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NONHOMOGENEOUS BIRTH PROCESSES AND l_∞-SPHERICAL DENSITIES, WITH APPLICATIONS IN RELIABILITY THEORY

MOSHE SHAKED
Department of Mathematics
University of Arizona
Tucson, AZ 85721
E-mail: shaked@math.arizona.edu

FABIO SPIZZICHINO
Department of Mathematics
University "La Sapienza"
00185 Rome, Italy
E-mail: spizzichino@uniroma1.it

FLORENTINA SUTER
Department of Mathematics
University of Bucharest
70109 Bucharest, Romania
E-mail: florent@math.math.unibuc.ro

In this article we characterize l_∞-spherical density functions by means of epoch times of nonhomogeneous pure birth processes. Some further properties of l_∞-spherical densities, such as Schur-concavity, positive dependence, and stochastic comparisons, are also given. The relationships of l_∞-spherical densities to notions of interest in reliability theory are highlighted.

1. INTRODUCTION

Let f be an exchangeable density function on $\mathbb{R}^n$ (i.e., $f(x_1, x_2, \ldots, x_n) = f(x_{\pi_1}, x_{\pi_2}, \ldots, x_{\pi_n})$ for any permutation $(\pi_1, \pi_2 \ldots, \pi_n)$ of $(1,2,\ldots,n)$). Then the function g, defined by

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$$g(x_1, x_2, \ldots, x_n) = \begin{cases} n! f(x_1, x_2, \ldots, x_n) & \text{if } x_1 \le x_2 \le \cdots \le x_n \\ 0, & \text{otherwise,} \end{cases} \tag{1.1}$$

is a density function with support in the n-dimensional cone $\{x_1 \le x_2 \le \cdots \le x_n\}$ (in fact, g is the density function of the corresponding order statistics). Conversely, f can be expressed by means of the above g as follows:

$$f(x_1, x_2, \ldots, x_n) = \frac{1}{n!}\, g(x_{\pi_1}, x_{\pi_2}, \ldots, x_{\pi_n}) \quad \text{if } x_{\pi_1} \le x_{\pi_2} \le \cdots \le x_{\pi_n}. \tag{1.2}$$

The purpose of this article is to study a special case of (1.1), namely the case when a density function g of a nonnegative random vector $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ is of the form

$$g(x_1, x_2, \ldots, x_n) = \begin{cases} \psi(x_n) & \text{if } 0 \le x_1 \le x_2 \le \cdots \le x_n \\ 0 & \text{otherwise} \end{cases} \tag{1.3}$$

for some nonnegative function ψ (i.e., the value of $g(x_1, x_2, \ldots, x_n)$ does not depend on the actual values of $x_1, x_2, \ldots, x_{n-1}$, but only whether or not they satisfy $x_1 \le x_2 \le \cdots \le x_{n-1} \le x_n$). In the main result of Section 4, it is shown that if $(X_1, X_2, \ldots, X_n)$ has a density of the form (1.3), then $X_1, X_2, \ldots, X_n$ must be the first n epoch times of some nonhomogeneous pure birth process.

Densities of the form (1.3) arise naturally in the Bayesian statistical theory of reliability as follows. Let $Z_1, Z_2, \ldots, Z_n$ be n exchangeable lifetimes with a Schur-constant density function; that is, suppose that the density function $f_{\boldsymbol{Z}}$ of $(Z_1, Z_2, \ldots, Z_n)$ is of the form

$$f_{\boldsymbol{Z}}(z_1, z_2, \ldots, z_n) = \psi(z_1 + z_2 + \cdots + z_n), \qquad (z_1, z_2, \ldots, z_n) \in \mathbb{R}^n_+; \tag{1.4}$$

see Spizzichino [22, Sects. 1.4 and 4.3] for a discussion about the intuitive meaning and the utilization of Schur-constant densities in the analysis of subjective probability models for lifetimes. If we let

$$X_i = Z_1 + Z_2 + \cdots + Z_i, \qquad i = 1, 2, \ldots, n, \tag{1.5}$$

then $(X_1, X_2, \ldots, X_n)$ has a density of the form (1.3). In the context of Bayesian reliability theory, the X_i's correspond to the sequential observation of failures, when only a single unit works, and immediate replacements occur at failure instances. From (1.4) and (1.5) it is seen that every property of X_i's with density (1.3) can be translated into a property for Z_i's with density (1.4).

As is indicated in (1.1) and (1.2), there corresponds to the g in (1.3) an exchangeable density function f, given by

$$f(x_1, x_2, \ldots, x_n) = \phi\Big(\max_{1 \le i \le n} x_i\Big), \qquad (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_+; \tag{1.6}$$

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here,

$$\phi \equiv \psi/n!. \tag{1.7}$$

Such densities are called spherical in the l_∞-norm, and the function ϕ is called the *generator* of the l_∞-spherical density. One useful property of such density functions is that they are easy to identify and construct. Rachev and Rüschendorf [15] obtained some de Finetti-type theorems for l_∞-spherical densities. Iglesias, Pereira, and Tanaka [9] analyzed them in some detail; in particular, they obtained some properties of these that are useful in the statistical analysis of life data. The main fact that we prove in Section 4—that (the ordered versions of) these are the density functions of the first n epoch times of a nonhomogeneous pure birth process—gives a useful meaning to such densities.

The article is organized as follows. In Sections 2 and 3, we recall the definition of nonhomogeneous pure birth processes, and we give some useful facts about them and about the associated multivariate conditional hazard rate functions. The relationship between (the ordered versions of) l_∞-spherical density functions and epoch times of nonhomogeneous pure birth processes is derived in Section 4. Positive dependence properties, and stochastic comparisons, of random vectors with l_∞-spherical density functions are given in Section 5. In particular, for random vectors with such densities, we describe in Section 5 a relationship between the positive aging properties of multivariate Polya frequency of order 2 (discussed in Shaked and Shanthikumar [19]), and of Schur concave survival functions (discussed in Barlow and Mendel [1] and in Bassan and Spizzichino [2]).

In this article, "increasing" means "nondecreasing" and "decreasing" means "nonincreasing."

2. NONHOMOGENEOUS PURE BIRTH PROCESSES

A counting process $N = \{N(t), t \geq 0\}$ is said to be a *nonhomogeneous pure birth process* with intensity (or rate) functions $r_i \geq 0$ if

$$\{N(t), t \geq 0\} \text{ has the Markov property,} \tag{2.1}$$

$$P\{N(t + \Delta t) = i + 1 | N(t) = i\} = r_i(t)\Delta t + o(\Delta t), \qquad i \geq 1, \tag{2.2}$$

$$P\{N(t + \Delta t) > i + 1 | N(t) = i\} = o(\Delta t), \qquad i \geq 1. \tag{2.3}$$

When all of the r_i's are identical, a nonhomogeneous pure birth process reduces to a nonhomogeneous Poisson process. We assume throughout that each r_i satisfies

$$\int_t^\infty r_i(u)\, du = \infty \quad \text{for all } t \geq 0; \tag{2.4}$$

this ensures that, with probability 1, the process has a jump after any time point t.

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Let $T_1 \le T_2 \le \cdots \le T_n$ be the first n epoch times of the point process N. The density function f of $(T_1, T_2, \ldots, T_n)$ can be written by means of the intensity functions as follows:

$$f(x_1, x_2, \ldots, x_n) = \left[\prod_{i=1}^{n} r_i(x_i)\right] \exp\left\{-\sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} r_i(u)\, du\right\},$$
$$0 \le x_1 \le x_2 \le \cdots \le x_n, \qquad \textbf{(2.5)}$$

where $x_0 \equiv 0$; see, for example, Snyder [21, p. 248]. Conversely, we have the following result.

PROPOSITION 2.1: *Let $T_1 \le T_2 \le \cdots \le T_n$ be the first n epoch times of a nonhomogeneous pure birth process with intensity functions r_i, $i = 1,2,\ldots,n$, and let f be the density function of $(T_1, T_2, \ldots, T_n)$. Then, the intensity functions can be expressed by means of f as follows:*

$$r_j(t) = \frac{\displaystyle\int_{u_{j+1}=t}^{\infty} \cdots \int_{u_n=u_{n-1}}^{\infty} f(x_1,\ldots,x_{j-1},t,u_{j+1},\ldots,u_n)\, du_n \cdots du_{j+1}}{\displaystyle\int_{u_j=t}^{\infty} \int_{u_{j+1}=u_j}^{\infty} \cdots \int_{u_n=u_{n-1}}^{\infty} f(x_1,\ldots,x_{j-1},u_j,u_{j+1},\ldots,u_n)\, du_n \cdots du_{j+1}\, du_j},$$

(2.6)

whenever $0 \le x_1 \le \cdots \le x_{j-1} \le t$, $j = 1,2,\ldots,n$.

PROOF: By plugging (2.5) into the numerator of the right-hand side of (2.6) we get

$$\int_{u_{j+1}=t}^{\infty} \cdots \int_{u_n=u_{n-1}}^{\infty} f(x_1,\ldots,x_{j-1},t,u_{j+1},\ldots,u_n)\, du_n \cdots du_{j+1}$$
$$= \int_{u_{j+1}=t}^{\infty} \cdots \int_{u_n=u_{n-1}}^{\infty} \left[\prod_{i=1}^{j-1} r_i(x_i)\right] \exp\left\{-\sum_{i=1}^{j-1} \int_{x_{i-1}}^{x_i} r_i(u)\, du\right\} r_j(t) \left[\prod_{i=j+1}^{n} r_i(u_i)\right]$$
$$\times \exp\left\{-\int_{x_{j-1}}^{t} r_j(u)\, du - \int_{t}^{u_{j+1}} r_{j+1}(u)\, du - \sum_{i=j+2}^{n} \int_{u_{i-1}}^{u_i} r_i(u)\, du\right\} du_n \cdots du_{j+1}.$$

Similarly, the denominator of the right-hand side of (2.6) is

$$\int_{u_j=t}^{\infty} \int_{u_{j+1}=u_j}^{\infty} \cdots \int_{u_n=u_{n-1}}^{\infty} f(x_1,\ldots,x_{j-1},u_j,u_{j+1},\ldots,u_n)\, du_n \cdots du_{j+1}\, du_j$$
$$= \int_{u_j=t}^{\infty} \int_{u_{j+1}=u_j}^{\infty} \cdots \int_{u_n=u_{n-1}}^{\infty} \left[\prod_{i=1}^{j-1} r_i(x_i)\right] \exp\left\{-\sum_{i=1}^{j-1} \int_{x_{i-1}}^{x_i} r_i(u)\, du\right\} r_j(u_j) \left[\prod_{i=j+1}^{n} r_i(u_i)\right]$$
$$\times \exp\left\{-\int_{x_{j-1}}^{u_j} r_j(u)\, du - \int_{u_j}^{u_{j+1}} r_{j+1}(u)\, du - \sum_{i=j+2}^{n} \int_{u_{i-1}}^{u_i} r_i(u)\, du\right\} du_n \cdots du_{j+1}\, du_j.$$

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Thus, the right-hand side of (2.6) is equal, after some simplification, to

$$r_j(t)\left(\int_{u_{j+1}=t}^{\infty}\cdots\int_{u_n=u_{n-1}}^{\infty}\left[\prod_{i=j+1}^{n} r_i(u_i)\right]\right.$$
$$\left.\times\exp\left\{-\int_t^{u_{j+1}} r_{j+1}(u)\,du-\sum_{i=j+2}^{n}\int_{u_{i-1}}^{u_i} r_i(u)\,du\right\}du_n\cdots du_{j+1}\right)$$
$$\left(\int_{u_j=t}^{\infty}\int_{u_{j+1}=u_j}^{\infty}\cdots\int_{u_n=u_{n-1}}^{\infty}\left[\prod_{i=j}^{n} r_i(u_i)\right]\right.$$
$$\left.\times\exp\left\{-\int_t^{u_j} r_j(u)\,du-\sum_{i=j+1}^{n}\int_{u_{i-1}}^{u_i} r_i(u)\,du\right\}du_n\cdots du_{j+1}\,du_j\right)^{-1}. \qquad \textbf{(2.7)}$$

A straightforward computation now shows that both expressions enclosed in large parentheses in (2.7) are equal to 1. ■

Note that the right-hand side of (2.6) does not depend on the actual values $x_1, x_2, \ldots, x_{j-1}$, but only on whether or not they satisfy $x_1 \le x_2 \cdots \le x_{j-1} \le t$.

Consider now a nonhomogeneous pure birth process, as defined in (2.1)–(2.3), with intensity functions $r_1, r_2, \ldots, r_n$ that satisfy (2.4). With every intensity function r_i, associate a survival function $\bar{H}_i$, and a density function h_i, defined by

$$\bar{H}_i(x) \equiv \exp\left\{-\int_0^x r_i(u)\,du\right\}, \qquad x \ge 0, \qquad \textbf{(2.8)}$$

$$h_i(x) \equiv r_i(x)\exp\left\{-\int_0^x r_i(u)\,du\right\}, \qquad x \ge 0. \qquad \textbf{(2.9)}$$

With this notation, f in (2.5) can be rewritten, after some substitution and simplification, as

$$f(x_1, x_2, \ldots, x_n) = \frac{h_1(x_1)h_2(x_2)\cdots h_{n-1}(x_{n-1})}{\bar{H}_2(x_1)\bar{H}_3(x_2)\cdots\bar{H}_n(x_{n-1})}\,h_n(x_n),$$
$$0 \le x_1 \le x_2 \le \cdots \le x_n. \qquad \textbf{(2.10)}$$

Nonhomogeneous pure birth processes are named "relevation counting processes" in Pellerey, Shaked, and Zinn [14], in Belzunce, Lillo, Ruiz, and Shaked [3], and in Belzunce and Shaked [4]. Some applications in reliability theory are described in the cited references. Another interpretation of these processes in reliability theory, by means of load sharing, is described in the following remark.

Remark 2.2: The first n epoch times of a nonhomogeneous pure birth process have a useful interpretation in reliability theory as follows. Consider a system of n identical components in parallel that are subjected to a nonnegative time-dependent load $L_1(\cdot)$. The load is initially shared equally among the n components in the sense that the failure rate of each component at time t, given that all the components are alive

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then, is $L_1(t)/n$. After the first failure of a component, the surviving $n-1$ components are subjected to a (possibly different) nonnegative time-dependent load $L_2(\cdot)$, and this load is shared equally among these $n-1$ components in the sense that the failure rate of each component at time t, given that $n-1$ components are alive then, is $L_2(t)/(n-1)$. In general, given that $i-1$ components have failed already, the surviving $n-i+1$ components are subjected to a nonnegative time-dependent load $L_i(\cdot)$, and this load is equally shared among these $n-i+1$ components in the sense that the failure rate of each component at time t, given that $n-i+1$ components are alive then, is $L_i(t)/(n-i+1)$, $i=1,2,\ldots,n$.

Note that the failure intensity of the first failure is $n[L_1(\cdot)/n] = L_1(\cdot)$, and, in general, the failure intensity of the ith failure is $(n-i+1)[L_i(\cdot)/(n-i+1)] = L_i(\cdot)$, $i=1,2,\ldots,n$. Thus, the n failure times of the components are the first n epoch times of a nonhomogeneous pure birth process with intensity functions $r_i(\cdot) = L_i(\cdot)$, $i=1,2,\ldots,n$. In other words, studying the first n epoch times of a nonhomogeneous pure birth process is equivalent to studying the lifetimes of n components of a load-sharing system, as above.

If $L_1(\cdot) = L_2(\cdot) = \cdots = L_n(\cdot)$, then we get a variation of the load-sharing model with the linear breakdown rule, described in Schechner [17]; in this case the n failure times of the components are the first n epoch times of a nonhomogeneous Poisson process with intensity function $L_1(\cdot)$. If the L_i's are constant functions, then we get a special case of the model discussed in Freund [8] [for $n=2$], in Ross [16], and in Norros [13].

It is worthwhile to mention that sequential order statistics, as described in Cramer and Kamps [6,7], can be seen (from the discussion in [6, p. 305] or [7, p. 308]) to have the same distribution as the first epoch times of a nonhomogeneous pure birth process.

3. MULTIVARIATE CONDITIONAL HAZARD RATE FUNCTIONS OF ORDERED VARIABLES

Let $X_1, X_2, \ldots, X_n$ be nonnegative random variables with an absolutely continuous distribution function. The multivariate conditional hazard rate functions that are associated with these random variables are defined, for example, in Shaked and Shanthikumar [18]. When the random variables are almost surely ordered as

$$X_1 \le X_2 \le \cdots \le X_n, \tag{3.1}$$

then the corresponding multivariate conditional hazard rate functions simplify considerably. In this section we describe these simplified functions, and we point out some facts about them.

The first hazard rate function λ_1 (which is actually the initial unconditional hazard rate function of X_1) is given by

$$\lambda_1(t) = \lim_{\Delta \downarrow 0} \frac{P[X_1 \le t+\Delta | X_1 > t]}{\Delta}, \qquad t \ge 0.$$

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For $2 \le j \le n$, the conditional hazard rate function of X_j, given that $X_1 = x_1, X_2 = x_2, \ldots, X_{j-1} = x_{j-1}$, denoted by $\lambda_{j|1,2,\ldots,j-1}(\cdot|x_1, x_2, \ldots, x_{j-1})$, is defined by

$$\lambda_{j|1,2,\ldots,j-1}(t|x_1, x_2, \ldots, x_{j-1}) = \lim_{\Delta \downarrow 0} \frac{P[X_j \le t + \Delta | X_1 = x_1, X_2 = x_2, \ldots, X_{j-1} = x_{j-1}, X_j > t]}{\Delta}, \qquad t \ge x_{j-1}$$

(by (3.1) we may assume that the x_i's above satisfy $x_1 \le x_2 \le \cdots \le x_{j-1}$). The density function f of $(X_1, X_2, \ldots, X_n)$ can be written by means of the multivariate conditional hazard rate functions as follows:

$$f(x_1, x_2, \ldots, x_n) = \lambda_1(x_1) \left[\prod_{i=2}^{n} \lambda_{i|1,2,\ldots,i-1}(x_i|x_1, x_2, \ldots, x_{i-1}) \right] \times \exp\left\{ -\sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} \lambda_{i|1,2,\ldots,i-1}(u|x_1, x_2, \ldots, x_{i-1})\, du \right\}, \quad 0 \le x_1 \le x_2 \le \cdots \le x_n, \tag{3.2}$$

where $x_0 \equiv 0$; see, for example, Snyder [21, p. 248]. We also have the following converse (which is an analog of Proposition 2.1).

PROPOSITION 3.1: *Let $X_1, X_2, \ldots, X_n$ be n random variables which satisfy $X_1 \le X_2 \le \cdots \le X_n$ almost surely. Then the multivariate conditional hazard rate functions $\lambda_{\cdot|\cdot}(\cdot|\cdot)$ of $(X_1, X_2, \ldots, X_n)$ can be expressed by means of the density function f of $(X_1, X_2, \ldots, X_n)$ as follows:*

$$\lambda_{j|1,2,\ldots,j-1}(t|x_1, x_2, \ldots, x_{j-1}) = \frac{\displaystyle\int_{u_{j+1}=t}^{\infty} \cdots \int_{u_n=u_{n-1}}^{\infty} f(x_1, \ldots, x_{j-1}, t, u_{j+1}, \ldots, u_n)\, du_n \cdots du_{j+1}}{\displaystyle\int_{u_j=t}^{\infty} \int_{u_{j+1}=u_j}^{\infty} \cdots \int_{u_n=u_{n-1}}^{\infty} f(x_1, \ldots, x_{j-1}, u_j, u_{j+1}, \ldots, u_n)\, du_n \cdots du_{j+1}\, du_j}, \tag{3.3}$$

whenever $0 \le x_1 \le \cdots \le x_{j-1} \le t$, $j = 1, 2, \ldots, n$.

The proof of Proposition 3.1 is similar to the proof of Proposition 2.1. We omit the details.

4. EPOCH TIMES OF NONHOMOGENEOUS PURE BIRTH PROCESSES AND l_∞-SPHERICAL DENSITIES

Let $T_1 \le T_2 \le \cdots \le T_n$ be the first n epoch times of a nonhomogeneous pure birth process as described in Section 2. From (2.1)–(2.3) and the definitions of $\lambda_{\cdot|\cdot}(\cdot|\cdot)$ in Section 3, it is immediately seen that the multivariate conditional hazard rate functions of these T_i's are $\lambda_1(\cdot) = r_1(\cdot)$ and $\lambda_{j|1,2,\ldots,j-1}(\cdot|x_1, x_2, \ldots, x_{j-1}) = r_j(\cdot)$,

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$j = 2,3,\ldots,n$, where the r_i's are the intensity functions of the nonhomogeneous pure birth process. Thus, it is seen that (3.2) and (3.3) are generalizations of (2.5) and (2.6), respectively. In fact, we have the following observation.

PROPOSITION 4.1: *Let $X_1, X_2, \ldots, X_n$ be n random variables which satisfy $X_1 \le X_2 \le \cdots \le X_n$ almost surely. If the multivariate conditional hazard rate functions $\lambda_{\cdot|\cdot}(\cdot|\cdot)$ of $(X_1, X_2, \ldots, X_n)$ are such that $\lambda_{j|1,2,\ldots,j-1}(t|x_1, x_2, \ldots, x_{j-1})$ does not depend on the actual values of $x_1, x_2, \ldots, x_{j-1}$, except, of course, that they satisfy $x_1 \le x_2 \le \cdots \le x_{j-1}, j = 2,3,\ldots,n$, then there exists a nonhomogeneous pure birth process $N = \{N(t), t \ge 0\}$ whose first n epoch times $T_1 \le T_2 \le \cdots \le T_n$ satisfy*

$$(T_1, T_2, \ldots, T_n) =_{\text{st}} (X_1, X_2, \ldots, X_n).$$

Furthermore, the intensity functions r_j of N are given by

$$r_1(t) = \lambda_1(t), \qquad t \ge 0,$$
$$r_j(t) = \lambda_{j|1,2,\ldots,j-1}(t|x_1, x_2, \ldots, x_{j-1}), \qquad t \ge 0, j = 2,3,\ldots,n,$$

where, in the latter equation, the x_i's are selected such that $x_1 \le x_1 \le \cdots \le x_{j-1} \le t$.

PROOF: Note that under the stated conditions on the $\lambda_{\cdot|\cdot}(\cdot|\cdot)$'s, (3.2) simplifies to (2.5); that is, $(T_1, T_2, \ldots, T_n)$ and $(X_1, X_2, \ldots, X_n)$ have the same density functions. ■

Of particular interest for us, in this article, is the case in which, for any arbitrary $x_n > 0$, the density f of $T_1 \le T_2 \le \cdots \le T_n$, given in (2.5), is such that $f(x_1, x_2, \ldots, x_{n-1}, x_n)$ does not depend on the particular values $x_1, x_2, \ldots, x_{n-1}$, but only on whether or not they satisfy $x_1 \le x_2 \le \cdots \le x_{n-1} \le x_n$. In other words, of particular interest for us is the case in which f of (2.5) is of the form (1.3) for some generator ψ. In this section, we focus our attention on this special case and, in particular, provide some characterizations of it. First, we state and prove the following basic result.

THEOREM 4.2: *Let $X_1, X_2, \ldots, X_n$ be n random variables such that $X_1 \le X_2 \le \cdots \le X_n$ almost surely. If the density function g of $(X_1, X_2, \ldots, X_n)$ is of the form (1.3), then there exists a nonhomogeneous pure birth process $N = \{N(t), t \ge 0\}$ whose first n epoch times $T_1 \le T_2 \le \cdots \le T_n$ satisfy*

$$(T_1, T_2, \ldots, T_n) =_{\text{st}} (X_1, X_2, \ldots, X_n).$$

PROOF: Let the multivariate conditional hazard rate functions of $X_1, X_2, \ldots, X_n$ be denoted by $\lambda_{\cdot|\cdot}(\cdot|\cdot)$. Substituting (1.3) into (3.3), we see that

$$\lambda_{j|1,2,\ldots,j-1}(t|x_1, x_2, \ldots, x_{j-1}) = \frac{\displaystyle\int_{u_{j+1}=t}^{\infty} \cdots \int_{u_n=u_{n-1}}^{\infty} \psi(u_n)\, du_n \cdots du_{j+1}}{\displaystyle\int_{u_j=t}^{\infty} \int_{u_{j+1}=u_j}^{\infty} \cdots \int_{u_n=u_{n-1}}^{\infty} \psi(u_n)\, du_n \cdots du_{j+1}\, du_j}$$

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whenever $0 \le x_1 \le \cdots \le x_{j-1} \le t$, $j = 1,2,\ldots,n$. We thus see that the multivariate conditional hazard rate functions of $X_1, X_2, \ldots, X_n$ satisfy the conditions of Proposition 4.1, and the stated result follows. ■

We now proceed to characterize l_∞-spherical densities of the form (1.3). In light of Theorem 4.2 one may wonder whether there exists a (partial) converse to it; namely, given a nonhomogeneous pure birth process, under what conditions on the intensity functions r_i do we have that the joint density function of the first n epoch times is of the form (1.3)? We answer this question in the next theorem, where we give the simple-to-check necessary and sufficient condition (4.1). Recalling definitions (2.8) and (2.9), let m_i denote the mean associated with $\bar{H}_i$, provided it exists; that is, $m_i = \int_0^\infty \bar{H}_i(x)\,dx$, $i = 1,2,\ldots,n$. The following result follows at once from (2.10).

THEOREM 4.3: *The density function f (given in (2.5)) of the epoch times $T_1 \le T_2 \le \cdots \le T_n$ of a nonhomogeneous pure birth process with intensity functions r_i, $i = 1,2,\ldots,n$, is of the form (1.3) if and only if*

$$h_i(x) = \bar{H}_{i+1}(x)/m_{i+1}, \qquad x \ge 0,\ i = 1,2,\ldots,n-1, \tag{4.1}$$

where h_i and $\bar{H}_i$ are defined in (2.8) and (2.9). In such a case, the generator ψ in (1.3) satisfies $\psi = h_n/(m_2 m_3 \cdots m_n)$.

Theorem 4.3 suggests a procedure for constructing nonhomogeneous pure birth processes with epoch times having a joint density of the form (1.3). One can start with an arbitrary survival function $\bar{H}_n$ and then derive the other $\bar{H}_i$'s recursively using (4.1). (Example 4.4 illustrates this idea when $n = 2$.) Alternatively, one can start from a density h_1 and derive the other h_i's recursively. The latter procedure is more complex. In the latter procedure, h_1 cannot be arbitrary—it must be decreasing, bounded, and its derivatives must satisfy various monotonicity properties; for example, h_1 may be chosen to be any function that is the derivative of a completely monotone function with bounded derivatives of order up to n. We omit the details. The intensity functions of the resulting process can then be computed by $r_i = h_i/\bar{H}_i$, $i = 1,2,\ldots,n$.

Example 4.4: Let $\bar{H}_2$ be the Weibull survival function given by $\bar{H}_2(x) = \exp\{-x^\alpha\}$, $x \ge 0$, where $\alpha > 0$ is some constant. The corresponding mean is $m_2 = \Gamma(1 + \alpha^{-1})$ and the corresponding hazard rate and density functions are given by $r_2(x) = \alpha x^{\alpha-1}$ and $h_2(x) = \alpha x^{\alpha-1}\exp\{-x^\alpha\}$, $x \ge 0$. Following (4.1), define now the density function h_1 by $h_1(x) = \bar{H}_2(x)/m_2 = \exp\{-x^\alpha\}/\Gamma(1 + \alpha^{-1})$, $x \ge 0$. The corresponding survival function is given by $\bar{H}_1(x) = \int_x^\infty h_1(u)\,du$, $x \ge 0$, and the corresponding hazard rate function is given by $r_1(x) = h_1(x)/\bar{H}_1(x)$, $x \ge 0$. From Theorem 4.3, we see that the first two epoch times of the nonhomogeneous pure birth process with the first two intensity functions being r_1 and r_2, as above, have a joint density of the form (1.3) with generator ψ given by $\psi(x) = h_2(x)/m_2 = \alpha x^{\alpha-1}\exp\{-x^\alpha\}/\Gamma(1 + \alpha^{-1})$, $x \ge 0$.

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There are other examples of nonhomogeneous pure birth processes for which (4.1) holds (see Examples 4.6–4.8). Before we describe these examples, it is worthwhile to give an alternative formulation of Theorem 4.3. Let us denote

$$R_i(x) \equiv \int_0^x r_i(u)\,du, \qquad x \geq 0,\ i = 1,2,\ldots,n.$$

From (2.5) we see that the density function f, of the first n epoch times $T_1 \leq T_2 \leq \cdots \leq T_n$ of a nonhomogeneous pure birth process with intensity functions r_i, $i = 1,2,\ldots,n$, can be rewritten as

$$f(x_1,x_2,\ldots,x_n) = \exp\left\{\sum_{i=1}^{n} \log r_i(x_i) - \sum_{i=1}^{n} R_i(x_i) + \sum_{i=1}^{n-1} R_{i+1}(x_i)\right\},$$
$$0 \leq x_1 \leq x_2 \leq \cdots \leq x_n. \qquad \textbf{(4.2)}$$

Taking this into account, we can give Theorem 4.3 a different, but useful, formulation. This is done in the following result, where we give the simple-to-check necessary and sufficient condition (4.3).

THEOREM 4.5: *Let $T_1 \leq T_2 \leq \cdots \leq T_n$ be the first n epoch times of a nonhomogeneous pure birth process with intensity functions r_i, $i = 1,2,\ldots,n$. If*

$$\log r_i(x) - R_i(x) + R_{i+1}(x) = k_i, \qquad x \geq 0,\ i = 1,2,\ldots,n-1, \qquad \textbf{(4.3)}$$

where the k_i's are some constants, then the joint density of $(T_1,T_2,\ldots,T_n)$ is of the form (1.3). In this case the generator ψ in (1.3) satisfies

$$\psi(x) \propto r_n(x)\exp\{-R_n(x)\}, \qquad x \geq 0. \qquad \textbf{(4.4)}$$

PROOF: If (4.3) holds, then (4.2) reduces to (1.3) with a generator ψ satisfying (4.4). ■

Example 4.6 (Poisson Process): Consider the pure birth process with intensity functions given by

$$r_i(t) = \theta, \qquad t \geq 0,\ i = 1,2,\ldots,$$

where $\theta > 0$ is a constant (i.e., a Poisson process). Then the h_i and $\bar{H}_i$ in (2.8) and (2.9) are given by $h_i(x) = \theta e^{-\theta x}$ and $\bar{H}_i(x) = e^{-\theta x}$, $x \geq 0$, and $m_i = \int_0^\infty \bar{H}_i(x)\,dx = 1/\theta$, $i = 1,2,\ldots$. Obviously, (4.1) holds. Note also that

$$R_i(x) = \int_0^x r_i(u)\,du = \theta x, \qquad x \geq 0,$$

and thus (4.3) holds. Indeed, the density function f of $T_1,T_2,\ldots,T_n$—the first n epoch times of N—is given by

$$f(x_1,x_2,\ldots,x_n) = \theta^n \exp\{-\theta x_n\}, \qquad 0 \leq x_1 \leq x_2 \leq \cdots \leq x_n,$$

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agreeing with (4.4); that is, f is of the form (1.3) with generator ψ given by $\psi(x) = \theta^n \exp\{-\theta x\}$, $x \geq 0$.

It is worthwhile to mention that from Theorem 4.5 it follows that the only nonhomogeneous Poisson process with the first n epoch times having a joint density of the form (1.3) is the homogeneous Poisson process. This can be seen by noting that the nonhomogeneous Poisson process is a special case of the nonhomogeneous pure birth process for which the R_i's in (4.3) do not depend on i. In this case, we see from (4.3) that the r_i's (which do not depend on i) must all be equal to the same constant.

Example 4.7 (Polya Process): Let $N = \{N(t), t \geq 0\}$ be a nonhomogeneous pure birth process with intensity functions r_i given by

$$r_i(x) = \frac{\lambda + i}{1 + x}, \qquad x \geq 0, i = 1,2,\ldots,$$

where $\lambda \geq 0$ is some constant. This process is a special case of the more general Polya process; see, for example, Chiang [5, p. 222]. Straightforward computations give

$$R_i(x) = \int_0^x r_i(u)\, du = (\lambda + i)\log(1 + x), \qquad x \geq 0, i = 1,2,\ldots,$$

$$\bar{H}_i(x) = \exp\{-R_i(x)\} = \frac{1}{(1 + x)^{\lambda+i}}, \qquad x \geq 0, i = 1,2,\ldots,$$

$$h_i(x) = \frac{d}{dx}\bar{H}_i(x) = \frac{\lambda + i}{(1 + x)^{\lambda+i+1}}, \qquad x \geq 0, i = 1,2,\ldots,$$

and

$$m_i = \int_0^\infty \bar{H}_i(x)\, dx = \frac{1}{\lambda + i - 1}, \qquad i = 1,2,\ldots.$$

It is now easy to verify that (4.1) holds (or that (4.3) holds with $k_i = \log(\lambda + i)$, $i = 1,2,\ldots$). Thus, from Theorem 4.5, we see that the joint density function of $(T_1, T_2, \ldots, T_n)$—the vector of the first n epoch times—is of the form (1.3) with generator ψ such that

$$\psi(x) \propto \frac{\lambda + n}{1 + x} \exp\{-(\lambda + n)\log(1 + x)\} = \frac{\lambda + n}{(1 + x)^{\lambda+n+1}}, \qquad x \geq 0.$$

The same conclusion can be obtained as well from Theorem 4.3.

Example 4.8: An example related to the previous one is obtained by letting $\lambda = l$, a positive integer, and

$$r_i(x) = \frac{l + i}{\beta + x}, \qquad x \geq 0, i = 1,2,\ldots,$$

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where $\beta > 0$ is some constant. Then,

$$R_i(x) = \int_0^x r_i(u)\,du = (l+i)\log\left(\frac{\beta+x}{\beta}\right), \qquad x \geq 0,\ i = 1,2,\ldots,$$

$$\bar{H}_i(x) = \exp\{-R_i(x)\} = \left(\frac{\beta}{\beta+x}\right)^{l+i}, \qquad x \geq 0,\ i = 1,2,\ldots,$$

$$h_i(x) = \frac{d}{dx}\bar{H}_i(x) = \frac{(l+i)\beta^{l+1}}{(\beta+x)^{l+i+1}}, \qquad x \geq 0,\ i = 1,2,\ldots,$$

and

$$m_i = \int_0^\infty \bar{H}_i(x)\,dx = \frac{\beta}{l+i-1}, \qquad i = 1,2,\ldots.$$

It is now easy to verify that (4.1) holds (or that (4.3) holds with $k_i = \log((l+i)/\beta)$, $i = 1,2,\ldots$). Thus, from Theorem 4.3, we see that the joint density function of $(T_1, T_2, \ldots, T_n)$—the vector of the first n epoch times—is of the form (1.3) with the generator ψ given by

$$\psi(x) = \frac{(l+n)!\beta^{l+1}}{l!(\beta+x)^{l+n+1}}, \qquad x \geq 0.$$

We remark that, technically, conditions (4.1) and (4.3) can be viewed as means to generate joint densities of the form (1.3). We recall from (1.4) and (1.5) that every Schur-constant density gives rise to a density of the form (1.3). It can be easily seen that a useful procedure for generating Schur-constant densities is to define

$$\psi(s) = \int_0^\infty \theta^n \exp\{-\theta s\}\pi(\theta)\,d\theta, \qquad s \geq 0,$$

where π is a probability density over $[0,\infty)$ and to let the resulting ψ be the generator in (1.4) (see, e.g., Theorem 1.44 of Spizzichino [22]); that is, a large class of densities g of the form (1.3) is defined by letting

$$g(x_1, x_2, \ldots, x_n) = \begin{cases} \displaystyle\int_0^\infty \theta^n \exp\{-\theta x_n\}\pi(\theta)\,d\theta & \text{if } 0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \\ 0 & \text{otherwise.} \end{cases}$$

In fact, the density in Example 4.7 is obtained by letting the above π be defined as

$$\pi(\theta) = \frac{1}{\Gamma(\lambda)}\theta^{\lambda-1}\exp\{-\theta\}, \qquad \theta \geq 0,$$

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and the density in Example 4.8 is obtained by letting the above π be defined as

$$\pi(\theta) = \frac{\beta^{l+1}}{l!}\,\theta^{l}\exp\{-\beta\theta\}, \qquad \theta \geq 0.$$

5. SOME PROPERTIES OF l_∞-SPHERICAL DENSITIES

In Theorems 4.2, 4.3, and 4.5 we obtained some useful properties of random variables, with support in $\{x_1 \leq x_2 \leq \cdots \leq x_n\}$, that have a joint density of the form (1.3). In this section we obtain some properties of exchangeable random variables $Y_1, Y_2, \ldots, Y_n$ that have an l_∞-spherical joint density; that is, a density of the form (1.6). Of course, the relationship (1.7) enables one to translate any result involving densities of the form (1.3) into a result involving l_∞-spherical densities, and vice versa. Before stating and proving the first result in this section, we need to recall some definitions.

For any vector $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, let $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$ denote the corresponding ordered elements. Recall (e.g., from Marshall and Olkin [12]) that $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)$ is said to *majorize* $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ (denoted $\boldsymbol{x} \prec \boldsymbol{y}$) if

$$\sum_{i=1}^{k} x_{(i)} \geq \sum_{i=1}^{k} y_{(i)}, \qquad k = 1,2,\ldots,n-1, \qquad \text{and} \quad \sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}.$$

An n-dimensional function ϕ is called *Schur-convex* (*Schur-concave*) if it is increasing (decreasing) with respect to the majorization order. In the sequel we will be concerned with density functions that are Schur functions. Note that such densities, as Schur functions, must be exchangeable. Indeed, any l_∞-spherical density is exchangeable.

A simple characterization of Schur-concavity for l_∞-spherical densities was already mentioned in Iglesias et al. [9] without a proof. In the following result, we detail the proof and we also prove the equivalence between this condition and a very strong condition of positive dependence, namely the multivariate total positivity of order 2. We recall that an n-dimensional random vector $\boldsymbol{X}$ (not necessarily exchangeable) is called *multivariate totally positive of order* 2 (MTP_2) if its density function f satisfies

$$f(\boldsymbol{x})f(\boldsymbol{y}) \leq f(\boldsymbol{x}\vee\boldsymbol{y})f(\boldsymbol{x}\wedge\boldsymbol{y}), \qquad \boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^n, \tag{5.1}$$

where $\boldsymbol{x}\vee\boldsymbol{y} \equiv (x_1 \vee y_1, x_2 \vee y_2, \ldots, x_n \vee y_n)$ and $\boldsymbol{x}\wedge\boldsymbol{y} \equiv (x_1 \wedge y_1, x_2 \wedge y_2, \ldots, x_n \wedge y_n)$; see Karlin and Rinott [10].

THEOREM 5.1: *Let $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_n)$ be a vector of random variables of the form (1.6) with generator ϕ. Then the following three properties are equivalent:*

(i) ϕ is a decreasing function.
(ii) $\boldsymbol{Y}$ is MTP_2.
(iii) $\boldsymbol{Y}$ has a Schur-concave density function.

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PROOF: Let $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ be two arbitrary vectors. Then, trivially,

$$\max_{1\le i\le n}(x_i \vee y_i) = \left(\max_{1\le i\le n} x_i\right) \vee \left(\max_{1\le i\le n} y_i\right) \tag{5.2}$$

and

$$\max_{1\le i\le n}(x_i \wedge y_i) \le \left(\max_{1\le i\le n} x_i\right) \wedge \left(\max_{1\le i\le n} y_i\right). \tag{5.3}$$

Let f be given by (1.6). If ϕ is decreasing, then

$$\begin{aligned} &f(x_1 \wedge y_1, x_2 \wedge y_2, \ldots, x_n \wedge y_n) f(x_1 \vee y_1, x_2 \vee y_2, \ldots, x_n \vee y_n) \\ &\quad = \phi\left(\max_{1\le i\le n}(x_i \wedge y_i)\right)\phi\left(\max_{1\le i\le n}(x_i \vee y_i)\right) \\ &\quad \ge \phi\left(\left(\max_{1\le i\le n} x_i\right) \wedge \left(\max_{1\le i\le n} y_i\right)\right)\phi\left(\left(\max_{1\le i\le n} x_i\right) \vee \left(\max_{1\le i\le n} y_i\right)\right) \\ &\quad = \phi\left(\max_{1\le i\le n} x_i\right)\phi\left(\max_{1\le i\le n} y_i\right) \\ &\quad = f(x_1, x_2, \ldots, x_n) f(y_1, y_2, \ldots, y_n), \end{aligned}$$

where the inequality follows by (5.2) and (5.3). So (i) $\Rightarrow$ (ii).

Conversely, suppose that (ii) holds. Then, by (5.1), we have

$$\begin{aligned} &\phi\left(\max_{1\le i\le n}(x_i \wedge y_i)\right)\phi\left(\max_{1\le i\le n}(x_i \vee y_i)\right) \\ &\quad \ge \phi\left(\max_{1\le i\le n} x_i\right)\phi\left(\max_{1\le i\le n} y_i\right) \\ &\quad = \phi\left(\left(\max_{1\le i\le n} x_i\right) \wedge \left(\max_{1\le i\le n} y_i\right)\right)\phi\left(\left(\max_{1\le i\le n} x_i\right) \vee \left(\max_{1\le i\le n} y_i\right)\right). \end{aligned}$$

From $\phi \ge 0$ and (5.2) we get

$$\phi\left(\max_{1\le i\le n}(x_i \wedge y_i)\right) \ge \phi\left(\left(\max_{1\le i\le n} x_i\right) \wedge \left(\max_{1\le i\le n} y_i\right)\right).$$

Let $t'' \ge t' \ge 0$ be arbitrary. Substituting $(x_1, x_2, \ldots, x_n) = (t', t'', \ldots, t'')$ and $(y_1, y_2, \ldots, y_n) = (t'', t', \ldots, t')$ in the above inequality, we obtain $\phi(t') \ge \phi(t'')$; that is, (ii) $\Rightarrow$ (i).

Next, note that $\max_{1\le i\le n} x_i = x_{(n)}$ is Schur-convex in $(x_1, x_2, \ldots, x_n)$. Therefore, if ϕ is decreasing, then $\phi(\max_{1\le i\le n} x_i)$ is Schur-concave in $(x_1, x_2, \ldots, x_n)$; that is, (i) $\Rightarrow$ (iii).

Finally, suppose that $\phi(x_{(n)})$ is Schur-concave. Let $t'' \ge t' \ge 0$ be arbitrary. Select two vectors $(x_1, x_2, \ldots, x_n) \prec (y_1, y_2, \ldots, y_n)$ such that $x_{(n)} = t'$ and $y_{(n)} = t''$. It follows that $\phi(t') \ge \phi(t'')$; that is, (iii) $\Rightarrow$ (i). ■

Remark 5.2: Theorem 5.1 gives a very simple way to obtain that an l_∞-spherical density function is Schur-concave. Schur-concavity of a density function implies the Schur-concavity of the associated distribution and survival functions; see Marshall

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and Olkin [11, p. 1193]. Marshall and Olkin [11] provide useful bounds on Schur-concave distribution and survival functions. Furthermore, Schur-concavity of a survival function can be seen as a condition of positive "aging" in a subjective probability approach; see Barlow and Mendel [1] and Bassan and Spizzichino [2]. Thus, Theorem 5.1 is useful in identifying distribution and survival functions to which the above bounds and aging notion apply.

Remark 5.3: As is mentioned in Remark 5.2, Schur-concavity of a survival function is a notion of positive aging. Another notion of positive aging is the notion of MPF_2 (multivariate Polya frequency of order 2) discussed in Shaked and Shanthikumar [19]. These two notions are of different spirit and are motivated by different points of view. In general, it is not clear what the relationship between these two multivariate aging notions is. However, Theorem 5.1 throws some light on this relationship. Shaked and Shanthikumar [19, Remark 6] showed that the MPF_2 property implies the positive dependence property of MTP_2. Therefore, by Theorem 5.1, for nonnegative random vectors with l_∞-spherical densities, the MPF_2 property is stronger than the property of having a Schur-concave survival function.

We now proceed to a result that compares random vectors with l_∞-spherical densities in a fairly strong stochastic order. Recall (see, e.g., Shaked and Shanthikumar [20, Sect. 4.E]) that an n-dimensional random vector $\boldsymbol{X}$ with a density function f is said to be smaller than an n-dimensional random vector $\boldsymbol{Y}$ with a density function g in the *multivariate likelihood ratio order* (denoted as $\boldsymbol{X} \leq_{\mathrm{lr}} \boldsymbol{Y}$) if

$$f(\boldsymbol{x})g(\boldsymbol{y}) \leq f(\boldsymbol{x} \wedge \boldsymbol{x})g(\boldsymbol{y} \vee \boldsymbol{y}), \qquad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n.$$

Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ have l_∞-spherical densities f and g, respectively, given by

$$f(x_1, x_2, \ldots, x_n) = \phi_1\Big(\max_{1 \leq i \leq n} x_i\Big), \qquad (x_1, x_2, \ldots, x_n) \in \mathbb{R}_+^n,$$

and

$$g(x_1, x_2, \ldots, x_n) = \phi_2\Big(\max_{1 \leq i \leq n} x_i\Big), \qquad (x_1, x_2, \ldots, x_n) \in \mathbb{R}_+^n,$$

for some generators ϕ_1 and ϕ_2. For simplicity, we assume that $\boldsymbol{X}$ and $\boldsymbol{Y}$ have the same support; that is, that $\{x : \phi_1(x) > 0\} = \{x : \phi_2(x) > 0\}$.

THEOREM 5.4: *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ have l_∞-spherical densities f and g as above with the same support. Then, $\boldsymbol{X} \leq_{\mathrm{lr}} \boldsymbol{Y}$ if and only if*

$$\phi_1 \text{ is decreasing and } \phi_2/\phi_1 \text{ is increasing on their common support.} \qquad \textbf{(5.4)}$$

For the proof of Theorem 5.4, we need the following lemma.

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LEMMA 5.5: *Let ϕ be a function which is decreasing on its support. Then*

$$\frac{\phi\Big(\max\limits_{1\le i\le n} y_i\Big)}{\phi\Big(\max\limits_{1\le i\le n}(x_i\wedge y_i)\Big)} \le \frac{\phi\Big(\max\limits_{1\le i\le n}(x_i\vee y_i)\Big)}{\phi\Big(\max\limits_{1\le i\le n} x_i\Big)}$$

for all x_i and y_i in the support of ϕ, $i=1,2,\ldots,n$.

PROOF: Note that $\max_{1\le i\le n}(x_i\vee y_i)$ must be equal either to $\max_{1\le i\le n} x_i$ or to $\max_{1\le i\le n} y_i$. In either case the above inequality follows from the stated monotonicity of ϕ. ■

PROOF OF THEOREM 5.4: First, assume that (5.4) holds. Let two points x_i and y_i be in the common support of ϕ_1 and ϕ_2, $i=1,2,\ldots,n$. Then

$$\begin{aligned}\frac{\phi_2\Big(\max\limits_{1\le i\le n} y_i\Big)}{\phi_1\Big(\max\limits_{1\le i\le n}(x_i\wedge y_i)\Big)} &= \frac{\phi_2\Big(\max\limits_{1\le i\le n} y_i\Big)}{\phi_1\Big(\max\limits_{1\le i\le n} y_i\Big)}\,\frac{\phi_1\Big(\max\limits_{1\le i\le n} y_i\Big)}{\phi_1\Big(\max\limits_{1\le i\le n}(x_i\wedge y_i)\Big)}\\ &\le \frac{\phi_2\Big(\max\limits_{1\le i\le n}(x_i\vee y_i)\Big)}{\phi_1\Big(\max\limits_{1\le i\le n}(x_i\vee y_i)\Big)}\,\frac{\phi_1\Big(\max\limits_{1\le i\le n}(x_i\vee y_i)\Big)}{\phi_1\Big(\max\limits_{1\le i\le n} x_i\Big)}\\ &= \frac{\phi_2\Big(\max\limits_{1\le i\le n}(x_i\vee y_i)\Big)}{\phi_1\Big(\max\limits_{1\le i\le n} x_i\Big)},\end{aligned}$$

where the inequality follows from Lemma 5.5 and the monotonicity of ϕ_2/ϕ_1; that is, $f(\boldsymbol{x})g(\boldsymbol{y}) \le f(\boldsymbol{x}\wedge\boldsymbol{y})g(\boldsymbol{x}\vee\boldsymbol{y})$.

Conversely, suppose that $\boldsymbol{X}\le_{\mathrm{lr}}\boldsymbol{Y}$; that is,

$$\phi_1\Big(\max_{1\le i\le n} x_i\Big)\phi_2\Big(\max_{1\le i\le n} y_i\Big) \le \phi_1\Big(\max_{1\le i\le n}(x_i\wedge y_i)\Big)\phi_2\Big(\max_{1\le i\le n}(x_i\vee y_i)\Big). \qquad \textbf{(5.5)}$$

Let $t''\ge t'\ge 0$ be arbitrary. Substituting $\boldsymbol{x}=(t',t'',\ldots,t'')$ and $\boldsymbol{y}=(t'',t',\ldots,t')$ in (5.5), we obtain $\phi_1(t'')\phi_2(t'')\le\phi_1(t')\phi_2(t'')$; that is, ϕ_1 is decreasing. Substituting $\boldsymbol{x}=(t',t'',\ldots,t'')$ and $\boldsymbol{y}=(t',t',\ldots,t')$ in (5.5), we obtain $\phi_1(t'')\phi_2(t')\le\phi_1(t')\phi_2(t'')$; that is, ϕ_2/ϕ_1 is increasing. ■

Example 5.6: Pairs of generators that satisfy (5.4) are easy to find and identify. For example, inspired by Example 4.6, we note that (5.4) holds if

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$$\phi_1(x) = \frac{1}{n!}\exp\{-\theta_1 x\} \quad \text{and} \quad \phi_2(x) = \frac{1}{n!}\exp\{-\theta_2 x\}, \qquad x \geq 0,$$

with $0 \leq \theta_2 \leq \theta_1$. These generators have also been considered in Example 3.30 of Spizzichino [22]. Inspired by Example 4.7, we also note that (5.4) holds if

$$\phi_1(x) \propto \frac{\lambda_1 + n}{n!(1+x)^{\lambda_1+n+1}} \quad \text{and} \quad \phi_2(x) \propto \frac{\lambda_2 + n}{n!(1+x)^{\lambda_2+n+1}}, \qquad x \geq 0,$$

with $0 \leq \lambda_2 \leq \lambda_1$. Finally, inspired by Example 4.8, we note that (5.4) holds if

$$\phi_1(x) = \frac{(l_1+n)!\beta_1^{l_1+1}}{l_1!(\beta_1+x)^{l_1+n+1}} \quad \text{and} \quad \phi_2(x) = \frac{(l_2+n)!\beta_2^{l_2+1}}{l_2!(\beta_2+x)^{l_2+n+1}}, \qquad x \geq 0,$$

with $0 \leq \beta_1 \leq \beta_2$ and integers $0 \leq l_2 \leq l_1$.

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