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A limit theorem for sojourns near indifferent fixed points of one-dimensional maps

MAXIMILIAN THALER
Institute of Mathematics, University of Salzburg, Hellbrunnerstraße 34, A-5020 Salzburg, Austria
(e-mail: Maximilian.Thaler@sbg.ac.at)

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Abstract. For a class of one-dimensional maps with two indifferent fixed points of the same order, a parallel to the arc-sine law for the number of positive partial sums of a real random walk is proved.

1. *Introduction*

In this paper we consider a class of one-dimensional maps with two indifferent fixed points of the same order giving rise to an infinite invariant measure. The purpose is to prove a distributional limit theorem for the number of visits to neighbourhoods of these points. Among the classical analogs there is the famous arc-sine law for the number of positive partial sums of a real random walk, discovered by Lévy [**Le**] and refined and extended in a large number of articles as for example in [**EK, ChF, An, Sp, La**]. We refer to [**F1**, Ch. III] and [**F2**, Ch. XII] for basic information and historical comments. Both the theorem we present and the structure of its proof were particularly inspired by [**La**].

The theorem proven in this paper applies, for example, to Boole's transformation $T : \mathbb{R} \to \mathbb{R}$, defined by

$$T(x) = x - 1/x.$$

This transformation is known to have several probabilistic properties in common with the simple symmetric random walk on the integers (see [**Aa1**]). Our result reveals a further interesting parallel as it asserts that

$$\lim_{n\to\infty} \nu\left(\left\{\frac{1}{n}\sum_{j=0}^{n-1} 1_{(0,\infty)} \circ T^j \le x\right\}\right) = \frac{2}{\pi}\arcsin\sqrt{x}, \quad 0 \le x \le 1,$$

for each probability measure ν absolutely continuous with respect to Lebesgue measure.

It also confirms a conjecture in [**In1**] and, together with Theorem 1 in [**La**], implies the main result in [**In2**].

We have structured the paper as follows. In §2 we state the limit theorem. In §3 we derive a recursion formula for the moments of the limiting distributions and discuss their connection with stable laws. The proof of the theorem is given in §4. The final section contains a brief discussion of the boundary cases.

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2. *Statement of the main result*

We shall consider maps $T : [0, 1] \to [0, 1]$ which fulfil the following conditions for some $c \in (0, 1)$:

(1) the restrictions $T : (0, c) \to (0, 1)$, $T : (c, 1) \to (0, 1)$ are increasing, onto and C^2, admitting C^2-extensions to the respective closed intervals;

(2) $T' > 1$ on $(0, c] \cup [c, 1)$; $T'(0) = T'(1) = 1$;

(3) T is convex (concave) on some neighbourhood of 0 (1).

Let $\mathcal{B}$ denote the Borel σ-field on $[0, 1]$, λ the Lebesgue measure on $\mathcal{B}$ and $\mathcal{D}$ the set of all λ-absolutely continuous probability measures on $\mathcal{B}$. As proved in [**T1, T2**], T is conservative and ergodic with respect to λ, and admits a σ-finite invariant measure μ equivalent to λ. The density $d\mu/d\lambda$ has a version h of the form

$$h(x) = h_0(x)\frac{x\,(1-x)}{(x - f_0(x))(f_1(x) - x)}, \quad x \in (0, 1),$$

where

$$f_0 = (T_{|_{[0,c]}})^{-1}, \quad f_1 = (T_{|_{[c,1]}})^{-1}$$

and h_0 is continuous and positive on $[0, 1]$. Moreover, T is pointwise dual ergodic, i.e. there exists a sequence $\{a_n\}$ of positive numbers, called the return sequence of T, so that

$$\frac{1}{a_n}\sum_{j=0}^{n-1}\hat{T}^j v \longrightarrow \int_0^1 v\,d\mu \quad \text{a.e.}$$

for all $v \in L_1(\mu)$ where $\hat{T}$ is the Perron–Frobenius operator with respect to the invariant measure μ (cf. [**Aa1, T3**]).

We shall be interested in the asymptotic distributional behaviour of the sequences

$$\left\{\sum_{j=0}^{n-1} 1_{[0,\gamma]} \circ T^j : n \geq 1\right\} \left(\left\{\sum_{j=0}^{n-1} 1_{[\gamma,1]} \circ T^j : n \geq 1\right\}\right), \quad 0 < \gamma < 1.$$

In view of the C^2-conditions in (1) the above formula for the invariant density shows that $\mu([0, \gamma]) = \mu([\gamma, 1]) = \infty$ for all $\gamma \in (0, 1)$.

Let $\{X_n\}$ be any sequence of real measurable functions defined on $[0, 1]$, X a real random variable and $\nu \in \mathcal{D}$. We use the notation

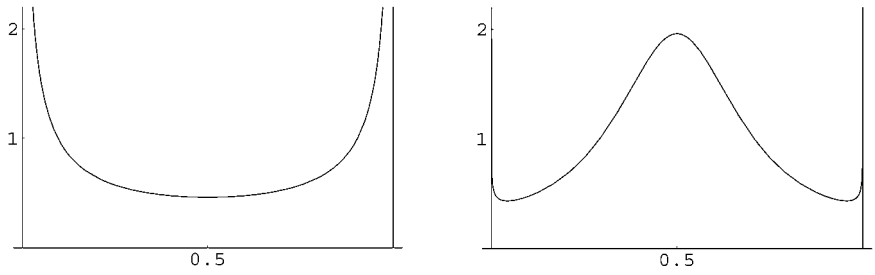
$$X_n \xrightarrow{\nu} X$$

if $\{X_n\}$ converges in distribution to X with respect to ν. If $X_n \xrightarrow{\nu} X$ for all $\nu \in \mathcal{D}$, we use

$$X_n \xrightarrow{\mathcal{L}} X.$$

Regarding the asymptotic distributional behaviour of the number of visits to measurable sets A with $\mu(A) < \infty$, we recall the following well-known result.

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(a) (b)

FIGURE 1. Graph of the density of $L_{\alpha,1/2}$ for (a) $\alpha = 0.4$ and (b) $\alpha = 0.8$.

THE DARLING–KAC THEOREM. [**Aa1, Aa2**] *If the return sequence* $\{a_n\}$ *is regularly varying with index* α ($\alpha \in [0, 1]$)*, then for all* $A \in \mathcal{B}$ *with* $\mu(A) < \infty$*,*

$$\frac{1}{a_n} \sum_{j=0}^{n-1} 1_A \circ T^j \xrightarrow{\mathcal{L}} \mu(A) Y_\alpha,$$

where Y_α *is a positive random variable distributed according to the normalized Mittag–Leffler law of order* α*, i.e.*

$$E(Y_\alpha^k) = k! \frac{(\Gamma(1+\alpha))^k}{\Gamma(1+k\alpha)}, \quad k \in \mathbb{N}_0.$$

As the example mentioned in the introduction shows, we encounter completely different laws if we consider measurable sets A with $\mu(A) = \infty$.

For $0 < \alpha, \beta < 1$, $b = (1-\beta)/\beta$, let $L_{\alpha,\beta}$ denote a random variable with (values in [0, 1] and) distribution function

$$G_{\alpha,\beta}(x) = \frac{1}{\pi\alpha} \operatorname{arccot}\left(\frac{((1-x)/x)^\alpha}{b \sin \pi\alpha} + \cot \pi\alpha \right), \quad 0 < x \le 1,$$

i.e. $L_{\alpha,\beta}$ has the density

$$g_{\alpha,\beta}(x) = \frac{b \sin \pi\alpha}{\pi} \frac{x^{\alpha-1}(1-x)^{\alpha-1}}{b^2 x^{2\alpha} + 2bx^\alpha (1-x)^\alpha \cos \pi\alpha + (1-x)^{2\alpha}}, \quad 0 < x < 1$$

(see Figure 1).

We shall see in §3 that these distributions occur in a natural way, and that

$$E(L_{\alpha,\beta}) = \beta \quad \text{and} \quad V(L_{\alpha,\beta}) = (1-\alpha)\beta(1-\beta), \quad 0 < \alpha, \beta < 1.$$

Our main result is the following analogue to Theorem 1 in [**La**].

THEOREM. *Let* $T : [0, 1] \to [0, 1]$ *satisfy the conditions (1)–(3). If*
(4a) $Tx - x$ *is regularly varying at zero with index* $p + 1$, $p \in (1, \infty)$*; and*
(4b) $(1-x) - T(1-x) \sim a^p (Tx - x)$ $(x \to 0)$ *for some* $a \in (0, \infty)$*,*

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then

$$\frac{1}{n}\sum_{j=0}^{n-1} 1_{[0,\gamma]} \circ T^j \xrightarrow{\mathcal{L}} L_{\alpha,\beta} \quad \text{for } 0 < \gamma < 1,$$

where

$$\alpha = \frac{1}{p} \quad \text{and} \quad \beta = \frac{T'(c^-)}{T'(c^-) + T'(c^+)/a}.$$

Conversely, if

$$\frac{1}{n}\sum_{j=0}^{n-1} 1_{[0,\gamma]} \circ T^j \xrightarrow{\nu} L$$

for some $\gamma \in (0,1)$, some $\nu \in \mathcal{D}$ and some random variable L which is not degenerated and not concentrated on the two-point set $\{0,1\}$, then the conditions (4a) and (4b) hold with

$$p = 1 + \frac{V(L)}{E(L(1-L))}, \quad a = \frac{E(L)}{1-E(L)}\frac{T'(c^+)}{T'(c^-)},$$

and thus

$$L \overset{\text{dist}}{=} L_{\alpha,\beta} \quad \text{with } \alpha = 1/p, \beta = E(L).$$

Comments. (i) Since

$$\operatorname{arccot} x = \begin{cases} \arcsin(1/\sqrt{1+x^2}), & x \geq 0, \\ \pi - \arcsin(1/\sqrt{1+x^2}), & x \leq 0, \end{cases}$$

$G_{\alpha,\beta}$ can be expressed by means of the arc-sine function, for example

$$G_{1/2,1/2}(x) = \frac{2}{\pi}\arcsin\sqrt{x}, \quad 0 \leq x \leq 1.$$

As illustrated by Figure 1, the densities $g_{\alpha,\beta}$ are not $\cup$-shaped in general. For example, $g_{\alpha,1/2}$ is $\cup$-shaped if and only if $\alpha \leq \cos(\pi\alpha/2)$.

(ii) Condition (4a) says that $Tx - x = \ell(x)x^{p+1}$, $0 < x < c$, where ℓ is slowly varying at zero, i.e. $\lim_{x\to 0} \ell(\kappa x)/\ell(x) = 1$ for all $\kappa > 0$. Hence the conditions (4a) and (4b) are obviously satisfied if

$$Tx - x \sim a_0\, x^{p+1} \ (x \to 0) \quad \text{and} \quad x - Tx \sim a_1(1-x)^{p+1} \ (x \to 1)$$

with $p > 1$ and constants $a_0, a_1 > 0$.

Regarding the necessary background from the theory of regular variation we refer to [**BGT**] and [**F2**].

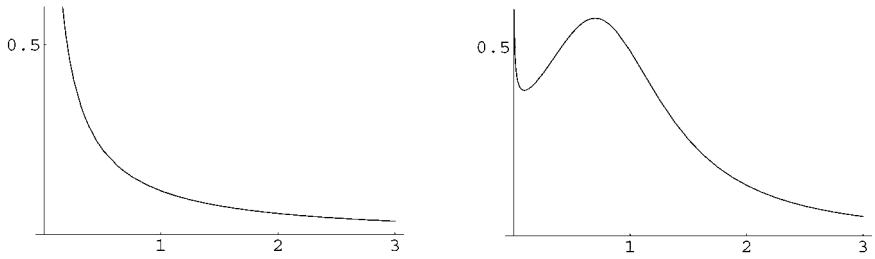
(iii) Suppose the conditions (1)–(3) and (4a), (4b) are satisfied. Then, for $\nu \in \mathcal{D}$ and $0 < \gamma < 1$,

$$\lim_{n\to\infty} \int_0^1 \left(\frac{1}{n}\sum_{j=0}^{n-1} 1_{[0,\gamma]} \circ T^j\right) d\nu = E(L_{\alpha,\beta}) = \beta.$$

Thus the parameter β describes the balance between the number of visits to $[0,\gamma]$ and $[\gamma,1]$.

By noting that $\lim_{n\to\infty}(1/n)\sum_{j=0}^{n-1} 1_A \circ T^j = 0$ λ-a.e. for all $A \in \mathcal{B}$ with $\mu(A) < \infty$, it is clear why our statements do not depend on $\gamma \in (0,1)$.

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(a) (b)

FIGURE 2. Graph of the density of $(1/L_{\alpha,1/2}) - 1$ for (a) $\alpha = 0.4$ and (b) $\alpha = 0.8$.

(iv) We mention alternative formulations of the limit theorem. Let $0 < \gamma_0, \gamma_1 < 1$, and let

$$S_n^{(0)} = \sum_{j=0}^{n-1} 1_{[0,\gamma_0]} \circ T^j, \quad S_n^{(1)} = \sum_{j=0}^{n-1} 1_{[\gamma_1,1]} \circ T^j, \quad n \geq 1.$$

Since

$$S_n^{(0)} + S_n^{(1)} = n \pm \sum_{j=0}^{n-1} 1_A \circ T^j,$$

where A is an interval with $\mu(A) < \infty$, we have

$$S_n^{(1)}/S_n^{(0)} - (n/S_n^{(0)} - 1) \to 0 \quad \lambda\text{-a.e.}$$

Therefore the convergence statement in the theorem is equivalent to

$$S_n^{(1)}/S_n^{(0)} \xrightarrow{\mathcal{L}} (1/L_{\alpha,\beta}) - 1.$$

The limiting distribution of the ratios is given by the density

$$y \mapsto \frac{b \sin \pi\alpha}{\pi} \frac{y^{\alpha-1}}{y^{2\alpha} + 2by^{\alpha} \cos \pi\alpha + b^2}, \quad y > 0$$

(see Figure 2).

This type of distribution also occurs if we consider the associated return time processes. Let φ_n denote the time of the nth visit to the interval $[0, \gamma]$ after time 0 where $\gamma \in (0, 1)$. Then the relation

$$\varphi_n \leq m \Longleftrightarrow \sum_{j=1}^{m} 1_{[0,\gamma]} \circ T^j \geq n \quad (n, m \in \mathbb{N})$$

shows that the convergence statement in the theorem is also equivalent to

$$\frac{1}{n}\varphi_n \xrightarrow{\mathcal{L}} 1/L_{\alpha,\beta}.$$

Examples. To get the arc-sine law for Boole's transformation

$$T(x) = x - 1/x, \quad x \in \mathbb{R} \setminus \{0\},$$

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we note that $T = \phi \circ S \circ \phi^{-1}$ where

$$\phi(x) = \frac{1}{1-x} - \frac{1}{x}, \quad x \in (0,1),$$
$$S(x) = \frac{x(1-x)}{1-x-x^2}, \quad x \in (0, 1/2)$$

and

$$S(x) = 1 - S(1-x), \quad x \in (1/2, 1).$$

It is readily seen that S belongs to our class of maps with $\alpha = \beta = 1/2$. Thus the limit theorem mentioned in the introduction is valid.

More generally, let $r > (\sqrt{5}-1)/2$ and let

$$T(x) = x - 1/x^r, \ x > 0; \quad T(x) = -T(-x), \ x < 0.$$

Then

$$\frac{1}{n}\sum_{j=0}^{n-1} 1_{(0,\infty)} \circ T^j \xrightarrow{\mathcal{L}} L_{\alpha,1/2} \quad \text{with } \alpha = 1/(r(r+1))$$

(cf. [**R**] and [**T3**, Example 4]).

A family of maps covering the whole parameter region is given by

$$T(x) = \begin{cases} x + (1-c)(x/c)^{p+1}, & 0 < x < c, \\ x - c((1-x)/(1-c))^{p+1}, & c < x < 1, \end{cases}$$

with $0 < c < 1$ and $p > 1$. We have

$$b = \frac{1+pc}{1+p(1-c)}\left(\frac{1-c}{c}\right)^{2/p},$$

so that indeed all parameters can be obtained by choosing p and c suitably. If $p > 1+\sqrt{5}$, different c-values may yield the same β. For example, if $p = 4$, $L_{\alpha,\beta}$ is symmetric for $c \in \{(2-\sqrt{3})/4, 1/2, (2+\sqrt{3})/4\}$.

3. *Formal and heuristic preparations*

In this section we prove a recursion formula for the moments of the limiting distributions and explain their connection with stable laws, which is the key to an intuitive understanding of our theorem. This will be clarified by sketching Lévy's original approach to the corresponding arc-sine law for the coin-tossing case. We shall, however, not make use of this connection in the proof of the theorem.

In [**La**] the limiting distributions are obtained by means of an inversion formula for Stieltjes transforms. Our approach is based on the above-mentioned recursion formula for the moments, to be derived directly from the densities. It is interesting to note that a closed formula for the distribution function is obtained by a plain integration procedure, whereas an advanced technique is required to calculate the expected value. We shall do this using complex analysis.

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For $0 < \alpha < 1$ let f_α denote the probability density

$$f_\alpha(y) = \frac{\sin \pi\alpha}{\pi} \frac{y^{\alpha-1}}{y^{2\alpha} + 2y^\alpha \cos \pi\alpha + 1}, \quad y > 0,$$

(cf. comment (iv) in §2). For $z \in \mathbb{C}$ let z^α be specified by

$$z^\alpha = r^\alpha e^{i\varphi\alpha} \quad \text{if } z = re^{i\varphi} \quad \text{and} \quad \varphi \in [0, 2\pi),$$

so that $z \mapsto z^\alpha$ is analytic on $\mathbb{C} \setminus [0, \infty)$, and $\arg z^\alpha \in (0, 2\pi\alpha)$ for $z \in \mathbb{C} \setminus [0, \infty)$. The formula in the following lemma is the basic tool for our purposes.

LEMMA. *If q is a rational function with poles $z_1, \ldots, z_m$ such that $z_j \notin [0, \infty)$, $1 \le j \le m$, and $\lim_{|z|\to\infty} |q(z)| = 0$, then*

$$\int_0^\infty q(y) f_\alpha(y)\, dy = \sum_{j=1}^m \operatorname{Res}(H, z_j),$$

where

$$H(z) = -q(z) \frac{z^{\alpha-1}}{z^\alpha + e^{i\pi\alpha}}, \quad z \in \mathbb{C} \setminus [0, \infty).$$

Proof. First of all, note that the integral exists and that

$$\frac{1}{y^{2\alpha} + 2y^\alpha \cos \pi\alpha + 1} = \frac{1}{2i \sin \pi\alpha} \left(\frac{1}{y^\alpha + e^{-i\pi\alpha}} - \frac{1}{y^\alpha + e^{i\pi\alpha}} \right), \quad y \ge 0,$$

so that the asserted identity becomes

$$\int_0^\infty q(y)\, y^{\alpha-1} \left(\frac{1}{y^\alpha + e^{-i\pi\alpha}} - \frac{1}{y^\alpha + e^{i\pi\alpha}} \right) dy = 2\pi i \sum_{j=1}^m \operatorname{Res}(H, z_j).$$

We calculate the integral choosing a path of integration which is commonly used to calculate Mellin transforms (see, e.g., [**Lg**]). Loosely speaking, for $0 < r < R$, the path of integration is $\gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ where

$$\gamma_1(y) = y,\ r \le y \le R; \quad \gamma_2(t) = Re^{it},\ 0 \le t \le 2\pi;$$
$$(-\gamma_3)(y) = e^{2\pi i} y,\ r \le y \le R; \quad \gamma_4(t) = re^{i(2\pi - t)},\ 0 \le t \le 2\pi.$$

The factor $e^{2\pi i}$ in γ_3 indicates the increase of the argument by 2π after running through γ_2.

We have

$$\int_{\gamma_1} H(z)\, dz + \int_{\gamma_3} H(z)\, dz = -\int_r^R q(y) \frac{y^{\alpha-1}}{y^\alpha + e^{i\pi\alpha}}\, dy + \int_r^R q(y) \frac{e^{2i\pi(\alpha-1)} y^{\alpha-1}}{e^{2i\pi\alpha} y^\alpha + e^{i\pi\alpha}}\, dy$$
$$= \int_r^R q(y) y^{\alpha-1} \left(\frac{1}{y^\alpha + e^{-i\pi\alpha}} - \frac{1}{y^\alpha + e^{i\pi\alpha}} \right) dy.$$

Since

$$\lim_{R\to\infty} \int_{\gamma_2} H(z)\, dz = \lim_{r\to 0} \int_{\gamma_4} H(z)\, dz = 0$$

and $z^{\alpha-1}/(z^\alpha + e^{i\pi\alpha})$ is analytic on $\mathbb{C} \setminus [0, \infty)$, Cauchy's Theorem yields the result. □

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With the aid of this lemma we easily get the desired information. We mention that we use the symbol $\binom{x}{n}$ ($x \in \mathbb{R}$, $n \in \mathbb{N}_0$) in the usual sense: $\binom{x}{n} := x(x-1)\cdots(x-n+1)/n!$, $n \in \mathbb{N}$; $\binom{x}{0} := 1$. In particular, $\operatorname{sign}\binom{x}{n} = (-1)^{n-1}\operatorname{sign} x$, $x < 1$, $n \geq 1$.

PROPOSITION 1. *Let $0 < \alpha, \beta < 1$, and let $L_{\alpha,\beta}$ be as in §2. Then, for all $r \in \mathbb{N}$,*

$$E(L_{\alpha,\beta}^r) = (-1)^r \beta \left\{ \sum_{j=0}^{r-1} (-1)^{j+1} \binom{\alpha}{r-j} E(L_{\alpha,\beta}^j) + \binom{\alpha-1}{r} \right\}.$$

In particular,

$$E(L_{\alpha,\beta}) = \beta \quad \text{and} \quad E(L_{\alpha,\beta}^2) = \beta(1-\alpha+\alpha\beta).$$

Proof. Let Y be a random variable with density f_α, and let $b = (1-\beta)/\beta$ as before. Then

$$L_{\alpha,\beta} \stackrel{\text{dist}}{=} \frac{1}{1+b^{1/\alpha}Y},$$

so

$$E(L_{\alpha,\beta}^r) = E\left(\frac{1}{1+b^{1/\alpha}Y}\right)^r, \quad r \geq 0.$$

Therefore we get for $x \in \mathbb{R}$, $|x| < 1$,

$$\sum_{r=0}^{\infty} E(L_{\alpha,\beta}^r) x^r = E\left(\sum_{r=0}^{\infty} \left(\frac{x}{1+b^{1/\alpha}Y} \right)^r \right) = 1 + E\left(\frac{x}{1-x+b^{1/\alpha}Y} \right).$$

Using the lemma we see that

$$\begin{aligned} E\left(\frac{x}{1-x+b^{1/\alpha}Y}\right) &= \operatorname{Res}\left(-\frac{x}{1-x+b^{1/\alpha}z} \frac{z^{\alpha-1}}{z^\alpha + e^{i\pi\alpha}}, -\frac{1-x}{b^{1/\alpha}} \right) \\ &= -\frac{x}{b^{1/\alpha}} \left(\frac{z^{\alpha-1}}{z^\alpha + e^{i\pi\alpha}} \right)_{z = e^{i\pi}(1-x)/b^{1/\alpha}} = x \frac{(1-x)^{\alpha-1}}{(1-x)^\alpha + b}. \end{aligned}$$

Therefore

$$\sum_{r=0}^{\infty} E(L_{\alpha,\beta}^r) x^r = \frac{(1-x)^{\alpha-1}+b}{(1-x)^\alpha + b},$$

i.e.

$$\sum_{r=0}^{\infty} \left(\sum_{j=0}^{r} (-1)^{r-j} \binom{\alpha}{r-j} E(L_{\alpha,\beta}^j) + b E(L_{\alpha,\beta}^r) \right) x^r = \sum_{r=0}^{\infty} (-1)^r \binom{\alpha-1}{r} x^r + b,$$

which is equivalent to the recursion formula to be proved. □

PROPOSITION 2. *Let X, Y be independent random variables, both of them distributed according to the one-sided stable law with Laplace transform $t \mapsto e^{-t^\alpha}$, $t \geq 0$ $(0 < \alpha < 1)$. Then Y/X has density f_α.*

Proof. We denote the Laplace transform of a random variable Z by φ_Z, i.e. $\varphi_Z(t) = E(e^{-tZ})$, $t \geq 0$. A standard calculation shows that

$$\varphi_{Y/X}(t) = \varphi_{1/X^\alpha}(t^\alpha), \quad t \geq 0,$$

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(see also [**F2**, Problem 10 in XIII.11]). The variable $1/X^\alpha$ has the Mittag–Leffler law specified by

$$\varphi_{1/X^\alpha}(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(1+k\alpha)} t^k, \quad t \geq 0,$$

(see [**F2**, Example (b) in XIII.8]). Therefore

$$\varphi_{Y/X}(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(1+k\alpha)} t^{k\alpha}, \quad t \geq 0.$$

To see that this is the Laplace transform of f_α we use the relation

$$\int_0^\infty e^{-ty} f_\alpha(y)\, dy = \lim_{n\to\infty} \int_0^\infty (1+ty/n)^{-n} f_\alpha(y)\, dy, \quad t \geq 0.$$

In view of the lemma we obtain, for $t > 0$ and $n \geq 1$,

$$\begin{aligned}\int_0^\infty (1+ty/n)^{-n} f_\alpha(y)\, dy &= \operatorname{Res}\left(-(1+tz/n)^{-n} \frac{z^{\alpha-1}}{z^\alpha + e^{i\pi\alpha}}, -\frac{n}{t}\right) \\ &= \left(\frac{n}{t}\right)^n \frac{g^{(n-1)}(-n/t)}{(n-1)!},\end{aligned}$$

where

$$g(z) = -\frac{z^{\alpha-1}}{z^\alpha + e^{i\pi\alpha}}.$$

If $z \in \mathbb{C} \setminus [0,\infty)$, $|z| > 1$, we have

$$g(z) = -\frac{1}{z(1+(-1/z)^\alpha)} = \sum_{k=0}^{\infty} (-1)^k (-1/z)^{1+k\alpha},$$

and thus

$$g^{(n-1)}(z) = \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(n+k\alpha)}{\Gamma(1+k\alpha)} (-1/z)^{n+k\alpha}, \quad n \geq 1.$$

Therefore, if $n > t$,

$$\int_0^\infty (1+ty/n)^{-n} f_\alpha(y)\, dy = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(1+k\alpha)} \frac{\Gamma(n+k\alpha)}{n^{k\alpha}(n-1)!} t^{k\alpha}.$$

Stirling's Formula shows that

$$\lim_{n\to\infty} \frac{\Gamma(n+k\alpha)}{n^{k\alpha}(n-1)!} = 1 \quad \text{for each } k \geq 0.$$

Since

$$\frac{\Gamma(n+k\alpha)}{n^{k\alpha}(n-1)!} \leq \Gamma(1+k\alpha) \quad \text{for all } k \geq 1/\alpha \text{ and all } n \geq 1,$$

the bounded convergence theorem yields

$$\int_0^\infty e^{-ty} f_\alpha(y)\, dy = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(1+k\alpha)} t^{k\alpha} \quad \text{for } t < 1,$$

and the proof is complete. □

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COROLLARY. *For* $0 < \alpha, \beta < 1$,

$$L_{\alpha,\beta} \stackrel{\text{dist}}{=} \frac{X}{X + b^{1/\alpha} Y} \quad (b = (1-\beta)/\beta)$$

where X, Y *are as in Proposition 2.*

Using this relation and arguments as in the proof of Corollary 2 on p. 303 in [**Le**] we can understand why the distributions $G_{\alpha,\beta}$ appear in our theorem. We outline this reasoning for a Markov chain analogue of the maps considered in this paper.

Let the state space be $\mathbb{Z}$, and let the transition probabilities $P(x, y)$ be specified by probability distributions $\{p^-_{n+1} : n \geq 1\}$, $\{p^+_{n+1} : n \geq 1\}$ and a weight $p \in (0, 1)$ as follows:

$$P(0, -n) = pp^-_{n+1}, \quad P(0, n) = (1-p)p^+_{n+1},$$
$$P(-n, -n+1) = P(n, n-1) = 1, \quad n \geq 1.$$

We ask for the asymptotic distributional behaviour of A^+_n / A^-_n, where A^+_n (A^-_n) denotes the number of visits to $\mathbb{Z}^+$ ($\mathbb{Z}^-$) up to the nth return to zero, assuming that the chain starts there. For convenience we count a visit to zero as a visit to $\mathbb{Z}^+$ ($\mathbb{Z}^-$) according to whether the preceding state is in $\mathbb{Z}^+$ ($\mathbb{Z}^-$).

Let τ_k ($k \geq 1$) denote the time of the kth return to zero, and let $\tau_0 = 0$. Let H_n be the number of time segments $(\tau_{k_j-1}, \tau_{k_j})$, $1 \leq k_j \leq n$, during which the process is in $\mathbb{Z}^+$, and let the segments $(\tau_{k'_j-1}, \tau_{k'_j})$, $1 \leq k'_j \leq n$, correspond to visits in $\mathbb{Z}^-$. Then

$$A^+_n / A^-_n = \left(\sum_{j=1}^{H_n} (\tau_{k_j} - \tau_{k_j-1}) \right) \Big/ \left(\sum_{j=1}^{n-H_n} (\tau_{k'_j} - \tau_{k'_j-1}) \right).$$

The time differences $\tau_k - \tau_{k-1}$, $k \geq 1$, are independent. Under the condition that the process is in $\mathbb{Z}^+$ during the interval (τ_{k-1}, τ_k), the length $\tau_k - \tau_{k-1}$ is distributed according to $\{p^+_{n+1} : n \geq 1\}$; otherwise it is distributed according to $\{p^-_{n+1} : n \geq 1\}$.

Assume now that

$$\sum_{k>n} p^+_k \sim d^+ / n^\alpha \ (n \to \infty) \quad \text{and} \quad \sum_{k>n} p^-_k \sim d^- / n^\alpha \ (n \to \infty)$$

for some $\alpha \in (0, 1)$ and some constants $d^-, d^+ > 0$. Then, under the above conditions, we have according to [**F2**, §XIII.6],

$$\frac{1}{m^{1/\alpha}} \sum_{j=1}^{m} (\tau_{k_j} - \tau_{k_j-1}) \stackrel{\text{dist}}{\longrightarrow} (d^+ \Gamma(1-\alpha))^{1/\alpha} Y$$

and

$$\frac{1}{m^{1/\alpha}} \sum_{j=1}^{m} (\tau'_{k_j} - \tau'_{k_j-1}) \stackrel{\text{dist}}{\longrightarrow} (d^- \Gamma(1-\alpha))^{1/\alpha} X,$$

where X, Y have the one-sided stable law referred to in Proposition 2. Since

$$(1/n) H_n \to 1 - p \quad \text{with probability 1,}$$

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the ratio A_n^+/A_n^- for large n will approximately be distributed as

$$(((1-p)nd^+)^{1/\alpha}Y)/((pnd^-)^{1/\alpha}X) = b^{1/\alpha}Y/X, \quad b = \frac{d^+}{d^-}\frac{1-p}{p}.$$

In view of our corollary, $A_n^-/\tau_n\ (= 1/(1 + A_n^+/A_n^-))$ will approximately be distributed as $L_{\alpha,\beta}$ for large n. Theorem 1 in [**La**] tells us that the limiting behaviour of the distributions is the same if we omit the condition of inspecting the process only at times of a return to zero.

4. *Proof of the theorem*

As mentioned before, it suffices to prove the theorem for a specific $\gamma \in (0, 1)$. We choose $\gamma = \gamma_0$ where γ_0 is the periodic point of T with period two satisfying $\gamma_0 \in (0, c)$ and $\gamma_1 := T(\gamma_0) \in (c, 1)$. Let

$$S_n = \sum_{j=0}^{n} 1_{[0,\gamma_0]} \circ T^j, \quad n \geq 0.$$

We shall prove the asserted convergence by the method of moments as in [**La**], yet avoiding the use of bivariate Laplace transforms. Our plan is to show that for all $\nu \in \mathcal{D}$ and all $r \in \mathbb{N}_0$ the transform

$$M_r(s) = \sum_{n=0}^{\infty} \left(\int_0^1 S_n^r \, d\nu \right) e^{-ns} \quad (s > 0)$$

satisfies

$$M_r(s) \sim r!\, E(L_{\alpha,\beta}^r) \left(\frac{1}{s}\right)^{r+1} \quad (s \to 0).$$

Since the moment sequences $\{\int_0^1 S_n^r \, d\nu\}$ are increasing, Karamata's Tauberian Theorem then yields the result. We put $0^0 := 1$, so that $M_0(s) = 1/(1-e^{-s})$ $(s > 0)$ for each $\nu \in \mathcal{D}$.

Let $U = (\gamma_0, \gamma_1)$, and let

$$U_0 = (c, \gamma_1)(= U \cap T^{-1}(0, \gamma_0)),$$
$$U_1 = (\gamma_0, c)(= U \cap T^{-1}(\gamma_1, 1)),$$

and

$$\varphi(x) = \min\{n \geq 1 : T^n(x) \in U\}, \quad x \in U.$$

Recall that $\hat{T}$ denotes the Perron–Frobenius operator of T with respect to the invariant measure μ, i.e.

$$\int_0^1 \hat{T}v \cdot u \, d\mu = \int_0^1 v \cdot u \circ T \, d\mu \quad \text{for all } v \in L_1(\mu) \text{ and all } u \in L_\infty(\mu).$$

The proof of the theorem will be divided into several lemmas. Lemma 1 contains the nucleus of the basic recursion formula.

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LEMMA 1. *Let T satisfy the conditions (1)–(3). Then, for $r \geq 1$ and $s > 0$,*

$$(1-e^{-s})\int_U \Big(\sum_{n=0}^{\infty}\Big(\sum_{k>n}\hat{T}^k 1_{U\cap\{\varphi=k\}}\Big)e^{-ns}\Big)\Big(\sum_{n=0}^{\infty}S_n^r e^{-ns}\Big)\,d\mu$$
$$= e^{-s}\sum_{j=0}^{r-1}\binom{r}{j}\int_U\Big(\sum_{n=1}^{\infty}n^{r-j}\hat{T}^{n+1}1_{U_0\cap\{\varphi=n+1\}}e^{-ns}\Big)\Big(\sum_{n=0}^{\infty}S_n^j e^{-ns}\Big)\,d\mu$$
$$+\sum_{n=1}^{\infty}n^r\mu(U_0\cap\{\varphi>n\})e^{-ns}.$$

Proof. Let $r \geq 1$ be fixed. For $2 \leq k \leq n$,

$$S_n = \begin{cases} k-1+S_{n-k}\circ T^k & \text{on } U_0\cap\{\varphi=k\},\\ S_{n-k}\circ T^k & \text{on } U_1\cap\{\varphi=k\},\end{cases}$$

and

$$S_n = \begin{cases} n & \text{on } U_0\cap\{\varphi>n\},\\ 0 & \text{on } U_1\cap\{\varphi>n\},\ n\geq 0.\end{cases}$$

Thus, taking into account the fact that $U\cap\{\varphi=1\}=\emptyset$ and $\hat{T}^k 1_{U\cap\{\varphi=k\}}=0$ a.e. on U^c for all $k\geq 1$, we get for $n\geq 0$,

$$\int_U S_n^r\,d\mu = \sum_{k=2}^{n}\int_{U_0\cap\{\varphi=k\}}(k-1+S_{n-k}\circ T^k)^r\,d\mu$$
$$+\sum_{k=2}^{n}\int_{U_1\cap\{\varphi=k\}}(S_{n-k}\circ T^k)^r\,d\mu + n^r\mu(U_0\cap\{\varphi>n\})$$
$$=\sum_{j=0}^{r-1}\binom{r}{j}\int_U\sum_{k=2}^{n}(k-1)^{r-j}\hat{T}^k 1_{U_0\cap\{\varphi=k\}}S_{n-k}^j\,d\mu$$
$$+\int_U\sum_{k=2}^{n}\hat{T}^k 1_{U\cap\{\varphi=k\}}S_{n-k}^r\,d\mu + n^r\mu(U_0\cap\{\varphi>n\}).$$

Therefore, for $s>0$,

$$\sum_{n=0}^{\infty}\Big(\int_U S_n^r\,d\mu\Big)e^{-ns}$$
$$= e^{-s}\sum_{j=0}^{r-1}\binom{r}{j}\int_U\Big(\sum_{n=1}^{\infty}n^{r-j}\hat{T}^{n+1}1_{U_0\cap\{\varphi=n+1\}}e^{-ns}\Big)\Big(\sum_{n=0}^{\infty}S_n^j e^{-ns}\Big)\,d\mu$$
$$+\int_U\Big(\sum_{n=2}^{\infty}\hat{T}^n 1_{U\cap\{\varphi=n\}}e^{-ns}\Big)\Big(\sum_{n=0}^{\infty}S_n^r e^{-ns}\Big)\,d\mu$$
$$+\sum_{n=1}^{\infty}n^r\mu(U_0\cap\{\varphi>n\})e^{-ns}.$$

Since $\mu_{|_{U\cap\mathcal{B}}}$ is invariant for the map induced on U, we have

$$\sum_{n=2}^{\infty}\hat{T}^n 1_{U\cap\{\varphi=n\}}=1 \quad \text{a.e. on } U,$$

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and therefore

$$1 - \sum_{n=2}^{\infty} \hat{T}^n 1_{U\cap\{\varphi=n\}} e^{-ns} = (1-e^{-s}) \sum_{n=0}^{\infty} \left(\sum_{k>n} \hat{T}^k 1_{U\cap\{\varphi=k\}} \right) e^{-ns} \quad \text{a.e. on } U.$$

This completes the proof of Lemma 1. □

Our next aim is to replace the transforms

$$\sum_{n=0}^{\infty} \left(\sum_{k>n} \hat{T}^k 1_{U\cap\{\varphi=k\}} \right) e^{-ns} \quad \text{and} \quad \sum_{n=1}^{\infty} n^{\ell} \hat{T}^{n+1} 1_{U_0\cap\{\varphi=n+1\}} e^{-ns} \quad (\ell \geq 1)$$

by transforms with constant coefficients. Recall that h denotes the version of the invariant density specified in §2, and that

$$f_0 = (T_{|[0,c]})^{-1}, \quad f_1 = (T_{|[c,1]})^{-1}.$$

Since

$$U_0 \cap \{\varphi = n\} = (c,1) \cap \bigcap_{j=1}^{n-1} T^{-j}(0,c) \cap T^{-n}U$$

and

$$U_1 \cap \{\varphi = n\} = (0,c) \cap \bigcap_{j=1}^{n-1} T^{-j}(c,1) \cap T^{-n}U,$$

T^n maps $U_0 \cap \{\varphi = n\}$ and $U_1 \cap \{\varphi = n\}$ monotonically onto U for all $n \geq 2$. The inverse functions are $f_1 \circ f_0^{n-1}$ and $f_0 \circ f_1^{n-1}$, respectively. Therefore we have

$$\hat{T}^n 1_{U_0\cap\{\varphi=n\}} = (h \circ f_1 \circ f_0^{n-1}/h)(f_1 \circ f_0^{n-1})' 1_U \text{ a.e.}$$

and

$$\hat{T}^n 1_{U_1\cap\{\varphi=n\}} = (h \circ f_0 \circ f_1^{n-1}/h)(f_0 \circ f_1^{n-1})' 1_U \text{ a.e.}$$

In what follows, $\hat{T}^n 1_{U_0\cap\{\varphi=n\}}$, $\hat{T}^n 1_{U_1\cap\{\varphi=n\}}$ denote the versions on the right-hand side. Since $f_0^n \to 0$ and $f_1^n \to 1$ uniformly on $[0,1]$,

$$\hat{T}^n 1_{U_0\cap\{\varphi=n\}} \sim (h(c) f_1'(0)/h)(f_0^{n-1})' \ (n \to \infty)$$

and

$$\hat{T}^n 1_{U_1\cap\{\varphi=n\}} \sim (h(c) f_0'(1)/h)(f_1^{n-1})' \ (n \to \infty)$$

uniformly on U.

The next lemma provides us with the necessary information on the asymptotics of $\{(f_0^n)'\}$ and $\{(f_1^n)'\}$. The identity in (b) is known as Abel's functional equation (see, e.g., [**K**], [**Z**, Appendix I]).

LEMMA 2. *Let $f : [0,1] \to \mathbb{R}$ be continuously differentiable, satisfying $0 < f(x) < x$, $f'(x) > 0$, $0 < x \leq 1$, $f'(0) = 1$, and let f be concave on $[0,\eta]$ for some $\eta > 0$. Then there exists a positive continuous function g on $(0,1]$ such that:*

(a) $(f^n)'(x) \sim g(x)(f^n(1) - f^{n+1}(1))$ $(n \to \infty)$ *uniformly on compact subsets of* $(0,1]$*; and*

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(b) $\int_{f(x)}^{x} g(y)\,dy = 1$ *for all* $x \in (0, 1]$.

Proof. Let

$$g_n(x) = \frac{(f^n)'(x)}{f^n(x) - f^{n+1}(x)}, \quad x \in (0, 1],\ n \geq 0,$$

and let

$$n(x) = \min\{n \geq 0 : f^n(x) \leq \eta\}, \quad x \in [0, 1].$$

Note that $n(x) = 0$ on $[0, \eta]$, and $n(x) \leq n(1) =: N$ for all $x \in [0, 1]$.

For $x \in (0, 1]$, $n \geq n(x)$ and $m \geq 0$,

$$f'(f^n(x))g_n(x) \leq g_{n+m}(x) \leq g_n(x). \tag{1}$$

To see this, use the mean value theorem to obtain

$$\frac{g_{n+m}(x)}{g_n(x)} = \frac{(f^{n+m})'(x)}{(f^m)'(\xi)(f^n)'(x)} \quad \text{with } f^{n+1}(x) < \xi < f^n(x).$$

Since $f^n(x) \leq \eta$ and f, f^m are concave on $[0, \eta]$,

$$\begin{aligned} f'(f^n(x)) \leq \frac{f'(f^n(x))}{f'(f^{n+m}(x))} &= \frac{(f^{n+m})'(x)}{(f^m)'(f^{n+1}(x))(f^n)'(x)} \\ &\leq \frac{g_{n+m}(x)}{g_n(x)} = \frac{(f^m)'(f^n(x))}{(f^m)'(\xi)} \leq 1. \end{aligned}$$

In particular, the sequence $\{g_n(x)\}_{n=n(x)}^{\infty}$ is decreasing for each $x \in (0, 1]$, and thus

$$\lim_{n\to\infty} g_n(x) =: g(x) \quad \text{exists for all } x \in (0, 1].$$

Letting $m \to \infty$ in (1) we get

$$f'(f^n(x))g_n(x) \leq g(x) \leq g_n(x), \quad x \in (0, 1],\ n \geq n(x). \tag{2}$$

Therefore, for $x \in (0, 1]$ and $n \geq N$,

$$0 \leq g_n(x) - g(x) \leq (1 - f'(f^n(x)))g_n(x) \leq (1 - f'(f^n(1)))g_N(x),$$

which shows that $g_n \to g$ uniformly on compact subsets of $(0, 1]$. Thus g is continuous on $(0, 1]$. The lower estimate in (2) shows that g is positive on $(0, 1]$. Taking into account the fact that the function $x - f(x)$ is increasing on $[0, \eta]$ we get for $x \in [f^{\ell}(1), 1]$, $\ell \geq 1$, and $n \geq N$,

$$\begin{aligned} (f^{\ell})'(f^n(1))(f^n(1) - f^{n+1}(1)) &\leq f^{n+\ell}(1) - f^{n+\ell+1}(1) \\ &\leq f^n(x) - f^{n+1}(x) \leq f^n(1) - f^{n+1}(1). \end{aligned}$$

Thus

$$\frac{f^n(x) - f^{n+1}(x)}{f^n(1) - f^{n+1}(1)} \to 1$$

uniformly on compact subsets of $(0, 1]$, which proves assertion (a). To get (b), let

$$I(x) = \int_{f(x)}^{x} g(y)\,dy, \quad 0 < x \leq 1.$$

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Since

$$g_n(f(x))f'(x) = g_{n+1}(x), \quad x \in (0, 1],\ n \geq 0,$$

we have

$$g(f(x))f'(x) = g(x), \quad x \in (0, 1].$$

Therefore I is constant on $(0, 1]$. The function g is decreasing on $(0, \eta]$ since the g_n are decreasing there. Moreover, choosing $n = 0$ in (2) we see that

$$\frac{f'(x)}{x - f(x)} \leq g(x) \leq \frac{1}{x - f(x)}, \quad x \in (0, \eta].$$

Thus, for $x \in (0, \eta]$,

$$I(x) \leq g(f(x))(x - f(x)) = g(x)(x - f(x))/f'(x) \leq 1/f'(x),$$

and

$$I(x) \geq g(x)(x - f(x)) \geq f'(x),$$

which proves that $I(x) = 1$ for all $x \in (0, 1]$. □

Applying Lemma 2 to the functions

$$f(x) = f_0(x) \quad \text{and} \quad f(x) = 1 - f_1(1 - x), \quad x \in [0, 1]$$

yields positive continuous functions g_0, g_1 defined on $(0, 1)$ such that:

(i) $(f_0^n)'(x) \sim g_0(x)(f_0^n(1) - f_0^{n+1}(1))$ $(n \to \infty)$ and $(f_1^n)'(x) \sim g_1(x)(f_1^{n+1}(0) - f_1^n(0))$ $(n \to \infty)$ uniformly on compact subsets of $(0, 1)$; and

(ii) $\int_{f_0(x)}^{x} g_0(y)\,dy = \int_{x}^{f_1(x)} g_1(y)\,dy = 1$ for all $x \in (0, 1)$.

Since $U = (f_0(\gamma_1), \gamma_1) = (\gamma_0, f_1(\gamma_0))$, (ii) shows that

$$\int_U g_0\,d\lambda = \int_U g_1\,d\lambda = 1.$$

It is crucial to note that these integrals are equal in order to identify the parameter β of the limiting distribution.

Now let

$$Q_0(s) = \sum_{n=0}^{\infty} f_0^n(1)e^{-ns},$$

$$Q_1(s) = \sum_{n=0}^{\infty} (1 - f_1^n(0))e^{-ns}$$

and

$$F_0(s) = \sum_{n=1}^{\infty} (f_0^{n-1}(1) - f_0^n(1))e^{-ns} \quad (s > 0).$$

Q_0 and F_0 are connected by

$$1 - F_0(s) = (1 - e^{-s})Q_0(s) \quad (s > 0).$$

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Note also that

$$\sum_{n=0}^{\infty} f_0^n(1) = \sum_{n=0}^{\infty}(1 - f_1^n(0)) = \infty,$$

reflecting the fact that $\mu([0, c]) = \mu([c, 1]) = \infty$. In fact, the estimates in [**T1**] show that

$$\sum_{n=0}^{\infty}(f_0^n)' \geq \kappa h \quad \text{on } (0, c) \text{ for some } \kappa > 0.$$

Therefore,

$$\sum_{n=0}^{\infty} f_0^n(1) = \int_0^1 \sum_{n=0}^{\infty}(f_0^n)'(x)\,dx \geq \kappa\mu([0, c]) = \infty.$$

The same reasoning yields divergence of the second series.

Lemma 2 enables us to obtain the following recursion formula.

LEMMA 3. *Let T satisfy the conditions (1)–(3). Then, for $r \geq 1$ and $s \to 0$,*

$$f_1'(0)Q_0(s)\sum_{n=0}^{\infty}\left(\int_U S_n^r g_0\,d\lambda\right)e^{-ns} + f_0'(1)Q_1(s)\sum_{n=0}^{\infty}\left(\int_U S_n^r g_1\,d\lambda\right)e^{-ns}$$
$$\sim (-1)^r\frac{f_1'(0)}{s}\left\{\sum_{j=0}^{r-1}(-1)^j\binom{r}{j}F_0^{(r-j)}(s)\sum_{n=0}^{\infty}\left(\int_U S_n^j g_0\,d\lambda\right)e^{-ns} + Q_0^{(r)}(s)\right\}.$$

Proof. We prove that, for $s \to 0$,

(a)

$$\sum_{n=0}^{\infty}\left(\sum_{k>n}\hat{T}^k 1_{U\cap\{\varphi=k\}}\right)e^{-ns}$$
$$\sim h(c)(f_1'(0)Q_0(s)g_0 + f_0'(1)Q_1(s)g_1)/h \quad \text{uniformly on } U;$$

(b)

$$\sum_{n=1}^{\infty} n^{\ell}\hat{T}^{n+1}1_{U_0\cap\{\varphi=n+1\}}e^{-ns}$$
$$\sim h(c)f_1'(0)(-1)^{\ell}F_0^{(\ell)}(s)g_0/h \quad \text{uniformly on } U\ (\ell \geq 1);$$

and

(c)

$$\sum_{n=1}^{\infty} n^r\mu(U_0 \cap \{\varphi > n\})e^{-ns} \sim h(c)f_1'(0)(-1)^r Q_0^{(r)}(s) \quad (r \geq 1).$$

Inserting these asymptotic equivalences into the formula in Lemma 1 yields the formula in Lemma 3.

Since

$$\hat{T}^k 1_{U\cap\{\varphi=k\}} = \hat{T}^k 1_{U_0\cap\{\varphi=k\}} + \hat{T}^k 1_{U_1\cap\{\varphi=k\}}$$
$$\sim h(c)(f_1'(0)(f_0^{k-1})' + f_0'(1)(f_1^{k-1})')/h \quad (k \to \infty)$$

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uniformly on U,

$$\sum_{k>n} \hat{T}^k 1_{U\cap\{\varphi=k\}} \sim h(c)(f_1'(0)g_0 f_0^n(1) + f_0'(1)g_1(1-f_1^n(0))/h \quad (n\to\infty)$$

uniformly on U. Taking into account the fact that

$$\lim_{s\to 0} Q_0(s) = \lim_{s\to 0} Q_1(s) = \infty$$

we get (a). For $\ell \geq 0$ we have

$$\sum_{n=1}^{\infty} n^\ell (f_0^{n-1}(1) - f_0^n(1))e^{-ns} = (-1)^\ell F_0^{(\ell)}(s) \quad (s>0)$$

and

$$\begin{aligned} n^\ell \hat{T}^{n+1} 1_{U_0\cap\{\varphi=n+1\}} &\sim h(c) f_1'(0) n^\ell (f_0^n)'/h \\ &\sim h(c) f_1'(0) n^\ell (f_0^{n-1}(1) - f_0^n(1)) g_0/h \quad (n\to\infty) \end{aligned}$$

uniformly on U. As $\lim_{s\to 0}(-1)^\ell F_0^{(\ell)}(s) = \infty$ for $\ell \geq 1$, (b) follows.

Finally,

$$\begin{aligned} \mu(U_0\cap\{\varphi>n\}) &= \int_U \sum_{k>n} \hat{T}^k 1_{U_0\cap\{\varphi=k\}}\, d\mu \sim h(c) f_1'(0)\left(\int_U g_0\, d\lambda\right) f_0^n(1) \\ &= h(c) f_1'(0) f_0^n(1) \quad (n\to\infty). \end{aligned}$$

Therefore, as $s\to 0$,

$$\begin{aligned} \sum_{n=1}^{\infty} n^r \mu(U_0\cap\{\varphi>n\}) e^{-ns} &\sim h(c) f_1'(0) \sum_{n=1}^{\infty} n^r f_0^n(1) e^{-ns} \\ &= h(c) f_1'(0)(-1)^r Q_0^{(r)}(s). \end{aligned}$$

□

In Lemma 3 the moments of S_n appear with respect to different probability measures. To eliminate this problem we use Aaronson's Compactness Theorem for distributional convergence [**Aa1, Aa2**].

LEMMA 4. *Let T satisfy the conditions (1)–(3). If*

$$\liminf_{n\to\infty} \int_0^1 \left(\frac{1}{n} S_n\right) d\nu_0 > 0 \quad \textit{for some } \nu_0 \in \mathcal{D},$$

then

$$\int_0^1 S_n^r\, d\nu \sim \int_0^1 S_n^r\, d\nu_0 \quad (n\to\infty) \quad \textit{for all } \nu \in \mathcal{D} \textit{ and all } r \geq 1.$$

Proof. Let $r \in \mathbb{N}$ and $\nu_1 \in \mathcal{D}$ be given, and let $\{n_k\}$ be an increasing sequence in $\mathbb{N}$ such that

$$\lim_{k\to\infty} \left(\int_0^1 S_{n_k}^r\, d\nu_1\right) \Big/ \left(\int_0^1 S_{n_k}^r\, d\nu_0\right) = \rho, \quad 0 \leq \rho \leq \infty.$$

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By Proposition 3.6.1 in [**Aa1**] there exists a subsequence $\{n'_k\}$ of $\{n_k\}$ and a random variable Y with values in $[0, 1]$ such that

$$\frac{1}{n'_k} S_{n'_k} \xrightarrow{\nu} Y \quad \text{for all } \nu \in \mathcal{D}.$$

Since $\{(1/n)S_n\}$ is bounded, we have

$$\lim_{k\to\infty} \int_0^1 \left(\frac{1}{n'_k} S_{n'_k}\right)^j d\nu = E(Y^j) \quad \text{for all } j \geq 0 \text{ and all } \nu \in \mathcal{D}.$$

Choosing $\nu = \nu_0$ we see that $E(Y) > 0$, and hence $E(Y^j) > 0$ for all $j \geq 1$. Therefore

$$\rho = \lim_{k\to\infty} \left(\int_0^1 \left(\frac{1}{n'_k} S_{n'_k}\right)^r d\nu_1 \right) \Big/ \left(\int_0^1 \left(\frac{1}{n'_k} S_{n'_k}\right)^r d\nu_0 \right) = 1.$$

This proves Lemma 4. □

The next lemma connects the asymptotic behaviour of T at the fixed points with that of the tails of the corresponding waiting times.

LEMMA 5. *Let T satisfy the conditions (1)–(3), and let $\alpha = 1/p$, $p > 1$, and $a > 0$. Then T satisfies the conditions (4a) and (4b) if and only if:*

(4a)$_1$ *$Q_0(s)$ is regularly varying at zero with exponent $\alpha - 1$; and*
(4b)$_1$ $Q_1(s) \sim Q_0(s)/a$ $(s \to 0)$.

Proof. Let

$$u_0(x) = \frac{x}{x - f_0(x)}, \quad u_1(x) = \frac{x}{f_1(1-x) - (1-x)}, \quad 0 < x \leq 1,$$

which are strictly decreasing on a neighbourhood of zero and satisfy

$$\lim_{x\to 0} u_0(x) = \lim_{x\to 0} u_1(x) = \infty.$$

Using standard facts from the theory of regular variation we see that the conditions (4a) and (4b) hold if and only if:

(4a)$_2$ u_0^{-1} is regularly varying at infinity with exponent $-\alpha$; and
(4b)$_2$ $u_1^{-1}(x) \sim u_0^{-1}(x)/a$ $(x \to \infty)$.

On the other hand, by Karamata's Tauberian Theorem (see, e.g., [**BGT, F2**]) and the monotonicity of $\{f_0^n(1)\}$ and $\{1 - f_1^n(0)\}$, the conditions (4a)$_1$ and (4b)$_1$ hold if and only if

(4a)$_3$ $\{f_0^n(1)\}$ is regularly varying with index $-\alpha$; and
(4b)$_3$ $1 - f_1^n(0) \sim f_0^n(1)/a$ $(n \to \infty)$.

In view of Lemma 3 in [**T4**], $x - f_0(x)(f_1(1-x) - (1-x))$ is regularly varying at zero with exponent $p+1$ if and only if $\{f_0^n(1)\}(\{1 - f_1^n(0)\})$ is regularly varying with exponent $-\alpha$. Arguments as in Remark (ii) at the end of [**T4**] show that in this case

$$f_0^n(1) \sim u_0^{-1}(n)/p^\alpha \quad (1 - f_1^n(0) \sim u_1^{-1}(n)/p^\alpha) \quad (n \to \infty).$$

(See also Lemma 4.8.6 in [**Aa1**].) These assertions show that Lemma 5 holds true. □

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Our final lemma is essentially a quotation of Theorem 2 in [**La**]. The second part is stated here in a slightly more general form, which is relevant for §5. (See also [**BGT**, §§1.3 and 1.7.3].)

LEMMA 6. *Let $f : (0, \eta) \to (0, \infty)$ $(\eta > 0)$ be continuously differentiable and let ρ be a real number.*

(a) *If $\lim_{x\to 0}[xf'(x)/f(x)] = \rho$, f is regularly varying at zero with index ρ.*

(b) *If f' is monotone and $v : (0, \eta) \to (0, \infty)$ is slowly varying at zero, then*

$$\lim_{x\to 0} \frac{f(x)}{x^\rho v(x)} = \theta \ (\theta \in [0, \infty)) \quad \textit{implies} \quad \lim_{x\to 0} \frac{f'(x)}{x^{\rho-1} v(x)} = \theta\rho.$$

Now assume T satisfies the conditions (1)–(3) and (4a), (4b). By Lemma 5,

$$Q_0(s) = (1/s)^{1-\alpha} L(1/s) \quad (s > 0)$$

and

$$Q_1(s) \sim (1/s)^{1-\alpha} L(1/s)/a \quad (s \to 0),$$

where L is slowly varying at infinity. Moreover,

$$1 - F_0(s) \sim s Q_0(s) = s^\alpha L(1/s) \quad (s \to 0).$$

Taking into account the fact that all derivatives of Q_0 and F_0 are monotone we get by induction, based on part (b) of Lemma 6,

$$Q_0^{(r)}(s) \sim r! \binom{\alpha - 1}{r} (1/s)^{r+1-\alpha} L(1/s) \quad (s \to 0) \quad \text{for all } r \geq 0,$$

$$F_0^{(\ell)}(s) \sim -\ell! \binom{\alpha}{\ell} (1/s)^{\ell-\alpha} L(1/s) \quad (s \to 0) \quad \text{for all } \ell \geq 1.$$

Introducing the notation $g = f_1'(0) g_0 + (f_0'(1)/a) g_1$, the recursion formula in Lemma 3 becomes

$$\sum_{n=0}^{\infty} \left(\int_U S_n^r g \, d\lambda \right) e^{-ns} \sim r!(-1)^r f_1'(0) \left\{ \sum_{j=0}^{r-1} (-1)^{j+1} \binom{\alpha}{r-j} \left(\frac{1}{s}\right)^{r-j} \right.$$
$$\left. \times \frac{1}{j!} \sum_{n=0}^{\infty} \left(\int_U S_n^j g_0 \, d\lambda \right) e^{-ns} + \binom{\alpha - 1}{r} \left(\frac{1}{s}\right)^{r+1} \right\} \quad (s \to 0).$$

In particular,

$$\sum_{n=0}^{\infty} \left(\int_U S_n g \, d\lambda \right) e^{-ns} \sim f_1'(0)/s^2 \quad (s \to 0).$$

Therefore,

$$\int_U S_n g \, d\lambda \sim f_1'(0)\, n \quad (n \to \infty).$$

This means that

$$\lim_{n\to\infty} \int_U \left(\frac{1}{n} S_n \right) d\nu_0 = f_1'(0) \Big/ \int_U g \, d\lambda,$$

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where

$$\frac{d\nu_0}{d\lambda} = \left(g \Big/ \int_U g d\lambda\right) 1_U,$$

and we can apply Lemma 4 to get

$$\int_U S_n^r g \, d\lambda \sim \left(\int_U g \, d\lambda\right) \int_0^1 S_n^r \, d\nu \quad (n \to \infty)$$

for all $r \geq 0$ and all $\nu \in \mathcal{D}$.

Let $\nu \in \mathcal{D}$ be fixed. The recursion formula now takes the shape

$$\frac{1}{r!}\sum_{n=0}^{\infty}\left(\int_0^1 S_n^r \, d\nu\right)e^{-ns} \sim (-1)^r \beta \left\{ \sum_{j=0}^{r-1}(-1)^{j+1}\binom{\alpha}{r-j}\left(\frac{1}{s}\right)^{r-j} \right.$$
$$\left. \times \frac{1}{j!}\sum_{n=0}^{\infty}\left(\int_0^1 S_n^j \, d\nu\right)e^{-ns} + \binom{\alpha-1}{r}\left(\frac{1}{s}\right)^{r+1} \right\} \quad (s \to 0),$$

where

$$\beta = f_1'(0) \Big/ \int_U g \, d\lambda = \frac{f_1'(0)}{f_1'(0) + f_0'(1)/a} = \frac{T'(c^-)}{T'(c^-) + T'(c^+)/a}.$$

Induction on r shows that

$$\frac{1}{r!}\sum_{n=0}^{\infty}\left(\int_0^1 S_n^r \, d\nu\right)e^{-ns} \sim m_r \left(\frac{1}{s}\right)^{r+1} \quad (s \to 0),$$

where the sequence $\{m_r\}$ is determined by $m_0 = 1$ and

$$m_r = (-1)^r \beta \left\{ \sum_{j=0}^{r-1}(-1)^{j+1}\binom{\alpha}{r-j} m_j + \binom{\alpha-1}{r} \right\}, \quad r \geq 1.$$

By Proposition 1 in §3,

$$m_r = E(L_{\alpha,\beta}^r), \quad r \geq 0.$$

Karamata's Tauberian Theorem and the monotonicity of $\{\int_0^1 S_n^r \, d\nu\}$ thus yield

$$\lim_{n\to\infty} \int_0^1 \left(\frac{1}{n} S_n\right)^r d\nu = E(L_{\alpha,\beta}^r) \quad \text{for all } r \geq 0,$$

which means that

$$\frac{1}{n} S_n \xrightarrow{\nu} L_{\alpha,\beta}.$$

Hence the proof of the first part of the theorem is complete.

Assume now that

$$\frac{1}{n} S_n \xrightarrow{\nu} L$$

for some $\nu \in \mathcal{D}$ where L has the required properties, i.e. $V(L) \neq 0$ and $E(L(1-L)) \neq 0$. Let $\beta = E(L)$. Then

$$\beta = \lim_{n\to\infty} \int_0^1 \left(\frac{1}{n} S_n\right) d\nu.$$

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Since L is not concentrated on 0 or on 1, we have $0 < \beta < 1$. In particular, Lemma 4 applies, and we get from Lemma 3

$$(f_1'(0)Q_0(s) + f_0'(1)Q_1(s)) \sum_{n=0}^{\infty} \left(\int_0^1 S_n^r \, dv \right) e^{-ns}$$
$$\sim (-1)^r \frac{f_1'(0)}{s} \left\{ \sum_{j=0}^{r-1} (-1)^j \binom{r}{j} F_0^{(r-j)}(s) \sum_{n=0}^{\infty} \left(\int_0^1 S_n^j \, dv \right) e^{-ns} + Q_0^{(r)}(s) \right\}$$

as $s \to 0$. Putting $r = 1$ and taking into account the fact that

$$\sum_{n=0}^{\infty} \left(\int_0^1 S_n \, dv \right) e^{-ns} \sim \beta \left(\frac{1}{s} \right)^2 \quad (s \to 0),$$

and

$$-F_0'(s) = e^{-s} Q_0(s) + (1 - e^{-s}) Q_0'(s) \quad (s > 0),$$

we obtain

$$\lim_{s \to 0} \frac{f_1'(0) Q_0(s)}{f_1'(0) Q_0(s) + f_0'(1) Q_1(s)} = \beta,$$

and thus

$$Q_1(s) \sim \frac{1 - \beta}{\beta} \frac{T'(c^-)}{T'(c^+)} Q_0(s) \quad (s \to 0).$$

Similarly, putting $r = 2$ and taking into account the fact that

$$\sum_{n=0}^{\infty} \left(\int_0^1 S_n^2 \, dv \right) e^{-ns} \sim 2E(L^2) \left(\frac{1}{s} \right)^3 \quad (s \to 0)$$

and

$$-F_0''(s) = -e^{-s} Q_0(s) + 2e^{-s} Q_0'(s) + (1 - e^{-s}) Q_0''(s) \quad (s > 0),$$

we obtain (after some calculation)

$$\lim_{s \to 0} \frac{s Q_0'(s)}{Q_0(s)} = \alpha - 1, \quad \text{where } E(L^2) = \beta(1 - \alpha + \alpha\beta).$$

Therefore, by Lemma 6(a), $Q_0(s)$ is regularly varying at zero with index $\alpha - 1$. Since $E(L^2) \notin \{\beta, \beta^2\}$, we have $\alpha \notin \{0, 1\}$. By Lemma 5, the conditions (4a) and (4b) are thus satisfied with

$$p = \frac{1}{\alpha} = 1 + \frac{V(L)}{E(L(1 - L))}, \quad a = \frac{\beta}{1 - \beta} \frac{T'(c^+)}{T'(c^-)}.$$

This finishes the proof of the theorem. □

5. *Concluding remarks: the boundary cases*

We conclude with a brief discussion of the boundary cases $\{\alpha, \beta\} \cap \{0, 1\} \neq \emptyset$.

Extending the family $G_{\alpha,\beta}$ continuously leads to

$$L_{\alpha,1} \equiv 1, \quad L_{\alpha,0} \equiv 0, \quad 0 \leq \alpha \leq 1$$

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and

$$L_{1,\beta} \equiv \beta, \quad \Pr(L_{0,\beta} = 1) = \beta = 1 - \Pr(L_{0,\beta} = 0), \quad 0 \le \beta \le 1.$$

The following version of our theorem also covers these cases. Throughout we assume T satisfies the conditions (1)–(3).

If:

(4a)$_4$ $\left\{\sum_{k=0}^{n} f_0^k(1)\right\}$ is regularly varying with index $1-\alpha$, $0 \le \alpha \le 1$; and
(4b)$_4$ $\lim_{n\to\infty} \left(\sum_{k=0}^{n} f_0^k(1)\right) \Big/ \left(\sum_{k=0}^{n}(1 - f_1^k(0))\right) = a$ with $0 < a < \infty$,

then

$$\frac{1}{n}\sum_{j=0}^{n-1} 1_{[0,\gamma]} \circ T^j \xrightarrow{\mathcal{L}} L_{\alpha,\beta} \quad \text{for } 0 < \gamma < 1$$

with

$$\beta = \frac{T'(c^-)}{T'(c^-) + T'(c^+)/a}.$$

If T satisfies condition (4b)$_4$ with $a = 0$ or $a = \infty$, the assertion also holds (where $1/0 = \infty$, $1/\infty = 0$ as usual).

Conversely, if

$$\frac{1}{n}\sum_{j=0}^{n-1} 1_{[0,\gamma]} \circ T^j \xrightarrow{\nu} L$$

for some $\gamma \in (0, 1)$, some $\nu \in \mathcal{D}$ and some random variable L which is not concentrated on 0 or 1, then (4a)$_4$ and (4b)$_4$ hold with

$$\alpha = \frac{E(L(1-L))}{E(L)(1-E(L))} \quad \text{and} \quad a = \frac{E(L)}{1-E(L)}\frac{T'(c^+)}{T'(c^-)}.$$

Proof. It only remains to consider the boundary cases. If (4a)$_4$ and (4b)$_4$ hold, the reasoning in the proof of the theorem also yields the assertion for $\alpha = 0$ and $\alpha = 1$. In fact, it suffices to prove convergence of the first two moments in these cases. The asymptotic relations for the derivatives of Q_0 and F_0 hold in the following sense:

$$\lim_{s\to 0} s^{r+1-\alpha} Q_0^{(r)}(s)/L(1/s) = r!\binom{\alpha-1}{r} \quad (r \ge 0)$$

and

$$\lim_{s\to 0} s^{\ell-\alpha} F_0^{(\ell)}(s)/L(1/s) = -\ell!\binom{\alpha}{\ell} \quad (\ell \ge 1).$$

Here we use the fact that Lemma 6(b) is also valid for $\theta = 0$. (Note that $\binom{\alpha-1}{r} = 0$ if $\alpha = 1$, $r \ge 1$, and $\binom{\alpha}{\ell} = 0$ if $\alpha = 1$, $\ell \ge 2$ or $\alpha = 0$, $\ell \ge 1$.)

If (4b)$_4$ holds with $a = 0$, then $\lim_{s\to 0} Q_0(s)/Q_1(s) = 0$, and the formula in Lemma 3 shows that

$$f_0'(1)\,Q_1(s)\sum_{n=0}^{\infty}\left(\int_U S_n g_1\, d\lambda\right)e^{-ns} \sim \frac{f_1'(0)}{s^2}Q_0(s) \quad (s \to 0).$$

Therefore

$$\lim_{s\to 0} s^2 \sum_{n=0}^{\infty}\left(\int_U S_n g_1 d\lambda\right)e^{-ns} = 0,$$

which implies $(1/n)S_n \to 0$ in measure (with respect to λ).

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Applying this result to the fixed point 1 we see that $(1/n)S_n \to 1$ in measure if $a = \infty$. Notice that regular variation plays no role in these cases.

The proof of the last statement remains unchanged except for the final conclusion. □

Suppose, in particular, T satisfies the conditions (1)–(3), and

$$Tx - x \sim a_0\, x^{p_0+1} \quad (x \to 0)$$

and

$$x - Tx \sim a_1(1-x)^{p_1+1} \quad (x \to 1),$$

with $a_0, a_1 > 0$ and

(i) $p_0 = p_1 = 1$; or

(ii) $1 \le p_0 < p_1$ (cf. [**In1**]).

The above reasoning yields

$$\frac{1}{n}S_n \to \begin{cases} T'(c^-)/(T'(c^-) + T'(c^+)/a) & \text{in measure in case (i), and} \\ 0 & \text{in measure in case (ii).} \end{cases}$$

Under the additional assumption that $T(x) + T(1-x) = 1$, $x \in [0, 1]$, it follows from a result in [**AaN**] that in case (i) the sequence $\{(1/n)S_n\}$ diverges a.e. (see also [**In3**]).

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References

[Aa1] J. Aaronson. *An Introduction to Infinite Ergodic Theory (Mathematical Surveys and Monographs, 50)*. American Mathematical Society, Providence, RI, 1997.

[Aa2] J. Aaronson. The asymptotic distributional behaviour of transformations preserving infinite measures. *J. Analyse Math.* **39** (1981), 203–234.

[AaN] J. Aaronson and H. Nakada. Trimmed sums for non-negative, ergodic stationary processes. *Preprint*. Tel Aviv University/Keio University, 2000.

[An] E. S. Andersen. On the number of positive sums of random variables. *Skand. Aktuarietidskr.* **32** (1949), 27–36.

[BGT] N. H. Bingham, C. M. Goldie and J. L. Teugels. *Regular Variation*. Cambridge University Press, Cambridge, 1987.

[ChF] K. L. Chung and W. Feller. On fluctuations in coin-tossing. *Proc. Natl. Akad. Sci. USA* **35** (1949), 605–608.

[EK] P. Erdös and M. Kac. On the number of positive sums of independent random variables. *Bull. Amer. Math. Soc.* **53** (1947), 1011–1020.

[F1] W. Feller. *An Introduction to Probability Theory and its Applications*. Vol. I. Wiley, New York, 1968.

[F2] W. Feller. *An Introduction to Probability Theory and its Applications*. Vol. II. Wiley, New York, 1971.

[In1] T. Inoue. Ratio ergodic theorems for maps with indifferent fixed points. *Ergod. Th. & Dynam. Sys.* **17** (1997), 625–642.

[In2] T. Inoue. Sojourn times in small neighbourhoods of indifferent fixed points of one-dimensional dynamical systems. *Ergod. Th. & Dynam. Sys.* **20** (2000), 241–257.

[In3] T. Inoue. Correction to 'Ratio ergodic theorems for maps with indifferent fixed points'. *Ergod. Th. & Dynam. Sys.* **21** (2001), 1273.

https://doi.org/10.1017/S0143385702000573 Published online by Cambridge University Press

[K] M. Kuczma. *Functional Equations in a Single Variable*. PWN—Polish Scientific Publishers, Warszawa, 1968.

[La] J. Lamperti. An occupation time theorem for a class of stochastic processes. *Trans. Amer. Math. Soc.* **88** (1958), 380–387.

[Lg] S. Lang. *Complex Analysis*. Springer, New York, 1993.

[Le] P. Lévy. Sur certains processus stochastiques homogènes. *Compositio Math.* **7** (1939), 283–339.

[R] C. Reichsöllner. On the order of magnitude of the invariant densities of real transformations. *PhD Thesis*. Salzburg, 1989 (in German).

[Sp] F. Spitzer. A combinatorial lemma and its application to probability theory. *Trans. Amer. Math. Soc.* **82** (1956), 323–339.

[T1] M. Thaler. Estimates of the invariant densities of endomorphisms with indifferent fixed points. *Israel J. Math.* **37** (1980), 303–314.

[T2] M. Thaler. Transformations on $[0, 1]$ with infinite invariant measures. *Israel J. Math.* **46** (1983), 67–96.

[T3] M. Thaler. A limit theorem for the Perron–Frobenius operator of transformations on $[0, 1]$ with indifferent fixed points. *Israel J. Math.* **91** (1995), 111–127.

[T4] M. Thaler. The Dynkin–Lamperti arc-sine laws for measure preserving transformations. *Trans. Amer. Math. Soc.* **350** (1998), 4593–4607.

[Z] R. Zweimüller. Ergodic theory of interval maps with indifferent fixed points. *PhD Thesis*. Erlangen, 2000.

https://doi.org/10.1017/S0143385702000573 Published online by Cambridge University Press