NEGATIVE PROBABILITY

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1. *Negative probability in quantum mechanics.* The appearance of negative probabilities in certain recent contributions to quantum theory (see, for example, (2))* makes it of some importance to examine their theoretical validity and meaning. The appropriate generalization of mathematical probability is given below without reference to any particular field of application; though it is in the highly abstract symbolism of physics that this extension is most likely to prove useful. Moreover, it is in the same field that the relation between the probability theory and its physical interpretation, while tending to agree with the natural viewpoint (cf. (2), p. 8) that the extended theory merely corresponds to a more elaborate accounting system involving debit as well as credit entries, is also a warning not to ascribe individual reality to negative entries in this theoretical book-keeping.

2. *Formal extension of mathematical probability.* The development here will be based on the exposition of orthodox probability theory given in a previous paper (1). A primitive probability distribution was denoted by the symbolic expression

$$p_r A_r \equiv p_1 A_1 + p_2 A_2 + \ldots + p_m A_m, \tag{1}$$

where the probabilities p_r were positive rational fractions less than unity, representing the proportions of members of a class A which were members of its subclasses A_r. The additive and multiplicative laws of probability followed at once from this definition. In an obvious generalization of this theory all real numbers between zero and unity were admissible, and the number of classes did not necessarily remain finite. The theoretical relation between these probability numbers and conceptual frequencies, by which their use as a theory for actual frequencies could be introduced, was also discussed.

Consider now the equation $$p = p_1 + p_2, \tag{2}$$

where p_1 and p_2 denote the first two probabilities in (1). This is a probability equation illustrating the addition theorem of probability. But alternatively, equation (2) might have represented an arbitrary arithmetical partition of the number p. For example, if p denoted the chance of a biassed penny giving 'heads', and for p_1 we write $\frac{1}{2}$, p_2 is a correcting term. If $p \geqslant \frac{1}{2}$, then $p_2 \geqslant 0$, and (2) is also a valid probability equation. We are naturally led to ask whether, if $p < \frac{1}{2}$, and $p_2 < 0$, we can still in any consistent sense regard (2) as a probability equation, provided we admit negative numbers into the domain of admissible probability numbers. Since a negative probability implies automatically a complementary probability greater than unity, we shall reconsider (1) with all restrictions on the values of the individual p_r removed, provided that the

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sum remains finite and equal to the conventional sum of unity. For those familiar with the correspondence between probability theory and the theory of measure, it is noted that the parallel extension in this more general form of probability theory corresponds to the use of an additive set function which is always the algebraic difference of two positive functions.

Since the formal structure for the distributions is unaltered, it is trivial to show that the probabilities in the extended theory must for consistency obey the same rules as before, that is, the addition and multiplication laws. In particular, the familiar method of deriving the distributional moments for binomial frequencies r_1 and r_2 corresponding to n independent trials for the event represented by equation (2), by forming the expectation

$$E\{\exp(r_1 t_1 + r_2 t_2)\} = (1 - p + p_1 e^{t_1} + p_2 e^{t_2})^n, \tag{3}$$

whence are obtained the well-known results

$$\bar{r}_2 \equiv E\{r_2\} = np_2,$$
$$\sigma^2(r_2) \equiv E\{(r_2 - \bar{r}_2)^2\} = np_2(1 - p_2),$$
$$\operatorname{cov}(r_1, r_2) \equiv E\{(r_1 - \bar{r}_1)(r_2 - \bar{r}_2)\} = -np_1 p_2,$$

etc., is also valid for p_2 negative.

3. *Limiting distributions and frequency ratios.* Having set up a consistent extended theory which still retains the same rules as the orthodox theory, we may next ask what physical interpretation these more general probability numbers can have. In the orthodox theory mathematical limit theorems provide the appropriate theoretical relations between probability numbers and limiting frequency ratios (see, for example, (1)). It is, however, not clear how r_2/n can tend to p_2 in any sense in the extended theory when p_2 is negative since the formal distribution of r_2 only covers the positive range 0 to n. The usual proof (e.g. on p. 9 of my previous paper) in fact breaks down for negative probabilities. Notice, moreover, that the partition or probability-generating function for the distribution of r_2 alone,

$$(1 - p_2 + p_2 z_2)^n,$$

consists of a number of terms of alternating sign for negative p_2, the sum of the absolute values of the coefficients being $(1 + 2 \mid p_2 \mid)^n$ and tending to infinity with n.

Thus probabilities in the original range 0 to 1, as we might reasonably expect, still retain their special significance. It is only these probabilities which we can immediately relate with actual frequencies; it is only these probabilities, for example, for which the theoretical frequency ratio r/n tends to p with probability one, as n tends to infinity.

4. *Extraordinary random variables and their characteristic functions.* While binomial distributions with negative probabilities give rise to improper distributions as n becomes infinite, their use is still possible through their moment-generating or characteristic functions, which still have proper limits. We have, for example, the cumulant function of $r_2/\sqrt{n}$ defined by

$$\log E\{\exp r_2 t/\sqrt{n}\} \to \sqrt{n}\, p_2 t + \tfrac{1}{2} p_2 q_2 t^2, \tag{4}$$

where $q_2 = 1 - p_2$. We may consistently think of $r_2/\sqrt{n}$ as a random normal or Gaussian variable with mean $\sqrt{n}\, p_2$ and standard deviation $\sqrt{(p_2 q_2)}$. For p_2 negative, this random

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variable has an imaginary standard deviation; it will be termed, using Eddington's convenient adjective ((3), p. 5), an *extraordinary* random variable. More generally than (4), we may consider the joint distribution of r_1 and r_2. The variable $r/\sqrt{n}$ becomes a normal random variable with two correlated components $r_1/\sqrt{n}$ and $r_2/\sqrt{n}$, of which $r_2/\sqrt{n}$ has an imaginary standard deviation, and imaginary correlation with $r_1/\sqrt{n}$.

One further result will be noted. The characteristic function equation

$$M(t) = M_1(t)\, M_2(t), \tag{5}$$

where $M(t) \equiv E\{\exp itx\}$, etc., implies that the random variable x is the sum of two independent components x_1 and x_2. Given $M(t)$ and $M_1(t)$, this equation has the solution

$$M_2(t) = M(t)\, M_1^{-1}(t), \tag{6}$$

which is of the same form as (5) if $M_1^{-1}(t)$ can be interpreted as the characteristic function of some random variable. Since $M_1(t)\, M_1^{-1}(t) = 1$, $M_1^{-1}(t)$ will be said to define the *reciprocal* distribution to $M_1(t)$. This distribution is only an admissible one in our extended theory, since it corresponds to an extraordinary independent random variable y, such that the random variable $x_1 + y$ is certainly zero. A binomial distribution with negative probability p_2 gives rise to a well-behaved reciprocal distribution $(q_2 + p_2 z)^{-n}$, which has the form of the so-called 'negative binomial distribution'. Even on the present theory, however, $M_1^{-1}(t)$ does not necessarily define a distribution function obtainable from it by Fourier inversion. An example is the reciprocal of the normal law, already discussed at the beginning of this section as the limiting form of the binomial.

5. *Summary.* It has been shown that orthodox probability theory may consistently be extended to include probability numbers outside the conventional range, and in particular negative probabilities. Random variables are correspondingly generalized to include extraordinary random variables; these have been defined in general, however, only through their characteristic functions.

This generalized theory implies redundancy, and its use is a matter of convenience. Eddington (3) has employed it in this sense to introduce a correction to the fluctuation in number of particles within a given volume.

Negative probabilities must always be combined with positive ones to give an ordinary probability before a physical interpretation is admissible. This suggests that where negative probabilities have appeared spontaneously in quantum theory it is due to the mathematical segregation of systems or states which physically only exist in combination.

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