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TRIAL AND ERROR MATHEMATICS I: DIALECTICAL AND QUASIDIALECTICAL SYSTEMS

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Abstract. We define and study quasidialectical systems, which are an extension of Magari's dialectical systems, designed to make Magari's formalization of trial and error mathematics more adherent to the real mathematical practice of revision: our proposed extension follows, and in several regards makes more precise, varieties of empiricist positions *à la* Lakatos. We prove several properties of quasidialectical systems and of the sets that they represent, called quasidialectical sets. In particular, we prove that the quasidialectical sets are Δ^0_2 sets in the arithmetical hierarchy. We distinguish between "loopless" quasidialectal systems, and quasidialectical systems "with loops". The latter ones represent exactly those coinfinite c.e. sets, that are not simple. In a subsequent paper we will show that whereas the dialectical sets are ω-c.e., the quasidialectical sets spread out throughout all classes of the Ershov hierarchy of the Δ^0_2 sets.

The needs that are pushing one to modify the theory taken as metamathematics have the same nature as those pushing physicists or any natural scientist[a].

Roberto Magari (1980)

[a] Our translation.

§1. Trial and error computation. The seminal studies by Gold (1965) and Putnam (1965) represent the earlier basis for many subsequent attempts at formalizing a procedure of computation allowed to "change its mind", by retracting any finite number of times its conclusion (the last "yes" or "no" is always to be the correct answer).

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In the mid 1970s, Magari and Jeroslow proposed two further different formal counterparts of the concept of "trial and error theory"; the sets of theorems of Jeroslow's "convergent experimental logics" turn out to coincide precisely with the collection of Δ_2^0 sets, as in the case of Putnam's trial and error predicates, whereas Magari's dialectical sets (where a "dialectical set" is the set of the so-called "final theses" of a dialectical system) form a strict subclass of the Δ_2^0 sets.

Thinking of Gödel's limitative results, Magari's purpose was in particular of introducing a kind of formal system, endowed with a certain degree of effectivity, and basically consisting of two actions: on the one hand removing contradictions when they arise, by removing some axioms, and on the other hand adding axioms until they do not give rise to contradictions. Between June and October 1973, the issue of Magari's "dialectical systems" and of Jeroslow's "experimental logics", together with the more general question of how we acquire mathematical knowledge by the underlying method of trial and error, were taken up in a brief exchange of (unpublished) letters between Magari, Kreisel and Jeroslow[1]. Kreisel reiterated his position according to which the trial and error approach neglects the way we actually find and explain axioms, the process of adding new axioms being rather connected with our wish of describing some mathematical object we have in mind: axioms, in Kreisel's view, emerge from a rigorous examination of informal notions. Kreisel's criticism echoed his well-known criticisms to so-called "pragmatist" and "positivist" mathematics, in line with his concept of *informal rigor* (Kreisel, 1967), according to which mathematics begins with the analysis of intuitive notions, and in laying down axioms, we first proceed by identifying properties of the involved entities, and eliminating doubtful properties from the intuitive notions. In his turn, Magari, in one of his letters to Kreisel, proposed to consider *logic* a *natural science*, thus supporting the idea that "genuine trials" do exist in mathematics[2]. We believe that nowadays the best way of understanding Magari's antisystematic positions is therefore in the light of the philosophically more aware Lakatos' "dialectical" reconstruction of history of mathematics, influenced by Popper's fallibilism: mathematics is quasiempirical and conjectural, and not growing by accumulation of eternal truths. Bridging the gap between mathematics and natural science was indeed the goal of Lakatos' search of a common fallibilist epistemological basis for them. (For an introduction to, and a discussion of, these issues, see (Lolli, 2008)).

As is also underlined in (Mancosu, 2008, p. 5) in the wake of Lakatos' dialectical philosophy of mathematics, several logicians called for an historically more grounded analysis of the development of mathematics (see for instance (Kitcher, 1983), and (Cellucci, 2000)). Trial and error machines have drawn attention in formal learning theory, and have been considered by Angluin and Smith in Angluin & Smith (1983), and Osherson, Stob, and Weinstein (Osherson, Stob, & Weinstein, 1991) (see also (Kelly, 1996)). Kugel (1986) takes into serious consideration the hypothesis that the human brain is a trial and error machine. The idea of a "multimachine theory of mind" (see (Copeland & Shagrir, 2013)) on the other hand, arose in Turing's later work, (Turing, 1947, 1996), where he outlines "the possibility of letting the machine alter its own instructions, mutating in fact from one machine into another, on the basis of some process of learning from experience, and nevertheless be such that one would have to admit that the machine was still doing very worthwhile calculations" (Turing, 1947). Making and correcting mistakes is intended

[1] The authors wish to thank Paolo Pagli for having made available this epistolary to them.

[2] (Magari, 1980) assimilated what he called "metamathematics" and "applied mathematics" (the latter in a sense that perhaps does not coincide with the usual one) to the "empirical sciences".

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as a manifestation of intelligence. Indeed: "If a machine is expected to be infallible, it cannot also be intelligent" (Turing, 1947). Making mistakes is furthermore a consequence of going through new methods: "This danger of the mathematician making mistakes is an unavoidable corollary of his power of sometimes hitting upon an entirely new method. This seems to be confirmed by the well known fact that the most reliable people will not usually hit upon really new methods." (Turing, 1996).

Several models of trial and error computation have been introduced more recently, for instance by Hintikka and Mutanen (Hintikka & Mutanen, 1988); see also Shapiro and McCarthy's concept of "projectability" (Shapiro & Mc Carthy, 1987), where a projection is a tentative value at a certain stage of the computation, subject to revision at a further stage. As already observed, some complexity classes of the arithmetical hierarchy have emerged in relation with the power of machine models considered in these studies, namely in most cases the levels Σ_2^0 and Δ_2^0 of the arithmetical hierarchy; for instance, van Leeuwen and Wiedermann (van Leeuwen & Wiedermann, 2012) have introduced another model, the so-called "red-green machines", a modern version of Turing's "noncircular machines": these machines, whose set of states is partitioned into two classes (*red* and *green*), recognize the sets in the class Σ_2^0, and accept the sets in Δ_2^0. Even sticking to dialectical systems or experimental logics, straightforward generalizations of these notions enable us to capture all Σ_2^0 sets: as observed by Gnani (1974), if we suitably modify the so-called "proposing function" of dialectical systems, which in the original definition is a computable permutation of ω, by allowing nonsurjectivity, then the possible sets of final theses coincide with the Σ_2^0 sets; on the other hand if we consider experimental logics where the so-called convergence property is not required, but only a condition called "stability", then the sets of theorems of modified experimental logics coincide with the whole class of Σ_2^0 sets.

Up to now Magari's "dialectical systems" have been somehow neglected. With the exception of a paper by Bernardi (1974), the above mentioned paper by Gnani (1974), and a more recent paper by Montagna, Simi, and Sorbi (Montagna, Simi, & Sorbi, 1996), where the concept of "dialectical system" is developed by assigning probabilities to provisional theses, Magari's work is mentioned only in Jeroslow's paper (Jeroslow, 1975) and in Hàjek's paper on Jeroslow's experimental logics (Hájek, 1977).

We take up here from Magari's paper. In Section 2, we recall Magari's original definition and some basic results; we also give a more convenient definition of a "dialectical system" in terms of enumeration operators. In Section 3, we introduce a more general notion of "dialectical system", namely that of a "quasidialectical system", led to this notion by a careful investigation of whether or not Magari's systems can be regarded as a satisfactory formalization of some of the most distinguishing features of empiricism in mathematics.

This paper is an introduction to quasidialectical systems, of which we outline some of the most useful general properties. More technical and mathematical results will appear in the paper (Amidei, Pianigiani, San Mauro, & Sorbi, to appear), where we show that the Turing degrees of the dialectical sets and of the quasidialectical sets coincide with the computably enumerable Turing degrees, and that the enumeration degrees of the dialectical sets and of the quasidialectical sets coincide with the Π_1^0 enumeration degrees, but the class of dialectical sets is properly contained in the class of quasidialectical sets: in fact, the dialectical sets belong to the class of the ω-c.e. sets, whereas one can find quasidialectical sets outside of any effectively indexable class of Δ_2^0 sets. As regards, the exact distribution of the dialectical sets and of the quasidialectical sets, we will prove in Amidei et al. (to appear) that for every finite level Σ_n^{-1}, $n > 0$, of the Ershov hierarchy there is a

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dialectical set $A \in \Sigma^{-1}_n \setminus \bigcup_{m<n} \Sigma^{-1}_m$, and there are ω-c.e. dialectical sets which do not lie in any finite level; on the other hand, for every notation $a \in O$ of a nonzero ordinal, one can find a quasidialectical set $A \in \Sigma^{-1}_a \setminus \bigcup_{b<_O a} \Sigma^{-1}_b$, where we refer here to Kleene's set O of ordinal notations, and Kleene's partial order $<_O$ on O: see e.g. Rogers (1967).

1.1. Background. On the technical side, no specific prerequisites are needed to read this paper, except perhaps for some introductory computability theory, for which our basic references are Cooper (2003); Rogers (1967); Soare (1987). Due to their importance throughout this paper and the forthcoming Amidei et al. (to appear), we only recall here some basic facts about enumeration operators, and Δ^0_2 sets. Any computably enumerable (c.e.) set Φ defines a mapping Φ taking sets of numbers to sets of numbers, namely,

$$\Phi(A) = \{x : (\exists \text{ finite } D)[\langle x, D\rangle \in \Phi \text{ and } D \subseteq A]\},$$

for every $A \subseteq \omega$, where ω denotes the set of natural numbers, and we identify finite sets with numbers, through their canonical indices. Such a mapping Φ is called an *enumeration operator*. A *computable approximation to* an enumeration operator Φ is a sequence $\{\Phi_s\}_{s\in\omega}$ of sets such that $\Phi_s \subseteq \Phi_{s+1}$, $\Phi = \bigcup_s \Phi_s$, and the relation $x \in \Phi_s$ is decidable in x, s. Since enumeration operators are nothing but c.e. sets, one can refer to a *uniformly* computable approximation $\{W_{e,s}\}_{e,s\in\omega}$ to the c.e. sets, where $\{W_e\}_{e\in\omega}$ is the standard indexing of the c.e. sets, the relation $x \in W_{e,s}$ is decidable in x, e, s, and for every e, $\{W_{e,s}\}_{s\in\omega}$ is a computable approximation to W_e (meaning that $W_{e,s} \subseteq W_{e,s+1}$, and $W_e = \bigcup_s W_{e,s}$) such that $W_{e,0} = \emptyset$ and every $W_{e,s}$ is finite. For more information on enumeration operators, see in particular Cooper (2003).

For any given set B, $B(x)$ denotes the value of the characteristic function of B on x. If $\{B_s\}_{s\in\omega}$ is a sequence of sets, and $x \in \omega$, we say that $\lim_s B_s(x)$ exists if there is t such that for every $s \geq t$, $B_s(x) = B_t(x)$. A set B lies in the class Δ^0_2 of the arithmetical hierarchy if and only if there is a computable sequence of sets $\{B_s\}$ (meaning that the relation $x \in B_s$ is decidable in x, s) such that, for every x, $\lim_s B_s(x)$ exists and $B(x) = \lim_s B_s(x)$, if and only if there is a 0-1 valued computable function $g(x, s)$ such that, for every x, $B(x) = \lim_s g(x, s)$.

§2. Dialectical systems. In this section, we give two definitions of a dialectical system. We first give (and comment on) Magari's original definition. Then we introduce the "revised" dialectical systems, which are more intuitive, and mathematically easier to work with, than dialectical systems. Then we show that dialectical systems and revised dialectical systems are equivalent. Once we have established this equivalence, revised dialectical systems will be identified *tout court* with dialectical systems, of which they will constitute our working definition.

2.1. Magari's dialectical systems. The basic ingredients of a dialectical system are a number c, called a *contradiction*; a computable function h (the *deduction function*) that tells us how to derive consequences from a finite set D (namely, the elements of the c.e. set $W_{h(D)}$); a *proposing function*, i.e. a computable function f that proposes axioms, to be accepted or rejected as provisional *theses* of the system: if up to a given stage we have accepted the axioms $f(i_1), \ldots, f(i_n)$, with $i_1 < \cdots < i_n$, and at this stage we see that we can derive c from $f(i_1), \ldots, f(i_m)$, for a least $m \leq n$, then we temporarily reject $f(i_m)$, still accept $f(i_1), \ldots, f(i_{m-1})$, and we are willing to add (perhaps again) $f(i_m + 1)$ to our

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working assumptions; on the other hand, if we see that c does not arise, then we are willing to add $f(i_n + 1)$ to our working assumptions.

2.1.1. Dialectical systems. We now give the precise definition, as given by Magari (1974). The power set of a set M will be denoted by 2^M: thus 2^ω denotes the class of sets of natural numbers; the symbol $2^{<\omega}$ will denote the class of finite sets of natural numbers; the elements of $2^{<\omega}$ will be often identified with their canonical indices.

DEFINITION 2.1. *A* dialectical system *is a triple* $d = \langle h, f, c\rangle$ *where*

1. $c \in \omega$;
2. h *is a computable function such that* $W_{h(\emptyset)} \neq \emptyset$, *and* $W_{h(\{c\})} = \omega$;
3. f *is a computable permutation of* ω.

REMARK 2.2. *The condition* $W_{h(\emptyset)} \neq \emptyset$, *states that our deduction function* h *does allow to derive something; the condition* $W_{h(\{c\})} = \omega$ *formalizes the familiar rule "ex falso quodlibet"; for the adequacy of the request that* f *be a permutation, see Magari (1974), and our Remark 2.6.*

Starting from h, one defines an algebraic closure operator $H : 2^\omega \longrightarrow 2^\omega$, together with a suitable computable approximation $\{K_s(D) : s \in \omega, D \in 2^{<\omega}\}$ to it. We recall that a function $F : 2^M \longrightarrow 2^M$ is a *closure operator on* M if, for all $X, Y \subseteq M$,

- $X \subseteq F(X)$;
- $F(X) \supseteq F(F(X))$;
- $X \subseteq Y \Rightarrow F(X) \subseteq F(Y)$.

A closure operator F is *algebraic* if $F(X) = \bigcup\{F(D) : D \subseteq X,\ D \text{ finite}\}$.

In order to define H and $\{K_s(D) : s \in \omega, D \in 2^{<\omega}\}$, one firstly views the computable function h as an enumeration operator[3] Φ,

$$\Phi = \{\langle x, D\rangle : x \in W_{h(D)}\},$$

and, using a computable approximation $\{\Phi_s\}_{s\in\omega}$ to Φ, defines a computable double sequence of c.e. sets $\{K_s(D) : s \in \omega,\ D \in 2^{<\omega}\}$, namely

$$K_0(D) = D$$
$$K_{s+1}(D) = \bigcup\{\Phi_{s+1}(E) : E \in 2^{<\omega},\ E \subseteq K_s(D)\} :$$

finally, one sets

$$H(X) = \bigcup\{K_s(D) : s \in \omega,\ D \in 2^{<\omega}, D \subseteq X\}.$$

For a proof that H is a closure operator, see Magari (1974). We call H the *closure operator associated with* d.

2.1.2. Dialectical sets. We now give the definition of a dialectical set, as proposed by Magari (1974).

[3] Magari (1974) however never talks of, nor uses, enumeration operators and their related machinery.

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DEFINITION 2.3. *Define two computable sequences s_n (where s_n is a finite sequence of pairs $\langle x, D\rangle$, with $x \in \omega$ and $D \in 2^{<\omega}$) and $A_n \in 2^{<\omega}$, by induction,*

- $s_0 = \langle 0, \{f(0)\}\rangle$; $A_0 = \emptyset$;
- *assume* $s_n = \langle\langle x_0, X_0\rangle, \ldots, \langle x_m, X_m\rangle\rangle$, *and let*

$$k = \begin{cases} \min\{i \leq m : c \in X_i\}, & \text{if there is such an } i, \\ m+1, & \text{otherwise}, \end{cases}$$

$$h = \begin{cases} k, & \text{if } k \leq m, \\ m, & \text{if } k = m+1. \end{cases}$$

Define

$$A_{n+1} = \bigcup_{i<k} X_i,$$

and

$$s_{n+1} = \langle\langle x_0, K_n(X_0)\rangle, \langle x_1, K_n(X_0 \cup X_1)\rangle, \ldots, \langle x_{k-1}, K_n(\bigcup_{i<k} X_i)\rangle, \langle x_h+1, \{f_{x_h+1}\}\rangle\rangle.$$

It is easy to see Magari (1974) that for every x, $\lim_n A_n(x)$ exists.

DEFINITION 2.4. *The set A_d, defined by*

$$A_d(x) = \lim_n A_n(x),$$

is called the dialectical set associated to, *or,* represented by, *d; the set A_d is also called the set of* final theses of *d. A set A is called* dialectical *if $A = A_d$ for some dialectical system d.*

The following is a useful characterization of A_d:

LEMMA 2.5 (Magari (1974)). *For every x,*

$$f(x) \in A_d \Leftrightarrow c \notin H(A_d \cap \{f(y) : y < x\} \cup \{f(x)\}).$$

Proof. See Magari (1974). □

REMARK 2.6. *Having defined what a dialectical set is, we are now in a position to better justify why, in the definition of a dialectical system $d = \langle h, c, f\rangle$, the proposing function f is taken to be a permutation. We first notice that injectivity is not a big issue here, as repetitions can always be eliminated (if the range is infinite); on the other hand, if f is not onto then the final theses would have to be searched in the range of f: as observed by Gnani (1974), the possible sets of final theses would coincide in this case with the Σ^0_2 sets, too many, to make them worth studying! In our opinion, however, the most convincing reason to ask for a surjective f lies in the fact that some of the most important and motivating examples of dialectical systems are in connection with formal theories, see Magari (1974) and Bernardi (1974). If T is a formal theory (for instance, Peano Arithmetic), then (under a suitable coding of the sentences in the language of T) one can take c to be any disprovable sentence, and h to be any computable function such that for every finite set D, $W_{h(D)}$ is the set of numbers that T can prove to be consequences of D. Consequently, H is the associated logical deduction operator, i.e. for any set $X \subseteq \omega$, $H(X)$ is the set of numbers that T can derive from X. Thus any choice of a computable*

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permutation f gives rise to a dialectical system $d = \langle h, c, f\rangle$. Clearly, the usual sentential connectives can be thought of as computable functions, say $\vee, \neg$ (just to mention two of them), so that for every $X \subseteq \omega$ and $x, y \in \omega$, $c \in H(\{x, \neg x\})$, $x \vee \neg x \in H(\emptyset)$, $H(X \cup \{x \vee y\}) = H(X \cup \{x\}) \cap H(X \cup \{y\})$, and other familiar deductive rules hold. These properties of H, and the fact that f is onto, suffice to show that if $c \notin H(\emptyset)$ then A_d is a completion *of T, in particular for every x, $A_d \cap \{x, \neg x\}$ has exactly one element.*

2.2. Revised dialectical systems. We now reformulate Magari's definition in a slightly, but more convenient, language. For convenience, in the rest of these notes, if f is the proposing function, then we denote $f(i)$ by f_i.

There is a precise way in which we can describe the closure operator H associated with a dialectical system, as an enumeration operator. As is known, enumeration operators can be uniformly composed: starting from an enumeration operator Ψ, let Ψ^n denote the composition of Ψ with itself, iterated n times, i.e. $\Psi^{n+1}(X) = \Psi(\Psi^n(X))$, where we start with

$$\Psi^0 = \{\langle x, \{x\}\rangle : x \in \omega\}.$$

Then the relation $\langle x, D\rangle \in \Psi^n$ (in n, x, D) is c.e. Now let

$$\Psi^\omega = \{\langle x, D\rangle : (\exists n)[\langle x, D\rangle \in \Psi^n]\}.$$

The following holds:

LEMMA 2.7. *If Ψ is an enumeration operator such that $\Psi(Y) \supseteq Y$ for every Y, then for every X, $\Psi^\omega(X)$ is the least fixed point Y of Ψ, with $Y \supseteq X$. Moreover, Ψ^ω is an enumeration operator, which is an algebraic closure operator. Conversely, if Ψ is an algebraic closure enumeration operator, then $\Psi = \Psi^\omega$.*

Proof. Notice,

$$\Psi^\omega(X) \subseteq \Psi(\Psi^\omega(X)),$$

as we assume that for every Y, $Y \subseteq \Psi(Y)$. On the other hand, if $x \in \Psi(\Psi^\omega(X))$ then there is a finite set D and a number n, such that

$$\langle x, D\rangle \in \Psi \,\&\, D \subseteq \Psi^n(X),$$

hence $x \in \Psi^{n+1}(X)$, and thus $x \in \Psi^\omega(X)$.

Assume now that $X \subseteq Y$ and $\Psi(Y) = Y$. By induction on n, it is easy to see that, for every n, $\Psi^n(X) \subseteq Y$, thus $\Psi^\omega(X) \subseteq Y$.

The two remaining claims making up the statement of the lemma are straightforward. □

Let $d = \langle h, f, c\rangle$ be a dialectical system, and let H be the closure operator associated with d.

LEMMA 2.8. *H is an enumeration operator, and for every set X, $H(X)$ is the least fixed point $Y \supseteq X$, of the enumeration operator $\Psi = \Phi \cup \{\langle x, \{x\}\rangle : x \in \omega\}$, where*

$$\Phi = \{\langle x, D\rangle : x \in W_{h(D)}\}.$$

Proof. In the following, $\{\Phi_s : s \in \omega\}$ is a computable approximation to Φ. First of all, as $Y \subseteq \Psi(Y)$, for every set Y, it is clear that $H(X) \subseteq \Psi(H(X))$. To show that

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$\Psi(H(X)) \subseteq H(X)$, notice that

$$\begin{aligned} x \in \Psi(H(X)) &\Rightarrow (\exists E \in 2^{<\omega})[E \subseteq H(X) \,\&\, x \in \Psi(E)](,) \\ &\Rightarrow (\exists E \in 2^{<\omega})(\exists D \in 2^{<\omega})(\exists s)[D \subseteq X \,\&\, E \subseteq K_s(D) \\ &\qquad \&\, [x \in \Phi_{s+1}(E) \vee x \in E]] \\ &\Rightarrow (\exists D \in 2^{<\omega})[D \subseteq X \,\&\, x \in K_{s+1}(D)] \\ &\Rightarrow x \in H(X). \end{aligned}$$

The second implication is motivated by the fact that for every $e \in E$, where E is a finite set, there are s, and a finite D such that $e \in K_s(D)$, but then, using the monotonicity properties of $K_s(D)$ (K is increasing in s, and in D: see Magari (1974)) it follows that there are at least s and a $\subseteq$-minimal D, such that $E \subseteq K_s(D)$.

By Lemma 2.7, it is now enough to show that $H(X) \subseteq \Psi^{\omega}(X)$. For every finite set D, using that $\Phi \subseteq \Psi$ and $Y \subseteq \Psi(Y)$ for every set Y, it is easy to see by induction on s that $K_s(D) \subseteq \Psi^{s+1}(D)$. Now, if $x \in H(X)$ then there are a stage s, and a finite set D such that $x \in K_s(D)$: this implies that there is a finite set E such that $x \in \Phi_{s+1}(E)$ and $E \subseteq K_s(D)$; hence $x \in \Psi(\Psi^{s+1}(D))$, which implies that $x \in \Psi^{\omega}(X)$. □

2.2.1. Revised dialectical systems: the definitions. In the next definition we define what we call a *revised dialectical system*. In Theorem 2.13 we will show that revised dialectical systems are equivalent to dialectical systems.

DEFINITION 2.9. *A* revised dialectical system *is a triple $d = \langle H, f, c\rangle$, where f is a computable permutation of ω, $c \in \omega$, and H is an enumeration operator, such that $H(\emptyset) \neq \emptyset$, $H(\{c\}) = \omega$, and H is an algebraic closure operator, i.e., H satisfies, for every $X \subseteq \omega$,*

- $X \subseteq H(X)$;
- $H(X) \supseteq H(H(X))$.

2.2.2. Revised dialectical sets. Let $d = \langle H, f, c\rangle$ be a revised dialectical system, and fix a computable approximation $\{H_s\}_{s\in\omega}$ to H. (Lemma 2.12(3) will establish independence on the approximation.)

DEFINITION 2.10. *Define by induction values for several computable parameters: A_s (a finite set), r_s (a function such that for every x, $r_s(x) = \emptyset$ or $r_s(x) = \{f_x\}$), $p(s)$ (a number, the greatest number m such that $r_s(m) \neq \emptyset$), $h(s)$ (a number). In addition, there are the derived parameters: $L_s(x) = \bigcup_{y<x} r_s(y)$, and, for every $i < p(s)$, $\chi_s(i) = \bigcup_{j\leq i} H_s(L_s(j))$.*

Stage 0. *Define $p(0) = 0$, $h(0) = 0$,*

$$r_0(x) = \begin{cases} \{f_0\} & \text{if } x = 0 \\ \emptyset & \text{if } x > 0, \end{cases}$$

and let $A_0 = \emptyset$.

Stage $s + 1$. Assume $p(s) = m$. We distinguish the following cases:

1. *there exists no $z \leq m$ such that $c \in \chi_s(z)$: in this case, let $p(s + 1) = m + 1$, and define*

$$r_{s+1}(x) = \begin{cases} r_s(x) & x \leq m \\ \{f_{m+1}\} & \text{if } x = m + 1 \\ \emptyset & \text{if } x > m + 1; \end{cases}$$

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2. *there exists a least* $z \leq m$ *such that* $c \in \chi_s(z)$*: in this case, let* $p(s+1) = z+1$*, and define*

$$r_{s+1}(x) = \begin{cases} r_s(x) & \text{if } x < z \\ \{f_{z+1}\} & \text{if } x = z+1 \\ \emptyset & \text{if } x = z \text{ or } x > z+1. \end{cases}$$

Finally define $h(s+1) = p(s+1)$ *if Clause (1) applies, otherwise,* $h(s+1) = p(s+1)-1$*, and let*

$$A_{s+1} = \bigcup_{i<h(s+1)} \chi_{s+1}(i) (= H_{s+1}(L_{s+1}(h(s+1)-1)))$$

(where $A_{s+1} = \emptyset$*, if* $h(s+1) = 0$*). The latter equality is justified by monotonicity with respect to inclusion of* H_{s+1}*.*

Figure 1 and Figure 2 illustrate how we go from stage s to stage $s+1$, according to Clause (1) and Clause (2), respectively, of the definition. In the pictures, it is understood that $r(v) = \{f_v\}$, if f_v is positioned above v, otherwise $r(v) = \emptyset$.

DEFINITION 2.11. *We call* A_s *the set of* provisional theses *of* d *at stage* s*. The set* A_d *defined as*

$$A_d = \{f_x : (\exists t)(\forall s \geq t)[f_x \in A_s]\}$$

is called the set of final theses *of* d*. We often write* $A_{d,s} = A_s$ *when we want to specify the revised dialectical system* d*, which we are dealing with. A set* $A \subseteq \omega$ *is called* revised dialectical *if* $A = A_d$ *for some revised dialectical system* d*.*

Although the sequence $\{A_{d,s}\}_{s\in\omega}$ depends on the chosen approximation to H, the set A_d does not, as follows from Lemma 2.12: the proofs of (1) and (2) of this lemma are

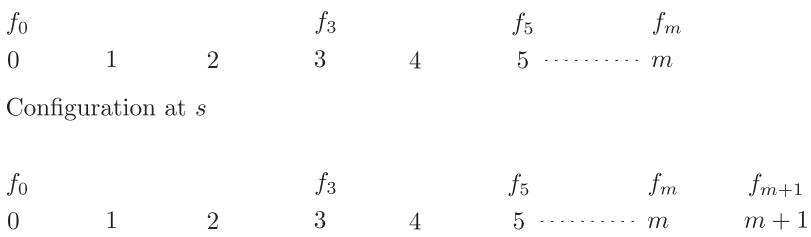


Fig. 1. From stage s to $s+1$ using Clause (1).

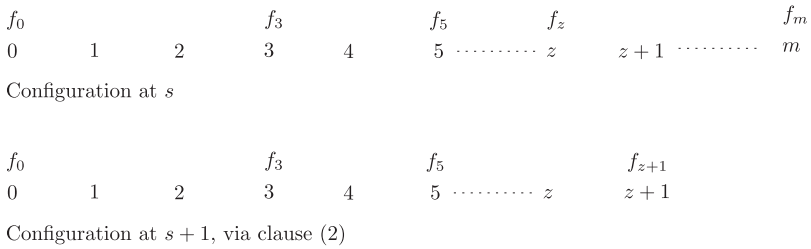


Fig. 2. From stage s to $s+1$ using Clause (2).

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postponed until Section 3, where we prove the same claims for quasidialectical systems, of which revised dialectical systems are particular cases.

LEMMA 2.12. *Let $d = \langle H, f, c\rangle$ be a revised dialectical system. For every x,*

1. *$A_d(x) = \lim_s A_s(x)$ exists;*
2. *$r(x) = \lim_s r_s(x)$ exists; $f_x \in A_d$ if and only if $r(x) = \{f_x\}$; and, letting $L(x) = \bigcup_{y<x} r(y)$, we have*
$$f_x \in A_d \Leftrightarrow c \notin H(L(x) \cup \{f_x\});$$
3. *$r(x)$, $L(x)$, and the set A_d do not depend on the chosen computable approximation to H.*

Proof. The proofs of items (1) and (2) are postponed until Section 3: (1) will be a consequence of Corollary 3.18; (2) will be a consequence of Lemma 3.8, Lemma 3.14, and Lemma 3.16.

Finally (3) easily comes from (2) by induction on x. Indeed, $L(0) = \emptyset$ and $f_0 \in A_d$ if and only if $r(0) = \{f_0\}$, if and only if $c \notin H(\{f_0\})$; assuming the claim for x, then $L(x+1) = L(x) \cup r(x)$, and we have $f_{x+1} \in A_d$ if and only if $r(x+1) = \{f_{x+1}\}$, if and only if $c \notin H(L(x+1) \cup \{f_{x+1}\})$. □

2.3. Our working definitions. It is not difficult to show that Magari's dialectical systems, and revised dialectical systems are equivalent, as they provide the same class of dialectical sets:

THEOREM 2.13. *We can uniformly go from a computable function h to an algebraic closure enumeration operator H, such that for every pair f, c, where f is a computable function and $c \in \omega$, if $d = \langle h, f, c\rangle$ is a dialectical system, then $d' = \langle H, f, c\rangle$ is a revised dialectical system such that $A_d = A_{d'}$.*

Conversely, we can uniformly go from an enumeration operator H, to a computable function h, such that for every pair f, c, where f is a computable function and $c \in \omega$, if $d' = \langle H', f, c\rangle$ is a revised dialectical system, then $d = \langle h, f, c\rangle$ is a dialectical system such that $A_{d'} = A_d$.

Proof. Given a computable function h, we can take H to be the algebraic closure enumeration operator associated with d: notice that we can always uniformly build such an H from h (take $H = \Psi^\omega$, where Ψ is built from h as in the proof of Lemma 2.8), even if we do not start from a dialectical system.

Conversely, if we start with an enumeration operator H', let h be a computable function such that $h(D)$ is a c.e. index of $H'(D)$, for every finite set D. Suppose now that $d' = \langle H', f, c\rangle$ is a revised dialectical system, and let $\Psi = \Phi \cup \{\langle x, \{x\}\rangle : x \in \omega\}$, where

$$\Phi = \{\langle x, D\rangle : x \in W_{h(D)}\}.$$

Then, due to the assumptions on H', $\Psi(D) = \Phi(D) = H'(D)$ for every finite set D, and $\Psi^\omega = \Psi = H'$. On the other hand, by Lemma 2.8 the algebraic closure operator H associated with $d = \langle h, f, c\rangle$, satisfies $H = \Psi^\omega$, thus $H = H'$.

The rest of the proof follows from the monotonicity of the enumeration operators, and their approximations, and of the operators K_s employed in Magari's definition. At stage $n+1$, there is an obvious parallel between the sequence

$$\langle\langle x_0, X_0\rangle, \langle x_1, X_1\rangle, \ldots, \langle x_m, X_m\rangle\rangle$$

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considered in Definition 2.1, and the sequence

$$\langle\langle x_0, \chi_n(x_0)\rangle, \langle x_1, \chi_n(x_1)\rangle, \ldots, \langle x_m, \chi_n(m)\rangle\rangle$$

which one can work out from Definition 2.9, where, in the latter case, $\langle x_0, \ldots, x_m\rangle$ is the list, in increasing order, of all i such that $r_n(i) \neq \emptyset$. If d and d' are respectively a Magari's dialectical system and a revised dialectical system which are related as above, then at stage s we might have $A_{d,s} \neq A_{d',s}$, but eventually $A_d = A_{d'}$, as is easily seen by Lemma 2.5 which characterizes A_d, and Lemma 2.12(2) which characterizes $A_{d'}$. □

In the rest of the paper, *dialectical systems* will always be presented as revised dialectical systems, i.e. as triples $\langle H, f, c\rangle$, where H is an algebraic closure enumeration operator; or even as triples $\langle H, f, c\rangle$ where H is just an enumeration operator such that $X \subseteq H(X)$, for every set X. In this latter case, using Theorem 2.13, we will in fact refer to $d = \langle H^\omega, f, c\rangle$. Notice that the request that $X \subseteq H(X)$ for all X does not in fact introduce any noneffectivity, as one can effectively go from H to the enumeration operator $H \cup \{\langle x, \{x\}\rangle : x \in \omega\}$ which satisfies the request.

Consistently, *dialectical sets* will be always presented as revised dialectical sets.

As an exemplification of our working definition, notice:

LEMMA 2.14 (Magari (1974)). *If $A \in \Pi^0_1$, and $A \neq \omega$, then there is a dialectical system d such that $A = A_d$.*

Proof. If $A = \emptyset$, then $A = A_d$ for every dialectical system $d = \langle H, f, c\rangle$ such that $c \in H(\emptyset)$. Otherwise, assume that $A \in \Pi^0_1$, $A \neq \emptyset$, and $\overline{A} \neq \emptyset$, where $\overline{A}$ denotes the complement of A. Choose $a \in A$ and $b \in \overline{A}$. Define

$$H = \{\langle y, \{x\}\rangle : x \notin A, y \in \omega\} \cup \{\langle a, \emptyset\rangle\} \cup \{\langle x, \{x\}\rangle : x \in \omega\}.$$

(We let $\langle a, \emptyset\rangle \in H$, to satisfy $H(\emptyset) \neq \emptyset$.) Notice that H is an algebraic closure operator. Indeed, let $X \subseteq \omega$: clearly $X \subseteq H(X)$; on the other hand, if $X \cap \overline{A} \neq \emptyset$ then $H(X) = \omega$, thus $H(X) = H(H(X))$; if $X \subseteq A$, then $H(X) = X \cup \{a\}$, giving again that $H(X) = H(H(X))$. Take $d = \langle H, f, b\rangle$, where f is the identity. It is easy to see that d is our desired dialectical system. □

Whereas every co-c.e. set, different from ω, is dialectical, it is interesting to observe (see (Magari, 1974)) that no c.e. set is dialectical, unless it is decidable. In view of the fact that dialectical sets represented by dialectical systems for a formal theory correspond to completions of the theory (see Remark 2.6), this is not surprising, as it corresponds to the well known result of logic, that every c.e. complete theory is decidable. Notice also that ω is not a dialectical set, as we can never have $c \in A_d$, for any dialectical system d.

§3. Quasidialectical systems. The goal of this section is twofold. Firstly, on a more philosophical side, we aim at investigating the relation between dialectical systems and a form of empiricism, concerning mathematical theories, that naturally embeds a certain notion of "revision". Secondly, our main strategy for such an investigation results in contrasting Magari's original systems with some modified systems (whose formulation is new) that are apt to formalize this informal idea of revision.

But the whole section is not just a philosophical detour. The main point here is to analyze to which extent dialectical systems can capture that kind of trial and error view

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of mathematics that is supported by Magari. The belief that such a point is a central one is shared by Magari himself. In his words[4], see Magari (1974):

> "Their [of dialectical systems] possible interest lies upon certain assumptions, which, as questionable as they might be, are nonetheless tenable, in the author's opinion, regarding the systems under consideration, for instance the assumption that the systems under consideration well schematize a way of proceeding, essentially admitted by the mathematical community."

Thus, Magari claims that the general significance of his proposal, as well as its possible conceptual fruitfulness, can be fully understood only when considering the philosophical frame in which dialectical systems are incorporated. It is fair to say that the philosophical frame that Magari has in mind consists of a (rather liberal) form of empiricism. That is, mathematical theories are by no means static compounds of eternal truth, but rather they do evolve in time, through a process in which axioms are (also) chosen by trial and error. It is clear that, at this level of vagueness, such a position is compatible with several different stances in the philosophy of mathematics (for instance, Lakatos famously opens his Lakatos (1976b) with reviewing many different points of view that, in his opinion, are somewhat sounded with such a view). Nonetheless, when introducing his dialectical systems, Magari did not aim at subscribing to any precise form of empiricism, but he rather intended to offer a formal notion by which a wide class of these forms can be characterized. To clarify this latter point, it is useful to spell out two methodological features that, although implicitly, emerge in Magari's proposal:

1. Dialectical systems are *extremely general*, in the sense that very few requirements are imposed to their basic components, i.e. the deduction operator H, the proposing function f, and the symbol of contradiction c. For instance, in general, H does not have to encode any well-known logical rule.[5]
2. Dialectical systems are *purely syntactical objects*. That is, there is no formal semantics for them. However, as we just said, there is an intended (informal) semantics, that takes them as a convenient representation of "a way of proceeding, essentially admitted by the mathematical community". The main contribution of this section is exactly that of pushing forward the boundaries of this intended semantics.

Thus one could formulate our main question, here, as follows: do dialectical systems really match Magari's informal intuition of a mathematical theory that, in choosing its axioms, proceeds by trial and error?

At first sight, a fully positive answer appears to be constrained by the lack, within the formalization of dialectical systems, of one of the key features of trial and error processes, namely some notion of revision by which our statements, in the presence of a possible problem, are not discarded but rather substituted. For instance, Lakatos' "monster barring" (see Lakatos (1976a)) provides an historical example of a way in which a mathematical hypothesis, when encountering a counterexample, might be refined, instead of being just removed. Dialectical systems seem to be unfit for such cases, since each contradiction imposes to discard the axiom, and no substitution, or refinement, is considered.

[4] Our translation.

[5] Dialectical systems that do obey specific constraints on H, reflecting the behavior of logical connectives, are studied in Bernardi (1974).

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Thus, we propose to modify Magari's original definition by introducing some new systems (that we call *quasidialectical systems*) apt to accommodate this idea of revision. Then, we will compare them to dialectical systems in terms of their expressiveness and information content, thus verifying whether such a notion of revision can be already embedded in Magari's systems. Furthermore, as we will see, the study of quasidialectical systems will be mathematically interesting *per se*.

Later on, we will of course provide a formal definition for these new systems, including some informal comments that might help in understanding their behavior. For the moment, let us just focus on two very general, and somewhat preliminary, aspects of our proposed formalization.

Quasidialectical systems extend standard dialectical systems with two additional symbols: c^- and f^-. Roughly, the role of f^- is that of replacing a certain axiom, that has produced some kind of problem, formally encoded by c^-, with another axiom. Thus, while c represents the mathematical contradiction (in both Magari's definition and ours), c^- corresponds to a large variety of possible problems that might lead a mathematician to replace an axiom. At the very high level of generality in which our presentation is pursued (such as Magari's one), the specific nature of these kinds of problems is disregarded. That is, we do not want to commit ourselves to the specific semantic status of c^-. On the contrary, our aim is to keep the intended meaning of c^- vague enough to incorporate a wide class of problems. These problems do not necessarily pertain to the formal side of the mathematical practice. Indeed, due to the generality of our proposal, they might include problems related to that kind of informal *desiderata* one can expect from an axiom, such as fruitfulness, or simplicity – or even psychological and aesthetic features, these latter being fully admissible as long as they can represent some reason to replace a given axiom.

This inclination towards generality also affects f^-. In particular, we do not ask for a unique notion of relevance between axioms and their possible substitutions via f^- that might hold for all systems. Roughly, the only condition in this respect is that f^- has to be computable. Being so liberal with respect to our definition of f^- might seem to contrast with our attempt of capturing an informal idea of revision. Indeed, once dismissed any notion of relevance from our formalization, how can we claim that our revision can be regarded as a whatsoever refinement? Again, the answer appeals to generality. Since the problems encoded by c^- are taken to be as general as possible, we do not want to restrict their possible solutions (i.e., the outputs of f^-) under any *a priori* rule of relevance.

This latter point suggests the following remark on f^-. For dialectical systems the proposing function f is given independently from the computation, i.e. the operator H: consistently with the fact that there is no restriction on the generality of f, there is no reason either to take its proposals as being somewhat linked to the behavior of the computation. On the other hand, f^- can be better understood as imposing a certain number of corrections to f: i.e., if some axiom raises a problem, we substitute this very axiom with a new one, and we force the system to go on from this new axiom.

Let us conclude by citing an additional phenomenon that represents a major difference between dialectical and quasidialectical systems. In general, these latter ones depend on how the operator H is approximated, in the sense that different approximations of the same system might yield to different sets of final theses. This is due to the fact that, within a quasidialectical system, a set of axioms X might derive both c and c^- (and in fact this is the case for basically every nontrivial system). Therefore, in such a case, there would be some approximations in which X derives c before c^-, whereas in other ones, X would derive c^- before c. As it will be clear hereinafter, this difference might also affect the corresponding

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sets of final theses. In particular, we will offer below an example of a quasidialectical system whose set of final theses depends on the choice of approximation to the operator H.

Enough of motivations, to business now!

3.1. Quasidialectical systems. The definition of a quasidialectical system is modeled on the definition of a dialectical system, as given in Section 2.3, following Theorem 2.13.

DEFINITION 3.1. *A* quasidialectical system q *is a quintuple* $q = \langle H, f, f^-, c, c^- \rangle$, *such that* $\langle H, f, c \rangle$ *is a dialectical system and* q *satisfies the following conditions:*

1. $c^- \in \omega$;
2. f^- *is a total computable function and* $c^- \notin \text{range}(f^-)$;
3. f^- *is* acyclic, *i.e., for every* x, *the* f^-*-orbit of* x *is infinite, where, for any function* g *and number* x, *we define the* g-orbit of x *the set*

$$\text{orb}_g(x) = \{x, g(x), g(g(x)), \ldots, g^n(x), \ldots\}.$$

If $c \neq c^-$ *then* q *is called a* proper *quasidialectical system.*

We call f^- the *revising function*, and c^- the *counterexample*[6].

REMARK 3.2. *A few words concerning Condition (3) above are in order. It seems reasonable to restrict ourselves to systems in which the operation of replacement is somewhat always enriching, in the following sense. Suppose we find some axiom unsatisfactory (again, this could be for a plenty of different reasons). Then we replace it. Later on, some problem occurs with this latter axiom, and thus we replace it too, with a third one. Now, if one aims at harmonizing the definition of* f^- *with some informal idea of "trial and error", in which knowledge is obtained through a process of refining subsequent proposals, then it is natural to ask that this third axiom is different from the first one we already replaced. Being acyclic just generalizes this intuition.*

It is important to notice that Condition (3) is noneffective. This may seems as strongly conflicting with Magari's dialectical systems in which all the basic components are computable. Still, instead of considering this as a downside of our formalization, we argue that this only expresses one of the well-known limitations, in terms of effectivity, that arise while working with formal systems. After all, mathematicians would also enjoy to know a priori if their systems are consistent or not – but, alas, because of Gödel, they just cannot.

Our definitions of *provisional theses* and *final theses* follow the ones given by Magari for dialectical systems. The difference consists, of course, in the role played by the additional symbols f^- and c^-. Informally, if at a given stage s we can derive c^- from a set of axioms $f_0, \ldots, f_n$, then we change f_n with $f^-(f_n)$. Thus c^- can be intended as imposing a revision (a weakening, perhaps) to some axiom, instead of simply dismissing it.

This only difference has (at least at first sight) significant consequences also with respect to aspects already considered in Magari's definition. Firstly, it is clear that a quasidialectical system, while computing, might be ready to admit repetitions in its set of axioms. In other words, the same axiom can enter the system many times. Moreover, while in a dialectical

[6] Our choice to use the word "counterexample" to refer to c^- is largely motivated by a matter of convenience. Indeed, due to the wide variety of possible meanings of c^-, there is no single word that might represent them all.

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system the "new" axiom proposed after a contradiction is always the successor (under the proposing function f) of the eliminated axiom, in our context the scenario is different. Indeed, the eliminated axiom, e.g. f_i, might have been object of some replacement of a previous axiom, f_j, i.e. $f_i = f^-(f_j)$. In this case, we would like to continue our computation from the successor of this f_j, i.e. f_{j+1}. This is because we might have discarded f_{j+1} due to some conflict with f_i. Thus, once f_i is put aside, f_{j+1} has to be tested again. Of course, the same holds if $f_j = f^-(f_k)$, for some f_k (in this case, we would take f_{k+1} as the continuation of our computation).

The last ingredient of the formalization of a quasidialectical system is given by the function $r_s(x)$. This function parallels the analogous function $r_s(x)$ which was used in the definition of (revised) dialectical systems, where, for every s, x we had either $r_s(x) = \emptyset$, or $r_s(x) = \{f_x\}$. For quasidialectical systems, $r_s(x)$ will always be a (possibly empty) string of axioms: its role (much more important here than for dialectical systems) is to take note, for the axioms considered at stage s, of all their history, that is to record all possible actions of f^-, at previous stages, that eventually have led to them.

Let us make, in the next section, these latter observations more formal.

3.2. Provisional and final theses for a quasidialectical system. Let $q = \langle H, f, f^-, c, c^-\rangle$ be a quasidialectical system, and let us fix a computable approximation $\alpha = \{H_s\}_{s\in\omega}$ to H. (As we will see, a major difference with respect to dialectical systems is that the set of final theses depends now on which computable approximation to the enumeration operator one chooses.)

DEFINITION 3.3. *Define by induction values for several computable parameters (which all depend on our choice of α): A_s (a finite set), r_s (a function such that for every x, $r_s(x)$ is a finite string of numbers, which is viewed as a "vertical" string, or* stack*), $p(s)$ (a number, the greatest number such that $r_s(p(s)) \neq \langle\,\rangle$, where the symbol $\langle\,\rangle$ denotes the empty string), $h(s)$ (a number). In addition, there are the derived parameters: $\rho_s(x)$ is the top of the stack $r_s(x)$, $L_s(x) = \{\rho_s(y) : y < x \text{ and } r_s(y) \neq \langle\,\rangle\}$, and, for every $i \leq p(s)$, $\chi_s(i) = \bigcup_{j\leq i} H_s(L_s(j))$.*

Stage 0. Define $p(0) = 0$, $h(0) = 0$,

$$r_0(x) = \begin{cases} \langle f_0\rangle & x = 0 \\ \langle\,\rangle & x > 0, \end{cases}$$

and let $A_0 = \emptyset$

Stage $s + 1$. Assume $p(s) = m$. We distinguish the following cases:

1. *there exists no $z \leq m$ such that $\{c, c^-\}\cap\chi_s(z) \neq \emptyset$: in this case, let $p(s+1) = m+1$, and define*

$$r_{s+1}(x) = \begin{cases} r_s(x) & \text{if } x \leq m \\ \langle f_{m+1}\rangle & \text{if } x = m+1 \\ \langle\,\rangle & \text{if } x > m+1; \end{cases}$$

2. *there exists a least $z \leq m$ such that $c \in \chi_s(z)$, and for all $z' < z$, $c^- \notin \chi_s(z')$: in this case, let $p(s + 1) = z + 1$, and define*

$$r_{s+1}(x) = \begin{cases} r_s(x) & x < z \\ \langle f_{z+1}\rangle & x = z+1 \\ \langle\,\rangle & x = z \text{ or } x > z+1; \end{cases}$$

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3. *there exists a least* $z \leq m$ *such that* $c^- \in \chi_s(z)$, *and for all* $z' \leq z$, $c \notin \chi_s(z')$: *in this case, let* $p(s+1) = z+1$, *and define, where* $\rho_s(z) = f_y$,

$$r_{s+1}(x) = \begin{cases} r_s(x) & x < z \\ r_s(x)^\frown\langle f^-(f_y)\rangle & x = z \\ \langle f_{z+1}\rangle & x = z+1 \\ \langle\,\rangle & x > z+1. \end{cases}$$

Finally define $h(s+1) = p(s+1)$, *if Clause (1) applies, otherwise* $h(s+1) = p(s+1)-1$, *and let*

$$A_{s+1} = \bigcup_{i<h(s+1)} \chi_{s+1}(i) (= H_{s+1}(L_{s+1}(h(s+1)-1))).$$

We call A_s *the set of* provisional theses *of* q *at stage* s. *The set* A_q^α *defined as*

$$A_q^\alpha = \{f_x : (\exists t)(\forall s \geq t)[f_x \in A_s]\}$$

is called the set of final theses *of* q *with respect to* α. *We often write* $A_s = A_{q,s}^\alpha$ *when we want to specify the quasidialectical system* q *and the chosen approximation to the enumeration operator. A set* $A \subseteq \omega$ *is called* quasidialectical *if* $A = A_q^\alpha$ *for some quasidialectical system* q, *and approximation* α *to the enumeration operator of* q; *and sometimes, we will say in this case that* A *is* represented *by the pair* (q, α).

Stacks, tops of stacks, and other stuff. A few remarks are needed after such a long definition.

Firstly, notice that in case a certain set of axioms derive, at a given stage, both c and c^-, then the system considers only c.

Secondly, in what follows it would be convenient to regard elements of the domain of r_s as *slots*. In view of their wide use in what follows, let us devote a few words to explaining how such slots have to be conceived of. At each stage s, whenever we say that x is the *slot of an axiom* f_y, we mean that $r_s(x)$ is a string having f_y as its rightmost element. The idea is that

$$r_s(x) = \langle f_x, f^-(f_x), \ldots, (f^-)^{(n)}(f_x)\rangle,$$

where the last element of the string, $f_y = (f^-)^{(n)}(f_x)$, has been obtained by stage s, through a sequence of replacements, starting from f_x, and dictated by f^-, due to case (3) of the quasidialectical procedure.

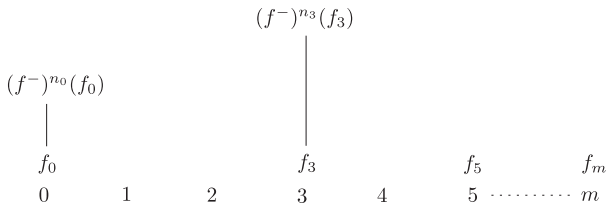
Figures 3–5 illustrate the various cases of Definition 3.3. The vertical strings above the various slots represent the various stacks $r(x)$ at the given stage. In each figure, only the relevant slots are depicted: it is of course understood that for each slot v to the right of the last one which is depicted, we have $r(v) = \langle\,\rangle$ (where for simplicity we omit to specify at which stage s the parameter $r_s(v)$ is evaluated). For every slot x, the set $L(x)$ consists of the strings which are at the top of the nonempty stacks $r(y)$, with $y < x$.

3.3. The dependence of the final theses from the approximations. We give an example to show how the definition of a quasidialectical set depends on how the relevant enumeration operator is approximated.

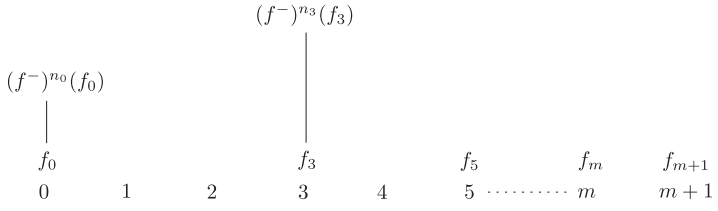
EXAMPLE 3.4. *Consider the quasidialectical system* $q = \langle H, f, f^-, c, c^-\rangle$, *where* $f_x = x$, $f^-(x) = x+2$, $c = 1$, $c^- = 2$, *and*

$$H = \{\langle y, \{2x+1\}\rangle : x, y \in \omega\} \cup \{\langle 0, \emptyset\rangle\} \cup \{\langle y, \{y\}\rangle : y \in \omega\},$$

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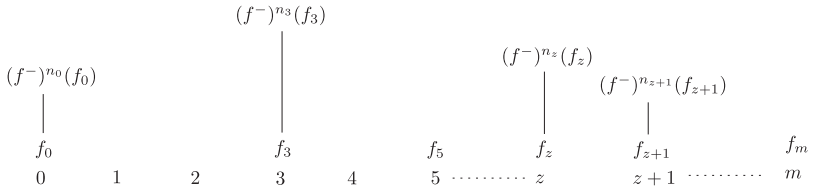


Configuration at s

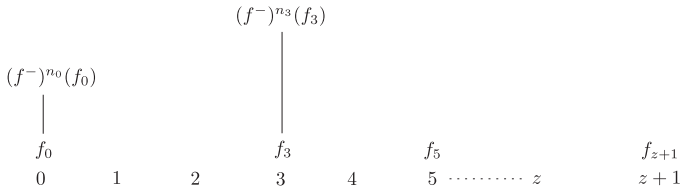


Configuration at $s+1$, via clause (1)

Fig. 3. From stage s to $s+1$ using clause (1).



Configuration at s



Configuration at $s+1$, via clause (2)

Fig. 4. From stage s to $s+1$ using Clause (2).

(the axiom $\langle 0, \emptyset\rangle \in H$ is to comply with the request that in a dialectical system $H(\emptyset) \neq \emptyset$). It is easy to see that H is a closure operator: if $y \notin H(X)$, then $y \neq 0$ and X does not contain any odd number, but then $H(X)$ does not contain any odd number either, thus $y \notin H(H(X))$. It is straightforward to see that there exist computable approximations α and β to H, such that in α for every x, the axiom $\langle c^-, \{2x+1\}\rangle$ comes before $\langle c, \{2x+1\}\rangle$, so that when processing $2x+1$, the pair (q, α) would use Clause (3) of the definition of provisional theses; on the contrary, in β for every x, the axiom $\langle c, \{2x+1\}\rangle$ comes before $\langle c^-, \{2x+1\}\rangle$, so that Clause (2) would be used. It is easy to see that these two approximations give rise to different quasidialectical sets, since $A_q^\alpha = \{0\}$, whereas, for instance $4 \in A_q^\beta$. Moreover, α gives rise to functions $r_s^\alpha(x)$, $\rho_s^\alpha(x)$, which have different "asymptotic" behavior from

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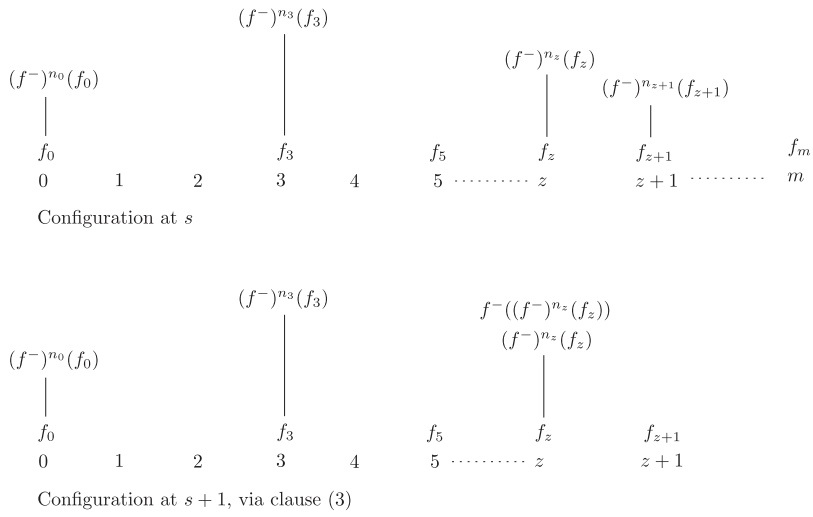


Fig. 5. From stage s to $s+1$ using Clause (3).

the functions $r_s^\beta(x)$, $\rho_s^\beta(x)$ yielded by β. For instance, we have that $\{\rho_s^\alpha(1) : s \in \omega\}$ is infinite. (If in addition $\alpha = \{H_s\}_{s\in\omega}$ satisfies that $\langle c^-, \{2x+1\}\rangle \in H_{x+2}$, then for every s, $r_s^\alpha(x) = \langle\,\rangle$, for all $x > 2$, i.e., for every $x > 2$, the axiom x is never "proposed".) On the contrary, $\{\rho_s^\beta(x) : s \in \omega\}$ is finite, for every x.

In view of this example, it could be objected that a quasidialectical system, rather than a quintuple $\langle H, f, f^-, c, c^-\rangle$ should be perhaps a quintuple $\langle \alpha, f, f^-, c, c^-\rangle$, where α is a computable approximation to an enumeration operator H. This is after all reasonable: we need first of all to be able to approximate H, in order to deduce with it.

We shall agree on the following definition:

DEFINITION 3.5. *An* approximated quasidialectical system *is a pair (q, α) where q is a quasidialectical system $q = \langle H, f, f^-, c, c^-\rangle$, and α is a computable approximation to H.*

Hence a set A is quasidialectical if there is an approximated quasidialectical system (q, α), such that $A = A_q^\alpha$.

LEMMA 3.6. *Every dialectical set A is quasidialectical, in fact there is a quasidialectical system q, such that for every computable approximation α to the enumeration operator of H, we have that $A = A_q^\alpha$.*

Proof. If $d = \langle H, f, c\rangle$ is a dialectical system, then let f^- be any acyclic computable function such that $c^- \notin \text{range}(f^-)$. It is clear that, letting $q = \langle H, f, f^-, c, c\rangle$, one has that $A_d = A_q^\alpha$, for any α: this follows from the fact that one never uses Clause (3) of Definition 3.3, as in this case $c^- = c$, and Clause (2) has right to way. □

3.4. General properties of quasidialectical systems. The fact that approximated quasidialectical systems (q, α), as the one in Example 3.4, do exist is not just a matter of curiosity. In fact, as we will see, pairs (q, α) that eventually propose all the axioms, and pairs that – on the contrary – might fail to propose some of them, do not even share the same class of representable sets. In order to distinguish these two classes of pairs, let us give a preliminary definition:

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DEFINITION 3.7. *Let* (q, α) *be an approximated quasidialectical system, and* y *a slot. We say that* (q, α) *has a* loop over y *if* $\{\rho_s(y) : s \in \omega\}$ *is infinite. If* (q, α) *has no loops, we call it* loopless.

Therefore, a loop can be visualized as expressing an infinite ascending stack of substitutions over some slot[7].

Example 3.4 shows that, for a quasidialectical system, having loops depends on how we approximate the enumeration operator of H.

Let us now show a result that applies to both quasidialectical systems with loops and loopless ones. It tells us when to expect stability for a given set of axioms.

LEMMA 3.8. *Let* (q, α) *be an approximated quasidialectical system, and* y *a slot. If for each* $x \leq y$*, the pair* (q, α) *has no loop over* x*, then* $\lim_s r_s(y)$ *exists, i.e. there is a stage* t *such that, for every* $s \geq t$*,* $r_s(y) = r_t(y)$.

Proof. The proof is by induction on y. For $y = 0$, the claim is obvious. Recall that $r_0(0) = \langle f_0 \rangle$. If there is no stage t such that $\{c, c^-\} \cap H_t(\{r_t(0)\}) \neq \emptyset$, then for every s, $r_s(0) = \langle f_0 \rangle$. If there is a stage t such that $c \in H_t(\{r_t(0)\})$, then for every $s \geq t$, $r_s(0) = \langle\,\rangle$. If there is no t such that $c \in H_t(\{r_t(0)\})$, but (as q has no loop above 0) there is a last t at which $c^- \in H_t(\{r_t(0)\})$, then for every $s \geq t$, we have that $r_s(0) = r_t(0)$.

Assume that the claim is true of y, and that the pair (q, α) has no loop over any $x \leq y+1$. Then by inductive hypothesis, there is a least stage t (necessarily, $t > 0$) such that for every $s > t$, and $x \leq y$, $r_s(x) = r_t(x)$, and let $L(y+1) = L_t(y+1)$. This implies also that $\{c, c^-\} \cap H(L(y+1)) = \emptyset$. By minimality of t, we have that $L_t(y+1) \neq L_{t-1}(y+1)$. We examine all possibilities that may have led to a change at stage t. First of all, it is not possible that $h(t) = k < y - 1$, as otherwise at stage $t+1$, through (1) we would change $r_{t+1}(k+2)$, contrary to the choice of t, as $k + 2 \leq y$. If $h(t) = y - 1$, then at t we set $r_t(y) = \langle f_y \rangle$, $r_t(y+1) = \langle\,\rangle$ but at $t+1$ we change this to $r_{t+1}(y+1) = \langle f_{y+1} \rangle$. If $h(t) = y$, then we set $r_t(y+1) = \langle f_{y+1} \rangle$. In all cases, we see that there is a least stage $t_0 \in \{t, t+1\}$ at which $r_{t_0}(y+1) = \langle f_{y+1} \rangle$. If there is a stage $s_0 \geq t_0$ such that $c \in H(L(y+1) \cup \{\rho_{s_0}(y+1)\})$ then for every $s \geq s_0$, we have that $r_s(y+1) = \langle\,\rangle$; otherwise, since we assume that there in no loop over $y+1$, we have that either there is no stage $s \geq s_0$ such that $c^- \in H(L(y+1) \cup \{\rho_s(y+1)\})$, in which case, for every $s \geq s_0$, we have $r_s(y+1) = \langle f_{y+1} \rangle$, or there is a last stage $s_1 \geq s_0$, such that $c^- \in H(L(y+1) \cup \{\rho_{s_1}(y+1)\})$, in which case $r_s(y+1) = r_{s_1}(y+1)$, for every $s \geq s_1$. □

The proof of the previous lemma shows also:

COROLLARY 3.9. *Let* (q, α) *be an approximated quasidialectical system and* y *be a slot. If* t *is the least stage such that, for every* $s \geq t$*, and* $x \leq y$*,* $r_s(x) = r_t(x)$*, then at either* $s_0 = t$ *or* $s_0 = t + 1$*, we have that* $r_{s_0}(y+1) = \langle f_{y+1} \rangle$.

Proof. Immediate. □

Intuitively, the last result might be understood as stating that there is no loss of information – in terms of the axioms proposed – in working after the stabilization of a given $L(x)$. Indeed, the result shows that any axiom f_x is proposed after stabilization of $L(x)$.

The following section provides a full characterization of quasidialectical systems with loops.

[7] Following this intuition, one can even quest the choice of calling such infinite stacks "loops". In doing so, we simply want to remark the familiar intuition of a system "going into loop".

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3.5. Characterizing quasidialectical systems with loops. To fit loops in our intuitive interpretation is not completely straightforward. Recall Magari's idea of dialectical systems as representing the behavior of a mathematician – or even of a mathematical community – while facing possible contradictions. According to this scenario, quasidialectical systems with loops would describe a mathematical community in which the overall progression of the theory is indeterminately interrupted by a never-ending refinement of a single axiom – a kind of behavior that might be jokingly compared with Kafkian bureaucracy.

Nonetheless, loops are not so pathological within the theory of quasidialectical systems.

On one hand, due to condition (3) in Definition 3.1, every f^--orbit of a given axiom is infinite. Thus, in principle one cannot rule out the possibility of building an infinite ascending stack over some axiom. Of course, whether or not this happens depends on the operator H, and how we approximate it.

On the other hand, it is worth considering quasidialectical systems with loops for at least two reasons. Firstly, it is not difficult to provide a full characterization of them: we give it in the current section. More importantly, even if at first sight quasidialectical systems with loops may appear, to some extent, stupid, it can be shown that they can represent sets (namely c.e. noncomputable sets) that are not representable by standard dialectical systems. As we will see, this is permitted by the fact that the information one can encode in a loop, is not necessarily trivial. Moral of the story: not all bureaucracy is pointless.

LEMMA 3.10. *Let (q, α) be an approximated quasidialectical system with loops. Then A^α_q is a c.e. set.*

Proof. Call b the least slot over which the pair (q, α) has a loop. By Lemma 3.8, there must be a stage t such that, for all $s \geq t$, $L_s(b) = L_t(b)$: call $X = L_t(b)$. Clearly $H(X)$ is a c.e. set, since X is finite. It is left to show that $A^\alpha_q = H(X)$. The inclusion $\supseteq$ is obvious, since for every $s \geq t$, $X \subseteq L_s(h(s))$. To show $A^\alpha_q \subseteq H(X)$, just notice that at every stage $s \geq t$ at which we add an axiom over b, we define the set of provisional theses to be $H_s(X)$. Thus no element not in $H(X)$ can be a final thesis. □

Recall that a c.e. set is said to be *simple*, if its complement is infinite, and does not contain any infinite c.e. set. As we can see through the next lemma, simplicity gives us a restraint on the kind of information that can be encoded within a loop.

LEMMA 3.11. *Let A be a c.e. set. Then there exists an approximated quasidialectical system (q, α) with loops such that $A^\alpha_q = A$ if and only if A is coinfinite and not simple.*

Proof. ($\Leftarrow$): If A is coinfinite and not simple, then there exists an infinite c.e. subset $B \subseteq \overline{A}$, such that $\overline{A} \setminus B$ contains at least two elements. Let $b = \min B$: we may assume without loss of generality that also $b = \min \overline{A}$. Then consider a quasidialectical system, $q = \langle H, f, f^-, c, c^-\rangle$, where f is the identity, f^- is any 1-1 computable function such that $\text{range}(f^-) \subseteq B$, $c \neq c^-$ and $c, c^- \in \overline{A} \setminus B$, and H satisfies $H(\emptyset) = A$, $c^- \in H(\{x\})$ if and only if $x \in B$. To take an appropriate H which is also an algebraic closure operator, take

$$H = \{\langle y, \{x\}\rangle : x \in B, y \in \omega, y \neq c\} \cup \{\langle a, \emptyset\rangle : a \in A\} \cup \{\langle x, \{x\}\rangle : x \in \omega\} :$$

to show that $H(X) = H(H(X))$, notice that if $X \cap B \neq \emptyset$ then $H(X) = \omega \setminus \{c\} = H(\omega \setminus \{c\})$, otherwise $H(X) = A = H(A)$. It is clear that whatever approximation α we work with, we have a loop over b, and clearly for every such α, $A_q = A^\alpha_q = A$.

($\Rightarrow$): Suppose that A is c.e. and there is an approximated quasidialectical system (q, α) with loops, and $A^\alpha_q = A$. Let b be the least slot such that there is a loop over b. It is

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immediate to see that $\mathrm{orb}_{f^-}(b)$ is an infinite c.e. set. We claim that $\mathrm{orb}_{f^-}(b) \subseteq \overline{A}$. So, suppose that some $f_y \in \mathrm{orb}_{f^-}(b)$ belongs to A. As $A = A_q^\alpha$, this means that $f_y \in A_q^\alpha$. By Lemma 3.8, there must be a stage t such that, for all $s \geq t$, $L_s(b) = L_t(b)$: call $X = L_t(b)$. So, as in the proof of the previous lemma, we would have that $f_y \in H(X)$. But since f_y belongs to the loop over b, we must have $c^- \in H(X \cup \{f_y\})$. On the other hand, as H is a closure operator, we have $X \subseteq H(X)$, so by $\{f_y\} \subseteq H(X)$, we get

$$H(X \cup \{f_y\}) \subseteq H(H(X)) = H(X).$$

Thus, at some stage $s > t$, we would see $c^- \in H_s(X)$, contrary to the fact that $L(b)$ does not change after t. □

REMARK 3.12. *Notice that the above proof shows in fact that if A is a coinfinite and not simple set, then there is a proper quasidialectical system q, such that for every computable approximation to the enumeration operator of H, one has that* $A = A_q^\alpha$.

The conjunction of the last two lemmas give us the following characterization theorem for quasidialectical systems with loops:

THEOREM 3.13. *The sets that are representable by approximated quasidialectical systems* (q, α) *with loops are exactly the c.e. sets that are coinfinite and not simple.*

Proof. Immediate. □

3.6. A locality result for loopless quasidialectical systems. In this section we point out some useful properties of loopless approximated quasidialectical systems, which show unexpected similarities of these systems with the dialectical systems. Lemma 3.14 below states a sort of locality result that informally expresses the following fact: even if a quasidialectical system, by means of the revising function f^-, might heavily modify the order in which axioms are tested, what really counts for an axiom f_x to be a final thesis is whether or not f_x has eventually x among its slots. Thus, the expressiveness of a quasidialectical system without loops, by which it might propose an axiom several times, ends up with a sort of redundancy: among all possible occurrences of f_x in the list of proposed axioms, what really counts is the one that has been proposed at slot x.

First of all, in view of Lemma 3.8, if the pair (q, α) is a loopless approximated quasidialectical system, then the corresponding parameters $r_s(x)$, $\rho_s(x)$, and $L_s(x)$ reach a limit with respect to s, so we are justified in defining, for every x,

$$r(x) = \lim_s r_s(x) \qquad \rho(x) = \lim_s \rho_s(x) \qquad L(x) = \lim_s L_s(x).$$

LEMMA 3.14. *Let* (q, α) *be a loopless approximated quasidialectical system. Then* $f_y \in A_q^\alpha$ *if and only if*

$$(\exists t)(\forall s \geq t)[r_s(y) = \langle f_y \rangle].$$

Notice that this is equivalent to saying that there exists a t such that $\rho_s(y) = f_y$ *for all* $s \geq t$.

Proof. ($\Leftarrow$): Let f_y be given. Under the assumption, and by Lemma 3.8, let t_0 be a stage such that for every $s \geq t_0$, $L_s(y+1) = L(y+1)$. Then for all $s \geq t_0$, we have that

$$f_y \in L(y+1) \subseteq L(h(s));$$

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let $t_1 \geq t_0$ be such that for all $s \geq t_1$, $L(y+1) \subseteq H_s(L(y+1))$ (we use here that $X \subseteq H(X)$ for every X): then for all $s \geq t_1$ we have

$$L(y+1) \subseteq H_s(L(y+1)) \subseteq H_s(L(h(s))) = A^{\alpha}_{q,s},$$

thus $f_y \in A^{\alpha}_q$.

($\Rightarrow$): Assume $f_y \in A^{\alpha}_q$, i.e. $f_y \in A^{\alpha}_{q,s}$, for all $s \geq t_0$, for some t_0. We first claim that there is a number i, such that, for every $s \geq t_0$, $i < h(s)$ and $f_y \in H_s(L_s(i))$. Since $f_y \in A^{\alpha}_{q,t_0}$, there is a least $i < h(t_0)$ such that $f_y \in H_{t_0}(L_{t_0}(i))$: we claim that this is the desired i. In order to prove the claim, assume that there is a least $s \geq t_0$ such that $f_y \in H_s(L_s(i))$, but $f_y \notin H_{s+1}(L_{s+1}(i))$: then $h(s+1) \leq i$, and thus $f_y \notin H_{s+1}(L_{s+1}(h(s+1)))$, i.e. $f_y \notin A^{\alpha}_{q,s+1}$, contrary to the fact that $s+1 \geq t_0$.

Therefore there is a least $x \leq i$ such that $f_y \in H(L(x))$. Now, let s_0 be the stage at which we propose f_y at slot y, as in Corollary 3.9. We would then remove f_y from $r(y) = \langle f_y \rangle$, only if at some later stage s we see $c \in H_s(L(y) \cup \{f_y\})$, or $c^- \in H_s(L(y) \cup \{f_y\})$. It follows that $f_y \notin H(L(y))$: otherwise, as in the proof of the left-to-right implication of Lemma 3.11, we would have $\{c, c^-\} \cap H(L(y)) = H(L(y) \cup \{f_y\}) \neq \emptyset$, contrary to the fact that $L(y)$ is the limit set. Therefore $L(y) \subset L(x)$, hence $\{c, c^-\} \cap H(L(x) \cup \{f_y\}) \neq \emptyset$: this implies $f_y \notin H(L(x))$, otherwise as in the proof of Lemma 3.11 we would conclude that $\{c, c^-\} \cap H(L(x)) \neq \emptyset$, contrary to the fact that $L(x)$ is the limit set. We have reached a contradiction, since $f_y \in H(L(x))$.

The claim about $\rho_s(y)$ is obvious, as by f^- being acyclic, we have that $\rho_s(y) = f_y$ if and only if the length of $r_s(y)$ is 1. □

Given an approximated quasidialectical system (q, α), such that for every x, $\rho(x) = \lim \rho_s(x)$ and $r(x) = \lim r_s(x)$ exist, let $L = \{\rho(x) : x \in \omega \text{ and } r(x) \neq \langle\,\rangle\}$ (where of course, $\rho_s(x)$, $r_s(x)$, and consequently $L(x)$, are taken with respect to α).

THEOREM 3.15. *If (q, α) is a loopless approximated quasidialectical system then $A^{\alpha}_q = L$. Moreover, for every y,*

$$r(y) \neq \langle\,\rangle \Rightarrow \text{range}(r(y)) \cap A^{\alpha}_q = \{\rho(y)\}.$$

Proof. Lemma 3.14 shows that $A^{\alpha}_q \subseteq L$. The converse is trivial, as by the very definitions, $\rho(y) \in A^{\alpha}_{q,s}$, for every big enough s.

Suppose now that $r(y) \neq \langle\,\rangle$, and by Lemma 3.8, let t_0 be the least stage after which $L_s(y)$ does not change any more. Assume $t_1 > t_0$, $\rho_{t_1}(y) \neq \rho(y)$, and $\rho_{t_1}(y) = f_z \in A^{\alpha}_q$. Now, $r(z) = \langle f_z \rangle$ by Lemma 3.14. Let $v = \max\{z, y\} + 1$, and let $t_2 \geq t_1$ be a stage after which for no $u \leq v$ does $r_s(u)$ change; but $f_z \in L(v)$, thus at some stage $s \geq t_2$ we would see $\{c, c^-\} \cap H(L(v)) \neq \emptyset$, contradiction, as $L(v)$ has already reached its limit. □

The following lemma can be presented as a natural companion of Lemma 2.5.

LEMMA 3.16. *Let (q, α) be a loopless approximated quasidialectical system. Then, $f_x \in A^{\alpha}_q$ if and only if neither $c \in H(L(x) \cup \{f_x\})$, nor $c^- \in H(L(x) \cup \{f_x\})$.*

Proof. ($\Rightarrow$): This follows from Lemma 3.14 which implies that $L(x+1) = L(x) \cup \{f_x\}$, so we cannot have $\{c, c^-\} \cap H(L(x+1))$, because $L(x+1)$ is the limit set.

($\Leftarrow$): This follows from the fact that after we propose, at stage s_0, f_x at slot x (see Corollary 3.9), and under the assumption that neither $c \in H(L(x) \cup \{f_x\})$, nor $c^- \in H(L(x) \cup \{f_x\})$, we never change $r(x)$, and thus $\rho(x) = f_x$, implying $f_x \in A^{\alpha}_q$ by Lemma 3.14 . □

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REMARK 3.17. *The proof of the previous theorem shows in fact that if* (q, α) *is an approximated quasidialectical system, such that there are no loops over any* $y < x$, *then* $f_x \in A_q^\alpha$ *if and only if neither* $c \in H(L(x) \cup \{f_x\})$, *nor* $c^- \in H(L(x) \cup \{f_x\})$.

COROLLARY 3.18. *For every approximated quasidialectical system* (q, α), *the quasidialectical set* A_q^α *is* Δ_2^0.

Proof. If (q, α) is an approximated quasidialectical system with loops, then A_q^α is c.e., thus Δ_2^0.

Thus, let q be a loopless approximated quasidialectical system: we show that the quasidialectical approximation $\{A_{q,s}^\alpha\}_{s\in\omega}$ provides a Δ_2^0 approximation to A_q^α. Recall that $A_{q,s}^\alpha = H_s(L_s(h(s) - 1))$. On the other hand, by Lemma 3.8, it is easy to see that $\lim_s p(s) = \infty$, and thus $\lim_s h(s) = \infty$. Now, let f_y be given, and let t_0 be a stage such that for all $s \geq t_0$, $h(s) \geq y + 1$, $L(y) = L_s(y) = L_{t_0}(y)$. If $f_y \in H_s(L_s(h(s) - 1))$ and $s \geq t_0$ then, by an already familiar argument, $\{c, c^-\} \cap H(L(y) \cup \{f_y\}) = \emptyset$, hence by Lemma 3.16, $f_y \in A_q^\alpha$. □

The following easy construction (accompanying Lemma 3.6) proves that dialectical systems can be viewed as proper loopless quasidialectical systems, in the strong sense that this identity is independent of how one approximates the enumeration operator of the quasidialectical system.

LEMMA 3.19. *From any dialectical system* $d = \langle H, f, c\rangle$, *and any number* $c^- \neq c$, *one can effectively build a quasidialectical system* q, *such that, if* $c^- \notin A_d$, *then for every approximation* α *to the enumeration operator of* q, *one has that* (q, α) *is loopless and* $A_q^\alpha = A_d$.

Proof. Let $d = \langle H, f, c\rangle$ be a dialectical system, and let $c^- \neq c$. Consider the quasidialectical system $q = \langle H^*, f, f^-, c, c^-\rangle$, where f^- is any acyclic computable function not having c^- in its range, and such that $f^-(c^-) = f^-(c) = a \in H(\emptyset)$, and let

$$H^* = \{\langle x, D\rangle : x \neq c^- \ \&\ \langle x, D\rangle \in H\} \cup \{\langle c^-, \{c^-\}\rangle, \langle c^-, \{c\}\rangle\}.$$

Let $X \subseteq \omega$. It is easy to see that $H^*(X) \subseteq H(X)$: indeed if $x \in H^*(X)$, and $x \neq c^-$ then, by definition of H^*, $x \in H(X)$; if $x = c^-$ then either $c^- \in X$, and thus $x \in H(X)$; or $c \in X$, and thus $x \in H(X)$. Also, it immediately follows from the definitions that, for $x \neq c^-$, if $x \in H(X)$ then $x \in H^*(X)$. Let us now show that H^* is a closure operator, if H is. Let $X \subseteq \omega$ be given: clearly, $X \subseteq H^*(X)$. Next, we want to show that $H^*(H^*(X)) \subseteq H^*(X)$. Assume $x \in H^*(H^*(X))$. If $x = c^-$, i.e. $c^- \in H^*(H^*(X))$ then there are two possibilities: either $c^- \in H^*(X)$ or $c \in H^*(X)$: in the former case, the claim is true; in the latter case, we have $c \in H(X)$, but then $c^- \in H(\{c\}) \subseteq H(H(X)) = H(X) \subseteq H^*(X)$. If $x \neq c^-$, then from $x \in H^*(H^*(X))$ we get $x \in H(H^*(X)) \subseteq H(H(X)) = H(X)$, giving that $x \in H^*(X)$, by definition of H^*.

Let us now consider any computable approximation to H^*, and any computable approximation to H: relatively to these approximations, we will distinguish r, L, as r^d, L^d, or r^q, L^q according to whether we deal with d or q. We now aim at showing that, if $c^- \notin A_d$, then $A_q^\alpha = A_d$: by induction on x, we in fact show that

$$f_x \in A_d \Leftrightarrow f_x \in A_q^\alpha,$$

and for each $y \leq x$, $r_s^q(y)$ reaches limit $r^q(y)$, and if $y \in \{c, c^-\}$, then $r^q(y) \in \{\langle\,\rangle, \langle y, a\rangle\}$.

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Suppose that the claim is true of all $y < x$. Assume now that $f_x \notin A_d$. Then $c \in H(L^d(x) \cup \{f_x\})$, from which it follows that $c \in H^*(L^d(x) \cup \{f_x\})$, and hence $c \in H^*(L^q(x) \cup \{f_x\})$, as by induction $L^d(x) \subseteq L^q(x)$. Now, it might be the case that also $c^- \in H^*(L^q(x) \cup \{f_x\})$. If, after the least stage at which $L^q(x)$ stabilizes, the event $c \in H^*(L^q(x) \cup \{f_x\})$ appears earlier than the event $c^- \in H^*(L^q(x) \cup \{f_x\})$ (i.e., depending on the approximation to H^*, c is enumerated into $H^*(L^q(x) \cup \{f_x\})$ before c^- is enumerated into $H^*(L^q(x) \cup \{f_x\})$) then $r^q(x) = \langle\,\rangle$, and $f_x \notin A^a_q$. If $c^- \in H^*(L^q(x) \cup \{f_x\})$ appears first, then we must examine the two possibilities: the first possibility is $c^- \in L^q(x) \cup \{f_x\}$, hence $c^- = f_x$ (as $L^q(x) \subseteq A_d$, by inductive assumption, since $a \in A_d$) and in this case it is easy to see that $r^q(x) = \langle c^-, a\rangle$; the second possibility is $c \in L^q(x) \cup \{f_x\}$, which similarly implies that $c = f_x$: thus in this case it is easy to see that $r^q(x) = \langle c, a\rangle$.

Vice versa, suppose that $f_x \notin A^a_q$: then (by Lemma 3.16 and Remark 3.17) $c^- \in H^*(L^q(x) \cup \{f_x\})$ or $c \in H^*(L^q(x) \cup \{f_x\})$. If $c^- \in H^*(L^q(x) \cup \{f_x\})$ appears first then, as above, $c^- \in L^q(x) \cup \{f_x\}$ or $c \in L^q(x) \cup \{f_x\}$: if $c^- = f_x$, then $r^q(x) = \langle c^-, a\rangle$ and $f_x \notin A_d$ by choice of c^-; notice that $c^- \in L^q(x)$ cannot occur since by the inductive hypothesis $L^q(x) \subseteq L^d(x) \cup \{a\}$, and thus if $c^- \in L^q(x)$ then we would have that $c^- \in A_d$; on the other hand, if $c = f_x$ then $r^q(x) = \langle c, a\rangle$ and $f_x \notin A_d$. It remains to consider the case in which $c \in H^*(L^q(x) \cup \{f_x\})$ appears first: but then $r^q(x) = \langle\,\rangle$; we also have $c \in H(L^q(x) \cup \{f_x\})$, thus by the inductive assumption $L^q(x) \subseteq A_d$, we have that $f_x \notin A_d$.

This shows the inductive step. □

COROLLARY 3.20. *Every dialectical set A, such that $\overline{A}$ has at least two elements, is represented by a loopless proper quasidialectical system (and the representation is independent of any computable approximation to the enumeration operator of the quasidialectical system).*

Proof. Let $d = \langle H, f, c\rangle$ be a dialectical system. If $\omega \setminus \{c\} \nsubseteq A_d$, then apply the previous lemma. The claim regarding $A = \omega$ is obvious. □

§4. Conclusions. In the forthcoming paper (Amidei et al., to appear), we will show that the information content of dialectical systems coincides with that of quasidialectical systems, in the sense that both the dialectical Turing degrees (i.e. the Turing degrees containing dialectical sets) and the quasidialectical Turing degrees (i.e. the Turing degrees containing quasidialectical sets) are exactly the c.e. Turing degrees; moreover, dialectical sets and quasidialectical sets give rise to the same class of enumeration degrees, namely the co-c.e. enumeration degrees. On one hand, these facts seem to suggest that dialectical systems are robust enough not to be superseded by the introduction of a more refined notion of revision. Indeed, the coincidence of the dialectical degrees with the quasidialectical degrees shows that we do not increase the computational power of standard dialectical systems, if by "computational power" of such a system one means its degree of relative computability. In other words, our notion of revision is, in this respect, already somehow encoded in Magari's original proposal.

On the other hand, the story is completely different if we address the problem of comparing the two systems from the point of view of the sets they might represent (instead of just being concerned with their degrees). As we know from Lemma 3.6, every dialectical set is quasidialectical, and in fact, by Corollary 3.20, every dialectical set whose complement has at least two elements, is represented by some loopless proper quasidialectical system. Now, by Theorem 3.13 it is clear that there are undecidable c.e. sets represented

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by quasidialectical systems with loops, whereas it is shown in (Magari, 1974) that every c.e. dialectical set is decidable. Therefore, if one allows loops, it is easy to see that there are quasidialectical sets which are not dialectical. But, even avoiding loops, one can show that loopless quasidialectical systems represent a class of sets, which is much larger than the one that is represented by dialectical systems. In fact, we will show in (Amidei et al., to appear) that the quasidialectical sets spread out throughout all classes of the Ershov hierarchy of the Δ_2^0 sets; in particular there are quasidialectical sets which are not ω-c.e., whereas every dialectical set is ω-c.e..

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BIBLIOGRAPHY

Amidei, J., Pianigiani, D., San Mauro, L., & Sorbi, A. Trial and error mathematics II: Dialectical sets and quasidialectical sets, their degrees, and their distribution within the class of limit sets. Preprint, 2014, submitted.

Angluin, D., & Smith, C. H. (1983). Inductive inference: Theory and methods. *ACM Computing Surveys (CSUR)*, **15**(3), 237–269.

Bernardi, C. (1974). Aspetti ricorsivi degli insiemi dialettici. *Bollettino della Unione Matematica Italiana. Series IV*, **9**, 51–61.

Cellucci, C. (2000). The growth of mathematical knowledge: An open world view. In Grosholz, E. and Breger, H., editors. *The Growth of Mathematical Knowledge*. Dordrecht: Kluwer, pp. 153–176.

Cooper, S. B. (2003). *Computability Theory*. Boca Raton, London, New York, Washington, DC: Chapman & Hall/CRC Mathematics.

Copeland, B. J., & Shagrir, O. (2013). Turing versus Gödel on computability and the mind. In Copeland B. J., Posy C. J., and Shagrir, O., editors. *Computability: Turing, Gödel, Church, and Beyond*. Cambridge, MA: MIT Press, pp. 1–33.

Gnani, G. (1974). Insiemi dialettici generalizzati. *Matematiche*, **XXIX**(2), 1–11.

Gold, E. M. (1965). Limiting recursion. *Journal of Symbolic Logic*, **30**, 28–48.

Hájek, P. (1977). Experimental logics and Π_3^0 theories. *Journal of Symbolic Logic*, **42**(4), 515–522.

Hintikka, J., & Mutanen, A. (1988). An alternative concept of computability. In Hintikka, J., editor. *Language, Truth, and Logic in Mathematics*. Dordrecht: Kluwer, pp. 174–188.

Jeroslow, R. G. (1975). Experimental logics and Δ_2^0 theories. *Journal of Philosophical Logic*, **4**(3), 53–267.

Kelly, K. (1996). *The Logic of Reliable Inquiry*. Oxford: Oxford University Press.

Kitcher, P. (1983). *The nature of mathematical knowledge*. New York: Oxford University Press.

Kreisel, G. (1967). Informal rigour and completeness proofs. In Lakatos, I., editor. *Problems in the Philosophy of Mathematics*. Amsterdam: North-Holland, pp. 138–171.

Kugel, P. (1986). Thinking may be more than computing. *Cognition*, **32**, 137–198.

Lakatos, I. (1976a). *Proofs and Refutations*. Cambridge: Cambridge University Press.

Lakatos, I. (1976b). A renaissance of empiricism in the recent philosophy of mathematics. *British Journal for the Philosophy of Science*, **27**(3), 201–223.

Lolli, G. (2008). Experimental methods in proofs. In Lupacchini, R., and Corsi, G., editors. *Deduction, Computation, Experiment*. Milan: Springer, pp. 65–79.

https://doi.org/10.1017/S1755020315000404 Published online by Cambridge University Press

Magari, R. (1974). Su certe teorie non enumerabili. *Annali di Matematica Pura ed Applicata. Series IV*, **XCVIII**, 119–152.
Magari, R. (1980). Natura empirica della metamatematica. Technical report, Dipartimento di Matematica, Università di Siena.
Mancosu, P. (2008). *The Philosophy of Mathematical Practice*. Oxford: Oxford University Press.
Montagna, F., Simi, G., & Sorbi, A. (1996). Logic and probabilistic systems. *Archive for Mathematical Logic*, **35**(4), 225–261.
Osherson, D. N., Stob, M., & Weinstein, S. (1991). A Universal Inductive Inference Machine. *Journal of Symbolic Logic*, **56**(2), 661–672.
Putnam, H. (1965). Trial and error predicates and the solution of a problem of Mostowski. *Journal of Symbolic Logic*, **30**, 49–57.
Rogers, Jr., H. (1967). *Theory of Recursive Functions and Effective Computability*. New York: McGraw-Hill.
Shapiro, S., & Mc Carthy, T. (1987). Turing projectability. *Notre Dame Journal of Formal Logic*, **28**(4), 520–537.
Soare, R. I. (1987). *Recursively Enumerable Sets and Degrees*. Perspectives in Mathematical Logic, Omega Series. Heidelberg: Springer-Verlag.
Turing, A. M. (1947). *Turing Digital Archive*. Lecture to London Mathematical Society, February 20, 1947.
Turing, A. M. (1996). Intelligent machinery, a heretical theory, c.1951. *Philosophia Mathematica. Series III*, **4**(3), 256–260.
van Leeuwen, J., & Wiedermann, J. (2012). Computation as an unbounded process. *Theoretical Computer Science*, **429**, 202–212.

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