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Truth, modality and intersubjectivity

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Quantum physics, together with the experimental (and slightly controversial) *quantum computing*, induces a twist in our vision of computation, and hence, since computing and logic are intimately linked, in our approach to logic and foundations. In this paper, we discuss the most mistreated notion of logic, *truth*.

1. Introduction

1.1. *Revisiting foundations*

Is there anything more frozen than 'foundations'? A quick glance at the list of the 'foundations of mathematics' at http://www.cs.nyu.edu/mailman/listinfo/fom shows a paradigm close to archaic astronomy: truth is a primitive (like the Earth), around which several systems and meta-systems gravitate (like the epicycles of Ptolemy). This being orchestrated by Doctors of the Law, in charge of the latest developments of Hilbert's program, that is, of a certain form of *finitism* that has been obsolete since Gdel's theorem (1931!), but that still holds an honourable position in this sort of *Jurassic Park*.

Let us put it bluntly: these people confuse foundations with *prejudices*. Of course, we cannot exclude the possiblility that the deep layers behave according to our preconceptions; but anyone thinking like this should draw the conclusions and quit. My personal bias, the one followed in this paper, is that the real *hypostases* are very different from our familiar (mis)conceptions: I shall thus propose a *disturbing* approach to foundations. This viewpoint is by no means 'non-standard', it is, on the contrary, most standard; but it relies on ideas developed in the last century and prompted by quantum physics, the claim being that *operator algebra is more primitive than set theory*.

1.2. *Sets versus operators*

In terms of foundations, the most impressive achievement of the turn of the century is not to be found within logic, not to mention the aforementioned *Jurassic Park*, but in the *non-commutative geometry* of Connes (Connes 1994), a paradigm that is violently anti-set-theoretic, and based on the following familiar result:

A commutative operator algebra is a function space.

Typically, a commutative C^*-algebra can be written $\mathbf{C}(X)$, the algebra of continuous functions on the compact X. Connes proposes that we consider non-commutative operator

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algebras as sorts of algebras of functions over... non-existing sets, which is an impressive blow against set-theoretic essentialism!

Logic is *a priori* a long way from the considerations internal to geometry; but this changes our ideas of a finite set, a point, a graph, and so on:

— The commutative, set-theoretic, world appears as a vector space equipped with a distinguished base. All operations are organised in relation to this base: in particular, they can be represented by linear functions whose matrices are diagonal in this base.
— The non-commutative world forgets the base. There is still one, but it is *subjective*, being the base that diagonalises the hermitian operator we are using: 'our' set-theory, so to speak. But, if two hermitians f and g have non-commuting 'set-theories', $f + g$ has a third set-theory bearing no relation to the previous ones.

Roughly speaking, the base is on the side of *particles*; while an operator is *wavelike*. If the latter is objective, the former, which corresponds to set-theory, is subjective.

1.3. *The three layers*

In Girard (2006a), I introduced three foundational layers: **-1**, **-2** and **-3**. This has nothing to do with playing around with iterated metas[†]; computationally speaking, the distinction can easily be explained using an example:

-1: The function φ sends integers ($\mathbb{N}$) to booleans ($\mathbb{B}$), which is traditionally expressed through the implication $\mathbb{N} \Rightarrow \mathbb{B}$.
-2: $\varphi(n) = T$ if n is prime, $\varphi(n) = F$ otherwise.
-3: φ implements the sieve of Eratosthenes.

Level **-1** deals with inputs/outputs; logically speaking, it corresponds to truth, logical consequence and satellite concepts such as consistency. Level **-2** considers proofs as functions and, more generally, as *morphisms* in an appropriate category. Finally, level **-3** deals with the dynamics, that is, with the *procedurality* of logical operations.

The central result of proof-theory, *cut-elimination*, reads as follows in the three layers:

-1: The absurd sequent, not being cut-free provable, is not provable at all, from which we get consistency.
-2: The Church–Rosser property (natural deduction, proof-nets) induces the *compositionality* of proofs, that is, the existence of an underlying category.
-3: The cut-elimination process can be expressed as the solution of a linear equation on the Hilbert space – the *feedback equation* (19) below.

Historically speaking, layer **-1** comes from the foundational discussion of *classical* logic; the view of proofs as functions (layer **-2**) must be ascribed to *intuitionism*; finally, the paradigm of proofs as actions (layer **-3**) is well adapted to *linear* logic. Quantum computing admits an interpretation of level **-2** (QCS below), but its spirit is mostly at level **-3**.

[†] A system rests on a meta-system, which in turn rests on a meta-meta-system... *Turtles all the way down*, as in the famous joke! The *Jurassic Park*, conscious of the problem, added one more turtle at the bottom, and so on. We get transfinite meta-turtles, 'predicative' or not, but does this make convincing foundations?

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1.4. *A failure: quantum logic*

According to Herodotus, Xerxes had the sea *beaten* for misbehaviour; *quantum logic* is, in its way, a punishment inflicted upon nature for making 'mistakes' of logic.

According to quantum logic, everything should stay the same, except the truth values; by the way, the idea that logic should be defined in terms of truth values, that is, at level **-1**, is spurious – such an assumption makes the departure from classical logic difficult, nay impossible. The boolean algebra $\{T, F\}$ is therefore replaced with the structure consisting of the closed subspaces of a given Hilbert space. Unfortunately, these subspaces do not get on together: any reasonable operation requires the *commutation* of the associated orthoprojections; typically, the intersection, which is easily defined as the product $\pi\pi'$ of the associated projections when they commute, but has no manageable definition otherwise. There are two ways of fixing this fundamental mismatch:

1 Abstract everything, and forget the Hilbert space: this leads to 'orthomodular lattices', that is, to nowhere.
2 Replace subspaces with their orthoprojections and close them under real linear combinations: this leads to hermitians and, eventually, to forgetting the logical nonsense about truth values. The second way was the one followed by von Neumann, who had the bad taste to introduce quantum logic, but soon corrected his mistake by the creation of what we now call *von Neumann algebras.*

For instance, the 'set-theories' of the hermitians $\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$ and $\begin{bmatrix}1/2 & 1/2\\ 1/2 & 1/2\end{bmatrix}$ correspond to the bases $\{\vec{X}, \vec{Y}\}$ and $\{\sqrt{2}/2(\vec{X}+\vec{Y}), \sqrt{2}/2(\vec{X}-\vec{Y})\}$, but the 'set-theory' of their sum $\begin{bmatrix}1/2 & 1/2\\ 1/2 & 3/2\end{bmatrix}$ does not belong in lattice theory, since it involves solving the algebraic equation $\lambda^2 - 2\lambda + 1/2 = 0$. In other words, the order structure of subspaces cannot cohabit with the basic quantum operation, *superposition.* This explain the failure of approach (1) and its replacement by (2).

This replacement assumes we relinquish the logical viewpoint; so can we establish a link between logic and quantum theory?

1.5. *Logic versus quantum*

Beyond any doubt, a relation should be established. Unfortunately, this vague question became: *find a logical explanation of quantum phenomena*, which eventually led to quantum 'logic'. In the same way, the vague question of the relation between the Earth and the planets was formulated as *find a geocentric explanation of celestial machanics* – this program was pursued for endless centuries and led to Ptolemy's notorious epicycles; another punishment inflicted upon nature, which was guilty of not following Joshua's Book.

In other words, what is there that is so good in logic that quantum physics should obey it? Can't we imagine that our conceptions about logic are wrong, in fact, so wrong that they are unable to cope with the quantum miracle? Indeed, the 'logical' treatment of the quantum world rests on the prejudice that the usual operator-theoretic

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approach is wrong; and logicians are happy toying with their own counter-explanations of quantum phenomena. In particular, they seem to believe in *hidden variables*, that is, in a thermodynamic explanation of quantum mechanics: otherwise, how can we explain the attempts to exhume the corpse of Gleason's theorem?

Instead of teaching logic to nature, it is more reasonable to learn from her. Instead of interpreting quantum theory in logic, we should interpret logic in quantum theory. This basically involves using operator algebras, the difficult part is finding the correct way to do so. We shall go beyond level **-1** (truth values). First, to level **-2** (functions, morphisms, categories), with *quantum coherent spaces*. There we shall meet a problem with infinite dimension: which will eventually force us to move to layer **-3**.

2. Quantum coherent spaces

2.1. *QCS*

If $\mathbf{H}$ is a complex Hilbert space of finite dimension n, then $\mathscr{L}(\mathbf{H})$, the space of endormorphisms of $\mathbf{H}$, has (complex) dimension n^2, and hence real dimension $2n^2$. Every $u \in \mathscr{L}(\mathbf{H})$ can be written uniquely as $u = h + ik$, with h,k *hermitian*; it follows that the real vector space $\mathscr{H}(\mathbf{H})$ of *hermitians* has dimension n^2. This space is indeed *euclidian*, that is, a real Hilbert space, when endowed with the bilinear form $\langle h \mid k\rangle := \mathrm{tr}(h \cdot k)$.

Definition 1. Two hermitians $h,k \in \mathscr{H}(\mathbf{H})$ are *polar* if and only if

$$x \perp y \quad :\Leftrightarrow \quad 0 \leqslant \mathrm{tr}(h \cdot k) \leqslant 1. \tag{1}$$

Given $\mathfrak{A} \subset \mathscr{H}(\mathbf{H})$, its *polar* $\sim\mathfrak{A}$ is defined as

$$\sim\mathfrak{A} \quad := \quad \{\mathfrak{k}; \forall \mathfrak{h} \in \mathfrak{A} \quad \mathfrak{o} \leqslant \mathrm{tr}(\mathfrak{h} \cdot \mathfrak{k}) \leqslant \mathfrak{1}\}. \tag{2}$$

A *quantum coherent space* (QCS) of *carrier* $\mathbf{H}$ is a subset $\mathfrak{X} \subset \mathscr{H}(\mathbf{H})$ equal to its bipolar.

QCSs yield a natural – categorical, that is, of level **-2** – interpretation for linear logic, and is also a reasonable candidate for the idea of a 'typed quantum algorithm'. This has already been developed in Girard (2004; 2007). We should mention a point here: linear logic admits a connective named 'tensor' and written $\otimes$, and this connective does not conform with the idea of *entanglement* at work in quantum physics and computing. However, this does not mean that QCS (and thus linear logic) are ill-adapted to quantum computing. Indeed, linear logic (and QCS) have *two* tensors, the other one being called 'par' and written $\parr$: it is this *cotensor* that actually deals with entanglement. The confusion arises from the fact that, algebraically speaking, tensor and cotensor coincide in finite dimensions; therefore, this apparent mismatch is purely a question of terminology.

2.2. *The adjunction*

The milestone in the relationship between QCS and linear logic is the interpretation of linear implication.

Theorem 1. There is a canonical isomorphism between the set of all *linear* maps from the QCS $\mathfrak{X}$ to the QCS $\mathfrak{Y}$ and a QCS $\mathfrak{X} \multimap \mathfrak{Y}$.

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If $\mathfrak{X},\mathfrak{Y}$ are of respective carriers $\mathbf{H},\mathbf{K}$, then $\mathfrak{X}\multimap\mathfrak{Y}$ is of carrier $\mathbf{H}\otimes\mathbf{K}$ and is defined as

$$\mathfrak{X}\multimap\mathfrak{Y} \quad := \quad \sim\{\mathfrak{h}\otimes\mathfrak{k};\mathfrak{h}\in\mathfrak{X},\mathfrak{k}\in\sim\mathfrak{Y}\}. \tag{3}$$

Given a linear map Φ from $\mathscr{H}(\mathbf{H})$ to $\mathscr{H}(\mathbf{K})$ sending $\mathfrak{X}$ into $\mathfrak{Y}$, its *skeleton* $\mathrm{sk}(\Phi)\in\mathscr{H}(\mathbf{H}\otimes\mathbf{K})$ is defined by

$$\langle \mathrm{sk}(\Phi)(x\otimes y)\mid w\otimes z\rangle \quad := \quad \langle(\Phi\cdot xw^*)(y)\mid z\rangle \qquad (x,w\in\mathbf{H}\quad y,z\in\mathbf{K}), \tag{4}$$

with $(xw^*)(y):=\langle y\mid w\rangle\cdot x$. By linearity,

$$\mathrm{tr}(\mathrm{sk}(\Phi)\cdot(h\otimes yz^*)) \quad := \quad \langle(\Phi(h))(y)\mid z\rangle \qquad (h\in\mathscr{H}(\mathbf{H})\quad y,z\in\mathbf{K}), \tag{5}$$

which shows how we can recover Φ from its skeleton.

Everything rests upon the 'application' $\varphi[h]$ of the skeleton $\varphi:=\mathrm{sk}(\Phi)$ to $h\in\mathfrak{X}$, which is characterised by the adjunction

$$\mathrm{tr}(\varphi[h]\cdot k)=\mathrm{tr}(\varphi\cdot(h\otimes k)) \qquad (h\in\mathscr{H}(\mathbf{H})\quad k\in\mathscr{H}(\mathbf{K})). \tag{6}$$

For instance, the *twist* $\sigma\in\mathscr{H}(\mathbf{H}\otimes\mathbf{H})$, defined by

$$\sigma(x\otimes y) \quad := \quad y\otimes x \tag{7}$$

is such that

$$\mathrm{tr}(h\cdot k)=\mathrm{tr}(\sigma\cdot(h\otimes k)) \qquad (h\in\mathscr{H}(\mathbf{H})\quad k\in\mathscr{H}(\mathbf{K})), \tag{8}$$

and hence $\sigma[h]=h$, which shows that it is the skeleton of the identity map.

2.3. *Coherent spaces*

QCSs are derived from the original[†] interpretation of linear logic: *coherent spaces* (Girard 2006). Roughly speaking, they appear as a *subjective* version of a QCS, obtained by focusing on a particular *base* of the carrier.

A finite-dimensional Hilbert space $\mathbf{H}$ can, given a base X, be written uniquely as $\mathbb{C}^X$. Moreover, if we restrict our attention to subsets $a,b\ldots$ of the base, the associated subspaces are represented by projections $\pi_a,\pi_b\ldots$ whose matrices are diagonal with entries equal to $0,1$. Observe that

$$\mathrm{tr}(\pi_a\pi_b)=\mathrm{tr}(\pi_{ab})=\sharp(a\cap b) \qquad (a,b\subset X), \tag{9}$$

so $\pi_a \mathrel{\downarrow\!\!\sim} \pi_b$ if and only if $\sharp(a\cap b)\leqslant 1$.

Definition 2. A *coherent space* with carrier X is a set $\mathfrak{X}$ of subsets of X equal to its bipolar, the polarity between subsets of X being defined as

$$a \mathrel{\downarrow\!\!\sim} b \quad \Leftrightarrow \quad \sharp(a\cap b)\leqslant 1, \tag{10}$$

and the adjunction (6) becomes

$$\sharp(\varphi[a]\cap b)=\sharp(\varphi\cap(a\times b)) \qquad (a\in\mathfrak{X}\quad \mathfrak{b}\in\sim\mathfrak{Y}). \tag{11}$$

[†] This is in a strong sense as linear logic issued from coherent spaces.

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We define the binary relation $\frown_{\mathfrak{X}}$ on the carrier X by

$$x \frown_{\mathfrak{X}} y \quad \Leftrightarrow \quad \{x, y\} \in \mathfrak{X}. \tag{12}$$

Theorem 2. $a \subset X$ belongs to $\mathfrak{X}$ if and only if a is a *clique* with respect to $\frown_{\mathfrak{X}}$

$$a \in \mathfrak{X} \quad \Leftrightarrow \quad \forall \mathfrak{x}, \mathfrak{y} \in \mathfrak{a}\ \mathfrak{x} \frown_{\mathfrak{X}} \mathfrak{y}, \tag{13}$$

which corresponds to the 'official' definition of coherent spaces.

The connectives ($\otimes, \parr$, &, $\oplus$) are, since they are logical, subjective. This is why they naturally involve the building (more generally, the maintenance) of distinguished bases. This explains why the objective, 'wavelike', artifacts (QCS) remained invisible for so long: they were indeed prompted by quantum computing. This means that every QCS naturally comes with its distinguished base, and that all logical constructions lead from cliques (that is, diagonal matrices with entries 0, 1) to cliques. There is, however, a remarkable exception, namely the *identity axiom* $A \multimap A$. If A stands for the QCS $\mathfrak{X}$, this axiom is interpreted by the *twist* (7), which is unitary, and hence by no means a projection. This interpretation therefore differs from the original one, which was formulated in terms of coherent spaces, and for which the identity axiom is the diagonal of $X \times X$, that is, corresponds to the projection obtained from the twist by chopping off all non-diagonal coefficients. Typically, if $\mathfrak{X}$ is of dimension 2, then the twist and the 'diagonal' are written

$$\sigma = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \delta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{14}$$

respectively. The distinction between σ and δ must be related to the following issues (see Girard (2004; 2007)):

— Eta-conversion: δ is an 'eta-expansion' of σ.
— Quantum measurement: δ is the result of the *reduction of the wave packet* applied to σ, or rather, the deterministic process of *preselection* associated with this non-deterministic operation.

Combining a rather obscure logical technicality, which has only yielded an afflictive literature, with one of the major discoveries of last century is unexpected. It shows that logicians have completely neglected what should have been their main interest: *the relation between object and subject*.

3. Perennialisation

3.1. *Perfect versus imperfect*

The major discovery of *linear logic* is the distinction between the *perfect* and *imperfect*: see Girard (2006a). As in natural language (the *imparfait* in French grammar), imperfection corresponds to repetition, that is, to *perenniality* and, eventually, to infinity. A specific connective, $!A$, the *exponential*, together with its dual $?A$, is in charge of perennialisation,

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which is expressed through various rules, typically *contraction*

$$!A \multimap !A \otimes !A\,, \tag{15}$$

which is the primal form of perenniality.

Technically speaking, the *perfect*, non-perennial, world is linear; which is expressed by the *linear* implication $A \multimap B$. On the other hand, the imperfect world relies on the usual (intuitionistic) implication $A \Rightarrow B$, which is not linear: it can be translated as $!A \multimap B$, which means that it allows constant, quadratic, polynomial dependencies. In *coherent Banach spaces* (CBS), which are a level **-2** interpretation of linear logic (Girard 2007), formulas are interpreted by Banach spaces and $!A \Rightarrow B$ is inhabited by analytic functions from the open ball $A_{<1}$ to the closed ball $B_{\leqslant 1}$.

3.2. *Obstacles to perennialisation*

Coherent spaces admit a natural exponentiation: if $\mathfrak{X}$ is a coherent space, the carrier of $!\mathfrak{X}$ consists of all *finite* cliques of $\mathfrak{X}$ and

$$a \subset_{!\mathfrak{X}} b \quad \Leftrightarrow \quad a \cup b \in \mathfrak{X}. \tag{16}$$

Long before coherent spaces, *Scott domains* yielded a topological interpretation of the imperfect implication $A \Rightarrow B$ by means of continuous maps from A into B.

Both interpretations have their limitations: for instance, Scott domains are a sort of childish topology in which the separately continuous are *ipso facto* continuous. Moving to the real thing, typically to *coherent Banach spaces*, poses problems: we noted above that implication corresponds to analytical maps from an *open* to a *closed* ball, and such maps do not compose. This indicates that there is a logical 'mistake' with regard to continuity in the rules for perenniality. In such a situation, one is faced with a dilemma:

— We can change the principles of topology and restrict ourselves to a castrated version of continuity, typically to Scott domains. This is the dominant viewpoint in logic, which leads nowhere.
— Or we can use the real thing and try to modify the rules of perenniality in a way that is compatible with normal mathematics. So far this has also led nowhere, but there is something promising here: changing the rules of exponentiation alters the rate of growth of definable functions. This phenomenon was first observed for the *light exponentials*: for instance, in the system **LLL** (Girard 2007), where definable functions are polytime. A connection with *complexity theory* is therefore expected.

3.3. *Geometry of interaction*

Returning to QCS, we quickly discover that the main obstacle to perennialisation is the limitation to finite dimension. Could we therefore allow infinite dimensional carriers? The answer is no, in the absence of a satisfactory *trace*: on the space $\mathscr{B}(\mathbf{H})$ of (bounded) endomorphisms of an infinite dimensional Hilbert space, only certain operators admit a trace, and are therefore styled 'trace-class'. Unfortunately, unitaries such as the *twist* are never trace-class, so the identity map does not belong here.

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It is time to recall that von Neumann algebras are meant to be generalisations of finite dimensional (matrix) algebras in which the trace is available: this is the case for vN algebras of type $\mathbf{II}_1$, amongst which is the celebrated *hyperfinite factor*[†] $\mathscr{R}$ of Murray and von Neumann (Connes 1994). Unfortunately, moving to type $\mathbf{II}_1$ does not solve our problem: a computation, made in the spirit of (8), yields the value $\mathrm{tr}(\sigma \cdot (h \otimes k)) = 0$, which shows that the identity map does not belong there either.

A change of paradigm is therefore necessary: this leads us to the *geometry of interaction* (GoI) (Girard 1989; 2007). To cut a long story short, this involves replacing the trace by the *determinant*, which makes sense, under reasonable hypotheses, in $\mathscr{R}$. If $\varrho(u) < 1$, we define

$$\det(I - u) \quad := \quad \exp(\mathrm{tr}(\ln(I - u))), \tag{17}$$

$\ln(I-u)$ being definable by the usual power series thanks to the hypothesis on the *spectral radius* $\varrho(u)$.

Everything rests on an analogue of the adjunctions (11) and (6). Typically,

$$\det(I - \varphi[u] \cdot v) = \det(I - \varphi \cdot (u \oplus v)), \tag{18}$$

which is a sort of logarithm of (6) – the trace becomes a determinant, the tensor product being replaced by a direct sum, corresponding to a decomposition $I = \pi + (I - \pi)$ of the identity into a sum of orthogonal projections.

Something close to (18) can indeed be achieved: $\varphi[u]$ can be defined as the solution of the *feedback equation*

$$\varphi[u] := (I - \pi) \cdot \varphi \cdot (I - u \cdot \varphi)^{-1} \cdot (I - \pi), \tag{19}$$

see Girard (2006b; 2007).

Unfortunately, (18) is slightly incorrect and must be replaced by

$$\det(I - \varphi[u] \cdot v) \cdot \det(I - \varphi \cdot u) = \det(I - \varphi \cdot (u \oplus v)). \tag{20}$$

The appearance of the scalar $\det(I - \varphi \cdot u)$ (a sort of truth value) radically modifies our approach to the question.

3.4. *The duality of GoI*

From now on we will use $\mathscr{R}$ to denote the *hyperfinite factor*.

Definition 3 (Projects). Let $\pi \in \mathscr{R}$ be a projection. A *project of base* π is the pair $\mathfrak{p} = (\alpha, \mathfrak{u})$ of a *wager* $\alpha \in \mathbb{C}$ and an *aim* $u \in \mathscr{R}$ such that $\|u\| < 1$ and $\pi u \pi = u$.

Definition 4 (Duality). Two projects $\mathfrak{p} = (\alpha, \mathfrak{u}), \mathfrak{q} = (\beta, \mathfrak{v})$ of the same base π are *polar*, denoted $\mathfrak{p} \overset{\downarrow}{\sim} \mathfrak{q}$, when the following hold[‡]:

1 $\varrho(uv) < 1$.
2 $\alpha\beta \det(I - uv) \neq 1$.

[†] $\mathscr{B}(\mathbf{H})$ is also a vN algebra, of trivial type $\mathbf{I}_\infty$.
[‡] $\varrho(.)$ denotes the *spectral radius*.

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We define, provided $\mathfrak{p} \downarrow \mathfrak{q}$, $\ll \mathfrak{p} \mid \mathfrak{q} \gg := \alpha\beta \det(\mathfrak{I} - \mathfrak{u}\mathfrak{v})$, when $\varrho(uv) < 1$; the scalar $\ll \mathfrak{p} \mid \mathfrak{q} \gg$ can be viewed as the degenerated project (of base 0), $(\ll \mathfrak{p} \mid \mathfrak{q} \gg, \mathfrak{o})$.

From this, we can make the following definition.

Definition 5 (Conducts). Let $\pi \in \mathscr{R}$ be a projection. A *conduct of base* π is a set $\mathbf{C}$ of projects of base π equal to its bipolar.

We can now reconstruct logic on this basis.

3.5. *The adjunction of GoI*

In the following, the bases π, ν of the conducts $\mathbf{C}, \mathbf{D}$, respectively, are orthogonal, that is, $\pi\nu = 0$.

Definition 6 (Linear implication). The conduct $\mathbf{C} \otimes \sim\mathbf{D}$ (of base $\pi + \nu$) is defined as

$$\mathbf{C} \otimes \sim\mathbf{D} \quad := \quad \sim\sim\{(\alpha\beta, u + v); (\alpha, u) \in \mathbf{C}, (\beta, v) \in \sim\mathbf{D}\}. \tag{21}$$

The conduct $\mathbf{C} \multimap \mathbf{D}$ (of base $\pi + \nu$) is then defined as its polar:

$$\mathbf{C} \multimap \mathbf{D} \quad := \quad \sim\{(\alpha\beta, u + v); (\alpha, u) \in \mathbf{C}, (\beta, v) \in \sim\mathbf{D}\}. \tag{22}$$

For the next theorem, remember the definition (19) of $\varphi[u]$.

Theorem 3 (Adjunction). $(\gamma, \varphi) \in \mathbf{C} \multimap \mathbf{D}$ if and only if

$$\forall (\alpha, u) \in \mathbf{C} \ (\varrho(\varphi u) < 1 \text{ and } (\alpha\gamma \det(I - \varphi u), \varphi[u]) \in \mathbf{D}). \tag{23}$$

Proof. The proof, essentially, relies on (20); see Girard (2007). □

4. Subjective truth

4.1. *Retrieving level -2*

The three layers **-1, -2, -3** are not some sort of Trinity, like the *hypostases* of neo-platonism. The idea is that everything rests upon layer **-3** and that **-1** and **-2** are only *surrogates*: the problem is therefore to retrieve them from layer **-3**.

The retrieval of layer **-2** consists of defining a *category* whose objects are conducts. It is immediate that the definition of linear implication yields a functional interpretation: 'functions' from $\mathbf{C}$ to $\mathbf{D}$. If $\mathbf{C}, \mathbf{D}, \mathbf{E}$ are of (pairwise orthogonal) bases π, ν, μ, respectively, and if $(\alpha, \varphi) \in \mathbf{C} \multimap \mathbf{D}$, $(\beta, \psi) \in \mathbf{D} \multimap \mathbf{E}$, then $(\gamma, \theta) \in \mathbf{D} \multimap \mathbf{E}$, with

$$\theta \quad := \quad (\pi\varphi + \nu)(I - \psi\varphi)^{-1}(\pi + \psi\nu) \tag{24}$$

$$\gamma \quad := \quad \alpha\beta \det(I - \varphi\psi). \tag{25}$$

(24) generalises (19), and the 'composition' thus defined is associative, see Girard (2007).

Unfortunately, this is not quite a category: this is due to the constraints on bases – for instance, one cannot form $\mathbf{C} \multimap \mathbf{C}$. In order to construct a category, one has to allow a certain amount of *delocation.*

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4.2. *The category of projects*

If $n > 0$, $\mathscr{M}_n(\mathscr{R})$ stands for the algebra of $n \times n$ matrices with coefficients in $\mathscr{R}$, that is, $\mathscr{M}_n(\mathbb{C}) \otimes \mathscr{R}$. As a vN algebra, $\mathscr{M}_n(\mathscr{R})$ is isomorphic to $\mathscr{R}$, and hence admits a unique trace. Indeed, the trace is made unique by the requirement that $\mathrm{tr}(I) = 1$. We shall make a different choice, namely, $\mathrm{tr}(I) = n$.

If $\mathtt{f} \in I(m,n)$ is an injective map from $m = 1,\dots,m$ into $n = 1,\dots,n$, it induces a $*$-isomorphism $\mathscr{M}_{\mathtt{f}}(\mathscr{R})$ from $\mathscr{M}_m(\mathscr{R})$ to $\mathscr{M}_n(\mathscr{R})$. This isomorphism preserves everything, except the identity; our convention about traces makes it preserve traces as well.

The following injective maps can be defined:

$\mathtt{s}$: $\mathtt{s} \in I(1,3) : \mathtt{s}(1) := 1$ (source).
$\mathtt{t}$: $\mathtt{t} \in I(1,3) : \mathtt{t}(1) := 3$ (target).
$\mathtt{p}$: $\mathtt{p} \in I(3,3) : \mathtt{p}(1) := 1, \mathtt{p}(2) := 3, \mathtt{p}(3) := 2$ (prefixing).
$\mathtt{q}$: $\mathtt{q} \in I(3,3) : \mathtt{q}(1) := 2, \mathtt{q}(2) := 1, \mathtt{q}(3) := 3$ (postfixing).
$\mathtt{r}$: $\mathtt{r} \in I(1,3) : \mathtt{r}(1) := 2$ (relay).

Definition 7 (Morphism). If $\mathbf{C}, \mathbf{D}$ are conducts of bases π, ν, respectively, a *morphism* from $\mathbf{C}$ to $\mathbf{D}$ is a project $\mathfrak{f} \in \mathtt{s}(\mathbf{C}) \multimap \mathtt{t}(\mathbf{D})$ of base

$$\begin{bmatrix} \pi & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \nu \end{bmatrix}.$$

Typically, the identity morphism of $\mathbf{C}$ of base π is

$$\left(1, \begin{bmatrix} 0 & 0 & \pi \\ 0 & 0 & 0 \\ \pi & 0 & 0 \end{bmatrix}\right).$$

Definition 8 (Composition). In order to compose the morphisms $\mathfrak{f} \in \mathfrak{Proj}(\mathbf{C}, \mathbf{D})$ and $\mathfrak{g} \in \mathfrak{Proj}(\mathbf{D}, \mathbf{E})$, we compose, following (24) and (25), $\mathtt{p}(\mathfrak{f}) \in \mathtt{s}(\mathbf{C}) \multimap \mathtt{r}(\mathbf{D})$ with $\mathtt{q}(\mathfrak{g}) \in \mathtt{r}(\mathbf{D}) \multimap \mathtt{t}(\mathbf{E})$.

These outlines of a categorical interpretation are enough to show that level **-2** can be retrieved from level **-3**.

4.3. *Retrieving level -1*

The *wager* is a sort of truth value, a wager equal to 1 meaning 'true': the interaction between projects $\mathfrak{p} \in \mathbf{C}$ and $\mathfrak{q} \in {\sim}\mathbf{C}$ yields a 'truth value' $\langle \mathfrak{p} \mid \mathfrak{q} \rangle$ that is necessarily distinct from 'true'. The intuition is therefore that a project of the form $\mathfrak{p} = (\mathfrak{i}, \mathfrak{u})$ is, so to speak, true: such a *wager-free* project corresponds to the familiar notion of *proof*. And we can expect to retrieve layer **-1** by saying that a conduct is *true* when it contains a wager-free project.

Unfortunately, this fails for a simple reason: wager-free projects are not closed under 'composition'. Typically, $\mathbf{C}, {\sim}\mathbf{C}$ may both contain wager-free projects. Indeed, the problem here is the scalar $\det(I - \varphi\psi)$ occurring in (25), which we have already met under the

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simplified form $\det(I - \varphi u)$ in (20). As this scalar is responsible for non-trivial wagers, we are seeking a way to enforce $\det(I - \varphi\psi) = 1$.

4.4. *Successful projects*

Definition 9 (Viewpoints). A *viewpoint* is a maximal commutative subalgebra $\mathscr{P} \subset \mathscr{R}$.

This is the same as saying that $\mathscr{P}$ is a subalgebra of $\mathscr{R}$ equal to its *commutant*.

It is instructive to consider the analogue of viewpoints in finite dimension. A maximal commutative subalgebra $\mathscr{P} \subset \mathscr{M}_n(\mathbb{C})$ consists of all matrices that are diagonal with respect to to a specific base. In other words, a viewpoint is the same as a base. In that case, the simplest option is to take the *canonical* viewpoint, that is, the one consisting of diagonal matrices.

Definition 10 (Success). Let $\mathscr{P}$ be a viewpoint. A project $\mathfrak{p} = (\alpha, \mathfrak{u})$ of base $\pi \in \mathscr{P}$ is *successful* (with respect to $\mathscr{P}$) if and only if:

1 $\mathfrak{p}$ is wager-free: $\alpha = 1$.
2 u is a *partial symmetry*: $u = u^* = a^3$.
3 u belongs to the *normaliser* of $\mathscr{P}$: $u\mathscr{P}u \subset \mathscr{P}$.

To say that the partial symmetry u belongs to the normaliser of $\mathscr{P}$ is the same as saying that whenever $v \in \mathscr{P}$ is a projection, uvu is still a projection of $\mathscr{P}$.

Returning to our finite dimensional analogy, and taking the canonical viewpoint for $\mathscr{P}$, the conditions on u read as follows: u is a symmetric matrice with entries equal to $0, 1$, and, moreover, there is at most one non-zero coefficient in any line (and hence in any column).

Definition 11 (Truth). Let $\mathscr{P}$ be a viewpoint. A conduct $\mathbf{C}$ of base $\pi \in \mathscr{P}$ is *true* when it contains a successful project. It is *false* when its negation $\sim\mathbf{C}$ is true.

Theorem 4 (Subjective coherence). Truth is closed under logical consequence: if $\pi, \nu, \mu \in \mathscr{P}$, and if $\mathbf{C}, \mathbf{D}, \mathbf{E}$ are of pairwise orthogonal bases π, ν, μ, respectively, then the truths of $\mathbf{C} \multimap \mathbf{D}$ and $\mathbf{D} \multimap \mathbf{E}$ entail the truth of $\mathbf{C} \multimap \mathbf{E}$.

Proof. The proof may be found in Girard (2007); see the corollary below for a simplified version. □

Corollary 4.1. $\mathbf{C}$ cannot be both true and false.

Proof. Assume that $\mathfrak{p} = (\mathfrak{1}, \mathfrak{u}) \in \mathbf{C}, \mathfrak{q} = (\mathfrak{1}, \mathfrak{v}) \in \sim\mathbf{C}$ are successful, and thus $\varrho(uv) < 1$ and $\det(I - uv) \neq 1$. Since they are successful, both $v \rightsquigarrow uvu$ and $v \rightsquigarrow vvv$ send projections in $\mathscr{P}$ to projections of $\mathscr{P}$. In particular, $(uv)^n I (vu)^n = (uv)^n (uv)^{n*}$ is a projection for all n, and hence is of norm 1 or 0. Since $\|(uv)^n\|^2 = \|(uv)^n (uv)^{n*}\|$, it follows that $(uv)^n$ is of norm 0 or 1. But $\varrho(uv) = \lim \|(uv)^n\|^{1/n} < 1$, so $(uv)^n$ cannot always be of norm 1. Hence uv is *nilpotent*. As in the finite dimension case, the determinant $\det(I - uv)$ is thus equal to 1, which is a contradiction. □

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It is quite easy to produce a conduct that is neither true nor false, and even easier to produce a conduct that is true with respect to a certain viewpoint $\mathscr{P}$ and false with respect to another viewpoint $\mathscr{Q}$. This illustrates the *subjective* character of our notion of truth. Of course this is unexpected, nay shocking, and calls for a detailed discussion.

4.5. *Modalities*

If we take the best, or rather least bad, modal logic $\mathbf{S_4}$, one is struck by the fact that the additional operations, such as the *necessity* $\Box A$, bring nothing new: typically, the erasure of all symbols $\Box, \Diamond$ preserves proofs. The situation can be summarised by the question:

What is the necessity of necessity?

The current answer is 'to write useless papers'. Indeed, there has been an industry of bad modal logics, people striving to get worse and worse systems: typical of these is $\mathbf{S_5}$, a system in which $\Box$ commutes with $\forall$, against all principles, such as cut-elimination.

Linear logic yielded another answer, through the *exponentials* $!A, ?A$, which are in charge of *perenniality*; these are indeed modalities, even if the symbols $\Box, \Diamond$ are not used, to avoid the infamous company of $\mathbf{S_5}$ and its likes. Indeed, linear modalities are something like '$\mathbf{S_4}$ + structural rules', that is, $!A$ behaves differently from A because $!A$ is the *perennialisation* of A; it is therefore impossible to erase the exponentials and preserve prooofs. Typically, the removal of modalities in *contraction*

$$!A \multimap !A \otimes !A \tag{15}$$

yields the incorrect

$$A \multimap A \otimes A. \tag{26}$$

Interpreting exponentials in GoI amounts to finding adequate hypotheses yielding contraction (15). The most obvious one is to restrict to projects whose wager equals 1 or 0. Without going into details, it is easy to see that contraction on $\mathfrak{p} = (\alpha, \mathfrak{u})$ supposes $\alpha^2 = \alpha$, that is, $\alpha = 1$ or $\alpha = 0$. Therefore, the rule of *promotion* 'from $A \multimap B$, derive $!A \multimap !B$', should only apply when the wager in the premise equals 0 or 1. However, cut-elimination supposes the *functoriality* of promotion. This requires that the class of proofs to which promotion applies should be closed under composition. Null wagers pose no problems, but wager-free projects are problematic, since they are not closed under composition. Here we stumble again on the problem encountered in Section 4.3, for which we know the answer, namely a restriction to *successful* projects. This means that $!A$ is subjective, since it depends on a *viewpoint* $\mathscr{P}$.

We eventually discover that the 'necessity' $!A$ is exactly an *affirmation*: $!A$ means that A is true with respect to a certain viewpoint $\mathscr{P}$; it should therefore be written $!_{\mathscr{P}}A$.

5. Subjective, but not subjectivistic !

5.1. *Truth according to Tarski*

Long ago, Tarski gave a definition of truth of the form

$$A \wedge B \textit{ is true iff } A \textit{ is true } \texttt{and } B \textit{ is true.}$$

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All other cases are treated in the same way. To sum up,

$$A \textit{ is true iff } A \texttt{ holds}.$$

Such a definition discouraged generations of mathematicians from even thinking about logic; moreover, logicians developed a sort of aesthetics[†] of the 'meta' ensuring that you must be dumb if you do not understand the depth of such 'definitions'.

But the king is naked, and one must say it: the arrogant essentialism of the Tarskian approach hides the absence of any interesting idea as to truth. It relies on a fantasy of *objectivity* reused by logical hustlers to develop systems of their own: typically,

$$A \text{ broccoli } B \textit{ is true iff } A \textit{ is true } \texttt{broccoli } B \textit{ is true}.$$

the logic of *broccoli*, which is not even edible! This is the triumph of discretionary definitions: the absence of a decent *subjective* dimension in the logical explanation eventually leads to *subjectivism.*

5.2. *Object and subject in logic*

One of the weirdest aspects of logic is that it has been rather shy about the relation between object and subject. The authority on the topic, Frege, distinguished between *sense* (subjective) and *denotation* (objective): the 'morning star' and the 'evening star' denoting the same object. This is compatible with the standard reading of incompleteness: there are *true* properties that we cannot *know*.

This relies on the assumption that the objective space exists independently of the subject: which is slightly reactionary after the discovery of quantum phenomena! Instead of incorporating the quantum viewpoint, logic tried to fill the gap between object and subject, so as to reduce, as much as possible, the pregnancy of the subject.

5.3. *Digression: the logic of dungeons*

This is especially conspicuous in junk logics (logics for AI); the authors of such atrocities usually know the tune, but not the words, of correct logic. Their productions are therefore typical of logical fantasies in that they all agree on one point: the distinction between object and subject is pointless. For instance, truth values have been assigned to cognitive situations, with the outcome that one can expect. . .

The worse paralogic ever produced is arguably the so-called *epistemic logic*. In this collection of childish riddles (the Baghdad cuckolds, etc.), one assumes that the objective world is well-constituted, that all questions have received answers, the only problem being to bridge things together: the hidden assumption here is the identification between *deduction* and *constatation*, which is, by the way, a possible definition of *totalitarism.*

Of course, this does not work, think of the *Houston* cuckolds: there is only one of them, but his intermediate initial is W., which means that, knowing the existence of one cukhold, moreover not one of his neighbours, he draws no conclusion; hence, his colleagues,

[†] In the same way, who dares to say that anybody can, like Csar, put his name on a compressed car, is styled 'reactionary'.

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well-told in epistemic logic, slay their innocent wifes. Which once more disproves the identification 'deduction = constatation', already refuted by incompleteness, indecidablity, in the narrower context of mathematical reasoning. In fact, no activity in the real world seems to match the ideal of this crystal-clear identification, except the sort of asymetric protocol in fashion in Guantanamo and similar places.

This digression enables one to introduce the expression 'intersubjectivity'. In epistemic logic, this is called 'common knowledge' and corresponds to the exchange of information between infallible and truthful partners: think of *Big Brother* and, more recently, the network of secret services and secret dungeons organised by the CIA.

5.4. *Intersubjectivity*

Intersubjectivity has definitely nothing to do with this totalitarian nightmare. If we agree that a single subject is something like the choice of a commutative algebra, intersubjectivity is the gathering of several of them, provided they commute. What I have called 'viewpoint' is therefore the (ideal) building of a complete intersubjectivity, this completeness being only a convenience.

Now, let us address the paradoxical aspects of our definition of truth. The point is that truth depends on the viewpoint $\mathscr{P}$. In particular, a theorem A may become false with respect to the 'wrong' viewpoint. This must not be taken as a sort of relativistic argument justifying the denial of evidence. When thinking of this *subjective paradox*, one must take into account that the viewpoint is part of the *meaning* that we ascribe to A. Provided we respect this intended meaning, nothing unpleasant or really shocking can occur; and if we depart from it, where is the paradox?

A theorem is not a decoration that one puts on a shelf, it is a tool, which can be used as a *lemma* to produce other theorems. It is the *use* of A through logical consequence that gives the actual *meaning* of A. Now, when I relate A and $A \multimap B$ to get B, I relate them with respect to their intended meaning; if distinct subjects have been in charge of A and B, then $A \multimap B$ makes sense only when these subjects 'recognise each other', that is, commute as commutative algebras. In other words, the meaning of A is determined by its intersubjective context, since it involves the creation of a *common viewpoint*.

Indeed, the subjective paradox is not very different from the various paradoxes induced through the rise of subjectivity in modern science. For instance, once the geocentric viewpoint had been relinquished, one could argue that speed, which has become relative to a Galilean reference frame, no longer makes sense; but, when studying the interaction of mechanical bodies, it is wise to choose a common reference!

To sum up:

$$\textit{Truth} = \textit{Modality} = \textit{Intersubjectivity}.$$

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