102.45 Revisiting a note on arithmetic and geometric means

Browsing through my run of back numbers of the *Gazette*, I came across a little gem, the note [1] (sent from a high school in China), where the following is proved.

> Let (a_n) be a sequence (finite or infinite) of positive numbers. Let A_n, G_n be, respectively, the arithmetic and geometric means of $a_1, a_2, \ldots, a_n$. Then $n(A_n - G_n)$ increases with n.

In [1], this is proved by considering the derivative of a somewhat elaborate function. However, it can be obtained as an immediate consequence of the well-known *weighted* AM-GM inequality. Recall that this states: if $w_j > 0$ for $1 \leqslant j \leqslant n$ and $\sum_{j=1}^{n} w_j = 1$, then for any choice of non-negative numbers a_j, we have $\prod_{j=1}^{n} a_j^{w_j} \leqslant \sum_{j=1}^{n} w_j a_j$. (In turn, this statement is delivered at once by Jensen's inequality applied to the concave function log, with no more effort than the ordinary AM-GM inequality.)

To derive the result of [1], let

$$\begin{aligned}\Delta_n &= n(A_n - G_n) - (n-1)(A_{n-1} - G_{n-1}) \\ &= a_n - nG_n + (n-1)G_{n-1}.\end{aligned}$$

Now $(G_n)^n = a_1 a_2 \ldots a_n = (G_{n-1})^{n-1} a_n$, so the weighted AM-GM inequality gives

$$G_n = (G_{n-1})^{1-\frac{1}{n}} a_n^{\frac{1}{n}} \leqslant \frac{n-1}{n} G_{n-1} + \frac{1}{n} a_n,$$

so $nG_n \leqslant (n-1)G_{n-1} + a_n$, hence $\Delta_n \geqslant 0$.

Reference

1. Gao Ling, Geometric and arithmetic means again, *Math. Gaz.* **69** (March 1985) pp. 39-40.

10.1017/mag.2018.125

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https://doi.org/10.1017/mag.2018.125 Published online by Cambridge University Press