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Centralizers of partially hyperbolic diffeomorphisms

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Abstract. It has been conjectured that a generic diffeomorphism on a compact manifold will have trivial centralizer. In this paper we show that the conjecture is true within certain classes of partially hyperbolic diffeomorphisms.

1. *Introduction*

Let $f \in \mathrm{Diff}^r(M)$ be a C^r diffeomorphism of a compact manifold M. The *centralizer* of f, denoted $\mathcal{C}(f)$, is the set of all C^r diffeomorphisms that commute with f:

$$\mathcal{C}(f) = \{g \in \mathrm{Diff}^r(M) : fg = gf\}.$$

$\mathcal{C}(f)$ will contain all integer powers of f. If f can be embedded as the time-1 map of a flow, then the centralizer will also contain all time-t maps of the flow. If f has a square root (a diffeomorphism whose composition with itself gives the map f) then this is in the centralizer as well. $\mathcal{C}(f)$ is closed under the operation of composition, and forms a topological group in the C^r topology. So we can ask questions about the structure of this group: Is it discrete? Is it finitely generated? We say that the centralizer of f is *trivial* if f commutes only with its integer powers. This paper addresses the following conjecture.

CONJECTURE 1.1. (Smale [**18**]) *Given a compact manifold M, there is an open, dense set of diffeomorphisms on M whose elements all have trivial centralizer.*

Recall that a subset of a metric space X is *residual* if it contains a countable intersection of open, dense sets. In a complete metric space, every residual subset is dense (but not necessarily open). We say that a property is *generic* in X if it holds for a residual subset of X.

Kopell has shown that the conjecture is true when M is a circle; in the set of all C^r diffeomorphisms of a circle (with $2 \leq r \leq \infty$), there is a C^1-open, C^r-dense subset with trivial centralizer [**9**]. On higher-dimensional manifolds, the conjecture has been proved for certain classes of diffeomorphisms. Palis and Yoccoz showed that

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the generic C^∞ structurally stable diffeomorphism has trivial centralizer (in the C^∞ topology) [**12**]. They have also shown that there is an open, dense subset of the C^∞ Anosov diffeomorphisms on a torus with trivial centralizers (again using the C^∞ topology) [**11**]. This stronger result has been proved only for Anosov diffeomorphisms on tori, but it was conjectured in [**11**] that similar techniques could be used to prove the same result for Anosov diffeomorphisms on an infranilmanifold. Katok has studied centralizers of surface diffeomorphisms that preserve an ergodic measure [**8**]. Others have studied centralizers of flows [**17, 7**].

In this paper we study the conjecture in the case of partially hyperbolic diffeomorphisms. Our main results are as follows.

THEOREM 1.2. *In the set of C^r partially hyperbolic diffeomorphisms of a compact manifold M ($r \geq 1$), there is a C^1-open and C^1-dense subset whose elements all have discrete centralizer.*

Moreover, we show that under some rather general assumptions, a partially hyperbolic diffeomorphism with a one-dimensional center will generically have trivial centralizer (see Theorem 3.2). This result has the following corollaries.

THEOREM 1.3. *Let $\mathcal{A}$ denote the set of all volume preserving Anosov flows on a compact manifold, M. (In particular, $\mathcal{A}$ contains the geodesic flow on the tangent space of any manifold with constant negative curvature.) Let ϕ be any flow in $\mathcal{A}$, and let ϕ_t denote the time-t map of ϕ. Then there is a neighborhood $U \subset \mathrm{Diff}^\infty(M)$ of ϕ_t such that the set of diffeomorphisms with trivial centralizer is C^∞ residual in U.*

The time-t map of an Anosov flow will never have trivial centralizer (the centralizer of ϕ_t will always contain the set $\{\phi_s : s \in \mathbb{R}\}$, so is not even discrete). But Theorem 1.3 tells us that the generic C^∞ perturbation of a volume preserving Anosov flow gives a diffeomorphism with trivial centralizer. Moreover, we can show that in most cases the centralizer of the time-t map of a volume preserving Anosov flow is exactly the set $\{\phi_s : s \in \mathbb{R}\}$.

PROPOSITION 1.4. *There is a residual subset $\mathcal{A}' \subset \mathcal{A}$ (in the C^r topology, $r \geq 1$) with the following property. For any flow $\phi \in \mathcal{A}'$, there is a residual subset $\mathcal{T}_\phi \subset \mathbb{R}$ such that if $t \in \mathcal{T}_\phi$ then $\mathcal{C}(\phi_t) = \{\phi_s : s \in \mathbb{R}\}$.*

A second corollary of Theorem 3.2 is within the class of partially hyperbolic skew products and their perturbations (see §2 for definitions).

THEOREM 1.5. *Let $f : \mathbb{T}^n \to \mathbb{T}^n$ be an Anosov diffeomorphism, and let $f_\phi : \mathbb{T}^n \times S^1 \to \mathbb{T}^n \times S^1$ be a C^∞ partially hyperbolic skew product that covers f, such that the leaves of the center foliation are the fibers $\{\{p\} \times S^1 : p \in \mathbb{T}^n\}$. Then there is a neighborhood $U \subset \mathrm{Diff}^\infty(\mathbb{T}^n \times S^1)$ of f_ϕ such that the set of diffeomorphisms with trivial centralizer is C^∞ residual in U.*

These results (and others) are described in greater generality in §3.

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2. *Background*

2.1. *Partial hyperbolicity.* A diffeomorphism f of a compact manifold M is *partially hyperbolic* if the tangent bundle to M splits into a continuous direct sum

$$TM = E^u \oplus E^c \oplus E^s,$$

where at least two of the subbundles are non-zero, and with the following properties. The subbundles are invariant under the tangent map Tf, and there is a constant $m \geq 1$ such that

$$\|Tf^m|_{E^s}\| \leq \tfrac{1}{2} \quad \text{and} \quad \|(Tf^m|_{E^u})^{-1}\| \leq \tfrac{1}{2},$$
$$\|Tf^m|_{E^s}\|\|(Tf^m|_{E^c})^{-1}\| \leq \tfrac{1}{2} \quad \text{and} \quad \|(Tf^m|_{E^u})^{-1}\|\|Tf^m|_{E^c})\| \leq \tfrac{1}{2}.$$

Some examples of partially hyperbolic diffeomorphisms. A discussion of the following examples can be found in [**4**, Ch. 1].

(1) *Anosov diffeomorphisms.* By definition, an *Anosov diffeomorphism* is a partially hyperbolic diffeomorphism with $E^c = 0$.

(2) *The time-t map of an Anosov flow.* A flow $\phi_t : M \to M$ is *Anosov* if TM splits into a $T\phi$-invariant sum

$$TM = E^u \oplus E^c \oplus E^s,$$

where E^c is the tangent bundle to the foliation by orbits of the flow, and there are constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|T\phi_t(v)\| \leq C\lambda^t\|v\| \text{ if } v \in E^s$$

and

$$\|T\phi_{-t}(v)\| \leq C\lambda^t\|v\| \text{ if } v \in E^u.$$

Every time-t map of an Anosov flow is a partially hyperbolic diffeomorphism, with a one-dimensional center bundle. (In particular, the time-t map of an Anosov flow is not an Anosov diffeomorphism.) The main examples of Anosov flows are the suspension flow of an Anosov diffeomorphism, and the geodesic flow on the unit tangent bundle of a manifold with negative sectional curvature.

(3) *Partially hyperbolic skew products.* Let f be a C^r Anosov diffeomorphism on $\mathbb{T}^n$, and let $\phi : \mathbb{T}^n \to \mathrm{Diff}^r(M)$ be a C^r map that assigns to each $x \in \mathbb{T}^n$ a C^r diffeomorphism, ϕ_x, of a compact manifold M. Then we can define a C^r skew product on $\mathbb{T}^n \times M$ by

$$f_\phi(x, y) = (f(x), \phi_x(y)).$$

The skew product f_ϕ is partially hyperbolic if there is a Tf_ϕ-invariant splitting:

$$T(\mathbb{T}^n \times M) = E^u \oplus TM \oplus E^s,$$

where $E^u \oplus E^s$ projects to the hyperbolic splitting for Tf on $\mathbb{T}^n$, and the inequalities for $\|Tf_\phi\|$ given above are satisfied.

(4) *Perturbations.* Partial hyperbolicity is a C^1-open property. So small C^1 perturbations of the above examples are also partially hyperbolic.

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2.2. *Foliations.* A *foliation* $\mathcal{F}$ of a compact manifold M is a partition of M into a disjoint union of k-dimensional *leaves*, satisfying the following properties.

(1) Each leaf is an injectively immersed, k-dimensional submanifold of M. Let $F(q)$ denote the leaf containing the point q.

(2) At every point p on the manifold, there is a *foliation chart*, that is, a homeomorphism

$$\phi : D^k \times D^{n-k} \to V,$$

where D^i is the open, i-dimensional disk, n is the dimension of M, V is a neighborhood of $p = \phi(0, 0)$, and such that

$$\phi\,(D^k \times \{y\}) \subset F(q),$$

where $q = \phi(0, y)$. The set $\phi\,(D^k \times \{y\})$ is called a *plaque* of $\mathcal{F}$.

We say that $\mathcal{F}$ is a C^r *foliation* if we can choose C^r charts for $\mathcal{F}$. The foliation is *invariant* under a diffeomorphism g if

$$g(F(q)) = F(g(q))$$

for every leaf $F(q)$ of $\mathcal{F}$. We will be interested in foliations that are tangent to the bundles E^s, E^c and E^u.

Definitions. A vector field E is *integrable* if there exists a foliation $\mathcal{F}$ such that $T\mathcal{F} = E$. We say that $T\mathcal{F}$ is *uniquely integrable* if every differentiable curve tangent to $T\mathcal{F}$ lies in a single leaf of $\mathcal{F}$.

Remarks.

- It follows from the definition that if $E = T\mathcal{F}$ is uniquely integrable, then $\mathcal{F}$ is the unique foliation tangent to E. However, the converse may not be true. An example is given in [**15**, p. 544] of a foliation $\mathcal{F}$ with two-dimensional leaves, with the property that any 2-disk tangent to $T\mathcal{F}$ is contained in a single leaf of $\mathcal{F}$ (so $\mathcal{F}$ is the unique foliation tangent to $T\mathcal{F}$). However, there is a differentiable curve, everywhere tangent to $T\mathcal{F}$, that does not lie wholly in any one leaf.
- It is not hard to show that if $\mathcal{F}$ is a C^1 foliation, then $T\mathcal{F}$ will be uniquely integrable.

If f is a partially hyperbolic diffeomorphism, then the unstable and stable bundles, E^u and E^s, are uniquely integrable. Their tangent foliations are the unstable and stable foliations, denoted $\mathcal{W}^u$ and $\mathcal{W}^s$. (The proof of this fact is essentially the same as the proof of the Stable Manifold Theorem. See for example [**16**, Theorem 10.1].) However, the center bundle, E^c, might not be integrable, and if it is, the tangent foliation might (in theory) not be unique. There are examples of partially hyperbolic diffeomorphisms with non-integrable center foliations (see [**20**]). But there is no known example of a partially hyperbolic diffeomorphism whose center bundle is integrable but not uniquely integrable.

THEOREM 2.1. (From [**6**, Theorem 6.8]) *If f is partially hyperbolic, then the leaves of $\mathcal{W}^c$ are C^1 submanifolds of M.*

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Remarks.

- If in addition f is plaque expansive, then the C^1 smoothness of the leaves of $\mathcal{W}^c$ persists under perturbations. A precise statement of this result and the definition of plaque expansiveness are given below.
- If f is plaque expansive and E^c is uniquely integrable, then any $\tilde{f}$ sufficiently C^1-close to f will also have a uniquely integrable center bundle. See Lemma 8.4.
- If $\mathcal{W}^c$ is a C^1 foliation (the foliation charts are C^1) then the leaves of $\mathcal{W}^c$ will be C^1 submanifolds. However, the converse is not necessarily true. In particular, partial hyperbolicity does not imply that the center foliation is C^1.

Recall that a sequence $\{x_i\}_{i\in\mathbb{Z}}$ is an *ϵ-pseudo-orbit* for f if, for every i, $d(f(x_i), x_{i+1}) < \epsilon$. We say that the ϵ-pseudo-orbit $\{x_i\}_{i\in\mathbb{Z}}$ *respects a foliation* $\mathcal{F}$ if, for every i, $f(x_i)$ and x_{i+1} lie in the same plaque of $\mathcal{F}$.

A map f is *expansive* if there is some $\epsilon > 0$ such that if $d(f^i(x), f^i(y)) < \epsilon$ for every integer i, then $x = y$. For example, the map $f(x) = 2x$ is expansive on $\mathbb{R}$. Any Anosov diffeomorphism is easily seen to be expansive. Plaque expansiveness is a generalization of the notion of expansiveness.

Definition. Let f be partially hyperbolic. Then f is *plaque expansive* if there is some $\epsilon > 0$ such that if

(1) $\{x_i\}_{i\in\mathbb{Z}}$ and $\{y_i\}_{i\in\mathbb{Z}}$ are two ϵ-pseudo-orbits that respect $\mathcal{W}^c$, and

(2) $d(x_i, y_i) < \epsilon$ for every integer i,

then x_i and y_i lie in the same leaf of $\mathcal{W}^c$ (for every $i \in \mathbb{Z}$).

Remarks.

- For example, we can think of an Anosov diffeomorphism as a partially hyperbolic diffeomorphism with trivial center bundle – each leaf of the center foliation consists of a single point. In this case, plaque expansiveness is equivalent to expansiveness.
- If $f : \mathbb{T}^n \to \mathbb{T}^n$ is an Anosov diffeomorphism, and $f_\phi : \mathbb{T}^n \times M \to \mathbb{T}^n \times M$ is a partially hyperbolic skew product over f, then f_ϕ is plaque expansive (as it projects to an expansive map on $\mathbb{T}^n$).
- If $\mathcal{W}^c_f$ is a C^1 foliation, then f will be stably plaque expansive; if $\tilde{f}$ is a C^1 diffeomorphism close to f, then $\tilde{f}$ is partially hyperbolic and plaque expansive [**6**, Theorems 7.2 and 7.3]. (But $\mathcal{W}^c_{\tilde{f}}$ might not be a C^1 foliation.)
- So our other main examples, the time-t map of an Anosov flow and its perturbations, are also plaque expansive diffeomorphisms.

Definitions. Let F and $\widetilde{F}$ be diffeomorphisms with invariant foliations $\mathcal{F}$ and $\widetilde{\mathcal{F}}$. Then $(F, \mathcal{F})$ is *leaf conjugate* to $(\widetilde{F}, \widetilde{\mathcal{F}})$ if there is a homeomorphism h that carries leaves of $\mathcal{F}$ to leaves of $\widetilde{\mathcal{F}}$, and for every leaf $L \in \mathcal{F}$, $\widetilde{F}h(L) = hF(L)$.

A partially hyperbolic diffeomorphism F is *structurally stable* relative to the foliation $\mathcal{W}^c_F$ if F has a neighborhood U such that for every $\tilde{F} \in U$, $E^c_{\tilde{F}}$ is integrable, and $(F, \mathcal{W}^c_F)$ is leaf conjugate to $(\widetilde{F}, \mathcal{W}^c_{\widetilde{F}})$.

THEOREM 2.2. (From [**6**, Theorems 6.8 and 7.1]) *Let f be partially hyperbolic and plaque expansive. Then given a small enough $\delta > 0$, there is a constant $\epsilon = \epsilon(\delta, f) > 0$ such that if $\tilde{f}$ is a C^1 diffeomorphism within C^1 distance δ of f, then:*

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(1) *$\tilde{f}$ is partially hyperbolic;*
(2) *the foliations $\mathcal{W}^c_f$ and $\mathcal{W}^c_{\tilde{f}}$ are leaf conjugate, via a homeomorphism $h : \mathcal{W}^c_f \to \mathcal{W}^c_{\tilde{f}}$;*
(3) *for every point p, $d(p, h(p)) < \epsilon$.*

3. *Statement of results*

3.1. *Discrete centralizers.* Let $\mathcal{PH}^r \subset \mathrm{Diff}^r(M)$ denote the set of C^r partially hyperbolic diffeomorphisms on a compact manifold M. We say that the centralizer of a diffeomorphism is *discrete* if there is some $\epsilon > 0$ such that if two diffeomorphisms, g_1 and g_2, are in $\mathcal{C}(f)$ and are ϵ-close (in the C^r topology), then $g_1 = g_2$.

THEOREM 3.1. *For each $r \geq 1$, there is a C^1-open and dense subset $\mathcal{V}$ of $\mathcal{PH}^r$ such that every $f \in \mathcal{V}$ has discrete centralizer. In particular, $f \in \mathcal{V}$ does not embed in a C^1 flow.*

3.2. *Trivial centralizers.* The question of *trivial* centralizers is not answered in such generality. But there is a large class of partially hyperbolic diffeomorphisms with trivial centralizer. The proof of the following theorem and its corollaries is given in §6.

THEOREM 3.2. *Let $\mathcal{U} \subset \mathcal{PH}^\infty$ be any C^∞-open set of C^∞ partially hyperbolic diffeomorphisms with the following properties. For each $f \in \mathcal{U}$:*
(1) *the center bundle is integrable, and $\mathcal{W}^c$ is one-dimensional;*
(2) *there is a closed, invariant center leaf for f, $\mathcal{W}^c(p_1)$;*
(3) *$\mathcal{W}^{cs}(p_1)$ and $\mathcal{W}^{cu}(p_1)$ are both dense in M (where p_1 is as in property 3);*
(4) *$\mathcal{W}^{cs}(p_1) = \bigcup_{q \in \mathcal{W}^c(p_1)} \mathcal{W}^s(q)$.*
Then the set of diffeomorphisms with trivial centralizer is C^∞ residual in $\mathcal{U}$.

Such sets exist (see §6.1). In particular, the set U contains the following examples, and their perturbations.

- If ϕ is an Anosov flow on M, and if the non-wandering set $NW(\phi)$ for ϕ is all of M, then $\mathcal{U}$ contains all time-t maps, ϕ_t. (See Proposition 6.1.) For example, if ϕ is the geodesic flow on the tangent space of a manifold with everywhere negative curvature, then ϕ is an Anosov flow with $NW(\phi) = M$, so any time-t map of ϕ is in U. If ϕ is any volume preserving Anosov flow then $NW(\phi) = M$.
- U contains all C^∞ partially hyperbolic skew products on $\mathbb{T}^n \times S^1$ that cover an Anosov diffeomorphism of $\mathbb{T}^n$, and such that the leaves of the center foliation are the fibres $\{\{p\} \times S^1 : p \in \mathbb{T}^n\}$. (See Proposition 6.2.)

So Theorems 1.3 and 1.5 are corollaries of Theorem 3.2. Moreover, we can fix *any* Anosov diffeomorphism f on $\mathbb{T}^n$ and consider the set of all partially hyperbolic diffeomorphisms on $\mathbb{T}^n \times S^1$ that project to f. All perturbations needed to give a trivial centralizer can be made in the fiber direction. So another corollary of the proof is the following.

COROLLARY 3.3. *Fix any Anosov diffeomorphism f on $\mathbb{T}^n$. Let $\mathcal{SP}_f(\mathbb{T}^n \times S^1)$ denote the set of C^∞ partially hyperbolic skew products on $\mathbb{T}^n \times S^1$ over f such that for each*

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$p \in \mathbb{T}^n$, $\mathcal{W}^c(p) = \{p\} \times S^1$. Then the set of diffeomorphisms with trivial centralizer is C^∞ residual in $\mathcal{SP}_f(\mathbb{T}^n \times S^1)$.

3.3. *Anosov flows.* If ϕ_t is the time-t map of a flow, then $\mathcal{C}(\phi_t) \supseteq \{\phi_s : s \in \mathbb{R}\}$. We would expect to have equality here most of the time. Let $\mathcal{A}$ be the set of all volume preserving Anosov flows on a compact manifold M. It is not difficult to show that for ϕ in a C^r residual subset of $\mathcal{A}$, and for a residual set of $t \in \mathbb{R}$, $\mathcal{C}(\phi_t) = \{\phi_s : s \in \mathbb{R}\}$. This result can also be stated in terms of the geodesic flow on the unit tangent bundle of a manifold with negative sectional curvature. For completeness, the proof is given at the end of §6.

3.4. *Skew products.* We also consider other classes of skew products that do not fit into the category defined in Theorem 3.2. All of the diffeomorphisms considered in Theorem 3.2 are partially hyperbolic, and have a one-dimensional center bundle. In §§7 and 8 we consider skew products with higher-dimensional center bundle. In §9, we study centralizers of skew products that are not partially hyperbolic. We will prove the following results in later sections.

THEOREM 3.4. *Let M be any compact manifold. For $r \geq 2$, let $\mathcal{SP}^r(\mathbb{T}^n \times M)$ denote the set of C^r partially hyperbolic skew products on $\mathbb{T}^n \times M$ that cover an Anosov diffeomorphism on $\mathbb{T}^n$, such that the leaves of the center foliation are the fibers $\{\{p\} \times M : p \in \mathbb{T}^n\}$ (see §2.1 for a definition). Then there is an open, dense subset $\mathcal{T} \subset \mathcal{SP}^r(\mathbb{T}^n \times M)$ (in the C^1 topology) with centralizers of the following form: if G commutes with $f_\phi \in \mathcal{T}$, then $G^k = (f_\phi)^j$ for some integers k and j.*

If f_ϕ is a skew product over $\mathbb{T}^2$, then this property holds generically under perturbations.

THEOREM 3.5. *If $f_\phi \in \mathcal{SP}(\mathbb{T}^2 \times M)$, then there is a C^1 neighborhood $U \subset \mathrm{Diff}^r(\mathbb{T}^2 \times M)$ containing f_ϕ, and a subset $V \subset U$, open and dense in U (in the C^1 topology) such that if $\widetilde{F} \in V$ and $G \in \mathcal{C}(\widetilde{F})$, then $G^i = (\widetilde{F})^j$ for some $i, j \in \mathbb{Z}$.*

Finally, in the case of skew products on $\mathbb{T}^n \times S^1$, it is possible to drop the hypothesis of partial hyperbolicity. Let $\mathcal{S}^r(\mathbb{T}^n \times S^1)$ denote the set of skew products of the form

$$f_\phi(x, a) = (f(x), \phi_x(a)),$$

where f is a C^r Anosov diffeomorphism of $\mathbb{T}^n$, and $\phi : \mathbb{T}^n \to \mathrm{Diff}^r(S^1)$ is a C^r map that assigns a C^r circle diffeomorphism ϕ_x to each $x \in \mathbb{T}^n$ (for $r \geq 2$). (Note that we are not assuming that f_ϕ is partially hyperbolic.)

THEOREM 3.6. *The set $\mathcal{T} = \{f_\phi \in \mathcal{S}^r(\mathbb{T}^n \times S^1) : f_\phi$ has trivial centralizer$\}$ is C^r residual in $\mathcal{S}^r(\mathbb{T}^n \times S^1)$ (for $r \geq 2$).*

4. *Some basic lemmas*

In this section, we state (largely without proof) some basic lemmas that will be used throughout.

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Notation. Let $\mathcal{W}^s(p)$ denote the leaf of the stable foliation $\mathcal{W}^s$ containing a point $p \in M$. The stable leaves can be defined asymptotically: there is a constant $0 < \lambda < 1$ such that

$$\mathcal{W}^s(p) = \{q \in M : \text{ for some } C, d(f^n(p), f^n(q)) \leq C\lambda^n \text{ for all } n \geq 0\}.$$

For a periodic point p, let $W^s(p)$ denote the stable manifold for p:

$$W^s(p) = \{q \in M : \lim_{n\to\infty} d(f^n(p), f^n(q)) = 0\}.$$

If f is a C^r diffeomorphism, then $W^s(p)$ is a C^r submanifold of M (and $\mathcal{W}^s(p)$ is a C^1 submanifold—see Theorem 2.1).

The unstable manifold $W^u(p)$ and unstable leaf $\mathcal{W}^u(p)$ are defined similarly. *A priori*, the sets $\mathcal{W}^s(p)$ and $W^s(p)$ can differ. The stable manifold may be a union of stable leaves.

LEMMA 4.1. *If p is periodic, then $W^s(p)$ is subfoliated by leaves of $\mathcal{W}^s$.*

Proof. The lemma follows directly from the definitions: if $q \in W^s(p)$, then $\mathcal{W}^s(q) \subset W^s(p)$. □

LEMMA 4.2. *Suppose f is C^1, partially hyperbolic, and commutes with a homeomorphism g. Then the following hold.*

(1) *g permutes the points of $\mathrm{Fix}^k(f)$, the fixed points for f^k:*

$$g(\mathrm{Fix}^k(f)) = \mathrm{Fix}^k(f).$$

(2) *g preserves stable and unstable manifolds:*

$$g(W^u(p)) = W^u(g(p)) \quad \text{and} \quad g(W^s(p)) = W^s(g(p)),$$

for any f-periodic point p.

(3) *If g is Lipschitz, then g preserves the stable and unstable foliations, $\mathcal{W}^s$ and $\mathcal{W}^u$:*

$$g(\mathcal{W}^u(p)) = \mathcal{W}^u(g(p)) \quad \text{and} \quad g(\mathcal{W}^s(p)) = \mathcal{W}^s(g(p)).$$

(4) *If g is a C^1 diffeomorphism, then Tg preserves the partially hyperbolic splitting for Tf:*

$$Tg(E^\alpha(p)) = E^\alpha(p), \quad \text{for } \alpha = u, c, s.$$

(5) *If g is a C^1 diffeomorphism and the center bundle, E^c, is uniquely integrable, then $\mathcal{W}^c$ is g-invariant:*

$$g(\mathcal{W}^c(p)) = \mathcal{W}^c(g(p)).$$

As stated in Lemma 4.2, every $g \in \mathcal{C}(f)$ permutes f-invariant orbits. The following lemma shows that for f in a residual set of C^1 diffeomorphisms, any $g \in \mathcal{C}(f)$ fixes the f-periodic orbits.

LEMMA 4.3. [**12**, Section 4.2] *Let $\mathrm{Diff}^1(N)$ be the set of all C^1 diffeomorphisms of a compact manifold N. For any $k > 0$, the set*

$$\{f : \text{if } g \in \mathcal{C}(f) \text{ and } f^k(p) = p, \text{ then } g(p) = f^n(p) \text{ for some } n \in \mathbb{Z}\}$$

is open and dense in $\mathrm{Diff}^1(N)$ (in the C^r topology).

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Proof of Lemma 4.3. By the Kupka–Smale theorem, it is an open, dense property that $\mathrm{Fix}^k(f)$, the set of fixed points for f^k, is finite. It is easy to see that g permutes the points in $\mathrm{Fix}^k(f)$. We may perturb f so that if p_1 and p_2 are both in $\mathrm{Fix}^k(f)$, then the linear maps $T_{p_1}f^k$ and $T_{p_2}f^k$ do not have the same eigenvalues unless $p_2 = f^n(p_1)$ for some $n \in \mathbb{Z}$. In particular, $T_{p_1}f^k$ and $T_{p_2}f^k$ are not conjugate matrices unless $p_2 = f^n(p_1)$. This is an open property. Now suppose that f has this property and commutes with g. Let p be in $\mathrm{Fix}^k(f)$. We will show that $g(p) = f^n(p)$ for some $n \in \mathbb{Z}$.

By commutativity,

$$f^k(g(p)) = g(f^k(p)),$$

and by the chain rule,

$$T_{g(p)}f^k.T_pg = T_pg.T_pf^k.$$

So the linear maps $T_{g(p)}f^k$ and T_pf^k are conjugate. Then by assumption, $g(p) = f^n(p)$ for some $n \in \mathbb{Z}$. □

5. *Discrete centralizers*

In this section we prove Theorem 3.1. Using Pugh's Closing lemma and the recent paper of Dolgopyat and Wilkinson [**5**], Theorem 3.1 is a direct corollary of the following theorem.

THEOREM 5.1. *Let $f \in \mathcal{U}^1$ and suppose that:*

(I) *for some $k \geq 1$, the set $\mathrm{Fix}^k(f)$ of fixed points for f^k is non-empty and finite;*

(II) *f is accessible (see below for a definition).*

Then f has discrete centralizer.

Definition. A partially hyperbolic diffeomorphism $f : M \to M$ is *accessible* if given any two points p and q in M, there is a path $\gamma : [0, 1] \to M$ such that:

(1) $\gamma(0) = p, \gamma(1) = q$;

(2) γ is a finite union of arcs, each of which lies completely inside either a leaf of $\mathcal{W}^s$ or a leaf of $\mathcal{W}^u$.

The path γ is called a u, s-path.

Proof of Theorem 3.1 from Theorem 5.1. Let $\mathcal{V}$ be the set of $f \in \mathcal{U}^r$ satisfying the following.

(I′) For some $k \geq 1$, the set $\mathrm{Fix}^k(f)$ of fixed points for f^k is non-empty and finite, and the fixed points for f^k are transverse.

(II′) f is stably accessible.

(I) and (II) are easily seen to be C^1-open. (I) is C^1-dense in $\mathrm{Diff}^r(M)$ by Pugh's Closing lemma and the Kupka–Smale theorem, and (II) was shown in [**5**] to be C^1-dense in $\mathcal{U}^r$. □

Proof of Theorem 5.1. Suppose $f \in \mathcal{U}^1$ satisfies (I) and (II). Let $TM = E^u \oplus E^c \oplus E^s$ be the partially hyperbolic splitting for f, and let $\mathcal{W}^u$ and $\mathcal{W}^s$ be the unstable and stable foliations for f, respectively.

LEMMA 5.2. *There is a constant $\epsilon > 0$ such that:*

(i) *if $p, q \in \mathrm{Fix}^k(f)$ satisfy $d(p, q) < \epsilon$, then $p = q$; and*

(ii) *for every $p \in M$ and $q \in \mathcal{W}^s(p) \cup \mathcal{W}^u(p)$, if $d(f^n(p), f^n(q)) < \epsilon$ for all $n \in \mathbb{Z}$, then $p = q$.*

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Proof of Lemma 5.2. It is possible to satisfy (i) because $\mathrm{Fix}^k(f)$ is a finite set.

Let d_u denote the distance along a leaf of $\mathcal{W}^u$. There is a constant $\lambda > 1$ such that if $q \in \mathcal{W}^u(p)$, then for every $n \in \mathbb{Z}$,

$$d_u(f^n(p), f^n(q)) > \lambda^n d_u(p, q). \tag{5.1}$$

Given $p \in M$ and $q \in \mathcal{W}^u(p)$, there is some N such that if $n > N$, then $f^{-n}(p)$ and $f^{-n}(q)$ lie in the same plaque of $\mathcal{W}^u$. By compactness of M, we can assume that each plaque has diameter at least ρ (for some $\rho > 0$). As the plaques of $\mathcal{W}^u$ are C^∞ submanifolds, there is a constant $C > 1$ such that

$$\frac{1}{C} d_u(p', q') < d(p', q') < C d_u(p', q') \tag{5.2}$$

whenever p' and q' lie in the same plaque of $\mathcal{W}^u$. By (5.1) and (5.2), given $\delta \ll \rho$ and $q \in \mathcal{W}^u(p)$, we can find $n > N$ such that

$$\delta < d_u(f^{-n}(p), f^{-n}(q)) < C\, d(f^{-n}(p), f^{-n}(q)).$$

Let $\epsilon = \delta / C$. The constant ϵ is independent of p and q, and if $p \in M$, $q \in \mathcal{W}^u(p)$, and $d(f^n(p), f^n(q)) < \epsilon$ for all $n \in \mathbb{Z}$, then $p = q$.

Similarly, there is a constant ϵ' such that for every $p \in M$ and $q \in \mathcal{W}^s(p)$, if $d(f^n(p), f^n(q)) < \epsilon'$ for all $n \in \mathbb{Z}$, then $p = q$. □

We now return to the proof of the theorem. Let $\epsilon > 0$ satisfying Lemma 5.2 be given, and suppose that $fg = gf$. We show that g is isolated in $\mathcal{C}(f)$; in fact, we prove that there is a C^0-open neighborhood $\mathcal{N}$ of g in $\mathrm{Diff}^1(M)$ such that $\mathcal{N} \cap \mathcal{C}(f) = \{g\}$. Suppose that g' is also in $\mathcal{C}(f)$ and that $d_{C^0}(g, g') < \epsilon$. We now show that $g = g'$. Let $h = g^{-1}g'$. Then h is in $\mathcal{C}(f)$ and $d_{C^0}(h, \mathrm{id}) < \epsilon$. The proof is reduced to showing that $h = \mathrm{id}$.

We have seen (Lemma 4.2) that h preserves the stable and unstable foliations:

$$h(\mathcal{W}^u(p)) = \mathcal{W}^u(h(p)) \quad \text{and} \quad h(\mathcal{W}^s(p)) = \mathcal{W}^s(h(p)).$$

(In fact we really only needed h Lipschitz here.)

Let $p_0 \in \mathrm{Fix}^k(f)$. Since h commutes with f^k, it permutes the points of $\mathrm{Fix}^k(f)$. Since $d_{C^0}(h, \mathrm{id}) < \epsilon$, it follows from (ii) that $h(p_0) = p_0$.

LEMMA 5.3. *If $h(q) = q$, then $h(x) = x$ for all $x \in \mathcal{W}^u(q) \cup \mathcal{W}^s(q)$.*

Proof. Let $x \in \mathcal{W}^u(q) \cup \mathcal{W}^s(q)$; without loss of generality, assume that $x \in W^u(q)$. Then $h(x) \in h(\mathcal{W}^u(q)) = \mathcal{W}^u(h(q)) = \mathcal{W}^u(q) = \mathcal{W}^u(x)$, since $h(q) = q$. On the other hand, the points $x, h(x)$ satisfy

$$d(f^n(x), f^n(h(x))) = d(f^n(x), h(f^n(x))) < d_{C^0}(h, \mathrm{id}) < \epsilon,$$

for all $n \in \mathbb{Z}$. By property (ii), it follows that $h(x) = x$, which completes the proof of the lemma. □

It now follows from (II) that $h = \mathrm{id}$. For any $y \in M$, there is a u, s-path from p_0 to y; by Lemma 5.3, h fixes all of the points along this path, and so $h(y) = y$. This completes the proof of Theorem 5.1. □

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Remarks. In fact, we have shown that if $f \in \mathcal{U}^1$ satisfies (I) and (II), then the group of Lipschitz homeomorphisms (in the uniform topology) commuting with f is discrete. In particular, f does not embed in a Lipschitz flow.

In the subclass $\mathcal{U}_1^r$ of $\mathcal{U}^r$ for which $\mathcal{W}^c$ is one-dimensional and contains a compact periodic leaf, the hypotheses (I) and (II) are in fact C^r-dense [**10**]. Hence there is a C^r-open and dense subset of $\mathcal{U}_1^r$ with discrete centralizers.

If we remove either hypothesis (I) or (II), the conclusion of Theorem 5.1 does not hold. If f is the time-t map of any mixing Anosov flow, then f satisfies (II) but not (I) and clearly the centralizer of f is not discrete. On the other hand, if f is the product of an Anosov diffeomorphism with the time-t map of any Morse–Smale flow (for t sufficiently small), then f satisfies (I) but not (II), and again the centralizer is not discrete.

6. *Proof of Theorem 3.2*

We will first show that open sets with the properties listed in Theorem 3.2 exist. The proof of Theorem 3.2 is given in §§6.2, 6.3 and 6.4. Section 6.5 contains a result about centralizers of time-t maps of Anosov flows.

6.1. *Examples.*

PROPOSITION 6.1. *Let $f = \phi_t$ be the time-t map of an Anosov flow on a compact manifold M, and suppose that $NW(f) = M$. Then there is a neighborhood $\mathcal{U} \subset \mathcal{P}H^\infty$ of f satisfying the hypotheses of Theorem 3.2.*

Proof of Proposition 6.1. The set of periodic orbits for an Anosov flow ϕ is dense in the non-wandering set for ϕ [**2**]. And if $NW(\phi_t) = M$ and $\mathcal{W}^c(p_1)$ is a closed, ϕ_t-invariant leaf, then $\mathcal{W}^{cs}(p_1)$ and $\mathcal{W}^{cu}(p_1)$ are dense in M [**13**, Theorem 1.3]. $\mathcal{W}^{cs}(p_1)$ is subfoliated by leaves of $\mathcal{W}^c$, each of which is mapped to itself by ϕ_t. The center-stable leaf $\mathcal{W}^{cs}(p_1)$ is a cylinder, and each $\mathcal{W}^c(q) \subset \mathcal{W}^{cs}(p_1)$ wraps around this cylinder, spiralling towards the invariant leaf $\mathcal{W}^c(p_1)$. Along each $\mathcal{W}^c(q)$, ϕ_t acts as a translation: there exists a constant $c > 0$ such that for every $q \in \mathcal{W}^{cs}(p_1)$, the distance between q and $\phi_t(q)$, measured along $\mathcal{W}^c(q)$, is greater than c.

There is an open set $V \subset \mathcal{W}^{cs}(p_1)$ containing $\mathcal{W}^c(p_1)$ that is subfoliated by leaves of $\mathcal{W}^s$, each of which intersects $\mathcal{W}^c(p_1)$ transversely. Given any $q \in \mathcal{W}^{cs}(p_1)$, there is some $n > 0$ such that $(\phi_t)^n(q) \in V$. So $(\phi_t)^n(q) \in \mathcal{W}^s(p)$ for some $p \in \mathcal{W}^c(p_1)$, and therefore $q \in \mathcal{W}^s(f^{-n}p) \subset \bigcup_{p \in \mathcal{W}^c(p_1)} \mathcal{W}^s(p)$. So $\mathcal{W}^{cs}(p_1) = \bigcup_{p \in \mathcal{W}^c(p_1)} \mathcal{W}^s(p)$.

By [**6**], this situation persists under perturbations. If $\tilde{f}$ is a partially hyperbolic diffeomorphism C^1-close to ϕ_t, then there are dense leaves $\mathcal{W}^{cs}_{\tilde{f}}(\tilde{p}_1)$ and $\mathcal{W}^{cu}_{\tilde{f}}(\tilde{p}_1)$, homeomorphic to $\mathcal{W}^{cs}(p_1)$ and $\mathcal{W}^{cu}(p_1)$ (see Theorem 2.2). The center-stable leaf, $\mathcal{W}^{cs}_{\tilde{f}}(\tilde{p}_1)$, is a cylinder, subfoliated by a center foliation homeomorphic to $\mathcal{W}^c_{\phi_t}$.

Restricted to $\mathcal{W}^c_{\tilde{f}}(q) \subset \mathcal{W}^{cs}_{\tilde{f}}(\tilde{p}_1)$, $\tilde{f}$ is very close to a translation. Each $q \in \mathcal{W}^{cs}_{\tilde{f}}(\tilde{p}_1)$ is attracted under iterates of $\tilde{f}$ towards a neighborhood $\widetilde{V}$ of $\mathcal{W}^c_{\tilde{f}}(\tilde{p}_1)$ which is subfoliated by leaves of $\mathcal{W}^s_{\tilde{f}}$. So as above, $\mathcal{W}^{cs}_{\tilde{f}}(\tilde{p}_1) = \bigcup_{p \in \mathcal{W}^c_{\tilde{f}}(\tilde{p}_1)} \mathcal{W}^s_{\tilde{f}}(p)$. □

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PROPOSITION 6.2. *Let f_ϕ be a partially hyperbolic skew product on $\mathbb{T}^n \times S^1$ that covers an Anosov diffeomorphism f on $\mathbb{T}^n$, and such that the leaves of $\mathcal{W}^c$ are the fibers $\{\{p\} \times S^1 : p \in \mathbb{T}^n\}$. Then there is a neighborhood $\mathcal{U} \subset \mathcal{P}H^\infty$ of f_ϕ satisfying the hypotheses of Theorem 3.2.*

Proof of Proposition 6.2. Every Anosov diffeomorphism on $\mathbb{T}^n$ has a fixed point. So if f_ϕ is a partially hyperbolic skew product on $\mathbb{T}^n \times S^1$ that covers an Anosov diffeomorphism f on $\mathbb{T}^n$, then there is a closed, invariant center leaf for f, $\mathcal{W}^c(p)$, where p projects to a fixed point for f. The center-stable leaf, $\mathcal{W}^{cs}(p)$, is dense in $\mathbb{T}^n \times S^1$ and foliated by center leaves which are all circles, and these circles are contracted towards $\mathcal{W}^c(p)$ under iterates of f_ϕ. In particular, $\mathcal{W}^{cs}(p) = \bigcup_{q \in \mathcal{W}^c(p)} \mathcal{W}^s(q)$. Similarly, $\mathcal{W}^{cu}(p)$ is dense in $\mathbb{T}^n \times S^1$, foliated by center leaves which are all circles, and these circles are contracted towards $\mathcal{W}^c(p)$ under iterates of $(f_\phi)^{-1}$.

If $\widetilde{F}$ is a C^∞ diffeomorphism C^1-close to f_ϕ, then the center foliation for $\widetilde{F}$ is leaf conjugate to the center foliation for f_ϕ (see Theorem 2.2). In particular, there is an invariant center leaf, $\mathcal{W}^c_{\widetilde{F}}(\tilde{p})$, for $\widetilde{F}$, close to $\mathcal{W}^c(p)$. Both $\mathcal{W}^{cs}_{\widetilde{F}}(\tilde{p})$ and $\mathcal{W}^{cu}_{\widetilde{F}}(\tilde{p})$ are dense in $\mathbb{T}^n \times S^1$. The center-stable leaf is subfoliated by leaves of $\mathcal{W}^c_{\widetilde{F}}$, each of which is attracted towards $\mathcal{W}^c_{\widetilde{F}}(\tilde{p})$ under iterates of $\widetilde{F}$. Since there is a neighborhood $V \subset \mathcal{W}^{cs}_{\widetilde{F}}(\tilde{p})$ of $\mathcal{W}^c_{\widetilde{F}}(\tilde{p})$ that is subfoliated by leaves of $\mathcal{W}^s_{\widetilde{F}}$, $\mathcal{W}^{cs}_{\widetilde{F}}(\tilde{p}) = \bigcup_{q \in \mathcal{W}^c_{\widetilde{F}}(\tilde{p})} \mathcal{W}^s_{\widetilde{F}}(q)$. □

6.2. *Some generic properties of diffeomorphisms in $\mathcal{U}$.* Let $\mathcal{U}$ be the set defined in the statement of Theorem 3.2. In this section, we give some generic properties of diffeomorphisms in $\mathcal{U}$. In the following section, we show that a diffeomorphism with the properties listed will have a trivial centralizer.

Notation. Let $\mathcal{W}^s(p)$, $\mathcal{W}^c(p)$ and $\mathcal{W}^u(p)$ denote the leaves of the stable, center and unstable foliations containing p. If p is periodic, then $W^s(p)$ and $W^u(p)$ will denote the stable and unstable manifolds for p. See §4 for definitions.

The following properties are generic in $\mathcal{U}$.

Property I. The intersection of the center stable and center unstable leaves at p_1, $\mathcal{W}^{cs}(p_1) \cap \mathcal{W}^{cu}(p_1)$, is dense in the center stable leaf, $\mathcal{W}^{cs}(p_1)$.

Let p be any point in $\mathcal{W}^{cs}(p_1)$, and let V be a small neighborhood of p. We will show that $\mathcal{W}^{cu}(p_1)$ intersects $\mathcal{W}^{cs}(p_1)$ inside the neighborhood V. We can assume that V is a small enough neighborhood so that there is a foliation chart for the foliation $\mathcal{W}^{cu}$ on V. As E^{cu} is transverse to the bundle E^{cs}, each leaf of $\mathcal{W}^{cu}$ that intersects V also intersects the center stable leaf $\mathcal{W}^{cs}(p_1)$. Finally, as we have assumed that the center unstable leaf $\mathcal{W}^{cu}(p_1)$ is dense in M, $\mathcal{W}^{cu}(p_1) \cap V \neq \emptyset$, so $\mathcal{W}^{cu}(p_1)$ intersects $\mathcal{W}^{cs}(p_1)$ inside the neighborhood V. Hence $\mathcal{W}^{cu}(p_1) \cap \mathcal{W}^{cs}(p_1)$ is dense in $\mathcal{W}^{cs}(p_1)$. This property holds for every diffeomorphism in U.

Property II. $f|_{\mathcal{W}^c(p_1)}$ has periodic points and they are all hyperbolic.

This property is C^1-open and C^∞-dense by the Kupka–Smale theorem, and because $\mathcal{W}^c(p_1)$ is a circle. Let $\mathcal{P} = \{p_1, p_2, \ldots, p_{2N}\}$ be the set of these periodic points.

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Note that if $q \in \mathcal{W}^{cs}(p_1)$, then $q \in \mathcal{W}^s(\overline{q})$ for some $\overline{q} \in \mathcal{W}^c(p_1)$. And $\overline{q} \in W^s(p_i)$ for some $p_i \in \mathcal{P}$. So $q \in W^s(p_i)$ also. Therefore $\mathcal{W}^{cs}(p_1)$ is a (disjoint) union of stable manifolds of points in $\mathcal{P}$,

$$\mathcal{W}^{cs}(p_1) = W^s(p_1) \sqcup \cdots \sqcup W^s(p_{2N}).$$

Notation. I will assume that the points $\mathcal{P}_u = \{p_1, p_3, \ldots, p_{2N-1}\}$ are sources for $f|_{\mathcal{W}^c(p_1)}$, and $\mathcal{P}_s = \{p_2, p_4, \ldots, p_{2N}\}$ are sinks for $f|_{\mathcal{W}^c(p_1)}$. If $\mathcal{W}^{cs}(p_1)$ is l-dimensional, then $\dim(W^s(p_i)) = l$ for $p_i \in \mathcal{P}_s$, and $\dim(W^s(p_i)) = l - 1$ if $p_i \in \mathcal{P}_u$. Moreover, $\bigcup_{p_i \in \mathcal{P}_s} W^s(p_i)$ is dense and open in $\mathcal{W}^{cs}(p_1)$.

Property III. Let n be the least period of the periodic points $\mathcal{P} = \{p_1, p_2, \ldots, p_{2N}\} \subset \mathcal{W}^c(p_1)$. Then for every $p_i \in \mathcal{P}$, the eigenvalues of $T_{p_i} f^n$ are distinct, and satisfy Sternberg's non-resonance condition. That is, if $\lambda_1, \ldots, \lambda_k$ are the eigenvalues for $T_{p_i} f^n$, then for every $1 \leq l \leq k$,

$$\lambda_l \neq \prod_{j=1}^{k} \lambda_j^{m_j}$$

for any choice of m_j with $\sum_{j=1}^{k} m_j \geq 2$.

This is clearly a C^1-open, C^∞-dense property.

Property IV. If $g \in \mathcal{C}(f)$ then for all $p_j \in \mathcal{P}$, $g(p_j) = f^k(p_j)$ for some $k \in \mathbb{Z}$.

This property is C^1-open and C^∞-dense in $\mathcal{U}$, by Lemma 4.3.

Property V. $\mathcal{W}^u(p_1) \cap W^s(p_i) \neq \emptyset$ for some $p_i \in \mathcal{P}_s$, and the intersection is transverse.

The center-stable leaf, $\mathcal{W}^{cs}(p_1)$, is dense in M, so comes arbitrarily close to the unstable leaf, $\mathcal{W}^u(p_1)$. As $\mathcal{W}^{cs}(p_1)$ and $\mathcal{W}^u(p_1)$ are tangent to the foliations E^{cs} and E^s respectively, and these foliations are transverse, the leaves must intersect,

$$\mathcal{W}^u(p_1) \cap \mathcal{W}^{cs}(p_1) \neq \emptyset.$$

$\bigcup_{p_i \in \mathcal{P}_s} W^s(p_i)$ is dense and open in $\mathcal{W}^{cs}(p_1)$. So for a dense set of diffeomorphisms in $\mathcal{U}$, $\mathcal{W}^u_f(p_1) \cap W^s_f(p_i) \neq \emptyset$ for some $p_i \in \mathcal{P}_s$. We need to show that this is an open property. This is proved in Lemma 6.6.

Let p_2 be a periodic point with this property.

Property VI. If g commutes with f, then $g|_{W^s(p_2)} = f^m$ for some $m \in \mathbb{Z}$ (where p_2 is the periodic point from Property V). In other words, $f|_{W^s(p_2)}$ has trivial centralizer.

It is non-trivial to show that this property is generic for diffeomorphisms in $\mathcal{U}$. This is proved in §6.4. First we show that a diffeomorphism with the six properties listed above will have trivial centralizer.

Let $\mathcal{T}$ denote the set of diffeomorphisms in $\mathcal{U}$ that satisfy Properties I through V.

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6.3. *If $f \in \mathcal{T}$, then the centralizer of f is trivial.* Fix a diffeomorphism $f \in \mathcal{T}$ and suppose that g commutes with f. We first need to show that g preserves leaves of the center foliation in $\mathcal{W}^{cs}(p_1)$ (this is not immediate, as we have not assumed that E^c is uniquely integrable).

LEMMA 6.3. *For any $q \in \mathcal{W}^{cs}(p_1)$, $g(\mathcal{W}^c(q)) = \mathcal{W}^c(g(q))$.*

Proof. The center stable leaf $\mathcal{W}^{cs}(p_1)$ is a disjoint union of stable manifolds,

$$\mathcal{W}^{cs}(p_1) = W^s(p_1) \cup \cdots \cup W^s(p_{2N}).$$

If g commutes with f, then $g(W^s(p_j)) = W^s(g(p_j)) = W^s(f^k(p_j)) = W^s(p_l)$ for some l. The second equality holds by assumption (property IV), and the third holds because the set $\{p_1, \ldots, p_{2N}\}$ is f-invariant. So g maps $\mathcal{W}^{cs}(p_1)$ to itself. Similarly, the center unstable leaf $\mathcal{W}^{cu}(p_1)$ is g-invariant. So their intersection, $\mathcal{W}^{cs}(p_1) \cap \mathcal{W}^{cu}(p_1)$, is g-invariant. These two leaves intersect along leaves of the center foliation, $\mathcal{W}^c$. If $\mathcal{W}^c(q)$ is a leaf in the intersection, then $g(\mathcal{W}^c(q)) \in \mathcal{W}^{cs}(p_1) \cap \mathcal{W}^{cu}(p_1) \subset \mathcal{W}^c$, so $g(\mathcal{W}^c(q)) = \mathcal{W}^c(g(q))$. The set of such leaves is dense in $\mathcal{W}^{cs}(p_1)$ (also by assumption—Property I). By continuity this holds for all $q \in \mathcal{W}^{cs}(p_1)$. □

In particular, $g(\mathcal{W}^c(p_1)) = \mathcal{W}^c(p_1)$. We have assumed that $\mathcal{W}^{cs}(p_1) = W^s(p_1) \sqcup \cdots \sqcup W^s(p_{2N})$, and that the points $\mathcal{P}_u = \{p_1, p_3, \ldots, p_{2N-1}\}$ are sources for $f|_{\mathcal{W}^c(p_1)}$, and $\mathcal{P}_s = \{p_2, p_4 \ldots, p_{2N}\}$ are sinks for $f|_{\mathcal{W}^c(p_1)}$. We have also assumed that $g|_{W^s(p_2)} = f^m$ for some $m \in \mathbb{Z}$, and therefore $g(p_i) = f^m(p_i)$ for every $p_i \in \mathcal{P}_s \cup \mathcal{P}_u$. Let $h = gf^{-m}$. So $h|_{W^s(p_2)} = \mathrm{id}$, and for each $p_i \in \mathcal{P}_s \cup \mathcal{P}_u$, $f^n(p_i) = p_i$ and $h(p_i) = p_i$.

Suppose that the circle $\mathcal{W}^c(p_1)$ has been oriented, and that $(p_i, p_{i+1}) \cap (p_j, p_{j+1}) = \emptyset$ for all $i \neq j$, where (p_i, p_{i+1}) denotes the open interval on $\mathcal{W}^c(p_1)$ between the points p_i and p_{i+1}. Since $h|_{W^s(p_2)} = \mathrm{id}$, we know that $h|_{[p_1, p_3]} = \mathrm{id}$. The first step will be to prove that $h|_{\mathcal{W}^c(p_1)} = \mathrm{id}$. Then we will show that $h = \mathrm{id}$ on all of $\mathcal{W}^{cs}(p_1)$. Since $\mathcal{W}^{cs}(p_1)$ is dense in M, this will imply that $h = \mathrm{id}$ on the entire manifold.

LEMMA 6.4. *If $h|_{[p_2, p_3]} = \mathrm{id}$, then $h|_{[p_3, p_4]} = \mathrm{id}$.*

Proof. We can reduce this lemma to a linear algebra question using the following theorem.

THEOREM 6.5. [**19**, Theorem 1] *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a C^∞ diffeomorphism defined in some neighborhood of the origin, such that $F(0) = 0$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of T_0F. Then if $\lambda_i \neq \lambda_1^{m_1} \cdots \lambda_n^{m_n}$ for every i and for all non-negative integers $m_1, \ldots, m_n$ with $\sum m_i > 1$, there is a neighborhood U of the origin and a C^∞ diffeomorphism α defined on U such that $\alpha^{-1} F \alpha = T_0F$.*

$W^u(p_3)$ is a C^∞ submanifold of M. So by Theorem 6.5, there is a neighborhood $V \subset W^u(p_3)$ containing p_3, and a C^∞ diffeomorphism $\alpha : V \to \mathbb{R}^l$ such that

$$\alpha f^n \alpha^{-1} = A,$$

where $A = T_{p_3} f^n|_{W^u(p_3)}$ is linear. We can extend the conjugacy α to all of $W^u(p_3)$ by letting $\alpha(q) = f^{nM} \alpha f^{-nM}(q)$, where M is large enough so that $f^{-nM}(q) \in V$.

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The center leaf $\mathcal{W}^c(p_1)$ is C^1 [**6**, Theorem 6.8], and so the curve $\gamma = \alpha((p_2, p_4)) \subset \mathbb{R}^l$ is C^1, and γ is A-invariant. Let $\gamma_1 = \alpha((p_2, p_3])$ and $\gamma_2 = \alpha([p_3, p_4))$. So $\gamma = \gamma_1 \cup \gamma_2$, and $\gamma_1 \cap \gamma_2 = \alpha(p_3) = 0$.

Let $B = \alpha h \alpha^{-1}$. Then B commutes with A. Kopell has shown that if C is a linear contraction whose eigenvalues satisfy Sternberg's non-resonance condition, and if D is a C^∞ diffeomorphism that commutes with C, then D is also linear [**9**, Theorem 6]. So the diffeomorphism B is a linear map. Moreover, since $h|_{(p_2,p_3]} = \mathrm{id}$, $B|_{\gamma_1} = \mathrm{id}$. We need to show that $B|_{\gamma_2} = \mathrm{id}$ also.

Since $T_{p_3} f^n|_{W^s(p_3)}$ is diagonalizable (the eigenvalues are distinct), we can assume that A has the form

$$A = \begin{pmatrix} \lambda_1 & & & & & & & \\ & \ddots & & & & & & \\ & & \lambda_k & & & & & \\ & & & a_1 & b_1 & & & \\ & & & -b_1 & a_1 & & & \\ & & & & & \ddots & & \\ & & & & & & a_j & b_j \\ & & & & & & -b_j & a_j \end{pmatrix}$$

and that λ_1 is the eigenvalue corresponding to the weak unstable (or center) direction, which lies along the x_1-axis in $\mathbb{R}^l$. By commutativity with A, B will have the same block diagonal form:

$$B = \begin{pmatrix} \mu_1 & & & & & & & \\ & \ddots & & & & & & \\ & & \mu_k & & & & & \\ & & & c_1 & d_1 & & & \\ & & & -d_1 & c_1 & & & \\ & & & & & \ddots & & \\ & & & & & & c_j & d_j \\ & & & & & & -d_j & c_j \end{pmatrix}.$$

The curve γ is a graph over the x_1-axis. It passes through the origin, and is tangent to the x_1-axis at the origin. Let $\pi(\gamma_1)$ denote the projection of γ_1 to the x_1-axis. Since $B|_{\gamma_1} = \mathrm{id}$ and B is linear, $B|_{\pi(\gamma_1)} = \mathrm{id}$ also, and by linearity, $B = \mathrm{id}$ on the entire axis. So $\mu_1 = 1$. (If γ_2 lies along the x_1-axis, then we are done.)

Suppose $q = (y_1, \ldots, y_l) \in \gamma_2$. We know that $B(q) = (y_1, z_1, \ldots, z_l) \in \gamma$. But γ intersects the hyperplane $\{x_1 = y_1\}$ in just one point, since γ is a graph over the x_1-axis. So $B(q) = q$, and therefore $h|_{[p_3,p_4]} = \mathrm{id}$. □

We can make repeated applications of Lemma 6.4 to conclude that $h|_{\mathcal{W}^c(p_1)} = \mathrm{id}$. Next we show that $h = \mathrm{id}$ on all of $\mathcal{W}^{cs}(p_1)$. We have seen that $h|_{\mathcal{W}^c(p_1) \cup W^s(p_2)}$ is the identity map, so in particular, $h(\mathcal{W}^s(y)) = \mathcal{W}^s(y)$ for every $y \in \mathcal{W}^c(p_1)$, and $h(\mathcal{W}^c(x)) = \mathcal{W}^c(x)$ for every $x \in W^s(p_2)$.

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Notation. Let $\mathcal{W}^s(y,\epsilon)$ denote the local stable manifold through y of radius ϵ. Similarly, let $\mathcal{W}^c(x,\epsilon)$ denote the local center manifold through x.

Let $V \subset \mathcal{W}^{cs}(p_1)$ be a small neighborhood of $\mathcal{W}^c(p_1)$ of the form

$$V = \bigcup_{y \in \mathcal{W}^c(p_1)} \mathcal{W}^s(y,\epsilon),$$

and such that if $q \in V$, then $\mathcal{W}^c(q) \cap W^s(p_2) \neq \emptyset$. Such a neighborhood exists because $\mathcal{W}^c(p_1) \cap W^s(p_2) \neq \emptyset$, and by continuity of the center foliation.

By continuity of h, there is a neighborhood $U \subseteq V$ of $\mathcal{W}^c(p_1)$ such that if $q \in U \cap \mathcal{W}^s(y,\epsilon)$ for some $y \in \mathcal{W}^c(p_1)$, then $h(q) \in \mathcal{W}^s(y,\epsilon)$. We will show that $h|_U = \mathrm{id}$. Since $\mathcal{W}^{cs}(p_1) = \bigcup_{i \leq 0} f^i(U)$, and since h commutes with f, this will prove that $h|_{\mathcal{W}^{cs}(p_1)} = \mathrm{id}$.

Pick any $x \in W^s(p_2) \cap U$ and let

$$F_x = \{q \in \mathcal{W}^c(x) \cap U : h(q) = q\}.$$

Since h is continuous, F_x is closed in $\mathcal{W}^c(x) \cap U$. To prove that $h|_U = \mathrm{id}$, it is enough to show that F_x is also open in $\mathcal{W}^c(x) \cap U$. If F_x is open and closed in $\mathcal{W}^c(x) \cap U$, then $F_x = \mathcal{W}^c(x) \cap U$. As x was arbitrary, $h|_U = \mathrm{id}$.

So suppose that $q \in F_x$. Given any $\delta_1 > 0$, there is some $\delta_2 \leq \delta_1$ such that if $q' \in \mathcal{W}^c(q,\delta_2)$, then $h(q') \in \mathcal{W}^c(q,\delta_1)$. Choose $\delta_1 > 0$ small enough so that for every $y \in \mathcal{W}^c(p_1)$, $\mathcal{W}^c(q,\delta_1) \cap \mathcal{W}^s(y,\epsilon)$ is either empty or a single point. If δ_2 is chosen as above and if $q' \in \mathcal{W}^c(q,\delta_2) \cap \mathcal{W}^s(y,\epsilon)$ for some $y \in \mathcal{W}^c(p_1)$, then $h(q') \in \mathcal{W}^c(q,\delta_1) \cap \mathcal{W}^s(y,\epsilon)$. So $h(q') = q'$, and therefore $q' \in F_x$. So F_x is open in $\mathcal{W}^c(x) \cap U$. □

6.4. *$\mathcal{T}$ is a residual subset of $\mathcal{U}$.* We have yet to prove that property V is open in $\mathcal{U}$, and that property VI is residual in $\mathcal{U}$. It follows from the following lemma that property V is open in $\mathcal{U}$.

LEMMA 6.6. *Suppose that $f \in \mathcal{U}$ satisfies Property II:*

$$\mathcal{W}^{cs}(p_1) = W^s(p_1) \sqcup \cdots \sqcup W^s(p_{2N}),$$

where $\mathcal{P} = \{p_1, \ldots, p_{2N}\}$ is the set of (hyperbolic) periodic points for $f|_{\mathcal{W}^c(p_1)}$. Then the following are true.

(1) *The periodic points $\mathcal{P} = \{p_1(f), \ldots, p_{2N}(f)\}$ persist under perturbations of f, and vary continuously with f.*
(2) *$\mathcal{W}^u_f(p_1) \cap \mathcal{W}^{cs}_f(p_1)$ is a countable set, $\{q_0, q_1, q_2, \ldots\}$.*
(3) *Each point $q_i(f) \in \mathcal{W}^u_f(p_1) \cap \mathcal{W}^{cs}_f(p_1)$ persists under perturbations of f, and varies continuously with f.*
(4) *Suppose that $q_i(f) \in \mathcal{W}^u_f(p_1) \cap W^s_f(p_j)$ for some $p_j(f) \in \mathcal{P}_s$. Then there is a neighborhood V_i of f such that if $\tilde{f} \in V_i$, then $q_i(\tilde{f}) \in \mathcal{W}^u_{\tilde{f}}(p_1) \cap W^s_{\tilde{f}}(p_j)$.*

Proof of Lemma 6.6. Part (1) follows immediately, as both periodic points are hyperbolic. Parts (2) and (3) are true because the submanifolds $\mathcal{W}^u_f(p_1)$ and $\mathcal{W}^{cs}_f(p_1)$ intersect transversely, and both vary continuously with f.

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If $p_j \in \mathcal{P}_s$, then $\dim(W^s(p_j)) = l = \dim(\mathcal{W}^{cs}(p_1))$, and $W^s(p_j)$ is open in $\mathcal{W}^{cs}(p_1)$. Suppose that $q_i(f) \in \mathcal{W}^u_f(p_1) \cap W^s_f(p_j)$. Since $q_i(f)$ and $W^s_f(p_j)$ both vary continuously with f, there is a neighborhood V_i of f such that if $\tilde{f} \in V_i$, then $q_i(\tilde{f}) \in \mathcal{W}^u_{\tilde{f}}(p_1) \cap W^s_{\tilde{f}}(p_j)$. So part (4) is true. □

Remark. We cannot expect to find a neighborhood V of f such that for every $q_i(f) \in \mathcal{W}^u_f(p_1) \cap W^s_f(p_j)$ and every $\tilde{f} \in V$, $q_i(\tilde{f}) \in \mathcal{W}^u_{\tilde{f}}(p_1) \cap W^s_{\tilde{f}}(p_j)$. The neighborhoods V_i from part (4) could get arbitrarily small.

Let $\widetilde{\mathcal{T}}$ be the set of all $f \in U$ that satisfy Properties I through V of §6.2. We have seen that $\widetilde{\mathcal{T}}$ is an open, dense subset of U. We need to show that the set of diffeomorphisms that also satisfy property VI are residual in $\widetilde{\mathcal{T}}$ (in the C^∞ topology). This type of argument was introduced in [**12**, §§4 and 5.4]. In those sections, Palis and Yoccoz consider the case of structurally stable diffeomorphisms with a periodic point p and a countable set of homoclinic points related to p. They show that, generically, if a diffeomorphism f satisfies these conditions, then restricted to a neighborhood of the periodic point, f will have trivial centralizer. Their method involves linearizing the diffeomorphism in a neighborhood of p and constructing from the set of homoclinic points an invariant set for the linear map. Any diffeomorphism g that commutes with f can be simultaneously linearized, and this linear map must preserve the same invariant set. In the generic case, this condition is strong enough to conclude that g must be a power of f. We adapt these arguments to the partially hyperbolic setting. In the setting of Theorem 3.2, we cannot assume to have a periodic point with homoclinic points. But there will be periodic points with heteroclinic intersections. We will construct a linear map by linearizing in neighborhoods of these periodic points, and we will construct an invariant set for this map from the set of heteroclinic points. Some extra care must be taken in light of the remark following the proof of Lemma 6.6.

6.4.1. *Localization.* It is sufficient to prove that given any $f_0 \in \widetilde{\mathcal{T}}$, property VI is generic in a neighborhood of f_0. Fix a diffeomorphism f_0 and let $\mathcal{N} \subset \widetilde{\mathcal{T}}$ be a neighborhood of f_0 such that if $f \in \mathcal{N}$ then:

(1) for every $1 \le i \le 2N$, $p_i(f)$ is close to $p_i(f_0)$, and has period n;
(2) for every $1 \le i \le 2N$, the stable manifold $W^s_f(p_i(f))$ has the same dimension as $W^s_{f_0}(p_i(f_0))$;
(3) the unstable leaf $\mathcal{W}^u_f(p_1(f))$ has the same dimension as $\mathcal{W}^u_{f_0}(p_1(f_0))$;
(4) $\mathcal{W}^u_f(p_1(f)) \cap \mathcal{W}^{cs}_f(p_1(f)) = \{q_0(f), q_1(f), \dots\}$.

We will also assume that $q_0(f_0) \in \mathcal{W}^u(p_1(f_0)) \cap W^s(p_2(f_0))$ (where $p_2(f_0)$ is a sink for $f_0|_{\mathcal{W}^c_{f_0}(p_1)}$), and that $q_0(f) \in \mathcal{W}^u_f(p_1(f)) \cap W^s_f(p_2(f))$ for every $f \in \mathcal{N}$.

We will show that Property VI holds for a residual subset of $\mathcal{N}$.

6.4.2. *Linearization.* In this subsection we assign a linear map $A(f)$ and an $A(f)$-invariant set $J(f)$ to each $f \in \mathcal{N}$.

For every $f \in \mathcal{N}$, $\mathcal{W}^u(p_1)$ is a C^∞ submanifold of M. So by Theorem 6.5, there is a neighborhood $V \subset \mathcal{W}^u_f(p_1)$ of p_1 such that $f^n|_V$ is conjugate to the linear map

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$T_{p_1}(f^n|_{\mathcal{W}^u(p_1)}) := A_1(f)$,

$$f^n|_V = \alpha^{-1} A_1 \alpha,$$

where $\alpha : V \to \mathbb{R}^k$ is C^∞. Moreover, $\alpha = \alpha(f)$ depends continuously on f [**1**]. We can extend α to all of $\mathcal{W}^u(p_1)$ by letting $\alpha(q) = A_1^M \alpha f^{-M}(q)$, where M is large enough so that $f^{-M}(q) \in V$.

Similarly, $f^n|_{W^s(p_2)}$ is conjugate to the linear map $T_{p_2}(f^n|_{W^s(p_2)}) = A_2(f)$ via a C^∞ diffeomorphism $\beta = \beta(f) : W^s(p_2) \to \mathbb{R}^l$,

$$f^n|_{W^s(p_i)} = \beta^{-1} A_2 \beta.$$

Let

$$A(f) = \begin{pmatrix} A_1(f) & 0 \\ 0 & A_2(f) \end{pmatrix}.$$

If $q_j \in \mathcal{W}^u(p_1) \cap W^s(p_2)$, then we will identify q_j with $\overline{q}_j = (\alpha(q_j), \beta(q_j)) \in \mathbb{R}^{k+l}$. Let

$$J(f) = \{ \overline{q}_j(f) = (\alpha(q_j), \beta(q_j)) : q_j \in \mathcal{W}^u_f(p_1) \cap W^s_f(p_2)\}.$$

Remarks.

- $J(f)$ is a countable subset of $\mathbb{R}^{k+l}$.
- Since both $\mathcal{W}^u(p_1)$ and $W^s(p_2)$ are invariant under f^n, their intersection is also f^n-invariant. So the set $J(f)$ is invariant under the linear map $A(f)$.
- Every point in $J(f)$ is isolated, because the submanifolds $\mathcal{W}^u(p_1)$ and $W^s(p_2)$ are C^∞ and intersect transversely.
- Every point $\overline{q}_j(f) \in J(f)$ varies continuously with f, since $q_j(f)$, $\alpha(f)$ and $\beta(f)$ are all continuous in f.
- Every point $\overline{q} \in \mathbb{R}^{k+l}$ corresponds to a pair of points, $\alpha^{-1}(\overline{q}) \in \mathcal{W}^u(p_1)$ and $\beta^{-1}(\overline{q}) \in W^s(p_2)$. For example, the origin in $\mathbb{R}^{k+l}$ corresponds to the points p_1 and p_2. A point $\overline{q}$ is in $J(f)$ if and only if $\alpha^{-1}(\overline{q}) = \beta^{-1}(\overline{q}) \in \mathcal{W}^u(p_1) \cap W^s(p_2)$.
- If it is clear what diffeomorphism I am referring to, I will use A and $\overline{q}_j$ to denote $A(f)$ and $\overline{q}_j(f)$.

Notation. Let $M_{k+l}(\mathbb{R})$ denote the set of $(k+l) \times (k+l)$ matrices with entries in $\mathbb{R}$. If $A \in M_{k+l}(\mathbb{R})$ and $J \subset \mathbb{R}^{k+l}$ is an A-invariant subset, then let

$$\mathcal{C}_J(A) = \{B \in M_{k+l}(\mathbb{R}) : AB = BA, \text{ and } B(J) = J\}.$$

Note that $\mathcal{C}_J(A)$ is a group. It is closed under composition, and contains the subgroup $\{A^m : m \in \mathbb{Z}\}$.

PROPOSITION 6.7. *Suppose that $f \in \mathcal{N}$ and $g \in \mathcal{C}(f)$. Let $A = A(f)$ and $J = J(f)$. If $\mathcal{C}_J(A) = \{A^m : m \in \mathbb{Z}\}$, then $g|_{W^s(p_2)} = f^m$ for some $m \in \mathbb{Z}$.*

Proof of Proposition 6.7. Suppose that $f \in \mathcal{N}$ and $\mathcal{C}_J(A) = \{A^m : m \in \mathbb{Z}\}$, where $A = A(f)$ and $J = J(f)$. If $g \in \mathcal{C}(f)$, then by assumption, $g(p_1) = f^k(p_1)$ for some k, and $g(p_2) = f^k(p_2)$ (property IV in the definition of $\tilde{\mathcal{T}}$). So $h = gf^{-k}$ is a diffeomorphism that commutes with f and fixes p_1 and p_2. In particular,

$$h(\mathcal{W}^u(p_1)) = \mathcal{W}^u(p_1),$$
$$h(W^s(p_2)) = W^s(p_2),$$

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and

$$h(\mathcal{W}^u(p_1) \cap W^s(p_2)) = \mathcal{W}^u(p_1) \cap W^s(p_2).$$

Let $B_1 = \alpha h \alpha^{-1}$ and $B_2 = \beta h \beta^{-1}$. Then B_1 commutes with A_1 and B_2 commutes with A_2. As in the proof of Lemma 6.4, we can conclude that the diffeomorphisms B_1 and B_2 are linear maps. Then

$$B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}$$

is a linear map that commutes with A and preserves the set $J(f)$. $B \in \mathcal{C}_J(A)$, so $B = A^j$ for some $j \in \mathbb{Z}$. It now follows that $h|_{\mathcal{W}^u(p_1) \cup W^s(p_2)} = f^{nj}$, and therefore $g|_{W^s(p_2)} = f^{nj+k} = f^m$. □

COROLLARY 6.8. *Suppose that $f \in \mathcal{N}$, $A = A(f)$ and $J = J(f)$. If $\mathcal{C}_J(A) = \{A^m : m \in \mathbb{Z}\}$, then $\mathcal{C}(f) = \{f^m : m \in \mathbb{Z}\}$.*

Proof. By Proposition 6.7, if $f \in \mathcal{N}$ and $\mathcal{C}_J(A) = \{A^m : m \in \mathbb{Z}\}$, then f satisfies property VI in the definition of $\tilde{\mathcal{T}}$. So $f \in \mathcal{T}$, and therefore by §6.3, $\mathcal{C}(f) = \{f^m : m \in \mathbb{Z}\}$. □

6.4.3. *Perturbations.* We will need to make small perturbations to the sets $J(f)$. The following standard lemma describes the type of perturbation we will make.

LEMMA 6.9. [**11**, Section 5.4] *Suppose that $f \in \mathcal{N}$ and $q_j(f) \in \mathcal{W}^u_f(p_1) \cap W^s_f(p_2)$. Let U be a neighborhood of $q_j(f)$, small enough so that $f^m(q_j(f)) \notin U$ for $m \neq 0$. Then given any point $q \in \mathcal{W}^u_f(p_1) \cup W^s_f(p_2)$ sufficiently close to q_j, there is a diffeomorphism $\tilde{f} \in \mathcal{N}$ with the following properties:*

(1) $q = q_j(\tilde{f}) \in \mathcal{W}^u_{\tilde{f}}(p_1) \cap W^s_{\tilde{f}}(p_2)$;

(2) $\tilde{f} = f$ *on* $M - U$;

(3) $\tilde{f}$ *is* C^∞*-close to* f.

COROLLARY 6.10. *Suppose that $f \in \mathcal{N}$ and $\overline{q}_j(f) \in J(f)$. Then given any finite subset $S = \{\overline{q}_{i_1}(f), \overline{q}_{i_2}(f), \ldots, \overline{q}_{i_n}(f)\} \subset J(f)$ not containing $A^m(\overline{q}_j(f))$ for any $m \in \mathbb{Z}$, and any point $\overline{q} \in \mathbb{R}^{k+l}$ sufficiently close to $\overline{q}_j$, there is a diffeomorphism $\tilde{f} \in \mathcal{N}$ with the following properties:*

(1) $\overline{q}_j(\tilde{f}) = \overline{q}$;

(2) $\overline{q}_{i_m}(\tilde{f}) = \overline{q}_{i_m}(f)$ *for all* $\overline{q}_{i_m}(f) \in S$;

(3) $A(\tilde{f}) = A(f)$;

(4) $\tilde{f}$ *is* C^∞*-close to* f.

Proof. Recall that there are C^∞ maps $\alpha(f) : \mathcal{W}^u_f(p_1) \to \mathbb{R}^k$ and $\beta(f) : W^s(p_2) \to \mathbb{R}^l$ such that $\overline{q}_j(f) = (\alpha(q_j), \beta(q_j))$ for some $q_j \in \mathcal{W}^u_f(p_1) \cap W^s_f(p_2)$. Every point $\overline{q} \in \mathbb{R}^{k+l}$ corresponds to a point $\alpha^{-1}(\overline{q}) \in \mathcal{W}^u_f(p_1)$ and a point $\beta^{-1}(\overline{q}) \in W^s_f(p_2)$. If $\overline{q}$ is close enough to $\overline{q}_i(f)$, then we can make small perturbations as in Lemma 6.9 to give a diffeomorphism $\tilde{f} \in \mathcal{N}$ satisfying Properties (1), (2) and (4). Moreover, these perturbations will not affect $T_{p_1} f^n$ or $T_{p_2} f^n$, so $A(\tilde{f}) = A(f)$. □

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COROLLARY 6.11. *If $K \subset \mathbb{R}^{k+l}$ is a bounded, nowhere dense subset, then*

$$\mathcal{R}_K = \{f \in \mathcal{N} : K \cap J(f) = \emptyset\}$$

is open and dense in $\mathcal{N}$.

Proof. Any bounded region $U \subset \mathbb{R}^{k+l}$ corresponds to two bounded regions, $\alpha^{-1}(U) \subset \mathcal{W}^u(p_1)$ and $\beta^{-1}(U) \subset W^s(p_2)$. Since $\mathcal{W}^u(p_1)$ and $W^s(p_2)$ are C^∞ submanifolds and intersect transversely, $\alpha^{-1}(U) \cap \beta^{-1}(U)$ contains at most a finite number of points. So $U \cap J(f)$ is finite, and this property persists under perturbations of f.

If $K \subset \mathbb{R}^{k+l}$ is bounded and nowhere dense, then K is contained in some bounded region U, and $U \cap J(f) = \{\overline{q}_{i_1}(f), \ldots, \overline{q}_{i_n}(f)\}$ is a finite set. Moreover, there is a neighborhood $V \subset \mathcal{N}$ of f such that if $\tilde{f} \in V$, then $U \cap J(\tilde{f}) = \{\overline{q}_{i_1}(\tilde{f}), \ldots, \overline{q}_{i_n}(\tilde{f})\}$. By Corollary 6.10, $\mathcal{R}_K$ is dense and open in $\mathcal{N}$. □

6.4.4. *The linear case.* To complete the proof of Theorem 3.2, it remains to prove the following proposition.

PROPOSITION 6.12. *There is a residual subset $\mathcal{R} \subset \mathcal{N}$ such that if $f \in \mathcal{R}$, $A = A(f)$ and $J = J(f)$, then $\mathcal{C}_J(A) = \{A^m : m \in \mathbb{Z}\}$.*

Proof of Theorem 3.2 from Proposition 6.12. To prove Theorem 3.2, it is enough to show that the set of diffeomorphisms with trivial centralizer is C^∞ residual in the set $\mathcal{N}$, defined in §6.4.1. This follows immediately from Corollary 6.8 and Proposition 6.12. □

Notation. The periodic points $\mathcal{P} = \{p_1, \ldots, p_{2N}\}$ all have period n, so if $q_j(f) \in \mathcal{W}^u(p_1) \cap W^s(p_k)$, then $\{f^{nm}(q_i) : m \in \mathbb{Z}\} \subset \mathcal{W}^u(p_1) \cap W^s(p_k)$. Fix a representative q_{j_i} for each f^n-orbit. So $\mathcal{W}^u(p_1) \cap \mathcal{W}^{cs}(p_1)$ is a disjoint union of orbits,

$$\mathcal{W}^u(p_1) \cap \mathcal{W}^{cs}(p_1) = \bigsqcup_{i \geq 0} \{f^{nm}(q_{j_i}) : m \in \mathbb{Z}\}.$$

Let q_{j_0} be the representative of the orbit $\{f^{nm}(q_0) : m \in \mathbb{Z}\}$. Recall that we have assumed that for every $f \in \mathcal{N}$, $q_0(f) \in \mathcal{W}^u_f(p_1) \cap W^s_f(p_2)$ (see §6.4.1).

If $q_{j_i}(f) \in \mathcal{W}^u(p_1) \cap W^s(p_2)$, then $\overline{q}_{j_i}(f) \in J(f)$. Let $\mathcal{O}(\overline{q}_{j_i})$ denote the A-orbit of $\overline{q}_{j_i}$:

$$\mathcal{O}(\overline{q}_{j_i}) = \{A^n(\overline{q}_{j_i}) : n \in \mathbb{Z}\}.$$

So

$$J(f) = \bigsqcup \mathcal{O}(\overline{q}_{j_i}(f)),$$

where the union is taken over all $q_{j_i}(f) \in \mathcal{W}^u(p_1) \cap W^s(p_2)$.

We will need the following result from linear algebra, which we state without proof.

LEMMA 6.13. *If $A : \mathbb{R}^m \to \mathbb{R}^m$ is an $m \times m$ matrix with m distinct, non-zero eigenvalues $\lambda_1, \ldots, \lambda_m$, then there is an open and dense subset of vectors $U \subset \mathbb{R}^m$ such that if $v \in U$, then $\{v, Av, \ldots, A^{m-1}v\}$ is a linearly independent set.*

COROLLARY 6.14. *There is an open, dense subset $\mathcal{R}_0 \subset \mathcal{N}$ such that if $f \in \mathcal{R}_0$, then $\mathcal{O}(\overline{q}_{j_0}(f))$ spans $\mathbb{R}^{k+l}$.*

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Proof. For any $f \in \mathcal{N}$, the eigenvalues of $A(f)$ are non-zero and distinct. So by Lemma 6.13, there is an open, dense set of points in $\mathbb{R}^{k+l}$ whose $A(f)$-orbits span $\mathbb{R}^{k+l}$. Let

$$\mathcal{R}_0 = \{f \in \mathcal{N} : \mathcal{O}(\overline{q}_{j_0}(f)) \text{ spans } \mathbb{R}^{k+l}\}.$$

Then $\mathcal{R}_0$ is dense in $\mathcal{N}$ by Corollary 6.10, and open because the point $\overline{q}_{j_0}(f)$ varies continuously with f. □

LEMMA 6.15. *Suppose that $f \in \mathcal{R}_0$ and $A = A(f)$. If $B \in M_{k+l}(\mathbb{R})$ and $C \in M_{k+l}(\mathbb{R})$, and if both B and C commute with A, then:*

(1) *$B = C$ if and only if $B(q_{j_0}) = C(q_{j_0})$;*

(2) *$B = A^m$ if and only if $B(\overline{q}_{j_0}) = A^m(\overline{q}_{j_0})$.*

Proof. For every $n \in \mathbb{Z}$,

$$B(A^n(\overline{q}_{j_0})) = A^n(B(\overline{q}_{j_0}))$$

and

$$C(A^n(\overline{q}_{j_0})) = A^n(C(\overline{q}_{j_0})).$$

So $B(\overline{q}_{j_0}) = C(\overline{q}_{j_0})$ if and only if $B(A^n(\overline{q}_{j_0})) = C((A^n(\overline{q}_{j_0}))$ for every $n \in \mathbb{Z}$. But $\mathcal{O}(\overline{q}_{j_0})$ spans $\mathbb{R}^{k+l}$, and B and C are linear, so $B = C$ if and only if $B(\overline{q}_{j_0}) = C(\overline{q}_{j_0})$.

Part (2) is a direct corollary of part (1). □

COROLLARY 6.16. *If $f \in \mathcal{R}_0$, $A = A(f)$ and $J = J(f)$, then $\mathcal{C}_J(A) = \{A^m : m \in \mathbb{Z}\}$ if and only if for every $B \in \mathcal{C}_J(A)$, $B(\overline{q}_{j_0}) \in \mathcal{O}(\overline{q}_{j_0})$.*

Proof. This is an immediate consequence of Lemma 6.15. □

Every $B \in \mathcal{C}_J(A)$ maps $\overline{q}_{j_0}$ into some $\mathcal{O}(\overline{q}_{j_i}) \subset J(f)$. For each $q_{j_i} \in \mathcal{W}^u(p_1) \cap \mathcal{W}^{cs}(p_1)$ (where $i \neq 0$), let $\mathcal{R}_i$ denote the set of all $f \in \mathcal{R}_0$ such that:

(1) $q_{j_i}(f) \in W^s_f(p_k)$ for some $p_k \in \mathcal{P}_s$; and

(2) if $q_{j_i}(f) \in W^s_f(p_2)$, then for every $B \in \mathcal{C}_{J(f)}A(f)$, $B(\overline{q}_{j_0}) \notin \mathcal{O}(\overline{q}_{j_i})$.

In light of Corollary 6.16, to prove Proposition 6.12 it remains to prove the following lemma.

LEMMA 6.17. *For every $i > 0$, the set $\mathcal{R}_i$ contains an open and dense subset of $\mathcal{R}_0$.*

Proof of Proposition 6.12 from Lemma 6.17. Let

$$\mathcal{R} = \bigcap_{i \geq 0} \mathcal{R}_i.$$

By Corollary 6.14 and Lemma 6.17, $\mathcal{R}$ is a residual subset of $\mathcal{N}$. Suppose that $f \in \mathcal{R}$, $A = A(f)$ and $J = J(f)$. If $B \in \mathcal{C}_J(A)$, then $B(\overline{q}_{j_0}) \in \mathcal{O}(\overline{q}_{j_0})$. So by Corollary 6.16, $\mathcal{C}_J(A) = \{A^m : m \in \mathbb{Z}\}$. □

The proof of Lemma 6.17 is given after the following lemma.

LEMMA 6.18. *There is an open, dense subset $\mathcal{R}_\mathcal{D} \subset \mathcal{R}_0$ such that if $f \in \mathcal{R}_\mathcal{D}$ and $B \in \mathcal{C}_{J(f)}(A(f))$, then $B^2 = A^k$ for some $k \in \mathbb{Z}$ if and only if $B = A^m$ for some $m \in \mathbb{Z}$.*

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Proof. Let

$$Q = \{D(\overline{q}_{i_0}) : D \in M_{k+l}(\mathbb{R}) \text{ and } D^2 = I\}$$

and

$$S = \{D(\overline{q}_{i_0}) : D \in M_{k+l}(\mathbb{R}) \text{ and } D^2 = A\}.$$

The set Q is a bounded, nowhere dense subset of $\mathbb{R}^{k+l}$ (because if $D^2 = I$, then $\|D\| = 1$, so Q is contained in a codimension 1 sphere centred at the origin). Since the eigenvalues of A are distinct, the set S is a finite collection of points. By Corollary 6.11, the set

$$\mathcal{R}_{\mathcal{D}} = \{f \in \mathcal{R}_0 : J(f) \cap (Q \cup S) = \{\overline{q}_{i_0}\}\}$$

is open and dense in $\mathcal{R}_0$.

Suppose that $f \in \mathcal{R}_{\mathcal{D}}$ and for some $B \in \mathcal{C}_{J(f)}(A(f))$ and some $k \in \mathbb{Z}$, $B^2 = A^k$. Depending on whether k is even or odd, there is some $n \in \mathbb{Z}$ such that either $(BA^{-n})^2 = I$ or $(BA^{-n})^2 = A$. So $BA^{-n}(\overline{q}_{i_0}) \in Q \cup S$, and $BA^{-n}(\overline{q}_{i_0}) \in J(f)$, because $J(f)$ is A- and B-invariant. By definition of $\mathcal{R}_{\mathcal{D}}$, $BA^{-n}(\overline{q}_{i_0}) = \overline{q}_{i_0}$, so by Lemma 6.15, $B = A^n$. □

Proof of Lemma 6.17. Fix some $q_{j_i} (\neq q_{j_0}) \in \mathcal{W}^u(p_1) \cap \mathcal{W}^{cs}(p_1)$. The set $\bigcup_{p_k \in \mathcal{P}_s} W^s(p_k)$ is open and dense in $\mathcal{W}^{cs}(p_1)$, and $q_{j_i}(f)$ varies continuously with f. So Property (1) in the definition of $\mathcal{R}_i$ is open and dense in $\mathcal{R}_0$.

Let

$$S_i = \{f \in \mathcal{R}_{\mathcal{D}} : q_{j_i}(f) \in \mathcal{W}^u_f(p_1) \cap W^s_f(p_2)\}.$$

If $B \in \mathcal{C}_J(A)$ and $B(\overline{q}_{j_0}) \in \mathcal{O}(\overline{q}_{j_i})$, then $BA^{-k}(\overline{q}_{j_0}) = \overline{q}_{j_i}$ for some $k \in \mathbb{Z}$ and $BA^{-k} \in \mathcal{C}_J(A)$. So it is enough to show that for f in an open, dense subset of S_i, if $B \in \mathcal{C}_J(A)$ then $B(\overline{q}_{j_0}) \neq \overline{q}_{j_i}$. By Lemma 6.15, there is a unique linear map $B_i \in M_{k+l}(\mathbb{R})$ that maps $\overline{q}_{j_0}$ to $\overline{q}_{j_i}$ and commutes with A. Since the points $\overline{q}_{j_0}(f)$ and $\overline{q}_{j_i}(f)$ are continuous in f, $B_i = B_i(f)$ will also vary continuously with f. We will show that for f in an open and dense subset of S_i, $B_i(f) \notin \mathcal{C}_{J(f)}(A(f))$; $B_i(f)$ will not preserve the set $J(f)$.

Suppose that $f \in S_i$ and that $B_i(f) \in J(f)$. Note that $\{B_i^m(\overline{q}_{j_0}) : m \in \mathbb{Z}\} \subset J(f)$.

CLAIM. *The three points $\overline{q}_{j_0}$, $\overline{q}_{j_i} = B_i(\overline{q}_{j_0})$, and $\overline{q}_k = B_i^2(\overline{q}_{j_0})$ lie in distinct A-orbits.*

Proof of Claim. We have assumed that $B_i(\overline{q}_{j_0}) = \overline{q}_{j_i} \notin \mathcal{O}(\overline{q}_{j_0})$, so $B_i \neq A^m$ for any $m \in \mathbb{Z}$. And as $f \in \mathcal{R}_{\mathcal{D}}$, $B_i^2 \neq A^k$ for any k. So $\overline{q}_k = B_i^2(\overline{q}_{j_0}) \notin \mathcal{O}(\overline{q}_{j_0})$. It remains to show that $\overline{q}_k \notin \mathcal{O}(\overline{q}_{j_i})$.

Since $\mathcal{O}(\overline{q}_{j_0})$ spans $\mathbb{R}^{k+l}$ and B_i is an invertible matrix, $\mathcal{O}(B_i(\overline{q}_{j_0})) = \mathcal{O}(\overline{q}_{j_i})$ also spans $\mathbb{R}^{k+l}$. In particular, $B_i = A^m$ if and only if $B_i(\overline{q}_{j_i}) = A^m(\overline{q}_{j_i})$ (see Lemma 6.15). But we have assumed that $B_i \neq A^m$. So $\overline{q}_k = B_i(\overline{q}_{j_i}) \notin \mathcal{O}(\overline{q}_{j_i})$. □

By Corollary 6.10, we can perturb f to obtain a diffeomorphism $\tilde{f}$, C^∞-close to f, such that:

(1) $A(f) = A(\tilde{f})$;
(2) $\overline{q}_{j_0}(f) = \overline{q}_{j_0}(\tilde{f})$, and $\overline{q}_{j_i}(f) = \overline{q}_{j_i}(\tilde{f})$, and therefore $B_i(f) = B_i(\tilde{f})$;
(3) there is a neighborhood U of $B_i^2(\overline{q}_{j_0})$ such that $U \cap J(\tilde{f}) = \emptyset$.

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Since $B_i^2(\overline{q}_{j_0}) \notin J(\tilde{f})$, $B_i(\tilde{f}) \notin \mathcal{C}_{J(\tilde{f})}(A(\tilde{f}))$. Moreover, since $B_i(\tilde{f})$, $A(\tilde{f})$ and $J(\tilde{f})$ are continuous in $\tilde{f}$, this property will persist under perturbations. So the property that $B_i(f) \notin \mathcal{C}_{J(f)}(A(f))$ holds for an open and dense subset of S_i. This completes the proof of Lemma 6.17. □

6.5. *Anosov flows.* Let $\mathcal{A}$ be the set of all volume preserving Anosov flows ϕ on a compact manifold M.

PROPOSITION 6.19. *There is a residual subset $\mathcal{A}' \subset \mathcal{A}$ (in the C^r topology, $r \geq 1$) with the following property. For any flow $\phi \in \mathcal{A}'$, there is a residual subset $\mathcal{T}_\phi \subset \mathbb{R}$ such that if $t \in \mathcal{T}_\phi$ then $\mathcal{C}(\phi_t) = \{\phi_s : s \in \mathbb{R}\}$.*

Remark. Proposition 6.19 can also be stated within the category of geodesic flows. Let M be a compact manifold, and let $\mathcal{R}$ denote the set of all Riemannian metrics on M with everywhere negative sectional curvature. For a metric $g \in \mathcal{R}$, let $\varphi : T_1(M) \to T_1(M)$ denote the geodesic flow on the unit tangent bundle to (M, g), and let φ_t denote the time-t map of φ. Then $\mathcal{C}(\varphi_t)$ contains the subgroup $\{\varphi_s : s \in \mathbb{R}\}$.

PROPOSITION 6.20. *There is a residual subset $\mathcal{R}' \subset \mathcal{R}$ such that if $g \in \mathcal{R}'$ and φ is the geodesic flow on $(T_1(M), g)$, then there is a residual subset $\mathcal{T}_g \subset \mathbb{R}$ with the following property. If $t \in \mathcal{T}_g$, then $\mathcal{C}(\varphi_t) = \{\varphi_s : s \in \mathbb{R}\}$.*

Proof of Proposition 6.19. The proposition will follow from the following theorem of Jungreis and Hirsch [**7**].

THEOREM 6.21. [**7**] *If $\phi \in \mathcal{A}$ and $f : M \to M$ is a homeomorphism such that*
(1) *f sends each periodic orbit of ϕ to itself, and*
(2) *f commutes with ϕ_t for all $t \in \mathbb{R}$,*
then $f = \phi_s$ for some $s \in \mathbb{R}$.

If $\phi \in \mathcal{A}$, then the set of periodic orbits is countable and dense in M. The residual subset $\mathcal{A}' \subset \mathcal{A}$ will be the set of volume preserving Anosov flows such that no two periodic orbits have the same period. (In the geodesic flow case, $\mathcal{R}'$ will be the set of Riemannian metrics g such that no two periodic orbits of the geodesic flow on $(T_1(M), g)$ have the same period.)

Suppose that $\phi \in \mathcal{A}'$. Let $S_\phi = \{s_1, s_2, \dots\}$ be the periods of the periodic orbits for ϕ. Let $\mathcal{T}_\phi \subset \mathbb{R}$ be the set of all real numbers that are irrationally related to every $s \in S$:

$$\mathcal{T}_\phi = \left\{ t \in \mathbb{R} : \frac{t}{s} \notin \mathbb{Q} \text{ for every } s \in S_\phi \right\}.$$

Note that $\mathcal{T}_\phi$ is residual in $\mathbb{R}$. Now suppose that $t \in \mathcal{T}_\phi$ and $f \in \mathcal{C}(\phi_t)$. We will show that:
(1) f sends each periodic orbit of ϕ to itself;
(2) f commutes with ϕ_s, for all $s \in \mathbb{R}$.
Then it follows from Theorem 6.21 that $f = \phi_s$ for some $s \in \mathbb{R}$.

Let $P_i = \{\phi_s(x) : 0 \leq s < s_i\}$ be a periodic orbit for ϕ with period s_i. As t is irrationally related to s_i, the set $\{\phi_{nt}(x) : n \in \mathbb{Z}\}$ is dense in P_i. By commutativity of f with ϕ_t, and by continuity of f and ϕ:

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(a) $f(P_i)$ is a closed curve;
(b) $f(\phi_{nt}(x)) = \phi_{nt}(f(x))$ for every integer n;
(c) the set $\{f(\phi_{nt}(x)) : n \in \mathbb{Z}\} = \{\phi_{nt}(f(x)) : n \in \mathbb{Z}\}$ is dense in $f(P_i)$; so
(d) $f(P_i)$ is invariant under the flow ϕ;
(e) if we orient the circles P_i and $f(P_i)$ in such a way that f preserves orientation, then by continuity the points $\{\phi_{nt}(f(x)) : n \in \mathbb{Z}\} \subset f(P_i)$ are in the same order as their preimages $\{\phi_{nt}(x) : n \in \mathbb{Z}\} \subset P_i$; and therefore
(f) $f(P_i)$ has the same period as P_i.

So $f(P_i)$ is also a closed orbit for ϕ, with the same period as P_i. As every closed orbit for ϕ has a different period, $f(P_i) = P_i$. This proves part (1).

The restriction of ϕ_t to any periodic orbit $P_i = \{\phi_s(x) : 0 \leq s < s_i\}$ is (conjugate to) an irrational circle rotation. It is not hard to show that the only homeomorphisms that commute with irrational rotations on a circle are rotations. In particular, for any periodic orbit P_i, $f|_{P_i} = \phi_{t_i}$ for some $t_i \in \mathbb{R}$. So the restriction of f to P_i commutes with ϕ_s, for any $s \in \mathbb{R}$. But the periodic orbits are dense in M, so for any $s \in \mathbb{R}$, f commutes with ϕ_s on a dense subset of M. By continuity, f commutes with ϕ_s on all of M, This proves part (2). □

7. *Partially hyperbolic skew products*

Proof of Theorem 3.4. Let f_ϕ be a partially hyperbolic skew product on $\mathbb{T}^n \times M$. The stable and unstable bundles, E^s and E^u, are tangent to smooth foliations $\mathcal{W}^s$ and $\mathcal{W}^u$. Each leaf $\mathcal{W}^s(x, y)$ in $\mathcal{W}^s$ is the graph of a C^r function over a stable leaf $\mathcal{W}^s_f(x)$ for f on $\mathbb{T}^n$ [**3**]. Similarly for the unstable leaves. The center bundle, E^c, is C^1 and therefore uniquely integrable. Each leaf $\mathcal{W}^c(x, y)$ in $\mathcal{W}^c$ is a copy of M.

Let $\mathcal{SP}^r(\mathbb{T}^n \times M)^k = \{f_\phi \in \mathcal{SP}^r(\mathbb{T}^n \times M) : f_\phi$ has the accessibility property, and a periodic point (x_0, y_0) with period $k\}$.

Again using the C^1 Closing Lemma and [**5**], Theorem 3.4 is a corollary of the following theorem.

THEOREM 7.1. *There is a set $\mathcal{T}^k$, C^1-open and C^r-dense in $\mathcal{SP}^r(\mathbb{T}^n \times M)^k$, such that if $f_\phi \in \mathcal{T}^k$ and $G \in \mathcal{C}(f_\phi)$, then $G^k = (f_\phi)^{km}$ for some $m \in \mathbb{Z}$.*

By the Closing lemma and [**5**], a C^1-open, dense subset of $\mathcal{SP}^r(\mathbb{T}^n \times M)$ is contained in $\bigcup \mathcal{SP}^r(\mathbb{T}^n \times M)^k$.

Proof of Theorem 7.1. Let G be a C^r diffeomorphism that commutes with $f_\phi \in \mathcal{SP}^r(\mathbb{T}^n \times M)$. We have seen (Lemma 4.2) that G preserves the stable, center and unstable foliations for f_ϕ. So G is also a skew product on $\mathbb{T}^n \times M$; $G = g_\beta$ for some C^r diffeomorphism $g : \mathbb{T}^n \to \mathbb{T}^n$ and C^r map $\beta : \mathbb{T}^n \to \mathrm{Diff}^r(M)$. And g commutes with f.

Let $\mathcal{T}^k$ be the set of all $f_\phi \in \mathcal{SP}^r(\mathbb{T}^n \times M)^k$ with the following properties.

(I) If $G = g_\beta$ commutes with f_ϕ, then $g = f^m$ for some $m \in \mathbb{Z}$.
(II) If G commutes with f_ϕ, then $G(x_0, y_0) = (f_\phi)^n(x_0, y_0)$ for some $n \in \mathbb{Z}$, where (x_0, y_0) is the periodic point for f_ϕ with period k.

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(I) is open and dense in $\mathcal{S}P^r(\mathbb{T}^n \times M)^k$ (in the C^r topology) by [**11**], and (II) is open and dense by Lemma 4.3.

LEMMA 7.2. *Suppose that H is a skew product on $\mathbb{T}^n \times M$ that commutes with f_ϕ and projects to the identity on $\mathbb{T}^n$. If $H(x, y) = (x, y)$ for some $(x, y) \in \mathcal{T}^n \times M$, then $H|_{\mathcal{W}^s(x,y)} = \mathrm{id}$ and $H|_{\mathcal{W}^u(x,y)} = \mathrm{id}$.*

Proof. By Lemma 4.2, $H(\mathcal{W}^s(x, y)) = \mathcal{W}^s(H(x, y)) = \mathcal{W}^s(x, y)$. So if $(w, z) \in \mathcal{W}^s(x, y)$, then $H(w, z) = (w, \tilde{z}) \in \mathcal{W}^s(x, y)$. But each stable leaf contains at most one point in each fiber. So $\tilde{z} = z$, and $H|_{\mathcal{W}^s(x,y)} = \mathrm{id}$. Similarly, $H|_{\mathcal{W}^u(x,y)} = \mathrm{id}$. □

Suppose that $f_\phi \in \mathcal{T}^k$ and $G = g_\beta$ commutes with f_ϕ, where $g = f^m$. Let $H = Gf_\phi^{-m}$. Then H is a skew product that commutes with f_ϕ and projects to the identity on $\mathbb{T}^n$. By definition of $\mathcal{T}^k$, $G(x_0, y_0) = (f_\phi)^n(x_0, y_0)$ for some n. So $H(x_0, y_0) = (f_\phi)^{n-m}(x_0, y_0)$, and $H^k(x_0, y_0) = (x_0, y_0)$. By the previous lemma, $H^k|_{\mathcal{W}^s(x_0,y_0)} = \mathrm{id}$ and $H^k|_{\mathcal{W}^u(x_0,y_0)} = \mathrm{id}$. Then using accessibility and the lemma, $H^k = \mathrm{id}$ on all of M. So $G^k = (f_\phi)^{km}$. □

8. *Perturbations of skew products*

In this section, we give a proof of Theorem 3.5. This is a result about perturbations of skew products over $\mathbb{T}^2$. The reason for assuming that the torus is two-dimensional is that Lemma 8.1 is not true more generally [**14**].

Proof of Theorem 3.5.

LEMMA 8.1. *Suppose that $f : \mathbb{T}^2 \to \mathbb{T}^2$ is an Anosov diffeomorphism, and $g : \mathbb{T}^2 \to \mathbb{T}^2$ is a homeomorphism commuting with f. Then $g^i = f^j$ for some $i, j \in \mathbb{Z}$.*

The proof of Lemma 8.1 will use the following results.

LEMMA 8.2. [**11**, Proposition 0] *If $A : \mathbb{T}^2 \to \mathbb{T}^2$ is a linear Anosov diffeomorphism with a fixed point p, and G is a homeomorphism that commutes with A and fixes p, then G is also linear.*

LEMMA 8.3. [**14**] *Let $A : \mathbb{T}^2 \to \mathbb{T}^2$ be a linear Anosov diffeomorphism. Then $\mathcal{C}(A) = \{G \in \mathrm{Diff}^\infty(\mathbb{T}^2) : GA = AG\}$ is isomorphic to $\mathbb{Z} \oplus \mathcal{F}$, where $\mathcal{F}$ is a finite commutative group.*

Proof of Lemma 8.1. If $f : \mathbb{T}^2 \to \mathbb{T}^2$ is an Anosov diffeomorphism, then f is conjugate to a linear Anosov diffeomorphism $A : \mathbb{T}^2 \to \mathbb{T}^2$ via a homeomorphism h:

$$A = hfh^{-1}.$$

If g is a homeomorphism that commutes with f, then $G = hgh^{-1}$ is a homeomorphism that commutes with A. Every Anosov diffeomorphism has a finite number of fixed points, and these fixed points are permuted by G. If p is a fixed point for A, then $G^n(p) = p$ for some $n \in \mathbb{Z}$. So by Lemma 8.2, G^n is a linear map. In particular, $G^n \in \mathcal{C}(A)$, and therefore $G^n = A^m F$, where $F^k = \mathrm{id}$ for some $k \in \mathbb{Z}$. Then $G^{nk} = A^{mk}$, and so $g^{nk} = f^{mk}$. □

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The center bundle for a partially hyperbolic skew product is uniquely integrable. We will show that this property still holds for perturbations of a skew product. I would like to thank Mike Shub for pointing out this argument.

LEMMA 8.4. *Let F be a C^1 partially hyperbolic diffeomorphism of a compact manifold such that $E^c(F)$ is uniquely integrable and $\mathcal{W}^c_F$ is C^1. If $\widetilde{F}$ is another C^1 diffeomorphism sufficiently C^1-close to F, then $\widetilde{F}$ is also partially hyperbolic, and $E^c(\widetilde{F})$ is uniquely integrable.*

Lemma 8.4 will follow from the following theorem [**6**]. (See §2.2 for definitions.)

THEOREM 8.5. (From [**6**, Theorems 6.8 and 7.1]) *Let F be a plaque expansive, C^1 partially hyperbolic diffeomorphism of a compact manifold N. There are constants $\delta = \delta(F)$ and $\epsilon = \epsilon(F, \delta)$ such that given any two C^1 diffeomorphisms G_1 and G_2 within C^1 distance δ of F:*

(1) *G_1 and G_2 are partially hyperbolic;*
(2) *there are foliations $\mathcal{W}^c_{G_1}$ and $\mathcal{W}^c_{G_2}$ tangent to $E^c_{G_1}$ and $E^c_{G_2}$;*
(3) *$(G_1, \mathcal{W}^c_{G_1})$ and $(G_2, \mathcal{W}^c_{G_2})$ are leaf conjugate via a homeomorphism $h : \mathcal{W}^c_{G_1} \to \mathcal{W}^c_{G_2}$;*
(4) *for every point $p \in N$, $d(p, h(p)) < \epsilon$.*

Moreover, $\mathcal{W}^c_{G_1}$ and $\mathcal{W}^c_{G_2}$ are the unique foliations tangent to $E^c_{G_1}$ and $E^c_{G_2}$ satisfying both Properties (3) and (4).

Proof of Lemma 8.4 from Theorem 8.5. Let F be a C^1 partially hyperbolic diffeomorphism such that $E^c(F)$ is uniquely integrable and $\mathcal{W}^c_F$ is C^1. Then F is stably plaque expansive, because the center foliation is C^1 ([**6**, Theorems 7.2 and 7.3], see §2.2). Let $\widetilde{F}$ be a C^1 diffeomorphism close to F (within C^1 distance δ, given by Theorem 8.5, and so that $\widetilde{F}$ is plaque expansive at $\mathcal{W}^c_{\widetilde{F}}$). Suppose that there are two foliations, $\mathcal{F}_1$ and $\mathcal{F}_2$, both tangent to $E^c(\widetilde{F})$.

Applying Theorem 8.5 to $\widetilde{F}$, there is a foliation $\mathcal{W}_1$ tangent to $E^c(F)$, such that $(F, \mathcal{W}_1)$ is leaf conjugate to $(\widetilde{F}, \mathcal{F}_1)$ via a homeomorphism $h_1 : \mathcal{W}_1 \to \mathcal{F}_1$, such that $d(p, h_1(p)) < \epsilon$ for every p. Similarly, there is a foliation $\mathcal{W}_2$ tangent to $E^c(F)$, such that $(F, \mathcal{W}_2)$ is leaf conjugate to $(\widetilde{F}, \mathcal{F}_2)$ via a homeomorphism $h_2 : \mathcal{W}_2 \to \mathcal{F}_2$, and $d(p, h_2(p)) < \epsilon$ for every p. By unique integrability of $E^c(F)$, $\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}^c_F$.

Now we have shown that the two foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ both satisfy conditions (3) and (4) of Theorem 8.5. By uniqueness, $\mathcal{F}_1 = \mathcal{F}_2$. □

Suppose that $f_\phi \in \mathcal{SP}^r(\mathbb{T}^n \times M)$. There is a neighborhood $U \subset \mathrm{Diff}(\mathbb{T}^n \times M)$ containing f_ϕ such that if $F \in U$ then:

(1) F is partially hyperbolic;
(2) $E^c(F)$ is uniquely integrable;
(3) $(F, \mathcal{W}^c_F)$ is leaf conjugate to $(f_\phi, \mathcal{W}^c_{f_\phi})$.

Since every stable leaf of f_ϕ is the graph of a C^r function over a stable leaf for f on $\mathbb{T}^n$, the skew product has the nice property that each center leaf intersects every stable leaf in at most one point. This property persists under C^1 perturbations. Similarly for the unstable leaves. Thus we can choose the following:

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(4) each leaf of $\mathcal{W}^c_F$ intersects every stable or unstable leaf for F in at most one point.

Finally, we let $V \subset U$ be the set of all diffeomorphisms $F \in U$ such that:

(I) F has the accessibility property;

(II) F has a periodic point p, and if $G \in \mathcal{C}(F)$, then $G(p) = F^k(p)$ for some $k \in \mathbb{Z}$.

As discussed in §7, V contains an open, dense subset of U (in the C^1 topology).

So suppose that $F \in V$ and $G \in \mathcal{C}(F)$. By assumption, $(F, \mathcal{W}^c_F)$ is leaf conjugate to $(f_\phi, \mathcal{W}^c_{f_\phi})$. Let h be the homeomorphism that maps leaves of $\mathcal{W}^c_{f_\phi}$ to leaves of $\mathcal{W}^c_F$. If $L \in \mathcal{W}^c_{f_\phi}$, then $h(L) \in \mathcal{W}^c_F$, and $Fh(L) = hf_\phi(L)$. So the map $h^{-1}Fh$ is a homeomorphism of $\mathbb{T}^n \times M$. It preserves the center foliation $\mathcal{W}^c_{f_\phi}$, and projects along leaves of $\mathcal{W}^c_{f_\phi}$ to the diffeomorphism f on $\mathbb{T}^n$.

If $G \in \mathcal{C}(F)$, then $h^{-1}Gh$ commutes with $h^{-1}Fh$. Note that $h^{-1}Gh$ is a homeomorphism, but is not necessarily a diffeomorphism. G preserves leaves of $\mathcal{W}^c_F$ (by Lemma 4.2 and unique integrability of $E^c(F)$). So $h^{-1}Gh$ preserves leaves of $\mathcal{W}^c_{f_\phi}$, and projects to a homeomorphism g on $\mathbb{T}^n$. By Lemma 8.1, $g^m = f^n$ for some $m, n \in \mathbb{Z}$.

Now let $H = G^{-m}F^n$. Then $h^{-1}Hh$ is a homeomorphism on $\mathbb{T}^n \times M$ that projects along leaves of $\mathcal{W}^c_{f_\phi}$ to the identity on $\mathbb{T}^n$, and H is a diffeomorphism that fixes leaves of $\mathcal{W}^c_F$.

From here the argument follows as in the proof of Theorem 3.4. Let p be the periodic point for F guaranteed by Property (II). Then $H(p) = F^{-mk+n}(p)$, and $H^i(p) = p$ for some integer i. If q is a point on $\mathcal{W}^u_F(p)$, then $H^i(q) \in \mathcal{W}^u_F(H^i(p)) = \mathcal{W}^u_F(p)$. And H fixes leaves of $\mathcal{W}^c_F$, so $H^i(q) \in \mathcal{W}^c_F(q)$. But then $H^i(q) = q$, as $\mathcal{W}^u_F(p) \cap \mathcal{W}^c_F(q)$ contains just one point. So H^i is the identity on $\mathcal{W}^u_F(p)$. Using the accessibility property and repeating this argument, we see that H^i is the identity map. So $G^{mi} = F^{ni}$. □

9. *Skew products that are not partially hyperbolic*

In this section, we give a proof of Theorem 3.6.

Notation. Regarding $\mathbb{T}^n \times S^1$ as a bundle with base $\mathbb{T}^n$, I will denote the fiber $\{x\} \times S^1$ by C_x. To simplify notation, let

$$\phi^m_x(a) = \phi_{f^{m-1}(x)} \circ \phi_{f^{m-2}(x)} \circ \cdots \circ \phi_{f(x)} \circ \phi_x(a).$$

Then

$$(f_\phi)^m(x, a) = (f^m(x), \phi^m_x(a)).$$

Proof of Theorem 3.6. Let $\mathcal{T}$ be the set of all $f_\phi \in \mathcal{S}^r(\mathbb{T}^n \times S^1)$ with the following properties.

(1) All periodic points for f_ϕ are hyperbolic.

(2) $f : \mathbb{T}^n \to \mathbb{T}^n$ has trivial centralizer.

(3) If $f^k(x) = x$ for some k, then:

- the circle diffeomorphism $(f_\phi)^k|_{C_x}$ has trivial centralizer;
- C_x contains f_ϕ-periodic points $\{(x, a_1), \ldots, (x, a_m)\}$;
- $\prod_{i=1}^m [(\phi^k_x)'(a_i)]^{j_i} \neq 1$ where each $j_i = 0$ or 1 and $\sum j_i \neq 0$.

PROPOSITION 9.1. *The set $\mathcal{T}$ is residual in $\mathcal{S}^r(\mathbb{T}^n \times S^1)$.*

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Proof. Let $f_\phi \in \mathcal{S}^r(\mathbb{T}^n \times S^1)$, and let $\mathrm{Per}(f) = \{x_0, x_1, x_2, \dots\}$ be the set of periodic points for $f : \mathbb{T}^n \to \mathbb{T}^n$. As f is uniformly hyperbolic, $\mathrm{Per}(f)$ is a countable set, and each $x_i = x_i(f)$ persists under perturbations of f. There is a neighborhood $\mathcal{N} \subset \mathcal{S}^r(\mathbb{T}^n \times S^1)$ containing f_ϕ such that if $\tilde{f}_{\tilde{\phi}} \in \mathcal{N}$, then:

- $\tilde{f}$ is Anosov;
- $\mathrm{Per}(\tilde{f}) = \{x_0(\tilde{f}), x_1(\tilde{f}), x_2(\tilde{f}), \dots\}$, where each $x_i(\tilde{f})$ has period k_i. □

We will show that $\mathcal{T} \cap \mathcal{N}$ is a residual subset of $\mathcal{N}$. As f_ϕ was arbitrary, this will prove that $\mathcal{T}$ is residual in $\mathcal{S}^r(\mathbb{T}^n \times S^1)$.

CLAIM. *For every $i \geq 0$, there is an open and dense subset $V_i \subset \mathcal{N}$ such that if $f_\phi \in V_i$, then:*

(a) *all periodic points for f_ϕ are hyperbolic;*
(b) *f is Anosov;*
(c) *the circle diffeomorphism $(f_\phi)^{k_i}|_{C_{x_i(f)}}$ satisfies the conditions given in (3).*

Proof of Claim. It follows from the Kupka–Smale theorem and [**11**] that Properties (a) and (b) hold for an open and dense subset of $\mathcal{N}$. The circle diffeomorphism $(f_\phi)^{k_i}|_{C_{x_i(f)}}$ varies continuously with f, so by [**9**], property (c) also holds for an open and dense subset of $\mathcal{N}$. □

Let $V = \bigcap_{i \geq 0} V_i$. Then V is a residual subset of $\mathcal{N}$, and $V \subset \mathcal{T}$. Proposition 9.1 is proved. □

PROPOSITION 9.2. *If $f_\phi \in \mathcal{T}$, then f_ϕ has trivial centralizer.*

Proof. Let G be a C^r diffeomorphism that commutes with $f_\phi \in \mathcal{T}$. □

LEMMA 9.3. *The diffeomorphism G preserves the set of fibers; for every $x \in \mathbb{T}^n$ there is a y such that $G(C_x) = C_y$.*

Proof of Lemma 9.3. Suppose that p_0 and p_1 are periodic points for f_ϕ, both contained in some periodic fiber C_x. I will assume for simplicity that x is fixed for f, and p_0 and p_1 are fixed for f_ϕ. Orient the circle, and suppose that the interval $[p_0, p_1] \subset C_x$ contains no other fixed points, and that if $q \in [p_0, p_1]$, then $(f_\phi)^i(q) \to p_0$ as $i \to \infty$, and $(f_\phi)^i(q) \to p_1$ as $i \to -\infty$.

$G(p_0)$ and $G(p_1)$ are also fixed points for f_ϕ, and G maps the interval $[p_0, p_1]$ to some continuous path between $G(p_0)$ and $G(p_1)$. Suppose that this path is not completely contained in a single fiber.

If $q \in [p_0, p_1]$, then $(f_\phi)^i(G(q)) = G(f_\phi^i(q)) \to G(p_0)$ as $i \to \infty$, and $(f_\phi)^i(G(q)) \to G(p_1)$ as $i \to -\infty$. Consider the projection of $G[p_0, p_1]$ to $\mathbb{T}^n$. The fixed points $G(p_0)$ and $G(p_1)$ project to some fixed points x_0 and x_1 (possibly not the same). The interval $[G(p_0), G(p_1)]$ projects to a continuous path (or loop) between these points. Every point on this path is a heteroclinic point, which approaches x_0 in forwards time and x_1 in backwards time. But the set of heteroclinic points for f on $\mathbb{T}^n$ is countable, so this is impossible. So G must map the whole interval $[p_0, p_1]$ to a single fiber. And G must map the whole fiber C_x to the same fiber.

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If p_0 and p_1 are periodic points for f_ϕ in some periodic fiber C_x, then p_0 and p_1 are fixed points for $(f_\phi)^n$ for some n. The same argument shows that $G(C_x) = C_y$ for some y. As periodic points are dense on $\mathbb{T}^n$, G preserves all fibers. □

Note. We did not need partial hyperbolicity to prove this lemma. The same argument will not necessarily work for skew products on $\mathbb{T}^n \times M$ for a manifold $M \neq S^1$. We needed an open, dense set of heteroclinic points in the fiber, and a countable set of heteroclinic points in the base.

COROLLARY 9.4. *The map G is a skew product over f^m for some $m \in \mathbb{Z}$.*

Proof. The diffeomorphism G preserves fibers, so it projects to a C^r diffeomorphism g on $\mathbb{T}^n$. The map g commutes with f, so $g = f^m$ for some m. □

Let $H = G(f_\phi)^{-m}$. Then H is a skew product over the identity map, and H commutes with f_ϕ:

$$H(x, a) = (x, \beta_x(a)),$$

for some C^r map $\beta : \mathbb{T}^n \to \mathrm{Diff}^r(S^1)$. If x is periodic for f with period k, then $(f_\phi)^k|_{C_x}$ and $H|_{C_x}$ are commuting diffeomorphisms of C_x. By definition of the set $\mathcal{T}$, $H|_{C_x} = (f_\phi)^{ki}|_{C_x}$ for some $i \in \mathbb{Z}$.

So if x is any periodic point for f, with period k, then there is some $i = i(x)$ such that

$$H(x, a) = (x, \beta_x(a)) = (f_\phi)^{ki}(x, a) = (x, \phi_x^{ki}(a))$$

for any $(x, a) \in C_x$.

We will show that H is the identity map. Suppose not. Since f-periodic points are dense in $\mathbb{T}^n$, there is an f-periodic point x_0 such that $H|_{C_{x_0}}$ is not the identity map. There are $2m$ fixed points for $(f_\phi)^J$ on C_{x_0}, for some $m \geq 1$ and $J \geq 1$. We assume that $J = 1$ (if not, replace f_ϕ with f_ϕ^J). We also assume that $m = 1$; there are exactly two fixed points for f_ϕ on C_{x_0}, (x_0, a_1) and (x_0, a_2). These are also the fixed points of H on C_{x_0}. The case $m > 1$ is similar, and is discussed at the end of this section.

By definition of $\mathcal{T}$, $\phi'_{x_0}(a_1) \neq 1$, $\phi'_{x_0}(a_2) \neq 1$, and $\phi'_{x_0}(a_1).\phi'_{x_0}(a_2) \neq 1$. We assume that

$$\phi'_{x_0}(a_1) < 1, \phi'_{x_0}(a_2) > 1 \quad \text{and} \quad \phi'_{x_0}(a_1).\phi'_{x_0}(a_2) < 1.$$

Let $U \subset \mathbb{T}^n$ be a neighborhood of x_0 such that the following hold.

(1) For each $x \in U$, there are exactly two fixed points for H on C_x, (x, a_1^x) and (x, a_2^x), close to the fixed points (x_0, a_1) and (x_0, a_2).

(2) There are constants $0 < \alpha < 1 < \nu$ such that for every $x \in U$,

$$0 < \phi'_x(a_1^x) < \alpha, \quad 1 < \phi'_x(a_2^x) < \nu \quad \text{and} \quad \alpha\nu \leq \gamma < 1 \quad \text{for some } \gamma. \tag{9.1}$$

(3) If $x \in U$ and $f(x) \in U$, then $f_\phi(x, a_i^x) = (f(x), a_i^{f(x)})$ for $i = 1, 2$.

Such a neighborhood exists because the maps $\beta : \mathbb{T}^n \to \mathrm{Diff}^r(S^1)$ and $\phi : \mathbb{T}^n \to \mathrm{Diff}^r(S^1)$ are C^1, and because f_ϕ permutes the fixed points of H.

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LEMMA 9.5. *Let x_0 be a fixed point of f and let U be any neighborhood of x_0. Then given any $A > 0$, the set*

$$\left\{x : f^k(x) = x \text{ for some } k \geq A, \text{ and at least } \left(\frac{A-1}{A}\right) k \text{ of the points } \{f^j(x) : 0 \leq j < k\} \text{ are in } U\right\}$$

is dense in $\mathbb{T}^n$.

Proof of Lemma 9.5. Let $V \subset \mathbb{T}^n$ be any open set. V contains a homoclinic point z for x_0. For some J, if $j \geq J$ then $f^j(z) \in U$ and $f^{-j}(z) \in U$. By the shadowing lemma, given any $\epsilon > 0$ there is an integer N such that if $K \geq N$ then the pseudo-orbit $\{f^j(z)\}_{j=-K}^{K}$ is ϵ-shadowed by a periodic point for f with period $2K$. Choosing ϵ small enough and K very large, there is a periodic point for f in V that spends $(A-1)/A$ of its orbit in U. □

To complete the proof of Theorem 3.6, we will choose a large A and show that if $f^k(x) = x$ for some $k \geq A$, and if $[(A-1)/A]k$ of the points $\{f^j(x) : 0 \leq j < k\}$ are in the neighborhood U defined above, then $H|_{C_x} = \mathrm{id}$. As these points are dense in $\mathbb{T}^n$, this will complete the proof that $H \equiv \mathrm{id}$, and $G = (f_\phi)^m$. The idea is that if a neighborhood U has been chosen as above, and if $x \in U$, then either most of the orbit of (x, a_1^x) is spent near (x_0, a_1), or most of the orbit of (x, a_1^x) is spent in neighborhoods of (x_0, a_1) and (x_0, a_2) (with equal time in each). Either way, if $H|_{C_x}$ is a very high power of f_ϕ, then under H there will be very strong contraction in the fibre direction at (x, a_1^x). But since H is a diffeomorphism, such contraction must be bounded, so we will obtain a contradiction.

Both f_ϕ and H are C^1,

$$T_{(x,a)}H = \begin{pmatrix} I & 0 \\ \partial\beta_x(a)/\partial x & \beta_x'(a) \end{pmatrix} \quad \text{and} \quad T_{(x,a)}f_\phi = \begin{pmatrix} T_x f & 0 \\ \partial\phi_x(a)/\partial x & \phi_x'(a) \end{pmatrix}.$$

By compactness of $\mathbb{T}^n \times S^1$, $\beta_x'(a)$ and $\phi_x'(a)$ are bounded. There are constants N and $M > 1$ such that

$$\frac{1}{N} < (\beta_x)'(a) < N \quad \text{and} \quad \frac{1}{M} < (\phi_x)'(a) < M \tag{9.2}$$

for all $(x, a) \in \mathbb{T}^n \times S^1$.

Choose A very large (to be determined later), and let $x \in U$ be a periodic point for f with period $k \geq A$. Suppose also that at least $[(A-1)/A]k$ of the points $\{f^j(x) : 0 \leq j < k\}$ are in U. So there are constants $0 \leq p \leq q \leq k$ such that:

(1) $f^j(x) \in U$ if $0 \leq j \leq p$ and if $q \leq j \leq k$;

(2) $p + (k - q) \geq [(A-1)/A]k$.

Since f_ϕ and H commute, f_ϕ permutes the fixed points of H. So $(f_\phi)^k(x, a_1^x) = (x, a_1^x)$ or (x, a_2^x), and therefore (x, a_1^x) and (x, a_2^x) have f_ϕ-period k or $2k$. Suppose they have period k. So $(f_\phi)^k(x, a_1^x) = (x, a_1^x)$.

Recall that by definition of U,

$$f^j(x, a_1^x) = (f^j(x), a_1^{f^j(x)}) \quad \text{for } 0 \leq j \leq p \text{ and for } q \leq j \leq k.$$

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In particular,

$$(\phi_x^p)'(a_1^x) = \prod_{j=0}^{p-1} \phi'_{f^j(x)}(a_1^{f^j(x)}) < \alpha^p$$

(by (9.1)), and

$$(\phi_{f^q(x)}^{k-q})'(a_1^{f^q(x)}) < \alpha^{k-q}.$$

So

$$\begin{aligned}(\phi_x^k)'(a_1^x) &< \alpha^p M^{q-p} \alpha^{k-q} \\ &\le \alpha^{(A-1)k/A} M^{(1/A)k} \\ &= (\alpha^{A-1} M)^{k/A}\end{aligned}$$

But $H(x, a_1^x) = (x, \beta_x(a_1^x)) = (f_\phi)^{ki}(x, a_1^x) = (x, \phi_x^{ki}(a_1^x))$ for some $i \ge 0$. So

$$(\beta_x)'(a_1^x) = (\phi_x^{ki})'(a_1^x) < (\alpha^{A-1} M)^{(k/A)i}.$$

If we assume that A has been chosen large enough so that $\alpha^{A-1} M < 1/N$, then this contradicts (9.2) unless $i = 0$. Therefore $\beta_x = \mathrm{id}$, and $H|_{C_x} = \mathrm{id}$.

Now suppose that the f_ϕ-period of (x, a_1^x) and (x, a_2^x) is $2k$. So $(f_\phi)^k(x, a_1^x) = (x, a_2^x)$, and $H|_{C_x} = (f_\phi)^{2ki}$ for some $i \ge 0$. As before, $f^j(x) \in U$ if $0 \le j \le p$ and if $q \le j \le k$, where $p + (k - q) \ge [(A-1)/A]k$. And since $(f_\phi)^k(x, a_1^x) = (x, a_2^x)$ and $(f_\phi)^{2k}(x, a_1^x) = (x, a_1^x)$,

$$f^j(x, a_1^x) = (f^j(x), a_1^{f^j(x)}) \quad \text{for } 0 \le j \le p \text{ and } k + q \le j \le 2k,$$

and

$$f^j(x, a_1^x) = (f^j(x), a_2^{f^j(x)}) \quad \text{for } q \le j \le k + p.$$

Therefore,

$$\begin{aligned}(\phi_x^{2k})'(a_1^x) &< \alpha^p M^{q-p} \mu^{(k+p)-q} M^{q-p} \alpha^{k-q} \\ &\le \alpha^{(A-1)k/A} \mu^{(A-1)k/A} M^{(2/A)k} \\ &= (\gamma^{A-1} M^2)^{k/A},\end{aligned}$$

where $\gamma = \alpha\mu < 1$ and $\beta_x = \phi_x^{2ki}$ for some $i \ge 0$. If A has been chosen large enough so that $\gamma^{A-1} M^2 < 1/N$, then $(\beta_x)'(a_1^x) < (1/N)^i$, which contradicts (9.2) unless $i = 0$. So $i = 0$, $\beta_x = \mathrm{id}$ and $H|_{C_x} = \mathrm{id}$.

Note. If we had started with (x_0, a_1) and (x_0, a_2) fixed for $(f_\phi)^J$ for some $J \ge 2$, then this argument would show that $H^J|_{C_x}$ is the identity map, for x in a dense subset of f-periodic points in $\mathbb{T}^n$. But if x is f-periodic with period k, then $f^k|_{C_x}$ is Morse–Smale and not a root of the identity map (by definition of $\mathcal{T}$), and $H|_{C_x} = (f_\phi)^{ki}$ for some $i \ge 0$. So $H^J|_{C_x}$ is the identity map if and only if $H|_{C_x}$ is the identity map.

If there are $2m$ fixed points for H on C_{x_0} for some $m \ge 1$, $\{(x_0, a_1), \ldots, (x_0, a_{2m})\}$, then let $U \subset \mathbb{T}^n$ be a neighborhood of x_0 such that:

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(1) for every $x \in U$, there are exactly $2m$ fixed points for H on C_x, $\{(x, a_1^x), \ldots, (x, a_{2m}^x)\}$;

(2) if $x \in U$ and $f(x) \in U$, then $f_\phi(x, a_j^x) = (f(x), a_j^{f(x)})$ for $1 \leq j \leq 2m$.

If $x \in U$ is periodic for f with period k, then $(f_\phi)^k$ permutes the points $\{(x, a_1^x), \ldots, (x, a_{2m}^x)\}$. We can assume without loss of generality that $(f_\phi)^k(x, a_j^x) = (x, a_{j+1}^x)$ for $1 \leq j < 2m$, and that $(f_\phi)^k(x, a_{2m}^x) = (x, a_1^x)$. So $H|_{C_x} = (f_\phi)^{2mki}$ for some $i \geq 0$, and therefore $\beta_x = \phi_x^{2mki}$. Since $\prod_{j=1}^{2m}[(\phi_{x_0})'(a_i)] \neq 1$, we may also assume that the neighborhood U has been chosen so that:

(3) there are constants $\alpha_1, \ldots, \alpha_{2m}$ such that for every $x \in U$, $\phi_x'(a_j^x) < \alpha_j$ for all $1 \leq j \leq 2m$, and $\prod_{j=1}^{2m} \alpha_j < \gamma$ for some $\gamma < 1$.

Let A be very large, and suppose that $x \in U$ has period $k \geq A$, and that at least $[(A-1)/A]k$ of the points $\{f^j(x) : 1 \leq j \leq k\}$ are in U. Then

$$\begin{aligned}(\phi_x^{2mk})'(a_1^x) &< \prod_{j=1}^{2m} (\alpha_j)^{(A-1)k/A} M^{k/A} \\ &< (M^{2m}\gamma^{A-1})^{k/A}.\end{aligned}$$

Since $\beta_x(a_i^x) = \phi_x^{2mki}(a_1^x)$ for some $i \geq 0$,

$$(\beta_x)'(a_i^x) < (M^{2m}\gamma^{A-1})^{(k/A)i}.$$

If A is large enough so that $M^{2m}\gamma^{A-1} < 1/N$, then this contradicts (9.1) unless $i = 0$. So it follows from Lemma 9.5 that $H = \mathrm{id}$. □

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