Math. Proc. Camb. Phil. Soc. (1991), **109**, 299
Printed in Great Britain

On the group determinant

By K. W. JOHNSON
The Pennsylvania State University, Ogontz Campus, 1600 Woodland Road, Abington, PA 19001, U.S.A.

(*Received* 1 *December* 1989; *revised* 23 *March* 1990)

1. *Introduction*

The original motivation for the introduction by Frobenius of group characters for non-abelian groups was the problem of the factorization of the group determinant corresponding to a finite group G. The original papers are [**5**] and [**6**] and a good historical survey of the work is given in [**7**] and [**8**]. If G is of order n, the *group matrix* X_G is defined to be the $n \times n$ matrix $\{x_{g,h}\}$ where $x_{g,h} = x_{gh^{-1}}$ for $g, h \in G$. Here the x_g, $g \in G$, represent variables. The *group determinant* Θ_G is defined to be $\det(X_G)$, and is thus a polynomial of degree n in the x_g. This determinant is the same, up to sign, as that of the matrix obtained from the unbordered multiplication table of G by replacing each element g by x_g. If there is no ambiguity Θ_G will be written as Θ.

The group determinant was subsequently relegated to a minor role by the discovery that the theory of matrix representations, and later that of algebras and modules, could be used to obtain proofs of Frobenius' results which were more accessible. Since then, of course, the theory of group representations and characters has become an important tool, but nevertheless there remain many aspects of the theory which are not fully understood (for example consider the many questions raised by Brauer in [**1**]). This paper takes a new look at one of the major sources of group representation theory, assuming the point of view that this could give a greater insight into its uses and limitations, as well as offering the prospect of additional tools.

A question which naturally springs to mind is the following: if the determinants of G and H are the same, does this imply that G and H are isomorphic? This may be compared to the group ring problem (see [**11**]). The theory of algebras with a *norm form* is relevant (see [**9**]). Although results are obtained here on the consequences for a pair of groups that they have the same determinant (for example they must form a Brauer pair), it is unknown whether such a pair must be isomorphic (but see the note added in proof at the end). One result given here is that if a group G has a representation of degree 2 there exists an algorithm which starting from the corresponding irreducible factor of the group determinant constructs an explicit matrix representation. Thus in the case where G has a faithful representation of degree 2 a single factor of the group determinant determines G completely.

In his original paper [**6**], Frobenius introduced functions from $G^{(k)}$ to $\mathbb{C}$, which for $k = 1$ are ordinary characters and which are called here k-characters. Similar functions have also appeared recently in the theory of matrix invariants. Given a representation ρ of degree f, the corresponding irreducible factor of the group determinant can be constructed from the f-character which corresponds to ρ. Even

https://doi.org/10.1017/S0305004100069760 Published online by Cambridge University Press

for the case $k = 2$ these characters give much more information about the group than the ordinary characters; for example it is well known that the dihedral and quaternion groups of order 8 have the same ordinary character tables whereas their unique non-trivial 2-characters are distinct. (Some care is needed to describe precisely the meaning of 'distinct'. See Section 2.) Moreover the k-characters corresponding to distinct irreducible characters χ_i, χ_j are orthogonal in the sense that

$$\sum_{\underline{g} \in G^{(k)}} \chi_i^{(k)}(\underline{g})\, \bar{\chi}_j^{(k)}(\underline{g}) = 0.$$

This leads to the construction of orthogonal arrays which may be thought of as higher analogues of group character tables. The columns of these tables are indexed by the orbits of the permutation group on $G^{(k)}$ generated by G (acting by conjugation) and the symmetric group S_k (acting by permuting factors) and the rows consist of the values of the k-characters together with 'degenerate' rows arising from lower characters. Examples are given of such tables.

A more modern view of this work is afforded by regarding the variable x_g as a probability on G, i.e. one supposes that x is a function from G to $[0, 1]$ with $\sum_{g \in G} x_g = 1$. The matrix X_G then becomes the Fourier transform $\hat{x}$ of x with respect to the regular representation of G (see [3], p. 7), and the irreducible factors of Θ_G are the Fourier transforms of the irreducible representations of G. It is well known that for any representation of G the Fourier transform satisfies

$$x * y = \hat{x} . \hat{y}, \tag{1·1}$$

where $(*)$ denotes convolution. In fact the determinant contains enough information to carry out the process of Fourier inversion described in [3].

The discussion here assumes that the field over which factorization takes place is $\mathbb{C}$. The paper is set out as follows. In Section 2 a summary of Frobenius' results is given. The k-characters are discussed, the orthogonality results are presented and a more detailed discussion of the 2-character table is given with examples. Section 3 contains the theorem on factors of degree 2. The results obtained on the information contained in an irreducible determinant factor of arbitrary degree are set out in Section 4 and in Section 5 there are given some problems and suggestions for further work.

2. *The results of Frobenius and higher orthogonalities*

Let T be the right regular representation of G on $\mathbb{C}G$, i.e.

$$T(g)\Big(\sum_{h \in G} a_h h\Big) = \sum_{h \in G} a_h hg.$$

As is pointed out in [7], if $\sigma(g)$ is the matrix representation of $T(g)$ with respect to the basis $\{g_1^{-1}, \ldots, g_n^{-1}\}$ of $\mathbb{C}G$, then

$$X_G = x_{g_1}\sigma(g_1) + x_{g_2}\sigma(g_2) + \ldots + x_{g_n}\sigma(g_n).$$

The standard theory of semi-simple algebras leads to the decomposition of $\mathbb{C}G$ into irreducible G-modules and this is equivalent to the reduction of X_G to a block diagonal matrix, which is again equivalent to the splitting of Θ_G into irreducible factors.

The following is a summary of the relevant results of Frobenius.

https://doi.org/10.1017/S0305004100069760 Published online by Cambridge University Press

(1) The number of irreducible factors of Θ is equal to the number of conjugacy classes of G.

(2) To each irreducible factor ϕ of Θ there is associated an irreducible character χ_ϕ of G: $\chi_\phi(e)$ is f, the degree of ϕ, and $\chi_\phi(g)$ is the coefficient of $x_e^{f-1}x_g$ in ϕ for $g \neq e$.

(3) Let χ be an irreducible character of G. Define the *k-character* $\chi^k(g_1, g_2, \ldots, g_k)$ recursively by

$$\chi^1(g) = \chi(g), \quad \chi^k(g_1, g_2, \ldots, g_k) = \chi(g_1)\chi^{(k-1)}(g_2, g_3, \ldots, g_k) - \chi^{(k-1)}(g_1 g_2, g_3, \ldots, g_k) - \ldots - \chi^{(k-1)}(g_2, g_3, \ldots, g_1 g_k). \quad (2{\cdot}1)$$

If no ambiguity arises the superscript on χ^k will be omitted. Then if f is the degree of χ, the corresponding irreducible factor ϕ_χ of Θ_G is given by

$$\phi_\chi = (1/f!)\sum \chi(g_1, g_2, \ldots, g_f)\, x_{g_1} x_{g_2} \ldots x_{g_f}, \quad (2{\cdot}2)$$

where the summation runs over all f-tuples of elements of G.

(4) If ϕ is an irreducible factor of Θ then the highest power of ϕ which divides Θ is f, the degree of ϕ.

(5) If a matrix representation ρ of G with character χ is given then ϕ_χ is the determinant of the Fourier transform $\sum_{g\in G} \rho(g)\, x_g$.

(1) is a consequence of the fact that $\mathbb{C}G$ is a direct sum of matrix algebras, and that its centre has the class sums as a basis. (2) can be most easily derived using (5), which appeared later in Frobenius' development. (5) follows from the decomposition of $\mathbb{C}G$ into irreducible G-modules. The proof of (4), which Frobenius initially regarded as the main result of the theory, can be reduced to standard character theory.

There is an alternative way of defining $\chi(g_1, g_2, \ldots, g_k)$ which has appeared in the theory of matrix invariants. The formulation below comes from [**4**]. Let σ be an element of S_k. Write σ as a product of disjoint cycles

$$\sigma = (a_1 \ldots a_{k_1})(b_1 \ldots b_{k_2})(c_1 \ldots c_{k_3}) \ldots,$$

where 1-cycles are included. Define the trace function $T_\sigma : (M_n(\mathbb{C}))^{(k)} \to \mathbb{C}$ by

$$T_\sigma(X_1, \ldots, X_k) = T(X_{a_1} \ldots X_{a_{k_1}})\, T(X_{b_1} \ldots X_{b_{k_2}})\, T(X_{c_1} \ldots X_{c_{k_3}}) \ldots, \quad (2{\cdot}3)$$

where T is the usual trace. Now let ρ be a representation of G which corresponds to χ, and let $X_{g_i} = \rho(g_i)$, for $i = 1, \ldots, n$. Then (2·3) becomes

$$T_\sigma(X_{g_1}, \ldots, X_{g_k}) = \chi(g_{a_1} \ldots g_{a_{k_1}})\, \chi(g_{b_1} \ldots g_{b_{k_2}})\, \chi(g_{c_1} \ldots g_{c_{k_3}}) \ldots,$$

and

$$\chi(g_1, \ldots, g_k) = \sum_{\sigma \in S_k} \operatorname{sign}(\sigma)\, T_\sigma(X_{g_1}, \ldots, X_{g_k}). \quad (2{\cdot}4)$$

In [**4**] it is stated that all relations among traces of $f \times f$ matrices are consequences of the relation

$$\sum_{\sigma \in S_{f+1}} \operatorname{sign}(\sigma)\, T_\sigma(X_1, \ldots, X_{f+1}) = 0. \quad (2{\cdot}5)$$

An equivalent definition of $\chi(g_1, \ldots, g_k)$ is given in [**6**], where it is also pointed out that if χ is an irreducible character of degree f, then $\chi(g_1, \ldots, g_{f+1}) = 0$ for all $g_1, \ldots, g_{f+1}$ in G.

A further interpretation of $\chi(g_1, \ldots, g_k)$ is as the value obtained by inserting the probability $x_e = 1$, $x_g = 0$ for $g \neq e$ into

$$\frac{\partial \phi_\chi}{\partial x_{g_1} \partial x_{g_2} \ldots \partial x_{g_k}}.$$

https://doi.org/10.1017/S0305004100069760 Published online by Cambridge University Press

The result on orthogonality follows.

THEOREM 2·1. *Let χ and ψ be distinct irreducible characters of G. Then*

$$\sum_{\underline{g}\in G^{(k)}} \chi^{(k)}(\underline{g})\,\bar{\psi}^{(k)}(\underline{g}) = 0. \tag{2·6}$$

Proof. Suppose that χ and ψ are irreducible characters of G. Using the formula (2·4) for $\chi^{(k)}(g_{i_1},\dots,g_{i_k})$ the sum (2·6) becomes

$$\sum_{(g_{i_1},\dots,g_{i_k})\in G^{(k)}} [(\sum_{\sigma\in S_k} \operatorname{sgn}(\sigma)\,\chi(g_{a_1}\cdots g_{a_{k_1}})\,\chi(g_{b_1}\cdots g_{b_{k_2}})\,\chi(g_{c_1}\cdots g_{c_{k_3}})\cdots)$$

$$(\sum_{\tau\in S_k} \operatorname{sgn}(\tau)\,\bar{\psi}(g_{a'_1}\cdots g_{a'_{k'_1}})\,\bar{\psi}(g_{b'_1}\cdots g_{b'_{k'_2}})\,\bar{\psi}(g_{c'_1}\cdots g_{c'_{k'_2}})\cdots)],$$

where

$$\sigma = (a_1\dots a_{k_1})(b_1\dots b_{k_2})(c_1\dots c_{k_3})\dots, \quad \text{and} \quad \tau = (a'_1\dots a'_{k'_1})(b'_1\dots b'_{k'_2})(c'_1\dots c'_{k'_3}).$$

The theorem holds if it can be shown that

$$\sum_{(g_{i_1},\dots,g_{i_k})\in G^{(k)}} [(\chi(g_{a_1}\cdots g_{a_{k_1}})\,\chi(g_{b_1}\cdots g_{b_{k_2}})\,\chi(g_{c_1}\cdots g_{c_{k_3}})\cdots)$$

$$(\bar{\psi}(g_{a'_1}\cdots g_{a'_{k'_1}})\,\bar{\psi}(g_{b'_1}\cdots g_{b'_{k'_2}})\,\bar{\psi}(g_{c'_1}\cdots g_{c'_{k'_3}})\cdots)] = 0, \tag{2·7}$$

where the sum corresponds to the cycle structure of σ and τ as described above. Without loss of generality it may be assumed that $g_{a_1} = g_{a'_1} = g_{i_1}$. Then the sum in (2·7) may be rearranged as

$$\sum_{(g_{i_2},\dots,g_{i_k})\in G^{(k-1)}} [(\chi(g_{b_1}\cdots g_{b_{k_2}})\,\chi(g_{c_1}\cdots g_{c_{k_3}})\cdots)$$

$$(\bar{\psi}(g_{b'_1}\cdots g_{b'_{k'_3}})\,\bar{\psi}(g_{c'_1}\cdots g_{c'_{k'_3}})\cdots)\sum_{g_{i_1}\in G} \chi(g_{i_1}g_{a_2}\cdots g_{a_{k_1}})\,\bar{\psi}(g_{i_1}g_{a'_2}\cdots g_{a'_{k'_1}})].$$

(2·7) will follow if it can be shown that

$$\sum_{g_{i_1}\in G} \chi(g_{i_1}g_{a_2}\cdots g_{a_{k_1}})\,\bar{\psi}(g_{i_1}g_{a'_2}\cdots g_{a'_{k'_1}}) = 0. \tag{2·8}$$

But (2·8) is equivalent to

$$\sum_{h\in G} \chi(h)\,\bar{\psi}(hg) = 0, \tag{2·9}$$

where χ and ψ are distinct and irreducible, and this is a standard character theory identity (cf. [2], p. 219, (31·6)). The theorem is proved. ∎

We now describe in detail the 2-character table of a group G. There are three types of rows. Let $\chi_1,\dots,\chi_n$ be the irreducible ordinary characters of G.

(*a*) For each χ_i of degree >1 the 2-character as described above. Explicitly

$$\chi_i^2(g,h) = \chi_i(g)\,\chi_i(h) - \chi_i(gh).$$

(*b*) For each χ_i define $\chi_i^{(2,+)}$ as follows:

$$\chi_i^{(2,+)}(g,h) = \chi_i(g)\,\chi_i(h) + \chi_i(gh).$$

https://doi.org/10.1017/S0305004100069760 Published online by Cambridge University Press

Table 1

The 2-character table for S_3

Let $a = (123)$ and $b = (12)$. The ordinary character table is

Class order		1	2	3
Representative		e	a	b
χ_1		1	1	1
χ_2		1	1	−1
χ_3		2	−1	0

The 2-character table is

Class order		1	2	2	4	3	6	6	12
Representative		(e,e)	(a,a^2)	(a,a)	(e,a)	(b,b)	(e,b)	(b,ab)	(a,b)
Character	Norm								
$\chi_1^{(2,+)}$	36	1	1	1	1	1	1	1	1
$\chi_1^0\chi_2$	72	2	2	2	2	−2	0	−2	0
$\chi_1^0\chi_3$	72	4	−2	−2	1	0	2	0	−1
$\chi_2^{(2,+)}$	36	1	1	1	1	1	−1	1	−1
$\chi_2^0\chi_3$	72	4	−2	−2	1	0	−1	0	1
$\chi_3^{(2,+)}$	108	6	3	0	−3	2	0	2	0
χ_3^2	36	2	−1	2	−1	−2	0	1	0
Orthogonal complement	54	0	3	−3	0	−2	0	1	0

(*c*) For any pair χ_i, χ_j of distinct irreducible characters define $\chi_i \circ \chi_j$ by

$$\chi_i \circ \chi_j(g,h) = \chi_i(g)\,\chi_j(h) + \chi_i(h)\,\chi_j(g).$$

The functions described in (*a*), (*b*) and (*c*) will be called *extended 2-characters*.

PROPOSITION 2·2. *The extended 2-characters are pairwise orthogonal, with respect to the inner product described in* (2·6) *with* $k = 2$.

Proof. The cases not covered by Theorem 2·1 proceed similarly and are left to the reader. ∎

PROPOSITION 2·3. *If* ψ *is any extended 2-character then*

(i) $\psi(g,h) = \psi(x^{-1}gx, x^{-1}hx)$ *for any* x *in* G, *and*
(ii) $\psi(g,h) = \psi(h,g)$.

Proof. This follows directly from the definitions above. ∎

Define the *2-classes* of G to be the orbits of the action of $\langle G, \sigma\rangle$ on $G \times G$ by $x(g,h) = (x^{-1}gx, x^{-1}hx)$ for x, g, h in G and $\sigma(g,h) = (h,g)$. In view of Proposition 2·3 it is natural to index the columns of the 2-character table of G by these classes.

Examples are given in Tables 1 and 2 of the 2-character tables for S_3 and the dihedral group of order 8.

The norm column refers to the sum $\sum_{x \in G\times G} |\psi(x)|^2$. As may be noted the extended 2-characters do not span the space of 2-class functions. In the case of S_3 the 'complementary' row is given below the table. In the dihedral case the complementary space is two-dimensional with basis vectors having zero entries except in columns 4–9.

The table for Q_8, the quaternion group of order 8 is obtained from Table 2 by reversing the signs in the entries of the last two rows in columns 6–9. Before

https://doi.org/10.1017/S0305004100069760 Published online by Cambridge University Press

https://doi.org/10.1017/S0305004100069760 Published online by Cambridge University Press

Table 2

The 2-character table for D_8

$D_8 = \{a, b : a^4 = b^2 = e\}$. The ordinary character table is

Class order	1	1	2	2	2
Representative	e	a^2	a	b	ab
χ_1	1	1	1	1	1
χ_2	1	1	1	−1	−1
χ_3	1	1	−1	1	−1
χ_4	1	1	−1	−1	1
χ_5	2	−2	0	0	0

The classes for the 2-character table are

Class	Representative	Class	Representative	Class	Representative
1	(e, e)	7	(b, a^2b)	13	(a^2, a)
2	(a^2, a^2)	8	(ab, ab)	14	(a^2, b)
3	(e, a^2)	9	(ab, a^3b)	15	(a^2, ab)
4	(a, a)	10	(e, a)	16	(a, b)
5	(a, a^3)	11	(e, b)	17	(a, ab)
6	(b, b)	12	(e, ab)	18	(b, ab)

The 2-character table

Class,		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Order		1	1	2	2	2	2	2	2	2	4	4	4	4	4	4	8	8	8
	Norm																		
$\chi_1^{(2,+)}$	64	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\chi_1^0\chi_2$	128	2	2	2	2	2	−2	−2	−2	−2	2	0	0	2	0	0	0	0	−2
$\chi_1^0\chi_3$	128	2	2	2	−2	−2	2	2	−2	−2	0	2	0	0	2	0	0	−2	0
$\chi_1^0\chi_4$	128	2	2	2	−2	−2	−2	−2	2	2	0	0	2	0	0	2	−2	0	0
$\chi_1^0\chi_5$	128	4	−4	0	0	0	0	0	0	0	2	2	2	−2	−2	−2	0	0	0
$\chi_2^{(2,+)}$	64	1	1	1	1	1	1	1	1	1	1	−1	−1	1	−1	−1	−1	−1	1
$\chi_2^0\chi_3$	128	2	2	2	−2	−2	−2	−2	2	2	0	0	−2	0	0	−2	2	0	0
$\chi_2^0\chi_4$	128	2	2	2	−2	−2	2	2	−2	−2	0	−2	0	0	−2	0	0	2	0
$\chi_2^0\chi_5$	128	4	−4	0	0	0	0	0	0	0	2	−2	−2	−2	2	2	0	0	0
$\chi_3^{(2,+)}$	64	1	1	1	1	1	1	1	1	1	−1	1	−1	−1	1	−1	−1	1	−1
$\chi_3^0\chi_4$	128	2	2	2	2	2	−2	−2	−2	−2	−2	0	0	−2	0	0	0	0	2
$\chi_3^0\chi_5$	128	4	−4	0	0	0	0	0	0	0	−2	2	−2	2	−2	2	0	0	0
$\chi_4^{(2,+)}$	64	1	1	1	1	1	1	1	1	1	−1	−1	1	−1	−1	1	1	−1	−1
$\chi_4^0\chi_5$	128	4	−4	0	0	0	0	0	0	0	−2	−2	2	2	2	−2	0	0	0
$\chi_5^{(2,+)}$	192	6	6	−6	−2	2	2	−2	2	−2	0	0	0	0	0	0	0	0	0
χ_5^2	64	2	2	−2	2	−2	−2	2	−2	2	0	0	0	0	0	0	0	0	0

explaining how the tables for D_8 and Q_8 are not equivalent it is necessary to define the appropriate concept. Let G be a group. A permutation π on the set G naturally induces a permutation $\pi^{(2)}$ on the elements of $G^{(2)}$ by $\pi^{(2)}((g,h)) = (\pi(g), \pi(h))$. In general the image of a 2-class will not be a 2-class. An *admissible* permutation of the columns of the 2-character table of G is defined to be any $\pi^{(2)}$ which permutes the 2-classes of G among themselves, i.e. for which the image of a 2-class is always a 2-class. Note that not all column permutations of the 2-character table are admissible since a 'diagonal' 2-class of the form $\{(a,a)\}$, where a runs through a conjugacy class of G must be taken by an admissible permutation into another diagonal 2-class. Then the 2-character tables of groups G and H are *equivalent* if and only if an admissible permutation of the 2-classes of H acting on the columns of the table for H and any permutation of the rows of the table for H produces a table which is identical to that of G. In this sense the tables for D_8 and Q_8 are not equivalent, although one may be obtained from the other by a non-admissible permutation of the columns. In particular the unique 2-characters which are represented by the last row of each table remain distinct under admissible permutations, as mentioned in the introduction.

3. *Factors of low degree*

Let G be a finite group and suppose that ϕ is an irreducible factor of Θ_G of degree 2.

THEOREM 3·1. *Given ϕ, an explicit 2-dimensional matrix representation of G can be constructed with character χ_ϕ.*

Proof. By result (5) of Frobenius $\phi = \det(\sum \rho(g)\, x_g)$, for some irreducible representation ρ. Without loss of generality it may be assumed that there is an element g such that $\rho(g)$ has distinct eigenvalues u and v, for otherwise all $\rho(g)$ would be scalar multiples of the identity matrix and ρ and hence ϕ would be reducible. Again without loss of generality it may be assumed that for the particular g in question $\rho(g)$ is diagonal.

Such an element g can be determined from ϕ. For the coefficients α of $x_e x_g$ and β of x_g^2 in ϕ are the same as in $\det(Ix_e + \rho(g)\, x_g)$, and hence these coefficients are respectively $u+v$ and uv. Now $u = v$ if and only if $\alpha^2 = 4\beta$, and a suitable g may be determined by a direct search. From now on it is assumed that g is a fixed element of G and that $\rho(g) = \operatorname{diag}(u, v)$.

LEMMA 3·2. *At least one element h of G can be found so that $\rho(h)$ has both off-diagonal elements non-zero.*

Proof. If all $\rho(h)$, $h \in G$, are upper triangular, then ρ is reducible. Hence if no h with the property in the statement of the lemma can be found there are two elements k_1 and k_2 such that

$$\rho(k_1) = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \quad \text{and} \quad \rho(k_2) = \begin{pmatrix} p & 0 \\ q & r \end{pmatrix}$$

with $bq \neq 0$. But then

$$\rho(k_1 k_2) = \begin{pmatrix} ap + bq & br \\ cq & cr \end{pmatrix}.$$

https://doi.org/10.1017/S0305004100069760 Published online by Cambridge University Press

If either c or r is 0, the corresponding representing matrices have zero determinant, so $h = k_1 k_2$ satisfies the conditions of the lemma. ∎

It may thus be supposed that there exists $h \in G$ with

$$\rho(h) = \begin{pmatrix} p & q \\ r & s \end{pmatrix},$$

with $qr \neq 0$. The coefficients of ϕ which involve only the variables x_e, x_g and x_h are the same coefficients as in $\det(Ix_e + \rho(g)x_g + \rho(h)x_h)$. This is

$$\det \begin{pmatrix} x_e + ux_g + px_h & qx_h \\ rx_h & x_e + vx_g + sx_h \end{pmatrix}.$$

Let γ, δ and ϵ be respectively the coefficients of $x_e x_h$, $x_g x_h$ and x_h^2 in ϕ. Thus

$$\gamma = p + s, \quad \delta = us + vp \quad \text{and} \quad \epsilon = ps - qr.$$

Since u and v are assumed to be known it follows that p and s can be found from the first two equations and hence qr can be calculated from the third. Hence a suitable h may be found by search on the coefficients of ϕ.

From now on it is assumed that h and g are fixed elements of G with

$$\rho(g) = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \quad \text{and} \quad \rho(h) = \begin{pmatrix} p & q \\ 1 & s \end{pmatrix}$$

where $u \neq v$. The set $\{u, v\}$ can be found explicitly from the coefficients of $x_e x_g$ and x_g^2 as above, and an arbitrary choice can be made as to the order in which they appear on the diagonal. The assumption that $r = 1$ may be made since otherwise the similarity transformation $D^{-1}\rho(k)D$ may be applied to all the matrices $\rho(k)$ where $D = \operatorname{diag}(1, r)$. From the discussion in the previous paragraph, p, q and s can be completely determined from the coefficients γ, δ and ϵ. Now take k to be an element of G distinct from e, g and h. Let

$$\rho(k) = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}.$$

LEMMA 3·3. *The entries P and S may be determined explicitly from ϕ, the element Q satisfies a quadratic equation which is determined by ϕ and a choice of Q determines a unique value for R.*

Proof. Suppose α_1, α_2, α_3 and α_4 are the respective coefficients of $x_e x_k$, $x_g x_k$, $x_h x_k$ and x_k^2 in ϕ. The entries P and S can be found as above from α_1 and α_2. The coefficients α_3 and α_4 are the respective coefficients of $x_h x_k$ and x_k^2 in

$$\det(\rho(h)x_h + \rho(k)x_k) = \det \begin{pmatrix} px_h + Px_k & qx_h + Qx_k \\ x_h + Rx_k & sx_h + Sx_k \end{pmatrix}.$$

Thus $\alpha_3 = pS + Ps - qR - Q$ and $\alpha_4 = PS - QR$. If $\mu = pS + Ps - \alpha_3$ and $\nu = PS - \alpha_4$ it is seen that Q satisfies the quadratic equation

$$x^2 - \mu x + \nu q = 0, \tag{(3·1)}$$

and that once a solution of (3·1) has been chosen R is determined uniquely. ∎

https://doi.org/10.1017/S0305004100069760 Published online by Cambridge University Press

It is not clear that it is possible to give a direct way of choosing appropriate solutions of the equations of the form (3·1), but the finite subgroups of $\mathrm{GL}_2(\mathbb{C})$ have been classified up to conjugation, and this may be used. The following appears in [**10**], but, as is indicated there, was originally a result of Klein.

PROPOSITION 3·4. *Any finite subgroup of* $\mathrm{GL}_2(\mathbb{C})$ *has a conjugate which is one of the following types.*

(*a*) *All elements are of the form*

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad or \quad \begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix}$$

(*dihedral type*).

(*b*) *The group is generated by matrices of the following types*:

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad and \quad \begin{pmatrix} \frac{1}{2}m(1-i) & \frac{1}{2}m(i-1) \\ \frac{1}{2}m(1+i) & \frac{1}{2}m(1+i) \end{pmatrix}$$

(*tetrahedral type*).

(*c*) *The group is generated by the matrixes in* (*b*) *together with* $\mathrm{diag}\,(nj, nj^{-1})$ (*octahedral type*).

(*d*) *The group is generated by the following set*:

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} \theta & 0 \\ 0 & \theta^{-1} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad and \quad \begin{pmatrix} \lambda & \mu \\ \mu & -\lambda \end{pmatrix},$$

where $\theta^{10} = 1,\ \lambda = 1/(\theta^2 - \theta^{-2})$ *and* $\mu^2 + \lambda^2 + 1 = 0$

(*icosahedral type*).

Essentially we will need only a small part of this characterization, namely that a finite non-abelian subgroup of $\mathrm{GL}_2(\mathbb{C})$ is conjugate to a subgroup containing elements g and h where h has a special form.

LEMMA 3·5. *Let* K *be a finite non-abelian subgroup of* $\mathrm{GL}_2(\mathbb{C})$. *Then a conjugate of* K *contains elements* h *and* g *where*

$$h = \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix} \quad and \quad g = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$$

with $u \neq v$.

Proof. From Proposition 3·4 there is a conjugate of K containing an element of M of the form

$$\begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix}.$$

If $D = \mathrm{diag}\,(1, (c/d)^{\frac{1}{2}})$ then

$$D^{-1}MD = \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}$$

where $m = (cd)^{\frac{1}{2}}$. In all the cases of the proposition there exists a diagonal element which does not commute with M (in case (*a*) the assumption that K is non-abelian must be used). Choose g to be this element. ∎

https://doi.org/10.1017/S0305004100069760 Published online by Cambridge University Press

The above algorithm may be modified by searching first for an element h with the coefficient of $x_h x_e$ equal to 0. The coefficient of x_h^2 will be assumed to be $-m^2$. Then a search is made for an element g such that if t is the coefficient of x_g^2 the coefficient of $x_g x_h$ is $-t-m^2$. If no such element g can be found then h is discarded and a new search is made. Since Lemma 3·5 ensures that two such elements can be found the search will eventually be successful.

Now assume that in the calculations preceding (3·1) the elements $\rho(g)$ and $\rho(h)$ are respectively

$$\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}.$$

In this case $\alpha_3 = -m(Q+R)$ and $\alpha_4 + PS = QR$ and both Q and R are solutions of (3·1), so a choice of solution produces either a matrix M or M^{T}. The calculation will thus give for each group element k either a unique matrix in the case of equal roots or a pair of matrices M and M^{T}. Define U to be the set of all matrices which arise from solutions of (3·1). Let the set $U_1 = \{N_1, \ldots, N_t\}$ of matrices be defined by

$$N_1 = I, \quad N_2 = \rho(g), \quad N_3 = \rho(h),$$

and the condition that U_1 contains all matrices of U corresponding uniquely to an element of G and all pairs $\{M_j, M_j^{\mathrm{T}}\}$ which correspond to two elements of G. All members of U_1 must be in a representing set of matrices for ρ where $\rho(g)$ and $\rho(h)$ are as above. The remaining matrices in U are pairs $\{M_i, M_i^{\mathrm{T}}\}$ where not both members of the pair can lie in such a representing set. Suppose that $U_2 = \{N_{t+1}, \ldots, N_n\}$ where the set $\{N_1, \ldots, N_n\}$ forms a representing set of matrices for ρ. It follows that $\{N_1, \ldots, N_t, N_{t+1}^{\mathrm{T}}, \ldots, N_n^{\mathrm{T}}\}$ also forms a representing set and $U = U_1 \cup U_2 \cup U_2^{\mathrm{T}}$ where $U_2^{\mathrm{T}} = \{M^{\mathrm{T}} : M \in U_2\}$. Now if $M_1 \in U_2$ and $M_2 \in U_2^{\mathrm{T}}$, then $M_1 M_2$ cannot lie in U. For if $M_1 M_2 = M_3$ lies in $U_1 \cup U_2$ then $M_2^{\mathrm{T}} = M_3^{\mathrm{T}}(M_1^{-1})^{\mathrm{T}}$ lies in $U_1 \cup U_2$, which gives a contradiction. On the other hand if M_3 lies in U_2^{T} it follows that $M_1^{\mathrm{T}} = (M_2^{-1})^{\mathrm{T}} M_3^{\mathrm{T}}$ and lies in $U_1 \cup U_2$, which also gives a contradiction. This leads to the following way of choosing U_2. Let N_{t+1} be chosen arbitrarily and then the remaining members of U_2 can be selected by testing their product with N_{t+1}. The element is included precisely if this product lies in U. A representing set of matrices for ρ is produced and the theorem is proved. ∎

4. *Factors of arbitrary degree*

Assume now that G is a finite group, Θ_G is its determinant and that ϕ is an arbitrary irreducible factor of Θ_G corresponding to the representation ρ of degree f. Note that ρ is necessarily irreducible.

LEMMA 4·1. *For any element g of G the characteristic equation of $\rho(g)$ is determined by ϕ.*

Proof. The coefficients in ϕ involving only x_e and x_g are the coefficients in $\psi(x_e, x_g) = \det(Ix_e + \rho(g)x_g)$. Thus $\psi(z, -1)$ is the characteristic equation of $\rho(g)$, and is obtained from ϕ by the substitutions $x_e = z$, $x_g = -1$, $x_h = 0$ for $h \neq g$.

The Molien series of a representation ρ of a group G is defined to be the series $M_\rho(z) = \sum a_i z^i$, where a_i is the dimension of the space of invariants of degree i. See **[12]** for a modern exposition of the theory.

https://doi.org/10.1017/S0305004100069760 Published online by Cambridge University Press

LEMMA 4·2. *The Molien series for G is determined by ϕ.*

Proof. From a result of Molien

$$M_\rho(z) = (1/|G|) \sum_{g \in G} \frac{1}{\det(I - z\rho(g))}.$$

It is readily seen using Lemma 4·1 that

$$M_\rho(z) = (1/|G|)\left(1/(1-z)^f + \sum \frac{1}{\phi(1, 0, \ldots, -z, 0, \ldots, 0)}\right),$$

where the summation is over all positions for the variable $-z$ except the first. ∎

PROPOSITION 4·3. *Assume that $\rho(h)$ has distinct eigenvalues for some $h \in G$. If a matrix representation $g \mapsto M(g)$ is in the equivalence class determined by ρ and $M(h)$ is diagonal, then the diagonal entries of $M(g)$ can be determined from ϕ for each $g \in G$.*

Proof. Suppose that $M(h) = \operatorname{diag}(\lambda_1, \ldots, \lambda_f)$ where $\lambda_i \neq \lambda_j$ for $i \neq j$. Consider an element $g \in G$ where $g \neq e, h$. Let

$$M(g) = (g_{ij})_{f \times f}.$$

The coefficients of ϕ involving x_e, x_h and x_g are by (5) of Section 2 the same as the coefficients in $\det(Ix_e + M(h)x_h + M(g)x_g)$. Before discussing these in detail we introduce the following notation. Let $\sigma_i(z_1, \ldots, z_{f-1})$ denote the ith symmetric function of the variables $z_1, \ldots, z_{f-1}$, so that

$$\sigma_1 = z_1 + z_2 + \ldots + z_{f-1} \quad \text{and} \quad \sigma_{f-1} = z_1 \ldots z_{f-1}.$$

Let $$\bar{\lambda} = (\lambda_1, \ldots, \lambda_f) \quad \text{and} \quad \bar{\lambda}_i = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_f),$$

and let $$\sigma_j(\bar{\lambda}_i) = \sigma_j(\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_f).$$

The following are the coefficients in ϕ of the monomials indicated:

$$\begin{aligned}
x_e^{f-1}x_g &: \quad g_{11} + g_{22} + \ldots + g_{ff} \\
x_e^{f-2}x_h x_g &: \quad \sigma_1(\bar{\lambda}_1)g_{11} + \ldots + \sigma_1(\bar{\lambda}_f)g_{ff} \\
x_e^{f-3}x_h^2 x_g &: \quad \sigma_2(\bar{\lambda}_1)g_{11} + \ldots + \sigma_2(\bar{\lambda}_f)g_{ff} \\
&\ldots \\
x_h^{f-1}x_g &: \quad \sigma_{f-1}(\bar{\lambda}_1)g_{11} + \ldots + \sigma_{f-1}(\bar{\lambda}_f)g_{ff}.
\end{aligned}$$

It follows that $g_{11}, \ldots, g_{ff}$ can be determined from ϕ provided that

$$\begin{pmatrix} 1 & \ldots & 1 \\ \sigma_1(\bar{\lambda}_1) & \ldots & \sigma_1(\bar{\lambda}_f) \\ & \ldots & \\ \sigma_{f-1}(\bar{\lambda}_1) & \ldots & \sigma_{f-1}(\bar{\lambda}_f) \end{pmatrix}$$

is non-singular. However it is an easy calculation to show that the determinant of the above matrix is $\prod_{i<j}(\lambda_i - \lambda_j)$. ∎

The pair G, H of groups is said to be a *Brauer pair* if the two groups have the same character table and if there is a 1:1 map between the conjugacy classes which preserves the power map $C \mapsto C^m$ for every m, where C is an arbitrary conjugacy class and $C^m = \{x^m : x \in C\}$.

https://doi.org/10.1017/S0305004100069760 Published online by Cambridge University Press

LEMMA 4·4. *If two groups have the same group determinant then they must form a Brauer pair.*

Proof. Suppose that G_1 and G_2 are groups such that there is a 1:1 map $\kappa: G_1 \mapsto G_2$ which induces the identity from Θ_{G_1} to Θ_{G_2}. Since the coefficient of $x_e^{f-1}x_g$ in an irreducible factor ϕ of Θ_{G_1} is $\chi_\phi(g)$ it follows that there is an induced 1:1 correspondence $\rho \mapsto \kappa\rho$ between the irreducible representations of G_1 and G_2, such that $\chi_\rho(g) = \chi_{\kappa\rho}(\kappa g)$. Suppose further that for $1 \leqslant t < k$ the elements g^t and $(\kappa g)^t$ lie in corresponding conjugacy classes. Let ϕ be an arbitrary irreducible factor of Θ_{G_1}.

LEMMA 4·5. *The coefficient of $x_e^{f-k}x_g^k$ in ϕ can be expressed in the form $((-1)^{k-1}/k)\chi(g^k)+P$, where P involves sums of products of terms of the form $\chi(g^t)$ where $t < k$.*

Proof. It is a result of Frobenius that $\chi(e, g_2, \ldots, g_p) = (f-p)\chi(g_2, \ldots, g_p)$ ([6], equation 23). Iterating this result it follows that

$$\chi(e, e, \ldots, e, g, g, \ldots, g) = ((f-k)!)\chi(g, g, \ldots, g)$$

where e occurs $f-k$ times on the left and g occurs k times on each side. From (2·2) the coefficient of $x_e^{f-k}x_g^k$ in ϕ is

$$\frac{1}{f!} \cdot \frac{f!}{k!(f-k)!}((f-k)!)\chi(g, g, \ldots, g)$$

(k occurrences of g). By Equation (2·4) no term of $\chi(g, g, \ldots, g)$ involves $\chi(g^k)$ apart from those arising from the $(k-1)!k$-cycles, which each have sign $(-1)^{k-1}$. This proves Lemma 4·5. ∎

Applying Lemma 4·5 it follows that for any irreducible representation ρ of G_1 we have $\chi_\rho(g^k) = \chi_{\kappa\rho}((\kappa g)^k)$ for arbitrary $g \in G_1$, since the coefficients of $x_e^{f-k}x_g^k$ in ϕ_ρ and of $x_e^{f-k}x_{\kappa g}^k$ in $\phi_{\kappa\rho}$ are equal for all g and k. Now if G_1 and G_2 do not form a Brauer pair there is some element h in G_1 and there is an integer j such that h^j and $(\kappa h)^j$ do not lie in corresponding conjugacy classes. But, by standard representation theory, given any pair (C_1, C_2) of distinct conjugacy classes of G_1 there is at least one irreducible representation ψ of G_1 such that if $r \in C_1$ and $s \in C_2$ then $\chi_\psi(r) \neq \chi_\psi(s)$. This implies that such a ψ can be found so that $\chi_\psi(h^j) \neq \chi_{\kappa\psi}((\kappa h)^j)$, which leads to a contradiction. hence G_1 and G_2 must form a Brauer pair and lemma 4·4 is proved. ∎

5. *Conclusions and comments*

It is hoped that the results here will persuade others that a revival of work on the group determinant can provide a new perspective regarding group representation theory. One of the many questions raised in [1] is that of how much information is necessary in addition to that of the character table in order to determine a group. This may be extended as follows.

Problem 1. To what extent does the knowledge of the k-characters determine a group?

Even in the case of 2-characters the following problem appears to be unsolved.

Problem 2. Find a pair of non-isomorphic groups whose 2-character tables are the same.

https://doi.org/10.1017/S0305004100069760 Published online by Cambridge University Press

Many other similar problems can be formulated.

Since the representation theory of the symmetric groups has had such a fruitful interaction with combinatorics, a natural question is whether there is a 'generic' way of constructing the factors of their determinants from the standard combinatorial tools. In this context it may be appropriate to note that the factor of an arbitrary group determinant corresponding to any representation ρ is a semi-invariant; more precisely

$$\phi_\rho(x_g, \ldots, x_h) = \det(\rho(c^{-1}))\,\phi_\rho(x_{gc}, \ldots, x_{hc}) = \det(\rho(c^{-1}))\,\phi_\rho(x_{cg}, \ldots, x_{ch}),$$

where $(g, \ldots, h)$ is an f-tuple of elements of G and c is an arbitrary element of G, f being the degree of ρ (see [**13**], section 52).

The results in Section 3 suggest that it would be worthwhile to examine the irreducible factors of degree 3 of Θ and also factors which arise from irreducible monomial representations with a view to constructing the corresponding matrix representations.

The author would like to thank Ed Formanek and Surinder Sehgal for helpful conversations, and the former also for supplying offprints and preprints of work on matrix invariant theory. Some of the work was financed by a research initiation grant from The Pennsylvania State University while the author was on an interchange program at University Park, Pennsylvania.

Added in proof. Formanek and Sibley have recently proved that groups with the same determinant must be isomorphic. Hoehnke has told the author that results of his imply that the k-characters for $k \leqslant 3$ completely determine a group.

REFERENCES

[1] R. Brauer. Representations of finite groups. In *Lectures in Modern Mathematics* (editor T. L. Saaty) (Wiley, 1963), pp. 133–175.

[2] C. W. Curtis and I. Reiner. *Representation Theory of Finite Groups and Algebras* (Interscience, 1962).

[3] P. Diaconis. *Group Representations in Probability and Statistics*. Lecture Notes – Monograph Series (Institute of Mathematical Statistics, Hayward, California, 1988).

[4] E. Formanek. The invariants of $n \times n$ matrices. In *Invariant Theory* (editor S. S. Koh), Lecture Notes in Math. vol. 1278 (Springer-Verlag, 1987), pp. 18–43.

[5] G. Frobenius. Über Gruppencharaktere. *Sitzungsber. Preuss. Akad. Wiss. Berlin* (1896), 985–1021. *Gesammelte Abhandlungen* (Springer-Verlag, 1968), pp. 1–37.

[6] G. Frobenius. Über die Primfaktoren der Gruppendeterminante. *Sitzungsber. Preuss. Akad. Wiss. Berlin* (1896), 1343–1382. *Gesammelte Abhandlungen* (Springer-Verlag, 1968), pp. 38–77.

[7] T. Hawkins. The origins of the theory of group characters. *Arch. Hist. Exact Sci.* **7** (1971), 142–170.

[8] T. Hawkins. New light on Frobenius' creation of the theory of group characters. *Arch. Hist. Exact Sci.* **12** (1974), 217–243.

[9] H.-J. Hoehnke. Über Beziehungen zwischen Probleme von H. Brandt aus der Theorie der Algebren und den Automorphismen der Normenform. *Math. Nachr.* **34** (1967), 229–255.

[10] C. Jordan. Mémoire sur les equations différentiales linéaires à intégrale algébrique. *J. Reine Angew. Math.* **84** (1878), 89–215.

[11] R. Sandling. The isomorphism problem for group rings: a survey. In *Orders and Their Applications*, Lecture Notes in Math. vol. 1142 (Springer-Verlag, 1985), pp. 256–288.

[12] R. P. Stanley. Invariants of finite groups and their applications to combinatorics. *Bull. Amer. Math. Soc.* (*N.S.*) **1** (1979), 475–511.

[13] H. Weber. *Lehrbuch der Algebra*, vol. 2 (Vieweg & Son, 1899).

https://doi.org/10.1017/S0305004100069760 Published online by Cambridge University Press