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UNCOUNTABLE COFINALITIES OF PERMUTATION GROUPS

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ABSTRACT

A sufficient criterion is found for certain permutation groups G to have uncountable strong cofinality, that is, they cannot be expressed as the union of a countable, ascending chain $(H_i)_{i\in\omega}$ of proper subsets H_i such that $H_iH_i \subseteq H_{i+1}$ and $H_i = H_i^{-1}$. This is a strong form of uncountable cofinality for G, where each H_i is a subgroup of G. This basic tool comes from a recent paper by Bergman on generating systems of the infinite symmetric groups, which is discussed in the introduction. The main result is a theorem which can be applied to various classical groups including the symmetric groups and homeomorphism groups of Cantor's discontinuum, the rationals, and the irrationals, respectively. They all have uncountable strong cofinality. Thus the result also unifies various known results about cofinalities. A notable example is the group $\mathrm{BSym}(\mathbb{Q})$ of all bounded permutations of the rationals $\mathbb{Q}$ which has uncountable cofinality but countable strong cofinality.

1. *Introduction*

Various authors have investigated cofinalities of particular groups G. The *cofinality* of a group G, denoted cf(G), is defined to be the smallest cardinal λ such that G can be written as the union of a chain of λ proper subgroups. Serre [**13**] considered groups of uncountable cofinality in his study of groups acting on trees. Sabbagh [**12**] showed that any ω_1-existentially closed group has uncountable cofinality. Macpherson and Neumann [**10**] proved this for the full symmetric group $\mathrm{Sym}(\Omega)$ of all permutations of a countable, infinite set Ω, Gourion [**7**] for the group $\mathrm{Aut}(\mathbb{Q},\leqslant)$ of all order-automorphisms of the rationals (as an ordered set), Hodges *et al.* [**8**] for the automorphism group of the random graph, and Thomas [**16**] for infinite dimensional linear groups. We [**5**] considered the homeomorphism groups of the Cantor space $\mathfrak{C}$, the rationals $\mathbb{Q}$, and the irrationals $\mathbb{I}$, with their natural topology, showing that their cofinality is uncountable. We refer the reader to the survey of Thomas [**17**] for further results in this direction.

This paper also deals with cofinality problems. It is stimulated by a very nice paper by Bergman [**2**] which sheds new light on the cofinality problems by looking at the following condition, which we call the *Bergman property* for the group G: for any generating subset E of G with $1\in E = E^{-1}$ there is a natural number $k\in\omega$ with $G = E^k$. Hence any chain $(E^k)_{k\in\omega}$ with union G terminates after finitely many steps. This provides the link to cofinalities of groups. We will say that G has *strong cofinality* λ and write $\mathrm{scf}(G)=\lambda$ if λ is the smallest cardinal such that G can be written as the union of a chain of λ proper subsets H_i such that for any $i\in\lambda$ we have $H_i = H_i^{-1}$ and there is $j\geqslant i$ with $H_iH_i\subseteq H_j$.

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We will see that either $\mathrm{scf}(G) = \omega$ or $\mathrm{scf}(G) = \mathrm{cf(G)}$. In [**6**] it is shown that G has uncountable strong cofinality if and only if G has the Bergman property and uncountable cofinality. Bergman [**2**] gives a new short proof that the cofinality of $\mathrm{Sym}(\Omega)$ is uncountable. We wonder if Bergman's consideration can be applied to other groups and would provide a uniform proof. Section 2 is devoted to establishing such a result. It provides a sufficient condition for a permutation group (G, Ω) to have uncountable cofinality. Moreover, given a fixed generating subset E of G of a certain type, we show that G has the Bergman property if and only if $G = E^k$ for some $k \in \omega$. In Section 3 we apply our main theorem, Theorem 2.8, to a large family of groups showing that they have uncountable strong cofinality. This includes the homeomorphism groups of the rationals, of the irrationals, and of Cantor's set with the usual topology, thereby sharpening the cofinality result mentioned above, and also the automorphism group of the reals as a Borel space; this partly answers a question of Bergman [**2**]. With different techniques (also from the theory of lattice-ordered groups), in Droste and Holland [**6**] it is shown that $\mathrm{Aut}(\mathbb{Q}, \leqslant)$, and more generally $\mathrm{Aut}(\Omega, \leqslant)$ for any doubly homogeneous chain $(\Omega, \leqslant)$ has uncountable strong cofinality as well. In this paper, we also show that the two notions of cofinality are not the same. We discuss in detail the group $\mathrm{BSym}(\mathbb{Q})$ of bounded permutations of $\mathbb{Q}$ and show that $\mathrm{BSym}(\mathbb{Q})$ has uncountable cofinality but not the Bergman property. It is tempting to try to show that the Jonsson groups constructed in ZFC by Shelah [**14**] and Olshanskii [**11**] are of the same form. Since these groups G in [**11**, **14**] have cardinality $\aleph_1$, and by definition all their proper subgroups are countable, it is immediate that $\mathrm{cf(G)} = \aleph_1$. However, Shelah [**14**] has also shown that any generating set E satisfies $E^{240} = G$, hence the Bergman property holds for Shelah's G as well (and thus $\mathrm{scf}(G) = \aleph_1$). Clearly, Olshanskii's Jonsson group G in [**11**, p. 391] also satisfies $\mathrm{cf(G)} = \aleph_1$ and it is unknown but very likely (using a natural generating system which comes element-wise from the generators of the cyclic groups which are adjoined by transfinite induction to obtain G) that $\mathrm{scf}(G) = \omega$; see [**11**, Theorem 35.1, p. 386].

2. *Cofinality of permutation groups*

In this section we present our main result describing permutation groups with uncountable cofinality. First let us recall these notions.

DEFINITION 2.1. Let G be a group that is not finitely generated.

(a) The *cofinality* of G, denoted $\mathrm{cf(G)}$, is the smallest cardinal λ for which there is a chain of proper subgroups $(U_i)_{i\in\lambda}$ of G such that $G = \bigcup_{i\in\lambda} U_i$.

(b) The *strong cofinality* of G, denoted $\mathrm{scf}(G)$, is the smallest cardinal λ for which there is a chain of proper subsets $(U_i)_{i\in\lambda}$ of G such that for each $i \in \lambda$, $U_i = U_i^{-1}$ and $U_i \cdot U_i \subseteq U_j$ for some $j \in \lambda$, and $G = \bigcup_{i\in\lambda} U_i$.

(c) G satisfies the *Bergman condition*, if for each subset $E \subseteq G$ with $1 \in E = E^{-1}$ and $G = \langle E \rangle$ there is $k \in \omega$ such that $G = E^k$.

Clearly, $\omega \leqslant \mathrm{scf}(G) \leqslant \mathrm{cf(G)}$. Moreover, we have the following.

PROPOSITION 2.2. *Let G be any uncountable group.*

(i) (Droste and Holland [**6**]) *G has uncountable strong cofinality if and only if G satisfies the Bergman condition and G has uncountable cofinality.*

(ii) $\mathrm{scf}(G) = \omega$ *or* $\mathrm{scf}(G) = \mathrm{cf(G)}$.

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Proof. (i) We include the short argument from [**6**] for the sake of completeness. First let G have uncountable strong cofinality. To show that G satisfies the Bergman condition, let $E \subseteq G$ with $1 \in E = E^{-1}$ and $G = \langle E \rangle$. Now apply the assumption to the chain of subsets $U_i = E^i$ $(i \in \omega)$ to obtain the result.

For the converse, let $G = \bigcup_{i\in\omega} U_i$ with subsets $U_i \subseteq G$ as in Definition 2.1(b). Then $G = \bigcup_{i\in\omega} \langle U_i \rangle$, so $\mathrm{cf(G)} > \omega$ implies that $G = \langle U_j \rangle$ for some $j \in \omega$. Now the Bergman condition shows that $G = U_j^k$ for some $k \in \omega$. Then $U_j^k \subseteq U_m$ for some $m \geqslant k$, so $G = U_m$.

(ii) Let $\lambda = \mathrm{scf}(G)$ be uncountable. Write $G = \bigcup_{i\in\lambda} U_i$ with a chain of subsets $(U_i)_{i\in\lambda}$ such that for each $i \in \lambda$ we have $U_i = U_i^{-1}$ and $U_i \cdot U_i \subseteq U_j$ for some $j \in \lambda$. We may assume that this chain is smooth, that is $U_j = \bigcup_{i<j} U_i$ for each limit-ordinal $j \in \lambda$. Then we obtain a subchain $I \subseteq \lambda$ of limit-ordinals such that each U_i $(i \in I)$ is a subgroup of G. Hence $G = \bigcup_{i\in I} U_i$, showing that $\mathrm{cf(G)} \leqslant \lambda$. Thus $\mathrm{scf}(G) = \mathrm{cf(G)}$. □

This result shows that the new notion of strong cofinality is, strictly speaking, superfluous: it suffices to consider cofinality and the Bergman condition. However, it is a useful technical notion permitting us to deal with cofinality and the Bergman condition uniformly.

We denote by (G, Ω) an infinite permutation group G acting on Ω. Our notation is mostly standard. If $\Sigma, \Gamma \subseteq \Omega$, then $\Sigma^* = \Omega \backslash \Sigma$ denotes the complement of Σ in Ω, and $\Sigma \dot\cup \Gamma$ denotes the disjoint union. Let $G_{\{\Sigma\}} = \{g \in G : \Sigma g = \Sigma\}$ denote the *setwise stabilizer* of Σ. If $g \in G$, then $\mathrm{supp}(g) = \{a \in \Omega : ag \neq a\}$ denotes the *support* of g. Let H be a subset of G. We say that $\Sigma \subseteq \Omega$ is *full for* H, if for each $g \in G_{\{\Sigma\}}$ there is $h \in H_{\{\Sigma\}}$ which coincides with g on Σ. If $g, h \in G$ then $g^h = h^{-1}gh$ is the conjugate of g by h.

In all of the following, let (G, Ω) be an infinite permutation group, and let $\mathfrak{K} \subseteq \mathfrak{P}(\Omega)$. Consider the following conditions.

(1) $\varnothing, \Omega \notin \mathfrak{K}$, and $\Sigma \in \mathfrak{K} \implies \Sigma^* \in \mathfrak{K}$.

(2) $\Sigma \in \mathfrak{K} \implies \exists\, \Sigma_1, \Sigma_2 \in \mathfrak{K} : \Sigma = \Sigma_1 \dot\cup \Sigma_2$.

(3) $\Delta, \Gamma, \Sigma \in \mathfrak{K}$, $\Gamma \dot\cup \Sigma \subseteq \Delta \implies \Gamma \dot\cup \Sigma \in \mathfrak{K}$.

(4) $\Omega = \Sigma_1 \cup \Sigma_2$, and $\Sigma, \Sigma_1, \Sigma_2 \in \mathfrak{K} \implies \Sigma \cap \Sigma_1 \in \mathfrak{K}$ or $\Sigma \cap \Sigma_2 \in \mathfrak{K}$.

(5) $\Gamma, \Sigma \in \mathfrak{K} \implies \exists\, g \in G : \Gamma g = \Sigma$.

(5′) $\Delta, \Gamma, \Sigma \in \mathfrak{K} \implies \exists\, g \in G : \Gamma g = \Sigma$ and $\Delta g \in \mathfrak{K}$.

(6) $\Sigma \in \mathfrak{K}$, $\Sigma = \Sigma_1 \dot\cup \Sigma_2 = \Gamma_1 \dot\cup \Gamma_2$ in $\mathfrak{K}$, $f_i \in G$ with $\Sigma_i f_i = \Gamma_i$ $(i = 1, 2) \implies f_1 \restriction \Sigma_1 \dot\cup f_2 \restriction \Sigma_2 \dot\cup \mathrm{id}_{\Sigma^*} \in G$.

(6′) $\Delta, \Sigma \in \mathfrak{K}, \Sigma = \Sigma_1 \dot\cup \Sigma_2 = \Gamma_1 \dot\cup \Gamma_2$ in $\mathfrak{K}$, $f_i \in G$ with $\Sigma_i f_i = \Gamma_i$ and $\Delta f_i \in \mathfrak{K}$ $(i = 1, 2) \implies g = f_1 \restriction \Sigma_1 \dot\cup f_2 \restriction \Sigma_2 \dot\cup \mathrm{id}_{\Sigma^*} \in G$ and $\Delta g \in \mathfrak{K}$.

(7) There are pairwise disjoint $\Gamma_i \in \mathfrak{K}$ $(i \in \omega)$ such that for all $f_i \in G_{\{\Gamma_i\}}$ $(i \in \omega)$ it follows that $\bigcup_{i\in\omega}(f_i \restriction \Gamma_i) \cup \mathrm{id}_{\Omega\backslash\bigcup_{i\in\omega}\Gamma_i} \in G$.

(8) *Bounded number of conjugates*: There is $n \in \omega$ such that whenever $\Sigma \in \mathfrak{K}$ and $f, g \in G$ with $\mathrm{supp}(f)$, $\mathrm{supp}(g) \subseteq \Sigma$ and $\Gamma \subseteq \mathrm{supp}(g)$ for some $\Gamma \in \mathfrak{K}$, then

$$f = (g_1')^{h_1} \cdot \ \ldots \ \cdot (g_n')^{h_n}$$

for some $h_i \in G$ with $\mathrm{supp}(h_i) \subseteq \Sigma$ and $g_i' \in \{g, g^{-1}\}$ $(i = 1, \ldots, n)$.

As a typical example, consider $G = \mathrm{Sym}(\Omega)$ and let $\mathfrak{K}$ comprise all moieties of Ω. Here a subset Σ of Ω is called a *moiety* if $|\Sigma| = |\Sigma^*| = |\Omega|$. Then (G, Ω) and $\mathfrak{K}$

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clearly satisfy conditions (1)–(7) and (5′), (6′). In particular, note that moieties are not closed under unions or intersection, but satisfy the weaker conditions (3) and (4). Condition (8) in general says that whenever $f, g \in G$ act on Σ (that is, are the identity on Σ^*) and g moves sufficiently many points, then f is a product of n conjugates of g or g^{-1}, for some uniform n, and with conjugators also acting on Σ. Note here that if Σ is a moiety, then $(\mathrm{Sym}(\Sigma), \Sigma) \cong (\mathrm{Sym}(\Omega), \Omega)$ and so it suffices to prove this condition with Σ replaced by Ω; this case of (8) follows from Proposition 3.1 below. Further examples will be given in Section 3.

For the rest of this section, let (G, Ω) be an infinite permutation group and $\mathfrak{K} \subseteq \mathfrak{P}(\Omega)$. Moreover, let $G = \bigcup_{i \in \omega} H_i$ be the union of an ascending chain of subsets H_i such that for each $i \in \omega$ we have $H_i = H_i^{-1}$ and $H_i^2 \subseteq H_j$ for some $j \geqslant i$.

LEMMA 2.3. *Assume* (1), (2), (5), (6) *and* (8). *If there is an* $i \in \omega$ *such that* $\Sigma \in \mathfrak{K}$ *is full for* H_i, *then there is* $j \in \omega$ *such that* $G_{\{\Sigma\}} \subseteq H_j$.

Proof. Split $\Sigma = \Sigma_1 \dot\cup \Sigma_2$ in $\mathfrak{K}$ by (2). By (5) there are $g_1, g_2 \in G$ such that $\Sigma_1 g_1 = \Sigma_2$ and $\Sigma_2 g_2 = \Sigma_1$. By (6) we have $g = g_1 \restriction \Sigma_1 \dot\cup g_2 \restriction \Sigma_2 \dot\cup \mathrm{id}_{\Sigma^*} \in \mathrm{G}$; hence $g \in G_{\{\Sigma\}}, \Sigma_1 g = \Sigma_2$ and $\mathrm{supp}(g) = \Sigma$. Fix n as in (8). Choose $j_1 \in \omega$ such that $g \in H_{j_1}$ and $j_2 \geqslant j_1$ such that $(H_i H_{j_1} H_i)^n \subseteq H_{j_2}$. We claim that any $p \in G_{\{\Sigma\}}$ with $p \restriction \Sigma^* = \mathrm{id}_{\Sigma^*}$ belongs to H_{j_2}.

By (8) there are $q_1, \ldots, q_n \in G$ with $\mathrm{supp}(q_j) \subseteq \Sigma$ such that

$$p = (g^{\pm 1})^{q_1} \cdot \ \ldots \ \cdot (g^{\pm 1})^{q_n}.$$

Since Σ is full for H_i, there are $h_j \in H_i$ such that $h_j \restriction \Sigma = q_j \restriction \Sigma$ $(j = 1, \ldots, n)$. Since $p \restriction \Sigma^* = \mathrm{id} \restriction \Sigma^* = \mathrm{g} \restriction \Sigma^*$, we obtain $p = (g^{\pm 1})^{h_1} \cdot \ \ldots \ \cdot \left(g^{\pm 1}\right)^{h_n}$ on Ω; hence $p \in H_{j_2}$.

Next we wish to derive the analogous claim for $p \in G_{\{\Sigma\}}$ with $p \restriction \Sigma = \mathrm{id}_\Sigma$. By (5) there is $g' \in G$ with $\Sigma g' = \Sigma^*$. Choose j_3 with $g' \in H_{j_3}$ and then $j_4 \geqslant j_3$ with $H_{j_3} \cdot H_{j_2} \cdot H_{j_3} \subseteq H_{j_4}$. Now if $p \in G_{\{\Sigma\}}$ with $p \restriction \Sigma = \mathrm{id}_\Sigma$, then $p' = p^{g'} \in G_{\{\Sigma\}}$ satisfies $p' \restriction \Sigma^* = \mathrm{id}_{\Sigma^*}$, so $p' \in H_{j_2}$ by the above. Thus $p \in H_{j_3} \cdot H_{j_2} \cdot H_{j_3} \subseteq H_{j_4}$ as claimed.

Now choose $j \in \omega$ such that $H_{j_2} \cdot H_{j_4} \subseteq H_j$. We show that $G_{\{\Sigma\}} \subseteq H_j$. Let $g \in G_{\{\Sigma\}}$ and consider $\Omega = \Sigma \dot\cup \Sigma^*$. By (2), split $\Sigma^* = \Gamma_1 \dot\cup \Gamma_2$ in $\mathfrak{K}$. Then $\Sigma \dot\cup \Gamma_1 = \Gamma_2^* \in \mathfrak{K}$. By (6), $g_1 = g \restriction \Sigma \cup \mathrm{id} \restriction \Gamma_1 \cup \mathrm{id}_{\Gamma_2} \in \mathrm{G}$, so $g_1 \restriction \Sigma^* = \mathrm{id}_{\Sigma^*}$. Similarly, $g_2 = g \restriction \Sigma^* \cup \mathrm{id}_\Sigma \in \mathrm{G}$. Then, by the above, $g = g_1 \cdot g_2 \in H_{j_2} \cdot H_{j_4} \subseteq H_j$. □

LEMMA 2.4. *Assume* (1), (2) *and* (5). *If* $\Sigma \in \mathfrak{K}$ *and* $G_{\{\Sigma\}} \subseteq H_i$ *for some* i, *then there is* $\Sigma' \in \mathfrak{K}$ *with* $\Sigma \cup \Sigma' = \Omega$, $\Sigma \cap \Sigma' \in \mathfrak{K}$ *and* $G_{\{\Sigma\}}, G_{\{\Sigma'\}} \subseteq H_j$ *for some* $j \in \omega$.

Proof. We decompose $\Sigma = \Gamma_1 \dot\cup \Gamma_2$ in $\mathfrak{K}$ by (2) and put $\Sigma' = \Gamma_2 \dot\cup \Sigma^* = \Gamma_1^* \in \mathfrak{K}$. Then $\Sigma \cap \Sigma' = \Gamma_2 \in \mathfrak{K}$. By (5) there is $g \in G$ with $\Sigma g = \Sigma'$, and there is $j \geqslant i$ such that $H_i^g \subseteq H_j$. Moreover $G_{\{\Sigma'\}} = G_{\{\Sigma\}}{}^g \subseteq H_i^g \subseteq H_j$. □

LEMMA 2.5. *Assume* (5) *and* (6). *Let* $\Sigma \in \mathfrak{K}$ *and let* $\Sigma = \Sigma_1 \dot\cup \Sigma_2 = \Gamma_1 \dot\cup \Gamma_2$ *be two decompositions in* $\mathfrak{K}$. *Then there is* $g \in G_{\{\Sigma\}}$ *with* $\Sigma_1 g = \Gamma_1$. *Moreover, if also* $\Delta \in \mathfrak{K}$ *is given and we have* (5′) *and* (6′), *we can choose* g *as above such that* $\Delta g \in \mathfrak{K}$.

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Proof. By (5) (respectively (5′)) there are $g_1, g_2 \in G$ such that $\Sigma_1 g_1 = \Gamma_1, \Sigma_2 g_2 = \Gamma_2$ (and $\Delta g_1, \Delta g_2 \in \mathfrak{K}$). By (6) (respectively (6′)) there is $g \in G_{\{\Sigma\}}$ such that $g \restriction \Sigma = g_1 \restriction \Sigma_1 \dot\cup g_2 \restriction \Sigma_2$ (and $\Delta g \in \mathfrak{K}$). Thus $\Sigma_1 g = \Gamma_1$. □

LEMMA 2.6. *Assume* (1), (3) *and* (4).

(a) *If* $A \subseteq B \subseteq \Omega$, *and* $\Sigma = B \dot\cup \Gamma$ *is a disjoint union with* $\Sigma, A, \Gamma \in \mathfrak{K}$, *then also* $B \in \mathfrak{K}$.

(b) *If* $\Omega = \Sigma_1 \cup \Sigma_2$ *and* $\Sigma, \Sigma_1, \Sigma_2, \Sigma_1 \cap \Sigma_2 \in \mathfrak{K}$, *then either* $\Sigma_1 \cap \Sigma, \Sigma_1 \cap \Sigma^* \in \mathfrak{K}$ *or* $\Sigma_2 \cap \Sigma, \Sigma_2 \cap \Sigma^* \in \mathfrak{K}$.

Proof. (a) We write $\Omega = \Sigma \dot\cup \Sigma^* = B \dot\cup (\Gamma \dot\cup \Sigma^*)$ with $\Gamma, \Sigma^* \in \mathfrak{K}$ and note that $\Gamma \dot\cup \Sigma^* \subseteq A^* \in \mathfrak{K}$. From (3) it follows that $\Gamma \dot\cup \Sigma^* \in \mathfrak{K}$ and hence $B = (\Gamma \dot\cup \Sigma^*)^* \in \mathfrak{K}$.

(b) If $\Omega = \Sigma_1 \cup \Sigma_2$, then by (4) we have $\Sigma \cap \Sigma_1 \in \mathfrak{K}$ or $\Sigma \cap \Sigma_2 \in \mathfrak{K}$. Now we may assume that $\Sigma \cap \Sigma_1 \in \mathfrak{K}$ and apply (4) for Σ^* to obtain $\Sigma_1 \cap \Sigma^* \in \mathfrak{K}$ or $\Sigma_2 \cap \Sigma^* \in \mathfrak{K}$.

If $\Sigma_1 \cap \Sigma^* \in \mathfrak{K}$ then the lemma holds, so let $\Sigma_2 \cap \Sigma^* \in \mathfrak{K}$ and apply (4) for $\Omega = \Sigma \cup \Sigma^*$ and $\Sigma_1 \cap \Sigma_2 \in \mathfrak{K}$. It follows that either $\Sigma_1 \cap \Sigma_2 \cap \Sigma \in \mathfrak{K}$ or $\Sigma_1 \cap \Sigma_2 \cap \Sigma^* \in \mathfrak{K}$. In the first case $\Sigma_2 = (\Sigma_2 \cap \Sigma) \dot\cup (\Sigma_2 \cap \Sigma^*)$ and by (a) we have $\Sigma_2 \cap \Sigma \in \mathfrak{K}$. In the second case $\Sigma_1 = (\Sigma_1 \cap \Sigma) \dot\cup (\Sigma_1 \cap \Sigma^*)$ and by (a) it follows that $\Sigma_1 \cap \Sigma^* \in \mathfrak{K}$. □

LEMMA 2.7. *Assume* (1), (2), (3), (4), (5′) *and* (6′). *Let* $\Sigma_1, \Sigma_2 \in \mathfrak{K}$ *such that* $\Sigma_1 \cup \Sigma_2 = \Omega$ *and* $\Sigma_1 \cap \Sigma_2 \in \mathfrak{K}$ *and put* $H = G_{\{\Sigma_1\}} \cup G_{\{\Sigma_2\}}$. *Then* H^4 *is transitive on* $\mathfrak{K}$. *In particular, for any* $\Delta \in \mathfrak{K}$ *we have* $G_{\{\Delta\}} \subseteq H^9$.

Proof. Let $\Sigma \in \mathfrak{K}$ be any element. By Lemma 2.6(b) we may assume that $\Sigma_1 \cap \Sigma, \Sigma_1 \cap \Sigma^* \in \mathfrak{K}$. By (2) we obtain $\Sigma_1 \cap \Sigma_2 = \Sigma_0 \dot\cup \Sigma_0'$ in $\mathfrak{K}$. Clearly

$$\Sigma_1 = (\Sigma_1 \cap \Sigma) \dot\cup (\Sigma_1 \cap \Sigma^*) = \Sigma_0 \dot\cup (\Sigma_2^* \dot\cup \Sigma_0').$$

By (3) we have $\Sigma_2^* \dot\cup \Sigma_0' \in \mathfrak{K}$, and by Lemma 2.5 there is $f_1 \in G_{\{\Sigma_1\}}$ such that $(\Sigma_1 \cap \Sigma) f_1 = \Sigma_0$ and $\Sigma f_1 \in \mathfrak{K}$. Hence $(\Sigma_1 \cap \Sigma^*) f_1 = \Sigma_2^* \dot\cup \Sigma_0'$ and $\Sigma f_1 \subseteq \Sigma_2 \backslash \Sigma_0'$. Then $\Sigma_2 = \Sigma f_1 \dot\cup \Gamma$ with $\Sigma_0' \subseteq \Gamma$, so $\Gamma \in \mathfrak{K}$ by Lemma 2.6(a). Also $\Sigma_2 = \Sigma_0 \dot\cup (\Sigma_0' \dot\cup \Sigma_1^*)$ and $\Sigma_0' \dot\cup \Sigma_1^* \in \mathfrak{K}$ by (3). By Lemma 2.5 there is $f_2 \in G_{\{\Sigma_2\}}$ with $\Sigma f_1 f_2 = \Sigma_0$. Thus $\Sigma f_1 f_2 = \Sigma_0$ and $f_1 f_2 \in G_{\{\Sigma_1\}} \cdot G_{\{\Sigma_2\}} \subseteq H^2$. Since Σ was arbitrary, H^4 is transitive on $\mathfrak{K}$.

For the final statement, let $\Delta \in \mathfrak{K}$. As shown, there is $h \in H^4$ with $\Sigma_1 h = \Delta$, so $G_{\{\Delta\}} = {G_{\{\Sigma_1\}}}^h \subseteq H^9$ as claimed. □

The following result describes for a class of permutation groups (G, Ω) a sufficient condition for uncountable cofinality. Moreover, it presents a single generating set whose properties determine whether G has the Bergman property.

THEOREM 2.8. *Let* (G, Ω) *be an infinite permutation group and let* $\mathfrak{K} \subseteq \mathfrak{P}(\Omega)$. *Now assume one of the following two cases.*

(I) *We have* (1), (2), (5), (6), (7), (8), *and for some finite subset* $F \subseteq \mathfrak{K}$, $E = \bigcup_{\Delta \in F} G_{\{\Delta\}}$ *generates* G.

(II) *We have* (1), (2), (3), (4), (5′), (6′), (7), (8), *and* $E = \bigcup_{\Sigma \in \mathfrak{K}} G_{\{\Sigma\}}$ *generates* G.

Then G *has uncountable cofinality. Moreover,* G *has the Bergman property if and only if* $G = E^m$ *for some* $m \in \omega$.

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Proof. Let $(H_i)_{i\in\omega}$ be an ascending chain of subsets of G such that for each $i \in \omega$ we have $H_i = H_i^{-1}$ and $H_i \cdot H_i \subseteq H_j$ for some $j \in \omega$ and $G = \bigcup_{i\in\omega} H_i$. We first show that $E \subseteq H_j$ for some $j \in \omega$. Choose Γ_n $(n \in \omega)$ in $\mathfrak{K}$ as in (7). Suppose that no Γ_n is full for H_n $(n \in \omega)$. Then for each $n \in \omega$, choose $f_n \in G_{\{\Gamma_n\}}$ which is not induced on Γ_n by any element of H_n. By (7) we have $f = \bigcup_{n\in\omega}(f_n \restriction \Gamma_n) \cup \mathrm{id}_{\Omega\setminus\cup_{n\in\omega}\Gamma_n} \in \mathrm{G}$. Thus $f \in H_n$ for some $n \in \omega$ and $f_n \restriction \Gamma_n = f \restriction \Gamma_n$, a contradiction. Hence for some $n \in \omega$, Γ_n is full for H_n, and by Lemma 2.3 we get $G_{\{\Gamma_n\}} \subseteq H_{j_1}$ for some $j_1 \in \omega$.

In case (I) let $F = \{\Delta_1, \dots, \Delta_k\}$. Choose $g_i \in G$ such that $\Gamma_n g_i = \Delta_i$ $(i = 1, \dots, k)$, so $G_{\{\Delta_i\}} = G^{g_i}_{\{\Gamma_n\}} \subseteq H^{g_i}_{j_1} \subseteq H_j$ for some $j \geqslant j_1$ and for all i. Thus $E \subseteq H_j$.

In case (II), by Lemma 2.4 there is $\Sigma' \in \mathfrak{K}$ with $\Gamma_n \cup \Sigma' = \Omega$, $\Gamma_n \cap \Sigma' \in \mathfrak{K}$ and $G_{\{\Gamma_n\}}, G_{\{\Sigma'\}} \subseteq H_{j_2}$ for some $j_2 \geqslant j_1$. Choose $j \in \omega$ with $H^9_{j_2} \subseteq H_j$. Then by Lemma 2.7 also $E \subseteq H_j$.

Now assume that each H_i is a subgroup of G. Then by the above $G = \langle E\rangle = H_j$ showing that G has uncountable cofinality.

For the final statement assume that $G = E^m$ for some $m \in \omega$. Then $H^m_j \subseteq H_{j'}$ for some j' showing that $G = E^m = H^m_j = H_{j'}$. Thus G has uncountable strong cofinality. Therefore by Proposition 2.2(i) G satisfies the Bergman property. Trivially, if $G \neq E^m$ for each m, then G does not satisfy the Bergman condition. □

To obtain a sufficient condition for uncountable strong cofinality we introduce four additional conditions.

(9) If $\Sigma \in \mathfrak{K}$, $f \in G$, $\Sigma \subseteq \mathrm{supp}(f)$, then there is $\Gamma \in \mathfrak{K}$ with $\Gamma \subseteq \Sigma$ and $\Gamma \cap \Gamma f = \varnothing$.

(10) There is $n \in \omega$ such that for all $g \in G$ there are $g_i \in G$ and $\Sigma_i \in \mathfrak{K}$ with $\Sigma_i \subseteq \mathrm{supp}(g_i)$ (for $i = 1, \dots, j$ and $j \leqslant n$) with $g = g_1 \dots g_j$.

(11) If $\Gamma, \Delta \in \mathfrak{K}$ and $g \in G$ with $\Delta \cap \Delta g = \varnothing$ and $\Delta \cup \Delta g \subseteq \Gamma$, then there is $h \in G_{\{\Gamma\}}$ with $\Delta gh = \Delta$.

(12) *The orbit condition:* If $\Delta \in \mathfrak{K}$ and $g \in G$ then $\Delta g \in \mathfrak{K}$.

Clearly, if condition (12) holds, then (5) implies (5′), and (6) implies (6′).

LEMMA 2.9. *Assume* (1), (2), (9), (10) *and either* (11), *or* (3), (5), (6) *and* (12). *Let* $E = \bigcup_{\Sigma\in\mathfrak{K}} G_{\{\Sigma\}}$ *and* n *be as in* (10). *Then* $G = E^{2n}$.

Proof. Let $g \in G$. By (10) we may assume that $\Sigma \subseteq \mathrm{supp}(g)$ for some $\Sigma \in \mathfrak{K}$; we must then show that $g \in E^2$. By (9) there is $\Gamma \in \mathfrak{K}$ with $\Gamma \subseteq \Sigma$ and $\Gamma \cap \Gamma g = \varnothing$. By (2) split $\Gamma = \Delta_1 \dot\cup \Delta_2$ in $\mathfrak{K}$. Then $\Delta_1 \cap \Delta_1 g = \varnothing$ and $\Delta_1 \dot\cup \Delta_1 g \subseteq \Delta_2^* \in \mathfrak{K}$. In case of (11) there is $h \in G_{\{\Delta_2\}}$ with $\Delta_1 gh = \Delta_1$, so $g = gh \cdot h^{-1} \in G_{\{\Delta_1\}} \cdot G_{\{\Delta_2\}} \subseteq E^2$. In case of (12) we have $\Delta_1 g \in \mathfrak{K}$. By (3) we have $\Gamma' = \Delta_1 \dot\cup \Delta_1 g \in \mathfrak{K}$. By Lemma 2.5 there is $h \in G_{\{\Gamma'\}}$ with $\Delta_1 gh = \Delta_1$. Proceeding as before we obtain $g \in E^2$. □

The following provides a sufficient condition for uncountable strong cofinality.

COROLLARY 2.10. *Let* (G, Ω) *be an infinite permutation group and let* $\mathfrak{K} \subseteq \mathfrak{P}(\Omega)$. *Assume* (1)–(10) *and either* (5′), (6′), (11), *or else* (12). *Then* G *has uncountable strong cofinality.*

Proof. Put $E = \bigcup_{\Sigma\in\mathfrak{K}} G_{\{\Sigma\}}$. By Lemma 2.9 we have $G = E^{2n}$ for some $n \in \omega$. By the orbit condition (12), axioms (5) and (6) imply (5′) and (6′). Now the result is immediate by Theorem 2.8. □

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Finally, we note that Theorem 2.8 can be easily refined to yield a more precise lower bound for the cofinality of G.

REMARK 2.11. Let κ be any infinite regular cardinal. If in condition (7) we replace the ordinal ω by κ, we obtain the following condition.

(7_κ) There are pairwise disjoint $\Gamma_i \in \mathfrak{K}$ $(i \in \kappa)$ such that for all $f_i \in G_{\{\Gamma_i\}}$ $(i \in \kappa)$ it follows that $\bigcup_{i\in\kappa}(f_i \restriction \Gamma_i) \dot\cup \mathrm{id}_{\Omega\setminus\cup_{i\in\kappa}\Gamma_i} \in \mathrm{G}$.

Clearly, if $(G, \Omega), \mathfrak{K}$ and E satisfy the assumptions of Theorem 2.8 with (7) replaced by (7_κ), then $\mathrm{cf(G)} > \kappa$. Moreover, if $G = E^m$ for some $m \in \omega$, then G has the Bergman property.

3. *Applications*

In this section we will apply Theorem 2.8 and Corollary 2.10 to specific groups. First we consider symmetric groups. We employ the following result.

PROPOSITION 3.1 (Droste and Göbel [**4**]). *Let Ω be any infinite set and $g \in \mathrm{Sym}(\Omega)$ such that $|\mathrm{supp}(g)| = |\Omega|$. Then any element of $\mathrm{Sym}(\Omega)$ is a product of four conjugates of g, that is $\mathrm{Sym}(\Omega) = (g^{\mathrm{Sym}(\Omega)})^4$.*

If Ω is countable, this result was obtained by Bertram [**3**] (however note that his argument was corrected and completed in [**4**, pp. 286, 288]).

COROLLARY 3.2 (Macpherson and Neumann [**10**], Bergman [**2**]). *Let Ω be any infinite set. Then $\mathrm{cf}(\mathrm{Sym}(\Omega)) > |\Omega|$ and $\mathrm{Sym}(\Omega)$ has the Bergman property.*

Proof. Let $\kappa = |\Omega|$ and $G = \mathrm{Sym}(\Omega)$. By Proposition 2.2(i) it suffices to show that $\mathrm{scf}(G) > \kappa$. Let $\mathfrak{K}$ comprise all moieties of Ω. As noted in Section 2, (G, Ω) and $\mathfrak{K}$ satisfy conditions (1)–(6), (7_κ) and clearly also conditions (9), (10) and (12). Condition (8) is immediate by Proposition 3.1. Now apply Corollary 2.10 and Remark 2.11. □

Next we consider particular topological spaces and their homeomorphism groups. Let $\mathbb{Q}$ be the space of all rational numbers, $\mathbb{I}$ be the space of irrational numbers and $\mathfrak{C}$ the Cantor space, each equipped with the natural topology. We strengthen the main result of [**5**], where we showed that the homeomorphism groups of these spaces have uncountable cofinality.

THEOREM 3.3. *Let G be the homeomorphism groups of $\mathbb{Q}, \mathbb{I}$ or $\mathfrak{C}$. Then G has uncountable strong cofinality, that is it satisfies the Bergman property and has uncountable cofinality.*

Proof. Let Ω be $\mathbb{Q}$, $\mathbb{I}$ or $\mathfrak{C}$ and let $\mathfrak{K}$ comprise all non-empty, proper clopen subsets of Ω. Clearly conditions (1)–(6) and (12) are satisfied. For condition (7), choose pairwise disjoint clopen subsets Γ_i with diameter $d(\Gamma_i)$ converging to 0 $(i \to \infty)$. By Truss [**18**, Theorem 2.1], if $1 \neq g \in G$, then any element of G is a product of four conjugates of g or g^{-1}. Since any $\Sigma \in \mathfrak{K}$ is homeomorphic to Ω, this implies condition (8). For conditions (9) and (10) observe that if $1 \neq g \in G$, by continuity there is $\Gamma \in \mathfrak{K}$ such that $\Gamma \cap \Gamma g = \varnothing$, in particular $\Gamma \subseteq \mathrm{supp}(g)$. Now apply Corollary 2.10. □

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Next we turn to measurable spaces. We consider $\mathbb{R}$ equipped with the Borel σ-field $B_{\mathbb{R}}$. A permutation g of $\mathbb{R}$ is called a *Borel automorphism* of $\mathbb{R}$, if $A \in B_{\mathbb{R}}$ if and only if $Ag \in B_{\mathbb{R}}$ for any subset $A \subseteq \mathbb{R}$. Let $\mathrm{Aut}(\mathbb{R}, B_{\mathbb{R}})$ denote the group of all Borel automorphisms of $\mathbb{R}$.

THEOREM 3.4. *The group* $\mathrm{Aut}(\mathbb{R}, B_{\mathbb{R}})$ *has uncountable strong cofinality.*

Proof. We call a subset $A \subseteq \mathbb{R}$ a Borel moiety, if $A \in B_{\mathbb{R}}$ and A is a moiety in $\mathbb{R}$ (as defined above). Let $\mathfrak{K}$ comprise all Borel moieties, and put $G = \mathrm{Aut}(\mathbb{R}, B_{\mathbb{R}})$. By a well-known result of Kuratowski [**9**, p. 451], any uncountable Borel subset of $\mathbb{R}$ has size continuum and is Borel-isomorphic to $\mathbb{R}$. It follows that $(G, \mathbb{R})$ and $\mathfrak{K}$ satisfy conditions (1)–(7), (10) and (12). Condition (8) follows from Shortt [**15**, Lemma 2.5 and its proof] (we may choose $n = 12$). Moreover, condition (9) is immediate by [**15**, Lemma 2.3]. Now apply Corollary 2.10. □

Theorems 3.3 and 3.4 yield automorphism groups satisfying (3) and (4) in Bergman [**2**, Section 3]. For later use we note the following.

LEMMA 3.5. *Let* $G = \prod_{i \in I} \mathrm{Sym}(\omega)$ *with any index set* I. *Then* $\mathrm{scf}(G) > \omega$.

Proof. Let $\Omega = \dot{\bigcup}_{i \in I} \Omega_i$ with $|\Omega_i| = \aleph_0$ for each $i \in I$, and put $K = \{g \in \mathrm{Sym}(\Omega) : \Omega_i g = \Omega_i \text{ for each } i \in I\}$. Then $G \cong K$. Let $\mathfrak{K}$ comprise all subsets $\Sigma \subseteq \Omega$ such that $\Sigma \cap \Omega_i$ is a moiety in Ω_i for each $i \in I$. Then (K, Ω) and $\mathfrak{K}$ satisfy conditions (1)–(3), (5)–(7), (9), (10) and (12) (but not (4), if $|I| \geqslant 2$). Applying Proposition 3.1 to each component Ω_i, it follows that condition (8) also holds. Moreover, in each Ω_i we may choose moieties Δ_i, Γ_i such that $\Omega_i = \Delta_i \cup \Gamma_i$ and $\Delta_i \cap \Gamma_i$ is a moiety. Let $E_i = \mathrm{Sym}(\Omega_i)_{\{\Delta_i\}} \cup \mathrm{Sym}(\Omega_i)_{\{\Gamma_i\}}$. It is easy to see that there is $g \in E_i^4$ interchanging Δ_i and Γ_i. By the proof of [**10**, Lemma 2.4] $\mathrm{Sym}(\Omega_i) = E_i^4$ (here we obtain $\mathrm{Sym}(\Omega_i) = E_i^{36}$ by Lemmas 2.7 and 2.9). Putting these moieties together, we can ensure condition (I) of Theorem 2.8 to obtain $\mathrm{scf}(G) > \omega$. □

Next we exhibit an example of a group G with uncountable cofinality but countable strong cofinality, that is without the Bergman property. Let

$$\mathrm{BSym}(\mathbb{Q}) = \{g \in \mathrm{Sym}(\mathbb{Q}) : \exists\, k \in \omega\ \forall\, x \in \mathbb{Q}\ |x - xg| \leqslant k\},$$

be the *bounded permutation group on* $\mathbb{Q}$.

THEOREM 3.6. $\mathrm{BSym}(\mathbb{Q})$ *has uncountable cofinality but not the Bergman property.*

Proof. Let $E = \{g \in \mathrm{BSym}(\mathbb{Q}) : \forall\, x \in \mathbb{Q}\ |x - xg| \leqslant 1\}$.

Claim 1: $\mathrm{BSym}(\mathbb{Q}) = \langle E \rangle$.

Indeed, let $g \in \mathrm{BSym}(\mathbb{Q})$ and $k \in \mathbb{Q}$ such that $|x - xg| \leqslant k$ for all $x \in \mathbb{Q}$. We show that $g \cdot h_1 \cdot h_2 = h_3$ for some $h_i \in \mathrm{BSym}(\mathbb{Q})$ such that $|x - xh_i| \leqslant \frac{2}{3}k$ for each $x \in \mathbb{Q}$ $(i = 1, 2, 3)$. Then $g = h_3 \cdot h_2^{-1} \cdot h_1^{-1}$, and induction implies $g \in \langle E \rangle$.

Enumerate $\mathbb{Q} = \{q_i : i \in \omega\}$. Using this enumeration, we define the permutations h_1, h_2 inductively as unions of finite approximations. At each stage, we ensure that the approximations (also denoted by h_1, h_2) are injective functions on their finite

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domains, that $|xgh_1h_2 - x| \leqslant \frac{2}{3}k$ for each $x \in (\mathrm{dom}(h_1))g^{-1}$ and that the domain of h_2 coincides with the range of h_1. Fix a stage, and let F be the union of the finite domains and ranges of h_1 and h_2 determined so far; then F is finite.

Assume that we want to define q_ih_1. Let $x = q_ig^{-1}$. Choose $x_1 \in \mathbb{Q}\backslash F$ such that $|x_1 - q_i|, |x - x_1| \leqslant \frac{2}{3}k$. Then put $q_ih_1 = x_1$ and $x_1h_2 = x_1$. (We could also ensure that $x_1h_2 \neq x_1$.) Clearly, h_1, h_2 remain injective and $|xgh_1h_2 - x| = |x_1 - x| \leqslant \frac{2}{3}k$. Working through the enumeration of $\mathbb{Q}$ ensures that the domain of h_1 will be $\mathbb{Q}$.

Next we wish to ensure that h_1, h_2 will be surjective. First consider h_1 and $q_i \in \mathbb{Q}$ so far not belonging to the range of h_1. Choose $x_1 \in \mathbb{Q}\backslash F$ such that $|q_i - x_1| < \frac{1}{3}k$. Put $y = x_1g^{-1}$. Then $|x_1 - y| \leqslant k$ and $|q_i - y| < \frac{4}{3}k$. Hence there is $x_2 \in \mathbb{Q}\backslash F$ such that $|q_i - x_2|, |x_2 - y| < \frac{2}{3}k$. Now put $x_1h_1 = q_i$ and $q_ih_2 = x_2$. Then $|ygh_1h_2 - y| = |x_2 - y| < \frac{2}{3}k$ as required.

Secondly, consider h_2 and $q_i \in \mathbb{Q}$ so far not belonging to the range of h_2. Choose $x_1 \in \mathbb{Q}\backslash Fg^{-1}$ with $|q_i - x_1| < \frac{1}{3}k$, and put $y = x_1g$. Then $y \notin F$. Again $|y - x_1| \leqslant k$ and $|y - q_i| < \frac{4}{3}k$. Thus there is $x_2 \in \mathbb{Q}\backslash F$ such that $|x_2 - y|, |q_i - x_2| < \frac{2}{3}k$. Then put $yh_1 = x_2$ and $x_2h_2 = q_i$. We get $|x_1gh_1h_2 - x_1| = |q_i - x_1| < \frac{1}{3}k$.

At each stage, any element getting into the range of h_1 by our construction also becomes a member of the domain of h_2. Since h_1 will be surjective, this implies that the domain of h_2 will also be all of $\mathbb{Q}$. Hence we obtain $h_1, h_2 \in \mathrm{BSym}(\mathbb{Q})$ as required, proving our claim.

Clearly, $\mathrm{BSym}(\mathbb{Q}) \neq E^n$ for each $n \in \omega$, so $\mathrm{BSym}(\mathbb{Q})$ does not have the Bergman property. Now let $A = \{g \in \mathrm{BSym}(\mathbb{Q}) : [4i, 4i+4)g = [4i, 4i+4) \text{ for each } i \in \mathbb{Z}\}$ and $B = \{g \in \mathrm{BSym}(\mathbb{Q}) : [4i+2, 4i+6)g = [4i+2, 4i+6) \text{ for each } i \in \mathbb{Z}\}$.

Claim 2: $E \subseteq A \cdot B \cdot A$.

Let $g \in E$. We claim that there are $h_1 \in A$ and $h_2 \in B$ such that $g \cdot h_1 \cdot h_2 \in A$. For this, we enumerate $\mathbb{Q}$ and define h_1, h_2 inductively, just as in the proof of Claim 1. We ensure at each stage for the partial mappings h_1, h_2 that if $x \in [4i, 4i+4)$ and xh_1 or xgh_1h_2 is defined, then xh_1 respectively xgh_1h_2 belongs to $[4i, 4i+4)$, and if $x \in [4i+2, 4i+6)$ and xh_2 is defined, then xh_2 also belongs to $[4i+2, 4i+6)$; surjectivity of h_1, h_2 is ensured analogously as above. This proves $g \in A \cdot B \cdot A$.

Now let $(H_i)_{i\in\omega}$ be a chain of subgroups of $\mathrm{BSym}(\mathbb{Q})$ such that $\mathrm{BSym}(\mathbb{Q}) = \bigcup_{i\in\omega} H_i$. Note that $A \cong \prod_{i\in\omega} \mathrm{Sym}(\omega) \cong B$, so $\mathrm{cf(A)} = \mathrm{cf(B)} > \omega$ by Lemma 3.5. By $A = \bigcup_{i\in\omega}(A \cap H_i)$ and $B = \bigcup_{i\in\omega}(B \cap H_i)$ it follows that $A, B \subseteq H_i$ for some $i \in \omega$. Claims 1 and 2 show that $\mathrm{BSym}(\mathbb{Q}) = \langle A \cup B\rangle$. Since H_i is a subgroup, we obtain $\mathrm{BSym}(\mathbb{Q}) = H_i$. Thus $\mathrm{BSym}(\mathbb{Q})$ has uncountable cofinality. □

In view of this result we would like to pose the following problem.

PROBLEM 3.7. Find a group G with countable cofinality and the Bergman property.

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