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Sequences with bounded logarithmic discrepancy

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1. *Introduction*

Let $\{\alpha\}$ denote the fractional part of the real number α. Write $\chi(x, y) = 1$ for $\{x\} < y$ and $\chi(x, y) = 0$ otherwise. A real sequence (x_n) is *uniformly distributed* (mod 1) if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi(x_n, y) = y \quad (0 < y < 1). \tag{1·1}$$

It is a consequence of (1·1) that $D_N = o(N)$,

where $$D_N = \sup_{0<y<1} \left| \sum_{n=1}^{N} \chi(x_n, y) - Ny \right|$$

is the *discrepancy* of the sequence (x_n).

The sequence $(\log n)$ has long been known not to be uniformly distributed (mod 1). Tsuji [7] considered the weighted sums

$$S_N(y) = \sum_{n=1}^{N} \frac{1}{n} \chi(x_n, y)$$

with this particular case in mind. A sequence with

$$\lim_{N\to\infty} \frac{1}{\log N} S_N(y) = y \quad (0 < y < 1)$$

is said to be *logarithmically uniformly distributed* (mod 1). Tsuji established this property for the sequence $(\log n)$, and Hlawka [3] extended this to the case

$$x_n = \alpha n + \beta \log n, \tag{1·2}$$

where α, β are real numbers ($\beta \neq 0$).

The logarithmic discrepancy of (x_n) is

$$D(N) = \sup_{0<y<1} \left| S_N(y) - y \sum_{n=1}^{N} \frac{1}{n} \right|.$$

In contrast to the discrepancy D_N (which must exceed $\frac{1}{100} \log N$ infinitely often: see [4]), the logarithmic discrepancy may be bounded. Tichy and Turnwald [5] showed that

$$D(N) < C_1(\alpha, \beta)$$

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for the sequence (1·2) whenever α is not a Liouville number. These authors conjectured that the last hypothesis is superfluous, and in the present paper we prove this conjecture. This supersedes the bound

$$D(N) < C_2(\beta)(\log\log N)^2$$

given in [5].

THEOREM 1. *Let α, β be real numbers with $\beta \neq 0$. The sequence* (1·2) *has bounded logarithmic discrepancy. In fact,*

$$D(N) < C_3(\beta). \tag{1·3}$$

We were naturally curious to discover what properties of the function $f(x) = \alpha x + \beta \log x$ were crucial in proving bounded discrepancy of $(f(n))$. This led us to Theorem 2 (below), which includes Theorem 1 as a special case.

A real valued function f defined on $[1, \infty)$ is said to be of class H if there are real numbers $1 = x_0 < x_1 < \ldots < x_H$ such that f is monotonic in each of the intervals $[x_{j-1}, x_j]$ $(j = 1, \ldots, H)$ and $[x_H, \infty)$.

THEOREM 2. *Let f be a real valued twice differentiable function on $[1, \infty)$. Suppose that there are positive constants c, K, δ and H with the following properties*: (i) *$x(f'(x) - \lambda)$ is of class H for every real λ*; (ii) *either*

$$cx^{-2} \leqslant f''(x) \leqslant Kx^{-1-\delta} \quad \text{or} \quad cx^{-2} \leqslant -f''(x) \leqslant Kx^{-1-\delta}$$

for $x \geqslant 1$. Then the logarithmic discrepancy of the sequence $(f(n))$ satisfies

$$D(N) < C_4(c, K, \delta, H). \tag{1·4}$$

We remark that $f(x) = \alpha x + \beta \log x$ is extremal in the sense that $f''(x) = -\beta x^{-2}$. (Of course, $x(f'(x) - \lambda)$ is linear.)

One could presumably prove a more complicated result of this type, in which a higher derivative of f than the second decreases at a suitable pace.

Once we had established (1·3), it seemed natural to ask whether the subsequence

$$\alpha p + \beta \log p \quad (p \text{ prime}) \tag{1·5}$$

had the corresponding property. This turns out to be the case. Not surprisingly, the proof is much more difficult than that of (1·3) or (1·4).

THEOREM 3. *Let α, β be real numbers with $\beta \neq 0$. The sequence* (1·5) *has bounded logarithmic discrepancy. In fact,*

$$D(N) < C_5(\beta).$$

Apart from the constants, all our results are best possible, because the logarithmic discrepancy of any sequence is at least $\frac{1}{2}$. This follows from the observation that

$$S_N(y) - y \sum_{n=1}^{N} \frac{1}{n}$$

has a jump of at least 1 at $y = \{x_1\}$.

2. *A weighted Erdös–Turán inequality*

A well known theorem of Erdös and Turán states that

$$D_N \leqslant \frac{N}{m+1} + 3 \sum_{h=1}^{m} \frac{1}{h} \left| \sum_{n=1}^{N} e(hx_n) \right|. \tag{2·1}$$

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A weighted version of this is used by Tichy and Turnwald [5]:

$$D(N) \leqslant \frac{1}{m+1}\sum_{n=1}^{N}\frac{1}{n}+3\sum_{h=1}^{m}\frac{1}{h}\left|\sum_{n=1}^{N}\frac{e(hx_n)}{n}\right|. \tag{2·2}$$

In (2·1) or (2·2), m may be any natural number.

Each of these bounds is a simple consequence of upper and lower bounds for $\chi(x,y)$;

$$L_m(x,y) \leqslant \chi(x,y) \leqslant U_m(x,y) \quad (m \geqslant 1; 0 < y < 1, x \text{ real}).$$

Here L_m and U_m are real valued trigonometric polynomials,

$$L_m(x,y) = \sum_{|h|\leqslant m} t(m,y,h)\,e(hx), \quad U_m(x,y) = \sum_{|h|\leqslant m} u(m,y,h)\,e(hx)$$

satisfying $$t(m,y,0) = y-\frac{1}{m+1}, \quad u(m,y,0) = y+\frac{1}{m+1}. \tag{2·3}$$

The construction of L_m and U_m is given in [1], lemma 2·7 and it is an easy deduction that

$$|t(m,y,h)| \leqslant \frac{3}{2|h|}, \quad |u(m,y,h)| \leqslant \frac{3}{2|h|} \quad (h \neq 0) \tag{2·4}$$

(cf. [1], (2·20)). We also observe that

$$|t(m,y,h)-t(n,y,h)| \leqslant \frac{2}{m} \quad \text{for} \quad n > m,$$

$$|u(m,y,h)-u(n,y,h)| \leqslant \frac{2}{m} \quad \text{for} \quad n > m. \tag{2·5}$$

To see this, consider (2·5), for instance. We have

$$\begin{aligned}|u(m,y,h)-u(n,y,h)| &= \left|\int_0^1 (U_m(x,y)-U_n(x,y))\,e(-hx)\,dx\right| \\ &\leqslant \int_0^1 |U_m(x,y)-U_n(x,y)|\,dx \\ &\leqslant \int_0^1 (U_m(x,y)-\chi(x,y))\,dx+\int_0^1 (U_n(x,y)-\chi(x,y))\,dx \\ &= \frac{1}{m+1}+\frac{1}{n+1} \leqslant \frac{2}{m}.\end{aligned}$$

The proof of (2·2) is as follows. We have

$$\chi(x_n,y) \leqslant U_m(x_n,y), \tag{2·6}$$

$$\sum_{n=1}^{N}\frac{\chi(x_n,y)}{n} \leqslant \sum_{n=1}^{N}\frac{U_m(x_n,y)}{n} = u(m,y,0)\sum_{n=1}^{N}\frac{1}{n}+\sum_{0<|h|\leqslant m} u(m,y,h)\sum_{n=1}^{N}\frac{e(hx_n)}{n}.$$

Using (2·3),

$$\sum_{n=1}^{N}\left(\frac{\chi(x_n,y)}{n}-\frac{y}{n}\right) \leqslant \frac{1}{m+1}\sum_{n=1}^{N}\frac{1}{n}+\sum_{0<|h|\leqslant m}\left|u(m,y,h)\sum_{n=1}^{N}\frac{e(hx_n)}{n}\right|.$$

There is an obvious corresponding lower bound, and (2·2) now follows on using (2·4).

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The novelty of our present approach lies in allowing m to depend on n in (2·6). Let us write $m = m(n)$. Then

$$\sum_{n=1}^{N} \frac{\chi(x_n, y)}{n} \leqslant \sum_{n=1}^{N} \frac{1}{n} U_{m(n)}(x_n, y)$$

$$= \sum_{n=1}^{N} \frac{u(m(n), y, 0)}{n} + 2\mathrm{Re} \sum_{n=1}^{N} \sum_{0<h\leqslant m(n)} u(m(n), y, h) \frac{e(hx_n)}{n},$$

since $$\overline{u(m, y, h)\, e(hx_n)} = u(m, y, -h)\, e(-hx_n).$$

Again applying (2·4),

$$\sum_{n=1}^{N} \left(\frac{\chi(x_n, y)}{n} - \frac{y}{n} \right) \leqslant \sum_{n=1}^{N} \frac{1}{n(m(n)+1)} + 2 \sum_{h>0} \left| \sum_{\substack{n\leqslant N \\ m(n)\geqslant h}} \frac{u(m(n), y, h)\, e(hx_n)}{n} \right|. \tag{2·7}$$

Although this is a kind of weighted Erdös–Turán bound, it is not obvious how to treat the exponential sum occurring in it. The following inequality is more manageable.

LEMMA 1. *Let $0 < \delta \leqslant 1$. Then the logarithmic discrepancy of the sequence (x_n) satisfies*

$$D(N) < C_6(\delta) + 24 \sum_{1\leqslant h\leqslant N^{\delta}} \frac{1}{h} \max_{A\geqslant h^{1/\delta}} \left| \sum_{n=A}^{N} \frac{1}{n} e(hx_n) \right|.$$

Proof. We apply (2·7) with

$$m(n) = [2^{k\delta}] \quad \text{for} \quad 2^k \leqslant n < 2^{k+1} \quad (k = 0, 1, \ldots).$$

This choice of $m(n)$ is convenient for the application of partial summation to the sum

$$S_h = \sum_{\substack{n\leqslant N \\ m(n)\geqslant h}} \frac{u(n, h)\, e(hx_n)}{n} = \sum_{A_h \leqslant n \leqslant N} \frac{u(n, h)\, e(hx_n)}{n}.$$

Here we have used the abbreviations

$$u(n, h) = u(m(n), y, h), \quad A_h = \min\{2^k : 2^{k\delta} \geqslant h\}.$$

We observe that

$$u(n, h) = u(n+1, h) \quad \text{for} \quad 2^k \leqslant n < 2^{k+1} - 1. \tag{2·8}$$

Thus $$S_h = \sum_{n=A_h}^{N} (u(n, h) - u(n+1, h)) \sum_{r=A_h}^{n} \frac{1}{r} e(hx_r) + u(N+1, h) \sum_{r=A_h}^{N} \frac{1}{r} e(hx_r).$$

Clearly $$\sum_{n=A_h}^{N} |u(n, h) - u(n+1, h)| + |u(N+1, h)| \leqslant \sum_{\substack{A_h \leqslant n \leqslant N \\ n = 2^{k+1}-1}} \frac{2}{n} + \frac{3}{2h} \tag{2·9}$$

from (2·5), (2·8). The right-hand side of (2·9) is bounded above by

$$\frac{4}{A_h} + \frac{3}{2h} < \frac{6}{h}.$$

It follows that

$$|S_h| \leqslant \frac{6}{h} \max_{A_h \leqslant n \leqslant N} \left| \sum_{r=A_h}^{n} \frac{1}{r} e(hx_r) \right| \leqslant \frac{12}{h} \max_{A\geqslant A_h} \left| \sum_{n=A}^{N} \frac{1}{n} e(hx_n) \right|.$$

Of course, $S_h = 0$ for $h > N^{\delta}$.

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Inserting this into (2·7), we find that

$$\sum_{n=1}^{N}\left(\frac{\chi(x_n,y)}{n}-\frac{y}{n}\right)\leqslant\sum_{n=1}^{N}\frac{1}{n(m(n)+1)}+24\sum_{1\leqslant h\leqslant N^{\delta}}h^{-1}\max_{A\geqslant N^{1/\delta}}\left|\sum_{n=A}^{N}\frac{e(hx_n)}{n}\right|$$

$$\leqslant 2^{\delta}\sum_{n\geqslant 1}n^{-1-\delta}+24\sum_{1\leqslant h\leqslant N^{\delta}}h^{-1}\max_{A\geqslant N^{1/\delta}}\left|\sum_{n=A}^{N}\frac{e(hx_n)}{n}\right|.$$

The corresponding lower bound is proved in the same way. This completes the proof of Lemma 1.

3. *Bounds for weighted exponential sums*

LEMMA 2. *Suppose that f satisfies the hypotheses of Theorem 2. Let h and ν be integers with $h\geqslant 1$. Let $B>A\geqslant 1$. Then*

$$\left|\int_A^B\frac{e(hf(x)-\nu x)}{x}dx\right|<C_7(c,H)\,h^{-\frac{1}{2}}.$$

Proof. The hypotheses of the theorem are satisfied if f is replaced by $-f$. Since

$$\int_A^B x^{-1}e(-hf(x)+\nu x)\,dx=\overline{\int_A^B x^{-1}e(hf(x)-\nu x)\,dx},$$

we assume henceforth that $f''(x)\geqslant cx^{-2}$.

Let $g(x)=f(x)-\nu x/h$. Then $[A,B]$ may be partitioned into at most $H+1$ intervals in each of which $xg'(x)$ is monotonic. It therefore suffices to show that

$$\left|\int_A^B x^{-1}e(hg(x))\,dx\right|<C_8(c)\,h^{-\frac{1}{2}} \tag{3·1}$$

under the hypotheses that

$$g''(x)\geqslant cx^{-2}, \tag{3·2}$$

$$xg'(x)\text{ is monotonic in }[A,B]. \tag{3·3}$$

Since g' is strictly increasing it has at most one zero in $[1,\infty)$.

Suppose first that g' has a zero in $[A,B]$, at a, say. Let

$$a'=\max(A,\tfrac{1}{2}a),\quad a''=\min(2a,B),$$

$$I_1=[A,a'),\quad I_2=[a',a''],\quad I_3=(a'',B].$$

Suppose that I_1 is non-empty. Then

$$\int_{I_1}\frac{e(hg(x))\,dx}{x}=\int_A^{a'}\frac{d(e(hg(x)))}{2\pi ihg'(x)\,x}=\left[\frac{e(hg(x))}{2\pi ihg'(x)\,x}\right]_A^{a'}-\int_A^{a'}\frac{e(hg(x))\,d}{2\pi ih}\left(\frac{1}{xg'(x)}\right)$$

with

$$g'(x)=-\int_x^a g''(t)\,dt\leqslant-\int_x^{2x}\frac{c}{t^2}dt=-\frac{c}{2x}.$$

Thus $xg'(x)\leqslant-\frac{1}{2}c$. From (3·3), we have

$$\int_A^{a'}\left|d\left(\frac{1}{xg'(x)}\right)\right|<\frac{2}{c}.$$

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In conjunction with the above integration by parts, this yields

$$\left|\int_A^{a'} \frac{e(hg(x))}{x}dx\right| < \frac{4}{\pi ch}. \tag{3·4}$$

By a similar method we get

$$\left|\int_{a''}^{B} \frac{e(hg(x))}{x}dx\right| < \frac{4}{\pi ch}. \tag{3·5}$$

For y in I_2, we define $$J(y) = \int_{a'}^{y} e(hg(x))\,dx.$$

Then $$|J(y)| \leqslant \frac{8y}{h^{\frac{1}{2}}c^{\frac{1}{2}}}$$

by (3·2) and lemma 4·4 of [6]. Thus

$$\int_{I_2} \frac{e(hg(x))}{x}dx = \left[\frac{J(y)}{y}\right]_{a'}^{a''} + \int_{a'}^{a''} \frac{J(y)}{y^2}dy$$

has modulus bounded above by

$$\frac{16}{h^{\frac{1}{2}}c^{\frac{1}{2}}} + \frac{8}{h^{\frac{1}{2}}c^{\frac{1}{2}}}\int_{\frac{1}{2}a}^{2a} \frac{1}{y}dy < \frac{30}{h^{\frac{1}{2}}c^{\frac{1}{2}}}. \tag{3·6}$$

The desired inequality (3·1) follows on combining (3·4)–(3·6).

Now suppose that g' does not vanish in $[A, B]$. There are two possibilities.

(*a*) $g'(x)$ is positive in $[A, B]$. In this case we show that

$$\left|\int_A^{\min(2A,\,B)} \frac{e(hg(x))}{x}dx\right| \leqslant \frac{30}{h^{\frac{1}{2}}c^{\frac{1}{2}}}$$

by the method used to obtain (3·6), while for $2A < x \leqslant B$ we have

$$g'(x) = g'(A) + \int_A^x g''(t)\,dt > g'(A) + \int_{\frac{1}{2}x}^x \frac{c}{t^2}dt \geqslant \frac{c}{x}.$$

We now complete the proof by arguing as we did to obtain (3·4).

(*b*) $g'(x)$ is negative in $[A, B]$. In this case the analogue of (3·6) holds for $[\max(A, \frac{1}{2}B), B]$ and the analogue of (3·4) applies to $[A, \max(A, \frac{1}{2}B)]$.

In all cases we obtain (3·1), and the proof of Lemma 2 is complete.

LEMMA 3. *Suppose that f satisfies the hypotheses of Theorem* 2. *Then, for any natural numbers h, A, B, we have*

$$\left|\sum_{n=A}^{B} \frac{e(hf(n))}{n}\right| < C_9(h^{\frac{1}{2}}A^{-\delta} + h^{-\frac{1}{2}}).$$

Here $C_9 = C_9(c, K, \delta, H)$.

Proof. We apply lemma 4·10 of [6] with $g(x) = x^{-1}$ and $hf(x)$ in place of $f(x)$. This gives

$$\sum_{n=A}^{B} \frac{e(hf(n))}{n} = \sum_{hf'(B)-\frac{1}{2}<\nu<hf'(A)+\frac{1}{2}} \int_A^B \frac{1}{x}e(hf(x) - \nu x)\,dx + O(A^{-1}\log(h|f'(A) - f'(B)| + 2)).$$

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the implied constant being absolute. (The upper and lower bounds in the summation over ν are to be reversed if $f'' > 0$.)

We observe that

$$|f'(A)-f'(B)| = \int_A^B |f''(x)|\,dx \leqslant \frac{KA^{-\delta}}{\delta}.$$

The O-term is thus bounded in modulus by $C_{10}A^{-1}\log 2h$, where $C_{10} = C_{10}(K,\delta)$.

As for the ν-sum, the number of terms is at most $C_{11}(hA^{-\delta}+1)$ where $C_{11} = C_{11}(K,\delta)$. Each term has modulus at most $C_7 h^{-\frac{1}{2}}$ by Lemma 2. Now Lemma 3 follows on combining these bounds.

We remark that in the case $f(x) = \alpha x + \beta \log x$, we may take $\delta = H = 1, c = K = |\beta|$. We then recover the estimate

$$\left| \sum_{n=A}^{B} \frac{e(h(\alpha n + \beta \log n))}{n} \right| < C_{12}(\beta)(h^{\frac{1}{2}}A^{-1} + h^{-\frac{1}{2}}) \tag{3·7}$$

given by Tichy and Turnwald ([5], theorem 3·1).

Proof of Theorem 2. By Lemma 1 and Lemma 3,

$$D(N) < C_6(\delta) + 24 \sum_{1 \leqslant h \leqslant N^\delta} \frac{1}{h} . 2C_9 h^{-\frac{1}{2}}.$$

The desired inequality follows at once.

4. *Proof of Theorem* 3

Throughout this section, implied constants depend at most on β.

We write

$$S(\gamma,\theta,A,B) = \sum_{A \leqslant p \leqslant B} \frac{\log p}{p} e(\gamma p + \theta \log p). \tag{4·1}$$

The most difficult part of the proof of our theorem lies in establishing the following estimate.

LEMMA 4. *Let γ be a real number, and suppose that h, A, B are positive integers with $h^2 \leqslant A < B$. Then, for all non-zero real β, we have*

$$S(\gamma, h\beta, A, B) \ll (\log h)^{-100}. \tag{4·2}$$

The number 100 could be replaced by an arbitrarily large positive integer.

Before proving this result we complete the proof of Theorem 3. Using the prime number theorem in the form

$$\pi(x) = \frac{x}{\log x - 1} + O\left(\frac{x}{(\log x)^3}\right),$$

we obtain

$$\frac{1}{n} = \frac{\log p_n - 1}{p_n} + O\left(\frac{1}{n(\log 2n)^2}\right).$$

Since

$$\sum_{n=1}^{\infty} \frac{1}{n(\log 2n)^2}$$

converges, we may replace n^{-1} by $(\log p_n - 1)p_n^{-1}$ in the definition of $D(N)$ and only alter the result by a constant.

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We write

$$S'(\gamma,\theta,A,B)=\sum_{A\leqslant p\leqslant B}\frac{1}{p}e(\gamma p+\theta\log p).$$

It then follows by partial summation from (4·2) that

$$S'(\gamma,h\beta,A,B)\ll(\log h)^{-100}(\log A)^{-1} \tag{4·3}$$

under the hypotheses of Lemma 4. Employing the argument of Section 2, we obtain

$$\begin{aligned}D(N)&\ll 1+\sum_{1\leqslant h\leqslant\sqrt{N}}\frac{1}{h}\max_{A>h^2}\left|\sum_{A\leqslant p\leqslant p_N}\frac{(\log p-1)}{p}e(h(\alpha p+\beta\log p))\right|\\&\ll 1+\sum_{1\leqslant h\leqslant\sqrt{N}}\frac{1}{h}(\max_{A\geqslant h^2}|S(h\alpha,h\beta,A,p_N)|+\max_{A\geqslant h^2}|S'(h\alpha,h\beta,A,p_N)|)\\&\ll 1\end{aligned}$$

from (4·2) and (4·3).

In order to prove Lemma 4 we will have to break up the summation range. The following three results are then applied to appropriate subsums.

LEMMA 5. *Let the hypotheses of Lemma* 4 *be given, and suppose that* a, q *are integers with*

$$|\gamma-a/q|<q^{-2}\quad\text{and}\quad(a,q)=1, q\geqslant 1. \tag{4·4}$$

Then

$$S(\gamma,h\beta,A,B)\ll h((\log B)^5q^{-\frac{1}{2}}+(\log A)^4(A^{-\frac{1}{5}}+A^{-\frac{1}{2}}q^{\frac{1}{2}})). \tag{4·5}$$

Proof. We first note that we may replace $\log p$ with $\Lambda(n)$, where Λ is the von Mangoldt function, and sum over all n in $[A,B]$ with additional error $O(A^{\frac{1}{2}})$. Since

$$|e(u)-e(v)|\leqslant 2\pi|u-v|,$$

it follows that

$$\left|\frac{e(h\beta\log n)}{n}-\frac{e(h\beta\log(n+1))}{n+1}\right|\leqslant(2\pi h|\beta|+1)\,n^{-2}.$$

The estimate (4·5) then follows by partial summation from Vaughan's bound (for example see [2], chapter 25):

$$\sum_{n\leqslant N}\Lambda(n)\,e(n\gamma)\ll(\log N)^4(Nq^{-\frac{1}{2}}+N^{\frac{4}{5}}+N^{\frac{1}{2}}q^{\frac{1}{2}}). \tag{4·6}$$

LEMMA 6. *Under the same hypotheses as Lemma* 5, *we have*

$$S(\gamma,h\beta,A,B)\ll(\log B)^5(h^{-\frac{1}{2}}+q^{-\frac{1}{2}})+(\log A)^4(h^{\frac{1}{2}}A^{-\frac{2}{3}}+A^{-\frac{1}{6}}+A^{-\frac{1}{2}}q^{\frac{1}{2}}). \tag{4·7}$$

Proof. As the form of the upper bound suggests, this is a hybrid estimate. We apply Vaughan's identity in a standard fashion and employ (3·7) to deal with certain sums; the techniques used to prove (4·6) are required for the remaining sums.

Let $X\geqslant 2$ and $\frac{1}{2}X\leqslant X_1\leqslant X$. We follow the argument presented in [2], chapter 24,

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in particular, (6) of page 142 with its accompanying remarks. We obtain (taking $U = V = X^{\frac{1}{3}}$ there),

$$\sum_{X_1 < n \leqslant X} \frac{\Lambda(n)}{n} e(n\gamma + h\beta \log n) \ll (\log X) \sum_{t \leqslant X^{\frac{1}{3}}} \frac{1}{t} \max_{w > X/2t} \left| \sum_{w \leqslant r \leqslant X/t} \frac{1}{r} e(rt\gamma + h\beta \log (rt)) \right|$$

$$+ (\log X)^3 \max_{X^{\frac{1}{3}} \leqslant M \leqslant X^{\frac{2}{3}}} \max_{X^{\frac{1}{3}} \leqslant j \leqslant X^{\frac{2}{3}}} \left(\frac{1}{X} \sum_{\substack{X^{\frac{1}{3}} \leqslant k \leqslant X/M \\ X \leqslant 4Mk}} |\Sigma^* e(\gamma m(j-k))| \right)^{\frac{1}{2}}$$

$$= S_1 + S_2, \text{ say.}$$

Here * represents the summation conditions:

$$M < m \leqslant 2M, \quad X_1 < mk \leqslant X, \quad X_1 < mj \leqslant X.$$

We remark that the $e(h\beta \log x)$ terms have disappeared in S_2 since they arise as $e(h\beta(\log mj - \log mk))$ which is, of course independent of m.

We apply (3·7) to S_1 and thus obtain

$$S_1 \ll (\log X) \sum_{t \leqslant X^{\frac{1}{3}}} \frac{1}{t} \left(h^{\frac{1}{2}} \left(\frac{X}{t} \right)^{-1} + h^{-\frac{1}{2}} \right) \ll X^{-\frac{2}{3}} h^{\frac{1}{2}} (\log X) + h^{-\frac{1}{2}} (\log X)^2. \tag{4·8}$$

By a familiar argument (see, for example [2], chapter 25), we infer that

$$S_2 \ll (\log X)^4 (q^{-\frac{1}{2}} + X^{-\frac{1}{6}} + X^{-\frac{1}{2}} q^{\frac{1}{2}}). \tag{4·9}$$

Combining (4·8) with (4·9) and summing over $X = 2^{-k}B$ $(k = 0, 1, \ldots; X > A)$ produces (4·7) as desired.

LEMMA 7. *Let the hypotheses of Lemma* 5 *be given, and suppose also that*

$$q \leqslant (\log A)^{1000}, \quad \gamma - a/q = \theta, \quad B|\theta| \leqslant (\log B)^{2000}.$$

Then
$$S(\gamma, h\beta, A, B) = \frac{\mu(q)}{\phi(q)} \sum_{n=A}^{B} \frac{1}{n} e(\theta n + h\beta \log n) + O(\exp(-c(\log A)^{\frac{1}{2}})). \tag{4·10}$$

Here $\phi(n)$ is Euler's totient function and c is a positive constant. The marriage of Lemma 7 and (3·7) gives rise to the following result.

COROLLARY. *Under the hypotheses of Lemma* 7, *we have*

$$S(\gamma, h\beta, A, B) \ll \frac{1}{\phi(q)} (A^{-\frac{1}{2}} h^{\frac{1}{2}} + h^{-\frac{1}{2}}) + \exp(-c(\log A)^{\frac{1}{2}}). \tag{4·11}$$

Proof. We first mention that the constant c in (4·10) is ineffective owing to our use of Siegel's theorem below. It is possible to obtain

$$S(\gamma, h\beta, A, B) \ll \frac{q^{\frac{1}{2}}}{\phi(q)} (A^{-\frac{1}{2}} h^{\frac{1}{2}} + h^{-\frac{1}{2}}) + \exp(-c(\log A)^{\frac{1}{2}})$$

with an effective constant and with q in a larger range. However this entails more work.

In order to establish (4·10) it suffices to prove that, for $A \leqslant N \leqslant B$,

$$\sum_{1 \leqslant n \leqslant N} \Lambda(n) n^{it} e(n\gamma) = \frac{\mu(q)}{\phi(q)} \sum_{1 \leqslant n \leqslant N} n^{it} e(n\theta) + O(N \exp(-c(\log N)^{\frac{1}{2}})) \tag{4·12}$$

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with $t = 2\pi h\beta$, and apply partial summation. Without loss of generality we may suppose that $t > 0$.

For χ a character mod q, we write

$$\psi(x, \chi, t) = \sum_{1 \leqslant n \leqslant x} \Lambda(n)\, n^{it} \chi(n).$$

We also put

$$\psi(x, t) = \sum_{1 \leqslant n \leqslant x} \Lambda(n)\, n^{it}, \quad T(N, \chi, t, \theta) = \sum_{n \leqslant N} \Lambda(n)\, n^{it} \chi(n)\, e(n\theta).$$

Writing χ_0 for the principal character, we have, as on page 121 of [2],

$$|\psi(x, \chi_0, t) - \psi(x, t)| \ll (\log q)(\log x). \tag{4·13}$$

By well-known properties of characters we then have

$$\sum_{\substack{1 \leqslant n \leqslant N \\ (n, q) = 1}} \Lambda(n)\, n^{it}\, e(n\gamma) = \frac{1}{\phi(q)} \sum_{\chi \bmod q} \chi(a)\, \tau(\bar{\chi})\, T(N, \chi, t, \theta), \tag{4·14}$$

where $\tau(\bar{\chi})$ denotes the Gauss sum of the conjugate character to χ. By partial summation we have

$$T(N, \chi, t, \theta) = \sum_{m \leqslant N} e(m\theta)\,(1 - e(\theta))\, \psi(m, \chi, t) + e((N+1)\,\theta)\, \psi(N, \chi, t). \tag{4·15}$$

We will demonstrate that, for $N^{\frac{1}{2}} \leqslant x \leqslant N$,

$$\psi(x, \chi, t) \ll x \exp(-3c(\log x)^{\frac{1}{2}}), \quad \text{for} \quad \chi \neq \chi_0, \tag{4·16}$$

and

$$\psi(x, t) = \sum_{1 \leqslant n \leqslant x} n^{it} + O(t + x \exp(-2c(\log x)^{\frac{1}{2}})). \tag{4·17}$$

Since $|1 - e(\theta)| \ll N^{-1}(\log N)^{2000}$ it follows from (4·15) and (4·16) that

$$T(N, \chi, t, \theta) \ll N \exp(-2c(\log N)^{\frac{1}{2}}) \quad \text{if} \quad \chi \neq \chi_0. \tag{4·18}$$

From (4·13), (4·17) and (4·15), with a reversal of the partial summation, we conclude that

$$T(N, \chi_0, t, \theta) = \sum_{n \leqslant N} n^{it}\, e(n\theta) + O(N \exp(-2c(\log N)^{\frac{1}{2}})). \tag{4·19}$$

(We recall that $t = O(N^{\frac{1}{2}})$.) Inserting (4·18) and (4·19) into (4·14) together with familiar results on $\tau(\chi)$ (for which see [2]), we obtain (4·12) as required.

To establish (4·16) we write $\eta = (\log x)^{-1}$. Also let ϵ be a positive constant such that no L-function of modulus q has a zero in the region

$$\sigma > 1 - 2\epsilon(\log T)^{-1}, \quad |t| \leqslant T,$$

where $T > \frac{1}{2}\exp(q^{\frac{1}{2000}})$. Such a constant exists by Siegel's theorem and results for the zero-free region of an L-function's complex zeros (see [2]). We put

$$E = \exp((\log N)^{\frac{1}{2}}),$$

$$T = \begin{cases} E & \text{if} \quad t \leqslant \frac{3}{4}E \\ \frac{1}{2}E & \text{if} \quad t > \frac{3}{4}E. \end{cases}$$

An application of the Perron formula, as given by lemma 3·19 of [6], then yields

$$\psi(x, \chi, t) = -\frac{1}{2\pi i} \int_{1+\eta-iT}^{1+\eta+iT} \frac{L'}{L}(s - it, \chi)\, \frac{x^s}{s}\, ds + O\left(\frac{x(\log x)^2}{T} + \log x\right). \tag{4·20}$$

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Assuming, as we may, that $3c < 1$, the error term in (4·20) is of a suitable size. We now shift the integral to the line $\sigma = 1-\epsilon(\log T)^{-1}$, encountering no poles of the integrand in the process. On the new line of integration and the two short horizontal line segments we have

$$\frac{L'}{L}(s-it,\chi) = \sum_{\rho}\frac{1}{s-\rho-it}+O(\log q(T+t)),$$

where the summation is over zeros of $L(s,\chi)$ with $|\mathrm{Im}\,(s-\rho-it)| < 1$. See [2], page 102; an additional condition there on $\mathrm{Im}\,(s-it)$ is not required for the result we need. Thus

$$\begin{aligned}\frac{L'}{L}(s-it,\chi) &\ll \epsilon^{-1}(\log T)\sum_{\rho}1+O(\log q(T+t))\\ &\ll (\log N)^2.\end{aligned}$$

The contribution from the horizontal line segments is thus bounded by

$$\frac{x}{T}(\log N)^2 \ll x\exp(-\tfrac{1}{2}(\log x)^{\frac{1}{2}}). \tag{4·21}$$

Also the integral on $\mathrm{Re}\,s = \sigma$ is bounded by

$$(\log N)^3 x^{\sigma} = x(\log N)^3\exp\left(-\frac{\epsilon\log x}{\log T}\right) \ll x\exp(-3c(\log x)^{\frac{1}{2}})$$

with $c = \min(\frac{1}{6},\frac{1}{4}\epsilon)$. In conjunction with (4·20) and (4·21) this furnishes us with (4·16).

To obtain (4·17) we work as previously, though we require only the de la Vallée–Poussin zero-free region for $\zeta(s)$. In contrast to the above we encounter a pole at $s-it = 1$ if $t \leqslant \frac{3}{4}E$. This contributes

$$\frac{x^{it+1}}{1+it} = \int_{2t}^{x} y^{it}\,dy+O(1) = \sum_{2t\leqslant n\leqslant x} n^{it}+O(1)$$

(using [6], lemma 4·8) $$= \sum_{1\leqslant n\leqslant x} n^{it}+O(t+1).$$

This leads to (4·17) when $t \leqslant \frac{3}{4}E$. For $t > \frac{3}{4}E$ the sum in (4·17) is smaller than the error term. This completes the proof of Lemma 6.

Proof of Lemma 4. Lemmas 5 and 6 correspond to minor arc estimates and Lemma 7 to a major arc estimate in the Hardy–Littlewood method ([2], chapter 26). By analogy with well-known terminology, we will call a summation range where we apply Lemma 5 or 6 a minor interval, and a summation range where we apply Lemma 7 a major interval.

Let Q be the set of denominators of the convergents to the continued fraction expansion for γ. Write

$$Q^{-}(X) = \max\{q : q\in Q, q\leqslant X\},$$

$$Q^{+}(X) = \min\{q : q\in Q, q > X\} \quad \text{if} \quad \{q : q\in Q, q > X\} \neq \varnothing.$$

If γ is a rational with denominator no more than X we put $Q^{+}(X) = \infty$.

We break the range $[A,B]$ into subintervals in the following fashion. Let

$$A_1 = A, \quad q = Q^{-}(A(\log A)^{-1000}).$$

If $q > (\log A)^{1000}$ we put $$A_2 = \min(B,\exp(q^{\frac{1}{500}}))$$

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and say that $[A_1, A_2)$ is a minor interval with denominator q. On the other hand, if $q \leqslant (\log A)^{1000}$ we let $r = Q^+(q)$ and put

$$A_2 = \min(B, r(\log r)^{2000}).$$

(Note that $r > A(\log A)^{-1000}$.) We then call $[A_1, A_2)$ a major interval with denominator q.

We proceed inductively. Suppose we have defined intervals up to $[A_j, A_{j+1})$ where $A_{j+1} < B$. Then, if $[A_j, A_{j+1})$ is a major interval with denominator q, we put

$$r = Q^+(q), \quad A_{j+2} = \min(B, \exp(r^{\frac{1}{500}})).$$

Then $[A_{j+1}, A_{j+2})$ is a minor interval with denominator r. However, if $[A_j, A_{j+1})$ is a minor interval with denominator q,

(i) if $q = Q^-(A_{j+1}(\log A_{j+1})^{-1000})$ put

$$r = Q^+(q), \quad A_{j+2} = \min(B, r(\log r)^{2000}),$$

then $[A_{j+1}, A_{j+2})$ is a major interval with denominator q; and

(ii) if $q < r = Q^-(A_{j+1}(\log A_{j+1})^{-1000})$ then put

$$A_{j+2} = \min(B, \exp(r^{\frac{1}{500}}))$$

and then $[A_{j+1}, A_{j+2})$ will be a minor interval with denominator r. We stop when $A_{j+2} = B$.

Now, in proving Lemma 4 we may suppose that A is so large that $(\log A)^{3000} < A$. Let q_1, q_2, q_3 be the denominators for the first three minor intervals (if there are that many). Let $[C, D)$ be the third minor interval. It follows that

$$C \geqslant \exp\left(\frac{A^{\frac{1}{500}}}{(\log A)^2}\right), \quad q_3 \geqslant \frac{A^2}{(2\log A)^{1000}},$$

by considering the 'worst case'

$$q_1 = (\log A)^{1000}, \quad A_2 = A^2, \quad q_2 = A(\log A)^{-1000}.$$

An application of (4·5) then yields the bound

$$S(\gamma, h\beta, C, D) \ll h(q_3^{-0.49} + (\log C)^{-496}).$$

Since the end points of the intervals and the denominators increase exponentially (indeed, much faster than exponentially) the contribution of all the minor intervals, excluding the first two, is

$$\ll h(q_3^{-0.49} + (\log C)^{-495}) \ll h(\log A)^{490} A^{-0.98} \ll (\log h)^{-100}. \tag{4·22}$$

Now let $[C, D)$ be a major interval with denominator q. By (4·11),

$$S(\gamma, h\beta, C, D) \ll \frac{(h^{\frac{1}{2}}C^{-1} + h^{-\frac{1}{2}})}{\phi(q)} + \exp(-c(\log C)^{\frac{1}{2}}).$$

Again, since q and C increase rapidly from one interval to the next, we obtain a much better bound than we need for the contribution of all the major intervals.

Finally, let $[C, D)$ be one of the first two minor intervals. Write

$$F = \max(C, \min(D, \exp(h^{\frac{1}{20}}))).$$

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We apply Lemma 6 to the interval $[C, F)$ to obtain, if $F > C$,

$$S(\gamma, h\beta, C, F) \ll h^{-\frac{1}{4}} + q^{-0\cdot 49} + (\log C)^4(h^{\frac{1}{2}}C^{-\frac{5}{3}} + C^{-\frac{1}{6}} + C^{-\frac{1}{5}}q^{\frac{1}{2}})$$

$$\ll h^{-\frac{1}{4}} + (\log A)^{-490} \ll (\log h)^{-100}.$$

We remark that it was vital that Lemma 6 should only be applied a bounded number of times. For the interval $[F, D)$, if non-empty, we utilize Lemma 5 to deduce that

$$S(\gamma, h\beta, F, D) \ll (\log D)^{25}q^{-\frac{1}{2}} + h(\log F)^4(F^{-\frac{1}{5}} + F^{-\frac{1}{2}}q^{\frac{1}{2}})$$

$$\ll q^{-0\cdot 45} + h(\log F)^4(F^{-\frac{1}{5}} + C^{\frac{1}{2}}(\log C)^{-500}F^{-\frac{1}{2}})$$

$$\ll (\log h)^{-100} + h(\log F)^{-496} \ll (\log h)^{-100}.$$

This completes the proof of Lemma 4.

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