(iii) For the hyperbola $yz - zx - xy = 0$ we have $l = -Y - Z$, $m = Z - X, n = -X + Y$, leading to

$$\phi = \alpha(-Y + Z)^2 + \beta(-X + 2Y + Z)^2 + \gamma(X - Y - 2Z)^2$$

and

$$\psi = (-X + 2Y + Z)(X - Y - 2Z) - (X - Y - 2Z)(-Y + Z)$$
$$-(-Y + Z)(-X + 2Y + Z).$$

Then when $A = B = C$ we find the least $\rho = R$ at the vertices of the hyperbola $A[1, 0, 0]$ and $\left[\frac{1}{5}, \frac{2}{5}, \frac{2}{5}\right]$.

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10.1017/mag.2018.25

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102.17 Dimensional analogues of two figurate identities

Definitions

Our figurate numbers are rows of unit line segments in one dimension, polygons dissected from close packings of circles in two dimensions, and polyhedra dissected from close packings of spheres in three. According to context we shall use a symbol to mean either a figure or the number it represents:

L_s will mean both s and that number of line segments arranged in a row;

T_s will mean both $\frac{1}{2}s(s + 1)$ and that number of circles arranged in an equilateral triangle;

Tet_s will mean both $\frac{1}{6}s(s + 1)(s + 2)$ and that number of spheres arranged in a regular tetrahedron;

Oct_s will mean both $\frac{1}{3}s(2s^2 + 1)$ and that number of spheres arranged in a regular octahedron;

s^2 will mean both that number and so many circles arranged in a rhombus, equivalent to a square in figurate terms;

s^3 will mean both that number and so many spheres arranged in a rhombohedron, equivalent to a cube in figurate terms.

By algebraic symmetry, k and n may be transposed throughout.

https://doi.org/10.1017/mag.2018.26 Published online by Cambridge University Press

Aim

Our object is to take two identities expressed in figurate form and try to establish their analogues in one, two and three dimensions. Hence the labels (I_1), (I_2), (I_3); (II_1), (II_2), (II_3). We shall offer proofs without words in one dimension (Figure 1), in two dimensions (Figure 2), and in three dimensions. The last case will require us to animate Figure 2 in two different ways.

In Figures 1 and 2, we make the arbitrary choice $k = 4$ and $n = 3$.

In one dimension

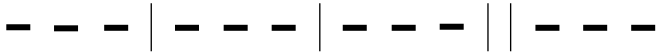


FIGURE 1

$$(\boldsymbol{I_1})\ L_{kn} = L_k L_n \qquad (\boldsymbol{II_1})\ L_{kn} = L_{k-1}n + 1L_n$$

In two dimensions

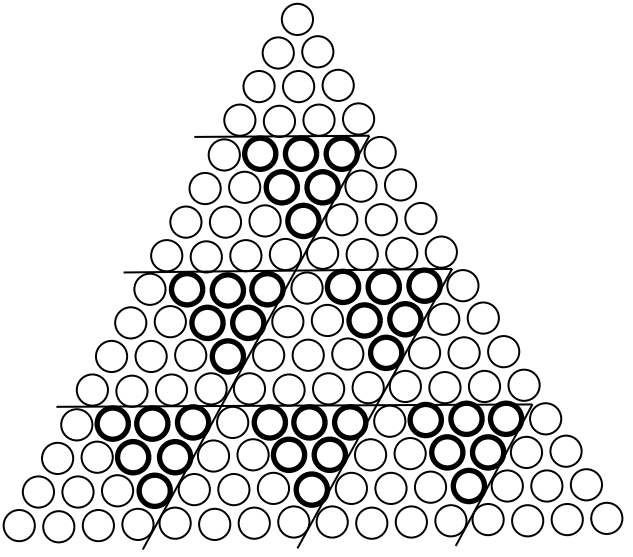


FIGURE 2

$$(\boldsymbol{I_2})\ T_{kn} = T_k T_n + T_{k-1} T_{n-1} \quad \text{(Compare [1])}$$

$$(\boldsymbol{II_2})\ T_{kn} = T_{k-1}n^2 + kT_n \quad \text{(Compare [2])}$$

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In three dimensions

(a) Figure 2 shows one of the 8 faces of the octahedron Oct_{kn}. The Oct_n (thin circles) occupy the same sites in Oct_{kn} that individual spheres do in Oct_k. They therefore account for $Oct_k Oct_n$ spheres in total.

(b) The interstices (thick circles) are filled by Tet_{n-1}. In successive layers from the centre there are 1, 3, 6, … , T_{k-1} for each face. They therefore account for $8Tet_{k-1}Tet_{n-1}$ spheres in total.

Combining (a), (b), we have

$(\boldsymbol{I_3})$ $Oct_{kn} = Oct_k Oct_n + 8Tet_{k-1}Tet_{n-1}$.

Alternatively, we now take Figure 2 as the base of Tet_{kn} and assemble the layers in k sets of n.

In the set extending from layer sn to layer $(s + 1)n$, taking s from 0 to $k - 1$, the following happen:

(c) We complete T_{k-s-1} rhombohedra, n^3. These therefore account for a total of $Tet_{k-1}n^3$ spheres. (Preserving the orientation of Figure 2, each successive rhombus is displaced by an identical vector in the direction of the upper right face.)

(d) We complete $(k - s)Tet_n$ upper right. These therefore account for a total of $T_k Tet_n$ spheres.

(e) But we must excise from each of the $(k - s - 1)$ rhombohedra overreaching the upper right face, Tet_{n-2}. (Figure 3 shows part of the layers $(sn - 1)$ and sn for $n = 6$, and how the rhombic section overlaps the boundary.) We must therefore subtract a total of $T_{k-1}Tet_{n-2}$ spheres.

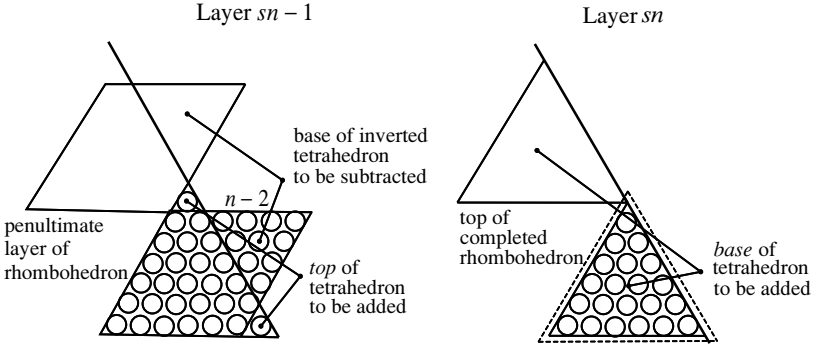


FIGURE 3

Combining (c), (d), (e), we have

$(\boldsymbol{II_3})$ $Tet_{kn} = Tet_{k-1}n^3 + T_k Tet_n - T_{k-1}Tet_{n-2}$.

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Comparison

$(\boldsymbol{I_1})\ L_{kn} = L_k L_n$

$(\boldsymbol{I_2})\ T_{kn} = T_k T_n + T_{k-1} T_{n-1}$

$(\boldsymbol{I_3})\ Oct_{kn} = Oct_k Oct_n + 8 Tet_{k-1} Tet_{n-1}.$

In one dimension there are no gaps between the L_n units. Thus there is only one term on the right hand side of (I_1). Comparing (I_2) and (I_3), we preserve the overall form, but Oct_s is not the 3-dimensional analogue of T_s and in (I_3) we have the coefficient 8 before the second term on the right.

$(\boldsymbol{II_1})\ L_{kn} = L_{k-1} n + 1 L_n$

$(\boldsymbol{II_2})\ T_{kn} = T_{k-1} n^2 + k T_n$

$(\boldsymbol{II_3})\ Tet_{kn} = Tet_{k-1} n^3 + T_k Tet_n - T_{k-1} Tet_{n-2}.$

Comparing (II_1), (II_2), (II_3), we preserve an exact dimensional analogy in both multiplier and multiplicand in the first two terms on the right but at the expense of having to add a third term in (II_3).

(I) and (II) can be expressed in many different ways according to the mode of dissection. It may be that readers can find analogous forms which are structurally simpler than ours. They may find useful the following identities (which they can readily confirm algebraically or geometrically):

$$Oct_s = Tet_s + 2Tet_{s-1} + Tet_{s-2}; \tag{1}$$

$$s^3 = 2Tet_{s-1} + Oct_s; \tag{2}$$

$$(1), (2) \Rightarrow s^3 = Tet_s + 4Tet_{s-1} + Tet_{s-2}. \tag{3}$$

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10.1017/mag.2018.26

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https://doi.org/10.1017/mag.2018.26 Published online by Cambridge University Press