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On the workspace of telescopic manipulators

Marco Ceccarelli

Dipartimento di Ingegneria Industriale, Università di Cassino, Via Di Biasio 43, 03043 CASSINO (FR), (Italy)
E-mail: cecca@ing.unicas.it

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SUMMARY
An analytical procedure is proposed for determining the workspace boundary of general telescopic manipulators. The algorithm has been developed by using the Theory of Envelopes providing an algebraic formulation. An analysis of the workspace of telescopic chain has been carried out to emphasize on a general topology and to investigate the effect of link parameters and telescopic stroke.

KEYWORDS: Telescopic manipulators; Workspace; Theory of Envelopes.

INTRODUCTION

Telescopic manipulator arms are widely used in robots since a suitable stroke on the last joint may provide great advantages both in workspace capability and robot dimensions. Workspace evaluation is recognized of great interest for robot evaluation and design, and both general and specific analysis algorithms have been developed in the recent past for robot manipulators. In fact, several procedures for workspace analysis of manipulators have been proposed: Iterative determination of reachable points by means of matrix formulation of direct problem;[1–3] or through probabilistic techniques;[4,5] or continuation methods;[6] dynamical evaluation of extreme configuration;[7,8] determination of boundary surfaces for Jacobian domain;[9–11] algebraic formulation for specific manipulators.[12,13]

However, telescopic arms have addressed specific interest only recently, and only some specific characteristics have been studied.[14]

In this paper we have investigated the general geometry of a telescopic robot arm to give a workspace characterization by using an algebraic algorithm which has been developed "*ad hoc*".

Indeed, the analytical treatment makes use of concepts and formulation from the Theory of Envelopes with the aim to provide a procedure which can be used to generalize in a near future an algorihm which has been proposed for n-R manipulators.[15] In particular, in this paper the formulation has been proposed in a form similar to the case of three-revolute chains, which have been previously studied.[16]

The developed algorithm has been provided in a very efficient numerical procedure for workspace analysis purposes. Thus a general geometry for the workspace of telescopic manipulators has been characterized as a "Cylindroid ring" and it has been described with its characteristics.

The numerical procedure has been useful also for a parametric study of the effect of the link parameters and the telescopic stroke on the workspace characteristics. Some numerical examples have been reported to illustrate the characteristic workspace topology and they have been provided with design purposes in a form which permits an easy comparison with the workspace of three-revolute[16] and four-revolute chains.[17]

TELESCOPIC ARMS AND WORKSPACE TOPOLOGY

Figure 1 shows the kinematic chain of a telescopic manipulator arm. In particular, the telescopic chain has been modelled in agreement with the kinematic diagram of Figure 1. A world frame OXYZ has been assumed as fixed with the manipulator base with the Z axis concident with the first joint axis and the X axis as the reference axis for the first joint coordinate. Moving frames $O_i\ X_i\ Y_i\ Z_i$ (i = 1, 2, 3) have been fixed on the links of the chain by assuming O_i in the center of the joint, Z_i as coinciding with the joint axis and X_i laying on the common normal between two consecutive joint axes. X_3 has been assumed to be parallel to X_2.

The link parameters are the link lengths a_1 and a_2, the link offsets d_1 and d_2, the twist angles α_1 and α_2. $d_1 = 0$ is

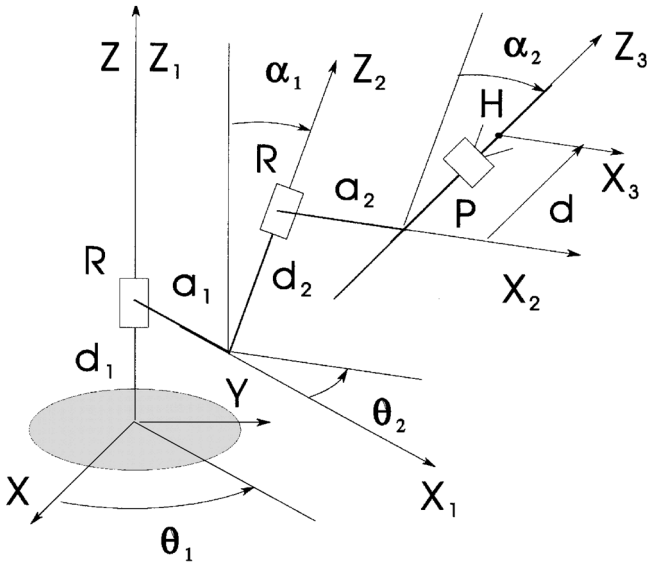


Fig. 1. Parameters of a RRP telescopic manipulator.

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assumed since it does not affect the workspace but only shifts it up or down. The kinetmatic parameters are the joint angles θ_1 and θ_2 of the revolute pairs and the joint stroke d of the sliding pair. Each joint coordinate starts from a line which is parallel to X_i axis of the previous link. The workspace is generated when a reference point H on the extremity of the chain is moved to reach all the possible positions because of the movements of the joints. However, from a descriptive geometry viewpoint the workspace of a telescopic arm can be characterized by looking at the figures which are generated by H with successive full movements of the joints, starting from the last up to first which is fixed to the manipulator base. Thus, movements of the sliding joint will generate a straight line segment. Then the second revolute pair of the chain allows it to perform a full rotation of the straight line segment which generates a cylindrical cone. Indeed, depending on the orientation of the straight-line segment with respect to the revolute joint axis we may have a cylinder, a cone or generally an hyperboloid. Finally, a full revolution of the first revolute joint will generate, by revolving the hyperboloid, a solid of revolution which can be generally depicted as a "Cylindroid Ring", as in Figure 2. A "Cylindroid Ring" shows a hollow bulk shape, and its cross-section is characterized by straight lines with possible cusps and two circular contours on the top and the bottom.

AN ANALYTICAL FORMULATION FOR THE WORKSPACE

The workspace volume can be thought as generated by the union of the tori which are traced by the points of the straight line segment because of the two revolutions. Thus this topology can be easy described in an algebraic form by using the torus equation.[18,19]

In particular, each point of the straight line segment is characterised by the value of the stroke d, and the corresponding torus may be written assuming $d_1 = 0$, with the hypothesis $\sin \alpha_1 \neq 0$ and with respect to OXYZ, by using the radial reach $r = (x^2 + y^2)^{1/2}$ and the axial reach z in the form

$$(r^2 + z^2 - A)^2 + (Cz + D)^2 + B = 0 \tag{1}$$

in which the so-called structural coefficients are expressed as

$$\begin{aligned} A &= a_1^2 + r_2^2 + (z_2 + d_2)^2 \\ B &= -4a_1^2 t_2^2 \\ C &= 2\frac{a_1}{\sin \alpha_1} \\ D &= -2a_1(z_2 + d_2)\frac{\cos \alpha_1}{\sin \alpha_1} \end{aligned} \tag{2}$$

where r_2 and z_2 are the radial and axial reaches with respect to $O_2 X_2 Y_2 Z_2$. r_2 and z_2 can be given straightforward from the geometry of the chain and they can be expressed as

$$\begin{aligned} r_2 &= (a_2^2 + d^2 \sin^2 \alpha_2)^{1/2} \\ z_2 &= d \cos \alpha_2 \end{aligned} \tag{3}$$

where the independent variable is the stroke parameter d.

Equations (1) to (3) can be used to determine the workspace volume of a given telescopic arm by scanning the stroke interval from a minimum value d_{min} to a maximum value d_{max}. Indeed, a torus is traced by using equations (1) to determine r by assuming z. This is obtained by scanning z within its range from a minimum value to a maximum one, which can be calculated by using the geometry of Figure 1 from

$$z = r_2 \sin \theta_2 \sin \alpha_1 + (d_2 + d \cos \alpha_2) \cos \alpha_1 + d_1 \tag{4}$$

Figure 2 shows an example of the workspace determination for a given telescopic manipulator by using the equations (1) to (4). From Figure 2 it is seen that the workspace boundary is composed of two different geometrical topologies: envelope segments and toroidal surfaces. The envelope segments are located in the lateral sides of the cross-section representation, and two toroidal surfaces are the top and the bottom covers, respectively. The cross-section shows a shape that justifies once more the name "Cylindroid Ring" we have given to the workspace volume of telescopic arms, since it recalls somehow a thick cylinder.

Moreover, the envelope segments lead to a formulation by using the Theory of Envelopes to obtain a more efficient computational algorithm for a workspace determination.

Indeed, the envelope segments are clearly generated by enveloping all the tori traced by H during the scanning process. Thus, we may use the envelope geometry also in an analytical form. In fact, from the Theory of Envelopes it is known that an analytical expression of the envelope can be obtained from equation (1) and its derivative with respect to the envelope parameter, which is the stroke d in this case.

Thus, differentiating equation (1) with respect to d it yields

$$-(r^2 + z^2 - A)A' + (Cz + D)D' + B' = 0 \tag{5}$$

where $'$ indicates the d-derivative operator.

Because of equations (2) and (3), an analytical expression for the derivatives of the structural coefficients can be straightforward obtained in the form

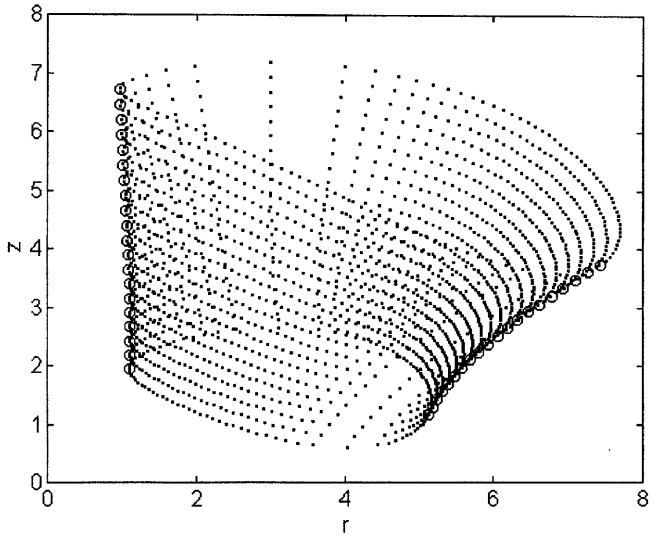


Fig. 2. Cross-section of a Cylindroid Ring workspace volume computed as union of generating tori for a telescopic RRP manipulator with $a_1 = 3$ u, $a_2 = 2$ u, $d_1 = 0$ u, $d_2 = 1$ u, $\alpha_1 = 30$ deg., a2 = 30 deg., $d_{min} = 1$ u, $d_{max} = 6$ u (u is a unit length).

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$$A' = 2(d + d_2 \cos \alpha_2$$
$$B' = -4a_1^2 d \sin^2 \alpha_2$$
$$D' = -2s_1 \cos \alpha_2 \frac{\cos \alpha_1}{\sin \alpha_1} \qquad (6)$$

Finally, by using equations (1) to (6) with suitable algebraic manipulations, the radial and axial reaches of the points of the envelope segments can be expressed with the hypothesis $A' \neq 0$ as

$$z = \frac{-B'D' \pm A'\sqrt{-[B'^2 + B(A'^2 + D'^2)]}}{(A'^2 + D'^2)C} - \frac{D}{C}$$

$$r = \sqrt{\frac{B' + (Cz + D)D'}{A'} + A - z^2} \qquad (7)$$

The sign of ambiguity exists to give the two envelope segments.

Equations (7) can be used to improve the numerical efficiency of the algorithm for workspace determination of telescopic arms since it permits to generate the workspace bounary together with only two toroidal surfaces, namely the top and bottom covers. Thus, a numerical algorithm has been developed by using equations (7) and equations (1) to (4) only twice for the aforementioned covers.

Figure 3 shows an example in which the covers of the cylindroid ring have been generated by using the upper branch of the top toroidal cover and the lower branch of the bottom toroidal cover only.

VOLUME AND CROSS-SECTION EVALUATION

A numerical evaluation of a workspace can be obtained by computing its cross-section area and its volume. In this paper, the scheme of Figure 4 has been used for a workspace numerical evaluation as well as to determine the workspace boundary.

The envelope branches and the toroidal top and bottom covers may intersect each other so that it may happen that parts of them get inside the workspace itself. Therefore, according to the sketch of Figure 4 it is easy to obtain the workspace boundary by selecting at each z_k only the maximum r_{maxk} and the minimum r_{mink} among the previously computed reaches r.

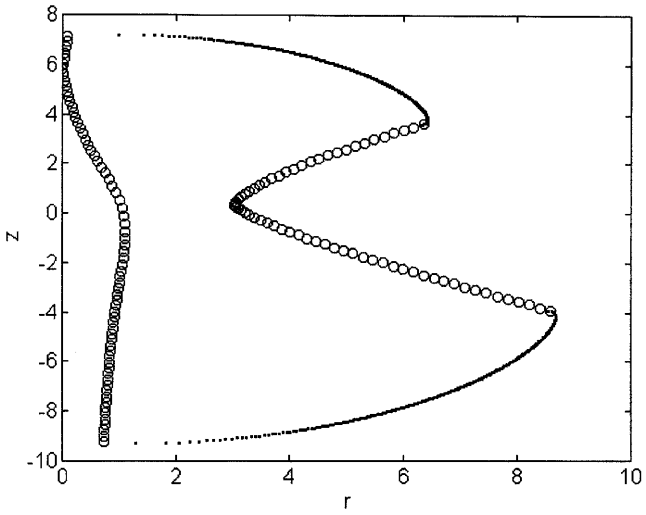


Fig. 3. Cross-section of a workspace boundary computed as tori envelope segments and toroidal covers for a telescopic RRP manipulator with $a_1 = 3$ u, $a_2 = 2$ u, $d_1 = 0$ u, $d_2 = 1$ u, $\alpha_1 = 30$ deg., $\alpha_2 = 30$ deg., $d_{min} = -10$ u, $d_{max} = 6$ u (u is a unit length).

Then it is possible to calculate the cross-section area A by means of the formula

$$A = \sum_{z_k = z_{min}}^{z_k = z_{max}} A_k = \sum_{z_k = z_{min}}^{z_k = z_{max}} \Delta z \, (r_{maxk} - r_{mink}) \qquad (8)$$

and the volume V by using the expression

$$V = \sum_{z_k = z_{min}}^{z_k = z_{max}} A_k \frac{\pi}{4} (r_{maxk} + r_{mink}) \qquad (9)$$

when a suitable scanning of the z interval from z_{min} to z_{max} with a step Δz is carried out.

A better numerical evaluation can be obtained with both smaller Δz and d scanning step. However, good results may be obtained by using for r_k the value r_{k-1} computed in previous step, when at the z_k computation may not calculate any r value because of the resolution of the d scanning process.

Thus, the determination of the workspace boundary makes it it easy to compute equations (8) and (9) for the cross-section area and the volume of the workspace of a given telescopic manipulator.

The workspace of the example of Figure 2 has been evaluated by calculating $A = 33.62$ u^2 and $V = 825.09$ u^3

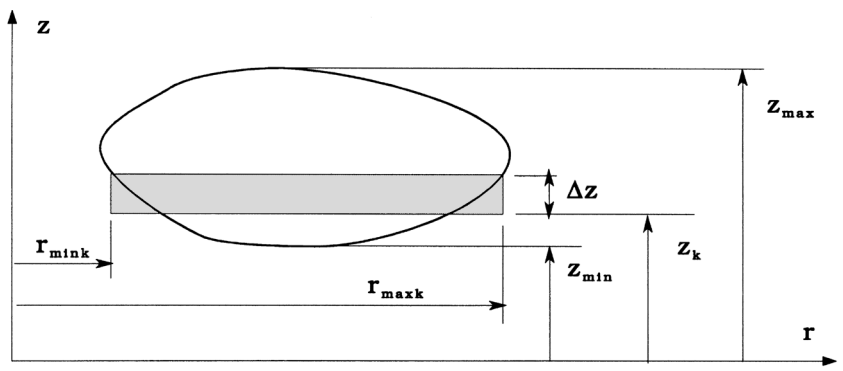


Fig. 4. A scheme for numerical evaluation of workspace cross-section area and volume.

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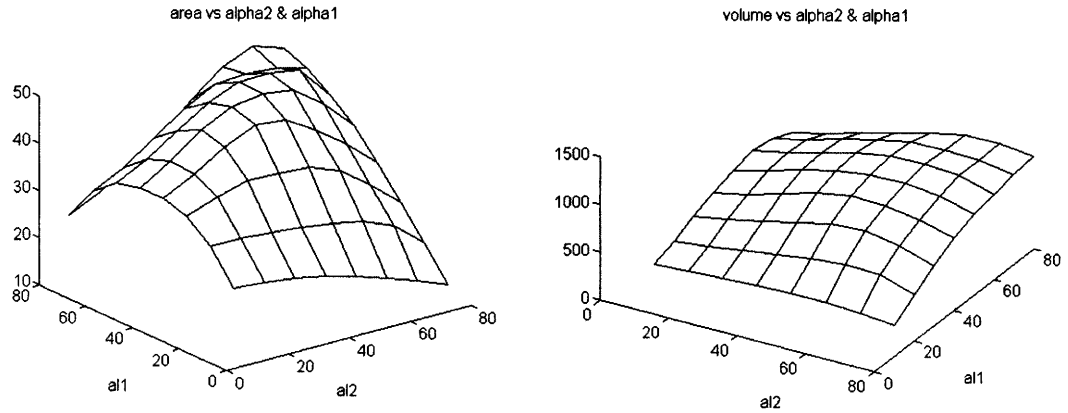


Fig. 5. Effect of twist angles α_1 and α_2 on the cross-section area and the volume of workspace for telescopic RRP manipulators with $a_1 = 1$ u, $a_2 = 2$ u, $d_1 = 0$ u, $d_2 = 1$ u, $d_{min} = -5$ u, $d_{max} = 5$ u (u is a unit length) (see Table 1).

when all the lengths have been expressed in terms of u as a unit length. For the case of Figure 3 the calculations have given $A = 83.65$ u^2 and $V = 1842.10$ u^3.

EFFECT OF LINK PARAMETERS AND TELESCOPIC STROKE

The proposed numerical procedure has been useful to carry out with no great computational efforts an investigation of the effect of the most important design manipulator parameters on the workspace. We have deduced some design charts for the shape of the Cylindroid Ring, its volume and cross-section area (Figures 5 to 7 and Tables I to III) with the aim of stressing the possible workspace topologies as a function of practical values of the design parameters. In partiuclar, Figure 5 and Table I show the effect of the twist angles. The angle α_1 gives a greater influence on the cross-section shape. This is numerically confirmed in Figure 5. Indeed, we may observe an optimal value for both the twist angles as more evident from the graph of the cross-section area Figure 5. The figures in Table I show the variety of possible shape for the cylindroid

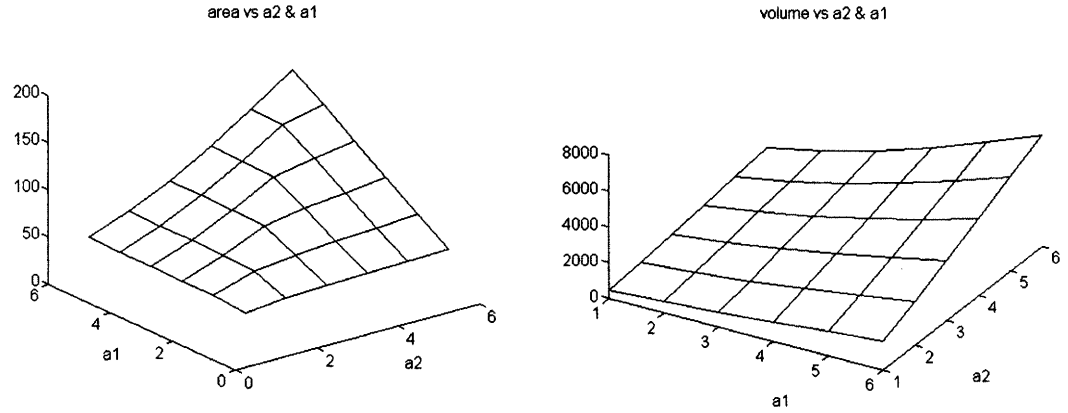


Fig. 6. Effect of the link parameters a_1 and a_2 on the cross-section area and the volume of workspace for telescopic RRP manipulators with $d_1 = 0$ u, $d_2 = 1$ u, $\alpha_1 = 30$ deg., $\alpha_2 = 30$ deg., $d_{min} = -5$ u, $d_{max} = 5$ u (u is a unit length) (see Table 2).

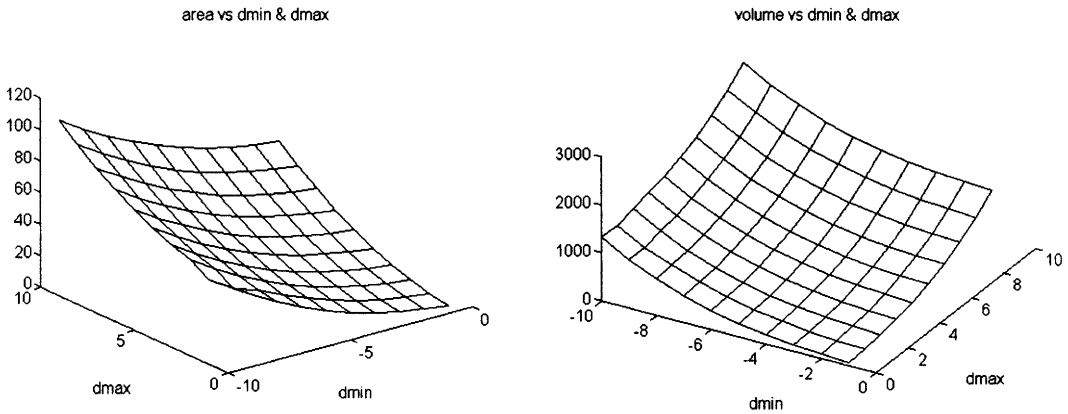
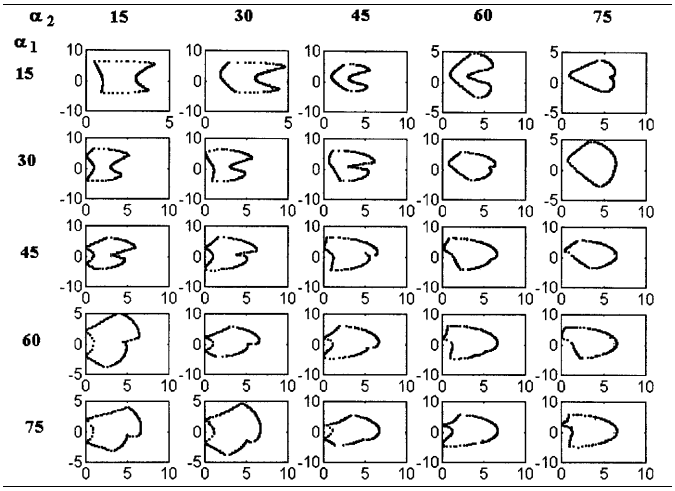


Fig. 7. Effect of the stroke values d_{min} and d_{max} on the cross-section area and the volume of workspace for telescopic RRP manipulators with $a_1 = 1$ u, $a_2 = 2$ u, $d_1 = 0$ u, $d_2 = 1$ u, $\alpha_1 = 30$ deg., $\alpha_2 = 30$ deg. (u is a unit length) (see Table 3).

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Table I. Effect of twist angles α_1 and α_2 on the cross-section shape of workspace for telescoic RRP manipulators with $a_1=1$ u, $a_2=2$ u, $d_1=0$ u, $d_2=1$ u, $d_{min}=-5$ u, $d_{max}=5$ u (u is a unit length) (see Figure 5).



ring workspace from a near dense cylinder (for example, for $\alpha_1=\alpha_2=15$ deg.) to a bulk ring (for $\alpha_1=30$ deg. and $\alpha_2=75$ deg.). Unusual shapes as the case e with $\alpha_1=15$ deg. and $\alpha_2=60$ deg. or $\alpha_1=60$ deg. and $\alpha_2=15$ deg. show a general topology well described as a cylindroid ring.

Figure 6 and Table II give an example of the prominence of a_1 on the workspace cross-section area. A greater effect of a_2 can be recognized on the workspace cross-section shape and volume. However, from Table II it seems that the shape form can be preserved. Both a_1 and a_2 may produce a hole in the workspace, although only large values of a_1 ensure the hole's existence.

Finally, Figure 7 and Table III report the effect of the extension of the telescopic stroke by looking at the different effect of d_{min} and d_{max}. It is seen that in this case d_{max} is prominent in affecting the workspace shape and volume. However, it has been found the fixed link encumberance has also great influence in a sense that the relative position of the whole stroke of d with respect to the first revolute joint axis may increase or decrease the effect of d_{min} and d_{max}.

In all the figures of Tables I to III it is possible to recognize the generating hyperboloid through its projection on the radial plane acrossing Z.

Table II. Effect of the link parameters a_1 and a_2 on the cross-section shape of workspace for telescopic RRP manipulators with $d_1=0$ u, $d_2=1$ u, $\alpha_1=30$ deg., $\alpha_2=30$ deg., $d_{min}=-5$ u, $d_{max}=5$ u (u is a unit length) (see Figure 6).

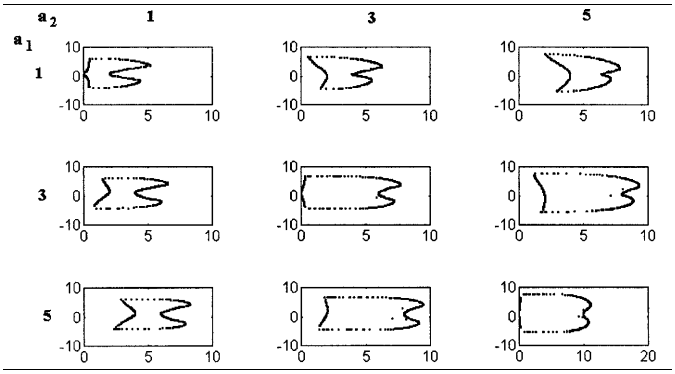
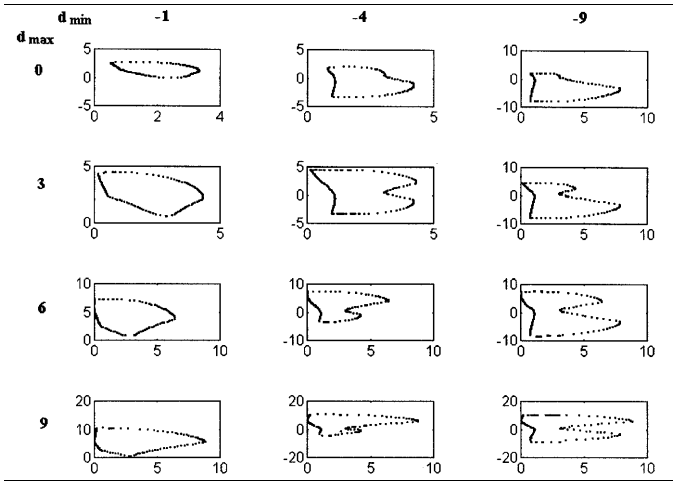


Table III. Effect of the stroke values d_{min} and d_{max} on the cross-section shape of workspace for telescopic RRP manipulators with $a_1=1$ u, $a_2=2$ u, $d_1=0$ u, $d_2=1$ u, $\alpha_1=0$ deg, $\alpha_2=30$ deg. (u is a unit length) (see Figure 7).



It is remarkable to note that usual and practical values of the design parameters do not make possible a void within the workspace, as the case of Tables I to III. This is why in this paper we may call the workspace of telescope manipulators a dense Cylindroid Ring.

However, a void, which is defined as a region of unreachable points within the workspace itself, may exist also for telescopic manipulators when particular values of the link parameters and the telescopic stroke are given. Figure 8 is an illustrative example. Indeed, it has been found that the void existence greatly depends on the stroke extension since a void in a ring can be easily filled up by only enlarging the stroke, as it is the case of industrial telescopic robots.

CONCLUSION

The workspace of some telescopic manipulators has been presented in this paper as a function of the link parameters

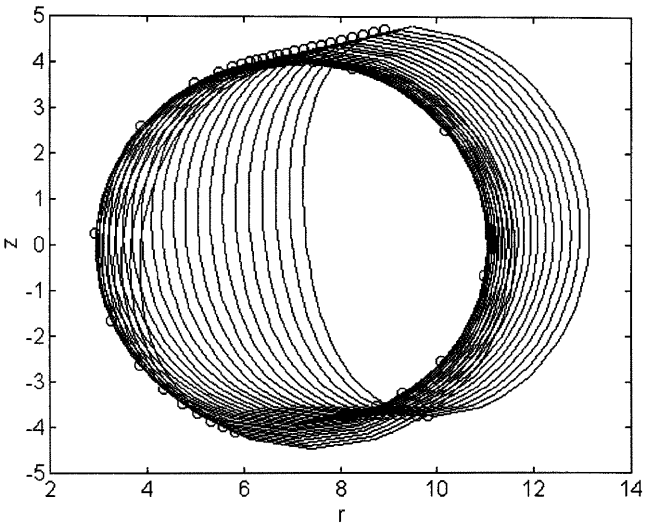


Fig. 8. Cross-section of a workspace volume with a void for a telescopic RRP manipulator with $a_1=7$ u, $a_2=4$ u, $d_1=0$ u, $d_2=2$ u, $\alpha_1=85.71$ deg., a2 = 18 deg., $d_{min}=-5$ u, $d_{max}=5$ u (u is a unit length).

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and the telescopic stroke.

A general topology has been presented and some design charts have been reported as a result of a suitable analysis for workspace determination, evaluation and classification. It is worth noting that the proposed algorithm makes use of an algebraic formulation which can also be useful to deduce analytical design criteria and procedures. Although the proposed formulation is based on the concept of an envelope surface the analytical development has been carried out by using a suitable procedure to obtain a useful and orignal algebraic algorithm. Some general shapes for the cylindroid ring workspace have been shown to characterize the possible workspace geometry of RRP telescopic manipulators.

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