clearly the sampled length and orientation are independent, with length distribution being F and orientations uniformly distributed on $[0,\pi)$.

On the other hand, suppose we randomly select one of the $N(A)$ segments which intersect with a convex A and observe its orientation and *complete* length. Since $N_{l,t}(A)$ and $N(A)-N_{l,t}(A)$ are independent Poisson variates, we have from (9) and (6) that, conditional upon $\mathscr{F} = \{N(A) \geqslant 1\}$, the probability that the observed length and orientation are $\leqslant l$ and $\leqslant t$, respectively, is given by

$$H(l,t|\mathscr{F}) = \frac{tF(l)\,m(A) + \int_0^l u\,dF(u)\int_0^t \alpha(y)\,dy}{\pi m(A) + \mu\mathscr{P}(A)}. \tag{15}$$

The special cases of (15) are somewhat more interesting. The marginal distributions of length and orientation given $\mathscr{F}$ are

$$H(l,\pi|\mathscr{F}) = \frac{\pi F(l)\,m(A) + \mathscr{P}(A)\int_0^l u\,dF(u)}{\pi m(A) + \mu\mathscr{P}(A)}, \tag{16}$$

$$H(\infty,t|\mathscr{F}) = \frac{tm(A) + \mu\int_0^t \alpha(y)\,dy}{\pi m(A) + \mu\mathscr{P}(A)}. \tag{17}$$

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When F has a density f, then the marginal *densities* can be written as

$$h(l,\pi|\mathscr{F}) = \frac{f(l)[\pi m(A)+l\mathscr{P}(A)]}{\pi m(A)+\mu\mathscr{P}(A)} \quad (l>0), \tag{18}$$

$$h(\infty,t|\mathscr{F}) = \frac{m(A)+\mu\alpha(t)}{\pi m(A)+\mu\mathscr{P}(A)} \quad (t\in[0,\pi)). \tag{19}$$

When $A = B_r$, these densities become

$$h(l,\pi|\mathscr{F}) = f(l)[\pi r+2l]/[\pi r+2\mu] \quad (l>0), \tag{20}$$

$$h(\infty,t|\mathscr{F}) = \pi^{-1} \quad (t\in[0,\pi)). \tag{21}$$

When A is itself a line-segment of length b, orientation ϕ, chosen independently of the LSP then (18) and (19) yield the more familiar forms

$$\begin{aligned} h(l,\pi|\mathscr{F}) &= lf(l)/\mu \quad (l>0). \\ h(\infty,t|\mathscr{F}) &= \tfrac{1}{2}\sin|t-\phi| \quad (t\in[0,\pi)). \end{aligned} \tag{22}$$

Furthermore, if one were to observe orientations *relative* to a (directed) line-segment A then the randomly selected segment from the LSP would have orientation density $h(\infty,t|\mathscr{F}) = \frac{1}{2}\sin t$. We have closely related results concerning sampling from those lines which cut a given rectifiable curve C of length q. Conditional upon $\mathscr{F} = \{Q(C)\geqslant 1\}$ we select randomly one of the $Q(C)$ intersection points and observe the associated line-segment. Since $Q_{l,t}(C)$ and $Q(C)-Q_{l,t}(C)$ are independent Poisson variates, the probability that this chosen segment has length $\leqslant l$ and orientation (at that particular intersection point) *relative to* $C \leqslant t$ is [from (13) and (14)] $\frac{1}{2}[1-\cos t]\int_0^l u\,dF(u)/\mu$. Thus, conditional upon $\mathscr{F}$, the observed length and orientation are independent and have the familiar density functions $lf(l)/\mu$ and $\frac{1}{2}\sin t$, respectively.

Numerous other interpretations of a similar character and based upon (3)–(14) can be found.

4. *Palm methods and ergodic theory.* In the preceding two sections we have given unbiased methods for sampling a line-segment having length and orientation distributions F and uniform respectively. (We have also given numerous sampling methods which give interesting biases to these distributions.) The unbiased sampling methods were proposed in the contexts of the Parker–Cowan process and the Coleman process. In this section we return to the general LSP and ask whether we can sample a line-segment whose length distribution is G and whose orientation distribution is uniform. In general, there appears to be no *realizable* sampling scheme, however there are two conceptually important (but *unrealizable*) sampling schemes which yield G as a length distribution. The first involves the so-called Palm methods and the second ergodic theoretic methods.

For our first method, suppose that we were lucky enough to have a realization where

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a segment mid-point was located at the origin. We show now that the length and orientation of this segment are distributed independently as G and uniform respectively.

For all realizations, let D_i be the distance from the origin to the ith nearest mid-point, L_i and T_i be the length and orientation of the associated line-segment and $D_i^*(l,t)$ be the distance from the origin to the ith nearest mid-point whose associated segment has length $\leqslant l$ and orientation $\leqslant t$. Denote $P\{D_i \leqslant x\}$ by $C_i(x)$, the joint probabilities $P\{D_i \leqslant x,\ L_i \leqslant l,\ T_i \leqslant t\}$ by $A_i(x;l,t)$ and $P\{D_i^*(l,t) \leqslant x\}$ by $C_i^*(x;l,t)$. We wish to show that, as $x \to 0$, $P\{L_1 \leqslant l,\ T_1 \leqslant t | D_1 \leqslant x\} \to tG(l)/\pi$; that is to say

$$\lim A_1(x;l,t)/C_1(x) = tG(l)/\pi. \tag{23}$$

Now the mid-points form an orderly, homogeneous point process of intensity τ. Also the mid-points having segments of length $\leqslant l$ and orientation $\leqslant t$ form an orderly point process of intensity $\tau tG(l)/\pi$. Applying Khintchine's theorem (see Leadbetter (8) for an appropriate two-dimensional version) we have that, as $x \to 0$, $C_1(x)/\pi x^2 \to \tau$ and $C_1^*(x;l,t)/\pi x^2 \to \tau tG(l)/\pi$. Also it is simple to show that $\sum_{i=1}^{\infty} C_i(x) = EM(B_x) = \tau\pi x^2$ and so $\sum_{i=2}^{\infty} C_i(x)/\pi x^2 \to 0$ as $x \to 0$. Now if $\mathscr{E}_i = \{L_i \leqslant l,\ T_i \leqslant t\}$ then

$$\{D^*(l,t) \leqslant x\} = \{D_1 \leqslant x, \mathscr{E}_1\} \cup \{D_2 \leqslant x, \bar{\mathscr{E}}_1, \mathscr{E}_2\} \cup \{D_3 \leqslant x, \bar{\mathscr{E}}_1, \bar{\mathscr{E}}_2, \mathscr{E}_3\} \cup \ldots .$$

Thus $C_1^*(x;l,t) = A_1(x;l,t)$ plus terms dominated by $\sum_{i=2}^{\infty} C_i(x)$ and so

$$A_1(x;l,t)/\pi x^2 \to \tau tG(l)/\pi.$$

Therefore (23) follows.

We now consider the second unrealizable method of sampling. If one has a *large* domain A, and we randomly select one of the segments which intersect with A (conditional upon there being at least one), observe its orientation and length (either the complete length or length within A, it matters not), then intuitively we might expect that its length and orientation distributions will be approximately G and uniform respectively. This intuitive hope can be converted into a mathematical statement when the LSP is assumed *ergodic*. We now make this assumption and develop the necessary mathematical argument.

Without loss of generality we consider the domain A to be a ball B_r of radius r. Let $J_r(l,t)$ be defined as the probability that, conditional upon $\mathscr{F}_r = \{N(B_r) \geqslant 1\}$, a segment randomly chosen from the $N(B_r)$ segments has complete length $\leqslant l$ and orientation $\leqslant t$. Clearly

$$J_r(l,t) = E[N_{l,t}(B_r)/N(B_r) | \mathscr{F}_r]. \tag{24}$$

Alternatively, if one decides to observe only the length *within* B_r, then the appropriate joint distribution is

$$J_r^*(l,t) = E[N_{l,t}^*(B_r)/N(B_r) | \mathscr{F}_r]. \tag{25}$$

We now prove that both J_r and J_r^* converge to $tG(l)/\pi$ as $r \to \infty$. To do this we must first explore some of the consequences of ergodicity.

If the LSP is ergodic then each of the three point processes formed by considering

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(a) the mid-points, or (b) the left-ends, or (c) the right-ends of segments are stationary ergodic point processes. Therefore as $r \to \infty$,

$$\frac{M(B_r)}{\pi r^2} \xrightarrow{\text{a.s.}} \tau, \tag{26}$$

$$\frac{L(B_r)}{\pi r^2} \xrightarrow{\text{a.s.}} \tau, \tag{27}$$

$$\frac{R(B_r)}{\pi r^2} \xrightarrow{\text{a.s.}} \tau, \tag{28}$$

where $L(B_r)$ and $R(B_r)$ are respectively the numbers of left-ends and right-ends within B_r.

We shall now show that as $r \to \infty$,

$$\frac{S(B_r)}{\pi r^2} \xrightarrow{\text{a.s.}} \tau\nu. \tag{29}$$

For $y \ll r$, we note the almost sure inequality

$$\int_{B_{r-y}} S(B_y + u)\, du \leqslant \pi y^2 S(B_r) \leqslant \int_{B_{r+y}} S(B_y + u)\, du, \tag{30}$$

where $B_y + u$ is a circle of radius y, centre u. Division throughout (30) by πr^2 and the application of Wiener's ergodic theorem [see Cowan (6) for a discussion of this theorem in the random geometry context] shows that both left and right sides (and hence the centre term) converge almost surely to $ES(B_y)$. Thus from (5), we have a proof of (29).

Next we prove that as $r \to \infty$

$$\frac{N(B_r) - T(B_r)}{\pi r^2} \xrightarrow{\text{a.s.}} 0, \tag{31}$$

that is, the normalized number of segments cutting the boundary converges almost surely to zero. Note that by the use of (6) and (8), stationarity alone implies convergence in mean to zero, the extra assumption of ergodicity strengthens the result to a.s. convergence. To prove this we note the inequality

$$N(B_r) - T(B_r) \leqslant y^{-1}[S(B_{r+y}) - S(B_r)] + [L(B_{r+y}) - L(B_r)] + [R(B_{r+y}) - R(B_r)],$$

which, when divided by πr^2, confirms (31) via the results (27), (28) and (29).

Now $N(B_r) - T(B_r) = Z_1 + Z_2$ where Z_i equals the number of segments cutting ∂B_r i times. Obviously $Z_i/\pi r^2 \xrightarrow{\text{a.s.}} 0$. We note that $L(B_r) + R(B_r) = Z_1 + 2T(B_r)$ and hence we have that $T(B_r)/\pi r^2 \xrightarrow{\text{a.s.}} \tau$. Therefore

$$\frac{N(B_r)}{\pi r^2} \xrightarrow{\text{a.s.}} \tau. \tag{32}$$

Suppose one were to consider the 'thinned' LSP formed by ignoring all line-segments of length $> l$ and orientation $> t$. The 'thinned' LSP will be ergodic with a new 'τ parameter' $t\tau G(l)/\pi$. Therefore a repetition of the foregoing argument yields the result (analogous to (32)),

$$\frac{N_{l,t}(B_r)}{\pi r^2} \xrightarrow{\text{a.s.}} t\tau G(l)/\pi. \tag{33}$$

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Now $N_{l,t}(B_r) \leqslant N^*_{l,t}(B_r) \leqslant N_{l,t}(B_r) + N(B_r) - T(B_r)$; thus from (31) and (33)

$$\frac{N^*_{l,t}(B_r)}{\pi r^2} \xrightarrow{\text{a.s.}} t\tau G(l)/\pi, \tag{34}$$

$$\frac{N_{l,t}(B_r)}{N(B_r)} \xrightarrow{\text{a.s.}} tG(l)/\pi, \tag{35}$$

and

$$\frac{N^*_{l,t}(B_r)}{N(B_r)} \xrightarrow{\text{a.s.}} tG(l)/\pi. \tag{36}$$

Clearly, (35) and (36) have relevance to the asymptotic behaviour of $J_r(l,t)$ and $J^*_r(l,t)$. The following lemma enables us to complete the proof that

$$\lim_{r\to\infty} J_r(l,t) = \lim_{r\to\infty} J^*_r(l,t) = tG(l)/\pi. \tag{37}$$

LEMMA 3. *If* $0 \leqslant X_r \leqslant Y_r$ *are random variables such that as* $r \to \infty$,

$$X_r \xrightarrow{\text{a.s.}} a \quad \textit{and} \quad Y_r \xrightarrow{\text{a.s.}} b > 0,$$

then $E(X_r/Y_r | Y_r > 0) \to a/b$.

Proof. Let $Z_r = X_r/Y_r$ if $Y_r > 0$; $= 1$ if $Y_r = 0$. Now $Y_r \xrightarrow{\text{a.s.}} b$, thus $P\{Y_r = 0\} \to 0$ and $Z_r \xrightarrow{\text{a.s.}} a/b$. Also $|Z_r| \leqslant 1$, so from bounded convergence $E(Z_r) \to a/b$. Therefore

$$E(Z_r | Y_r > 0)\, P\{Y_r > 0\} + P\{Y_r = 0\} \to a/b,$$

and hence $E(Z_r | Y_r > 0) \to a/b$.

Thus we have made precise the interpretation of G as an *ergodic* length distribution and of the uniform distribution as an *ergodic* orientation distribution.

REFERENCES

(1) AMBARTSUMIAN, R. V. Random fields of segments and random mosaics on a plane. *Proc. Sixth Symp. Math. Statist. Prob.* **3** (1970), 369–381.

(2) AMBARTSUMIAN, R. V. Convex polygons and random tesselations. *Stochastic geometry*, ed. Harding and Kendall (London, Wiley, 1974).

(3) COLEMAN, R. Sampling procedures for the lengths of random straight lines. *Biometrika* **59** (1972), 415–426.

(4) COLEMAN, R. The distance from a given point to the nearest end of one member of a random process of linear segments. *Stochastic geometry*, ed. Harding and Kendall (London, Wiley, 1974).

(5) COWAN, R. Properties of ergodic random mosaic processes. (To appear *Mathematische Nachrichten*.)

(6) COWAN, R. The use of the ergodic theorems in random geometry. *Suppl. Adv. Appl. Prob.* **10** (1978), 47–57.

(7) HALMOS, P. *Measure theory* (Princeton, N.J., Van Nostrand, 1956).

(8) LEADBETTER, M. R. On basic results of point process theory. *Proc. Sixth Symp. Math. Statist. Prob.* **3** (1970), 449–462.

(9) PARKER, P. and COWAN, R. Some properties of line-segment processes. *J. Appl. Prob.* **13** (1976), 96–107.

(10) SANTALÓ, L. A. Random processes of linear segments and graphs. *Proc. Buffon. bi-centenary conf. Paris.* (1977).

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