QUANTUM ELECTRODYNAMICS AND HILBERT SPACE THEORY

By J. G. TAYLOR

Received 28 September 1955

ABSTRACT. This paper describes an attempt to formulate quantum field theory, in particular quantum electrodynamics, in terms of Hilbert space theory. The work of Cook (1) is extended to give a precise description of non-interacting electrons and positrons. The hole interpretation is not required in this extension, and no subtraction formalism is required. It is shown that the formalism can never reduce to that of intuitive quantum field theory except by an abuse of language associated with the δ-function. Interaction cannot be introduced in a simple manner into the rigorous formalism, so it seems extremely difficult to develop the Hilbert space formalism for quantum field theory in any useful manner.

These difficulties indicate that an investigation of the Hilbert space basis of simple quantum theory is necessary before a rigorous mathematical formalism for intuitive quantum field theory can be developed.

1. *Introduction.* Physical theories may be regarded from two distinct points of view, that of the mathematician and that of the physicist. The physicist, who is concerned mainly with the agreement of his predictions with experimental results, invariably uses intuitive methods. The mathematician, who is concerned mainly with trying to rigorize the intuitive formalism developed by the physicist, tends to be insensitive to the problems of the physicist. This insensitivity takes the form of building mathematical models which have no direct relation to the physical theories they purport to imitate. It is far more useful to build models keeping in mind their relation to the physical theories. This is obviously more useful to the physicist, while to the mathematician this criterion gives direction to the model building. This criterion of usefulness guides the analysis described here.

This analysis is of an attempt to build a mathematical model for quantum field theory, in particular quantum electrodynamics, in terms of the theory of Hilbert space. The basic mathematical structure has been described by J. M. Cook (1), but was only applied explicitly by Cook to a system composed of photons. The first part of the paper is devoted to the application of this structure to a system of non-interacting electrons and positrons. The case of an external electromagnetic field acting on the electron-positron system is not considered in this paper. Certain results of independent interest are presented in this part. In particular, the total energy operator used for the system is usually that obtained by second quantization of the single electron energy operator H of Dirac, but the difficulties of the negative energy states require the hole interpretation and an ambiguous subtraction formalism. Moreover, arguments are put forward in § 3 to support the conclusion that the hole interpretation is not rigorously formalizable. It is shown here that all these difficulties may be circumvented by interpreting the negative energy states as positron states in the usual way (see (4), § 12), and by taking the total energy operator as the operator obtained by second quantization of $|H|$ and not by second quantization of H. The formalism so obtained is easily shown to be Lorentz invariant. Further, this replacement of H by $|H|$ in the total energy operator

https://doi.org/10.1017/S0305004100031807 Published online by Cambridge University Press

for the free electron-positron field is shown to be essentially the same as the procedure of symmetrization of the expression $\int \psi^*(\alpha p + \mu\beta)\psi\, d^3x$ and removal of the zero-point energy which is used in the intuitive field theory. This complete removal of the 'sea of negative-energy electrons' associated with the hole interpretation is eminently satisfactory, both from a mathematical and a physical viewpoint.

The next part of the paper is concerned with relating the Hilbert space formalism to that of intuitive quantum field theory. It is found that the rigorous formalism can never reduce by rigorous arguments to that of intuitive quantum field theory, but only by committing the 'abuse of language' of letting the delta distribution $\delta_{(0)}$ be regarded as a square-integrable function.

It might still be thought possible to obtain from the Hilbert space formalism experimental predictions which are in agreement with those obtained by intuitive methods from the present quantum field theory. In the last section of this paper it is shown that such predictions are not immediately obtainable. This follows from the observation that it is extremely difficult to introduce any interaction between the free fields in a rigorous manner in the Hilbert space formalism.

These two grave defects of quantum field theory based on Hilbert space formalism, namely, that the theory only reduces to the intuitive field theory by an abuse of language and that interaction cannot be immediately considered in it, lead one to investigate the usefulness of basing single-particle quantum theory on a Hilbert space formalism. It is hoped to describe the results of such an investigation in a later paper.

2. *The single-electron system.* The notation of Cook (1) will be used throughout this paper. The formalism developed by Cook was only applied explicitly to a system composed of photons. In order to apply the formalism to a system composed of electrons, it is first necessary to describe the notation which will be used for a system composed of a single electron obeying Dirac's relativistic wave equation, in the absence of an external electromagnetic field. This section is devoted to the required description.

The states of a single electron are elements of the tensor product space $R_a = L_2(E_3) \otimes C_a$, where C_a is a 4-dimensional Hilbert space and $L_2(E_3)$ is the space of square integrable functions defined on the 3-dimensional Euclidean space E_3. Let $\rho_\mu (\mu = 1, 2, 3, 4)$ be an orthonormal base of C_a. Then any $\phi \in R_a$ can be expressed uniquely in the form $\phi = \phi_\mu \otimes \rho_\mu$, where $\phi_\mu \in L_2(E_3)$. (The summation convention used is summation from 1 to 4 on Greek indices and from 1 to 3 on Latin indices except where it is explicitly stated to the contrary. Also $x = (x_1, x_2, x_3) \in E_3$.)

The time-dependent states of the electron are represented by elements of the form $\phi(t) = \phi_\mu(x, t) \otimes \rho_\mu$, where the functions $\phi_\mu(x, t)$ belong to $L_2(E_3)$. The time dependence of the functions $\phi_\mu(x, t)$ is determined by the requirement that $\phi(t)$ must satisfy the Dirac relativistic equation

$$(p_4 \otimes 1 + p_i \otimes \alpha_i + m1 \otimes \alpha_0)\phi = 0. \tag{1}$$

In equation (1), α_1, α_2, α_3, α_0 are the 4×4 Dirac spin matrices, considered as operators on the space C_a, and satisfy

$$\alpha_\mu \alpha_\nu + \alpha_\nu \alpha_\mu = 2\delta_{\mu\nu},$$

https://doi.org/10.1017/S0305004100031807 Published online by Cambridge University Press

where $\mu, \nu = 0, 1, 2, 3$. Also $p_4 = i\partial/\partial t$, $p_j = -i\partial/\partial x_j$ is the momentum operator for the electron, m is the mass of the electron, and for convenience units are chosen throughout the paper so that $\hbar = c = 1$. Let the linear operator H on R_a be defined by the equation

$$H = -p_j \otimes \alpha_j - m1 \otimes \alpha_0. \tag{2}$$

Then H (which is the energy operator for the electron) is a densely defined self-adjoint operator on R_a (Stone (5), p. 441). The time dependence of the element $\phi(t)$ is given by

$$\phi(t) = \exp(-iHt)\,\phi(0). \tag{3}$$

In equation (3), $\phi(0) = \phi_\mu \otimes \rho_\mu$, where $\phi_\mu = \phi_\mu(x) \in L_2(E_3)$ and $\phi(0)$ is the initial state at time $t = 0$, with $\exp(-iHt)$ a unitary operator on R_a (Stone (5), p. 232).

The required transformation properties of the wave functions $\phi(t)$ and the relativistic invariance of equation (1) are obtained by requiring that under those elements of the orthochronous Lorentz group (which does not reverse time) which are also positive (do not change right to left in space), the ρ_μ transform under the matrix $(\sigma \oplus \overline{\sigma}_c)_{\mathrm{tr}}$, where σ is the corresponding two-dimensional unimodular transformation and $\sigma_c = \sigma_{\mathrm{tr}}^{-1}$. The functions $\phi_\mu(x, t)$ are required to be invariant under any Lorentz transformation, so that $\phi'_\mu(x', t') = \phi_\mu(x, t)$, where the dashed and undashed quantities are determined in the transformed and initial coordinate frames respectively. Space reflexion is introduced by the matrix α_0 acting on C_a. (For further details, see H. Weyl (7), p. 147.)

The wave function $\phi(t)$ is just the four-component spinor function, with components $\phi_\mu(x, t)$, and these components have the required spinor transformation properties (after taking account of the transformation properties of the base ρ_μ).

The matrix elements $T_{\mu\nu}$, with respect to the base ρ_μ, of any linear operator T on C_a, are defined by $T\rho_\mu = T_{\nu\mu}\rho_\nu$. Then the functions

$$j_\mu(x, t) = \overline{\phi}_\xi(x, t)\,(\alpha_\mu)_{\xi\xi}\,\phi_\xi(x, t) \tag{4}$$

transform as the covariant components of a vector under the orthochronous Lorentz group, where $\mu = 1, 2, 3, 4$ and $\alpha_4 = 1$. The functions $j_\mu(x, t)$ are considered as the charge-current density, where $j_\mu \in L_1(E_3)$, and satisfy the conservation equation $\pm(\partial/\partial x_\mu) j_\mu = 0$ (Dirac (2), p. 259), where the $+$ sign is taken for $\mu = 4$, $x_4 = t$, and the $-$ sign for $\mu = 1, 2, 3$.

Cook ((1), p. 252), defined an expectation value on the single-photon space by the requirement that this expectation value is invariant under the Lorentz group. This expectation value determines an expectation value on the many-particle state space which is identical to that obtained from the indefinite metric introduced by S. N. Gupta (3), which removed gauge invariance difficulties.

In the case of the single electron space R_a, the usual inner product on R_a is not invariant under the Lorentz group. An invariant expectation value for any normal operator A, for the system in the state ϕ, may be defined to be $(A\phi, \alpha_0\phi)$. This indefinite metric would seem to have the property that if $\phi(x) = \exp(ipx)\,u$, where u is a constant spinor satisfying $-(p_j \otimes \alpha_j + m1 \otimes \alpha_0)\,u = Eu$, then the expectation value of H in the state ϕ is $|E|$. Certain difficulties associated with negative probabilities do, however, arise for this indefinite metric. Further, H does not have a real expectation value in

https://doi.org/10.1017/S0305004100031807 Published online by Cambridge University Press

every state of the system. For these reasons the non-invariant inner product on R_a will be used in the following to define expectation values.

The spin of the electron is determined by the spin operators $(\frac{1}{2})\sigma_j$ $(j = 1, 2, 3)$, which are self-adjoint operators on C_a (Dirac (2), pp. 254 et seq.). Particular matrix representations of $\sigma_j, \alpha_j, \alpha_0$ may be given, but will not be required here.

To finish these considerations on the single-electron system, the positive and negative energy components of any state ϕ will be defined, and certain useful relations concerning these components will be proved. Since H is a closed, densely defined, self-adjoint operator on R_a (Stone (5), p. 441), there exists a resolution of the identity $E(\lambda)$ for H (Stone (5), p. 176, Theorem 5·7). From the representation of H in momentum space, the continuous spectrum of H is the set $B(H) = (\lambda, |\lambda| \geqslant m)$, in the notation of (5), p. 129. Also $A(H) = C(H)$ is the null set, $D_E(H) = (\lambda, |\lambda| < m)$. Thus for any $\phi, g \in R_a$,

$$(\phi, g) = \int_{-\infty}^{+\infty} d(E(\lambda)\phi, g) = (\phi_+ + \phi_-, g); \tag{5}$$

where

$$(\phi_+, g) = \int_m^{\infty} d(E(\lambda)\phi, g),$$

$$(\phi_-, g) = \int_{-\infty}^{-m} d(E(\lambda)\phi, g).$$

The equation (5) defines a decomposition of ϕ into a positive energy part ϕ_+, and a negative energy part ϕ_-.

It will be useful to prove the following:

LEMMA 1. *Let $\phi \in R_a$, and let ϕ_+, ϕ_- be the positive- and negative-energy components of ϕ, respectively. Then if $\phi \in D_H$, the domain of H,*

$$H\phi_\pm = (H\phi)_\pm, \tag{6}$$

$$|H|\phi_\pm = \pm(H\phi)_\pm, \tag{7}$$

$$\exp(-i|H|t)\phi_\pm = (\exp(\mp iHt)\phi)_\pm. \tag{8}$$

The proof of the above equations is based on the identity (Stone (5), p. 174)

$$E(\lambda)E(\mu) \equiv E(\lambda) \quad (\lambda \leqslant \mu), \qquad E(\lambda)E(\mu) \equiv E(\mu) \quad (\lambda > \mu).$$

The proofs of the three equations are completely similar, and only the first will be given here. For any $\psi \in R_a$,

$$(H\phi_+, \psi) = \int_{-\infty}^{+\infty} \lambda\, d_\lambda(E(\lambda)\phi_+, \psi) = \int_{-\infty}^{+\infty} \lambda\, d_\lambda\left(\int_m^{\infty} d_\mu(E(\lambda)E(\mu)\phi, \psi)\right)$$

$$= \int_{-\infty}^{+\infty} \lambda\, d_\lambda \int_m^{\lambda} d_\mu(E(\mu)\phi, \psi) = \int_m^{\infty} \lambda\, d_\lambda(E(\lambda)d, \psi) = ((H\phi)_+, \psi),$$

and similarly for the negative-energy component.

3. *Many-electron systems.* The Fermi–Dirac quantization, as briefly described by Cook ((1), pp. 227–8), will be considered in more detail in this section. The only change from the notation of Cook will be that throughout this section $\omega(\phi)$ will be written in

https://doi.org/10.1017/S0305004100031807 Published online by Cambridge University Press

place of $\omega_A(\phi)$, since only the space $\mathscr{A}$ will be considered. In the remainder of the paper Cook's notation will be strictly adhered to.

The annihilation and creation operators $\omega^*(\phi)$, $\omega(\phi)$ may be decomposed into $\omega^*(\phi_\pm)$, $\omega(\phi_\pm)$, in accordance with the decomposition of equation (5), to give

$$\left.\begin{aligned}\omega(\phi) &= \omega(\phi_+)+\omega(\phi_-),\\ \omega^*(\phi) &= \omega^*(\phi_+)+\omega^*(\phi_-).\end{aligned}\right\} \tag{9}$$

Then the various commutator brackets are

$$\left.\begin{aligned}[\Omega(H),\omega(\phi_\pm)]^\sim &= \omega(H\phi_\pm),\\ [\Omega(H),\omega^*(\phi_\pm)]^\sim &= -\omega^*(H\phi_\pm),\\ [\Omega(I),\omega(\phi_\pm)]^\sim &= \omega(\phi_\pm),\\ [\Omega(I),\omega^*(\phi_\pm)]^\sim &= -\omega^*(\phi_\pm).\end{aligned}\right\} \tag{10}$$

The densely defined self-adjoint operators $\Omega(H)$ and $\Omega(I)$ are to be regarded as the total energy and total charge operators, respectively, for the system of non-interacting electrons (taking the charge of the electron to be equal to one). By the usual analysis of equation (10), based on hole theory, $\omega(\phi_+)$ may be regarded as an electron creation operator, and $\omega(\phi_-)$ as a positron annihilation operator, while $\omega^*(\phi_+)$ may be regarded as an electron annihilation operator and $\omega^*(\phi_-)$ as a positron creation operator (see, for example, Heitler (4), p. 114).

It would thus seem that $\omega(\phi)$, $\omega^*(\phi)$ may be regarded as analogous to the electron-positron field operators, given, for example, by Heitler ((4), p. 115, equation (1)). Using this analogy, one might attempt to develop the present formalism in a manner which is parallel to that of intuitive quantum field theory, and consider the time development of the system under the total-energy operator $\Omega(H)$. Two immediate and related difficulties are met with in this development of the formalism. First, it is not possible to define a non-trivial vacuum state f_0 for the system, and secondly, it is necessary to interpret the non-physical negative-energy states in terms of the states of a particle which has only positive-energy states.

That the first difficulty mentioned above is real will be shown in the next few lines. The electron-positron vacuum state f_0 is required to satisfy

$$\omega^*(\phi_+)f_0 = \omega(\phi_-)f_0 = 0 \tag{11}$$

for any $\phi \in R_a$. If $R_+(R_-)$ is the set of all $\phi_+(\phi_-)$ for $\phi \in R_a$, then R_a is the direct sum of the orthogonal vector sub-spaces R_+, R_-, so that $R_a = R_+ + R_-$. Hence a complete orthonormal set $\{\phi_n\}$ may be chosen in R_a, so that each ϕ_n is either in R_+ or in R_-. Then, in the notation of (1), p. 227, $\{(\phi_1^{n_1}\phi_2^{n_2}\ldots)_A\}$ is a complete orthonormal set in $\mathscr{A}$. If f_0 is expanded in terms of this set, the first condition of equation (11) is satisfied only if the expansion of f_0 contains no elements of $\{\phi_n\}$ which are in R_+. The second condition of equation (11) is satisfied only if the expansion of f_0 involves no term with only a finite number of elements of $\{\phi_n\}$ which are in R_-. Hence the only possible expression for f_0 is in the limiting form

$$f_0 = \underset{n}{\text{l.i.m.}} f_n = \underset{n}{\text{l.i.m.}}\,(n!)^{\frac{1}{2}} P_{a_n} \prod_{m=1}^{n} \otimes \phi_m, \tag{12}$$

https://doi.org/10.1017/S0305004100031807 Published online by Cambridge University Press

where, for each n, $\|f_n\| = 1$, and the elements $\phi_m \in R_-$ for all m. In other words, the vacuum state f_0 is that state in which every negative-energy state is occupied. Since the set $\{f_n\}$ is an orthonormal set, the sequence $\{f_n\}$ cannot converge to any element $f_0 \in \mathscr{A}$. The normalization condition on each f_n does not affect this conclusion, unless the trivially convergent case $\|f_n\| \to 0$ as $n \to \infty$, or $f_0 = 0$, is chosen. Hence there is no non-trivial state $f_0 \in \mathscr{A}$, such that f_0 satisfies equation (11).

The hole theory gives rise to imprecise subtraction procedures, and, moreover, the analysis of the preceding paragraph shows that it is not possible to define a non-trivial vacuum state for the system. The direct interpretation of the negative-energy states as positron states has been studied, from an algebraic view point, by J. G. Valatin (6). In this interpretation the negative-energy states are regarded as being the positive-energy states of the positron; this will be the interpretation adopted here. The single-particle states are still elements of R_a, but the energy operator is now defined to be $|H|$. This gives a positive expectation value to the energy in any positron state. This starting point is different from that of quantum field theorists (though the resulting algebraic structure is essentially the same as that developed by Valatin). The use of $|H|$ as the single-particle energy operator makes the development of the formalism extremely straightforward, as will be shown in the remainder of this section.

The non-interacting electron-positron system still has the state space $\mathscr{A}$. The total energy operator for the system is now $\Omega(|H|)$, which is a densely defined self-adjoint linear transformation on $\mathscr{A}$. The expectation value of $\Omega(|H|)$, in any state of the system, is always real and positive. To take $\Omega(|H|)$ as the total energy operator may not, at first sight, seem to be keeping close to the formalism of intuitive field theory. However, in intuitive field theory, the total energy operator, $\int \psi^*(\alpha p + \mu\beta)\psi\, d^3x$ (in the notation of (4), § 11), has to be symmetrized, and then the zero-point energy of the vacuum field removed. The final expression is then $\sum_{p,s}(n_p^+ + n_p^-)E_p$, which may be seen to be identical with the operator to which $\Omega(|H|)$ is reduced if it is allowed to act only on states which are of the form $\phi_{p_1} \otimes \ldots \otimes \phi_{p_n}$, where $|H|\phi_{p_i} = E_{p_i}\phi_{p_i}$. Since the elements ϕ_{p_i} do not belong to R_a, then $\Omega(|H|)$ cannot be regarded as identical to the field-energy operator of intuitive field theory, but only as reducing to this latter operator by an abuse of language.

It is evident that the total charge operator for the system is $Q = \Omega(I_+) - \Omega(I_-)$, where I_+, I_- are the projection operators on to R_+, R_- in R_a. Q is also a densely defined self-adjoint transformation on $\mathscr{A}$. The expectation values of Q can be positive or negative, and in a state Φ composed of n_e electrons and n_p positrons, so that $\Phi \in R_a^{(n_e+n_p)}$, $Q\Phi = (n_e - n_p)\Phi$.

The procedure outlined in the last two paragraphs solves the second difficulty mentioned above. To solve the first difficulty, the vacuum state is now simply defined as an element f_0 of $R^{(0)}$, which is the space of complex numbers. It is evident that f_0 is a state composed of no electrons and no positrons, and thus satisfies the physical requirements of a vacuum state. f_0 no longer satisfies equation (11); this causes no great difficulty, since the previous interpretations of $\omega(\phi_\pm)$, $\omega^*(\phi_\pm)$ have to be changed to fit in with the new total energy and total charge for the system. To reinterpret the roles

https://doi.org/10.1017/S0305004100031807 Published online by Cambridge University Press

of the operators ω, ω^* the various commutator brackets similar to those in equation (10) are required. These are

$$\left.\begin{aligned}[\Omega(|H|), \omega(\phi_\pm)]_\sim &= \pm\omega(H\phi_\pm),\\ [\Omega(|H|), \omega^*(\phi_\pm)]_\sim &= \mp\omega^*(H\phi_\pm),\\ [Q, \omega(\phi_\pm)]_\sim &= \pm\omega(\phi_\pm),\\ [Q, \omega^*(\phi_\pm)]_\sim &= \mp\omega^*(\phi_\pm).\end{aligned}\right\} \tag{13}$$

From a brief analysis of equation (13), it is evident that $\omega(\phi_+)$ ($\omega^*(\phi_+)$) may be regarded as an electron creation (annihilation) operator, and $\omega(\phi_-)$ ($\omega^*(\phi)$) may be regarded as a positron creation (annihilation) operator. The condition for the vacuum state f_0 now becomes

$$\omega^*(\phi_+)f_0 = \omega^*(\phi_-)f_0 = 0. \tag{14}$$

This equation is obviously satisfied by the above choice of f_0.

To construct the electron-positron field operators, some preliminary lemmas must be proved. The first lemma is associated with Fourier expansion on R_a. Let $u_s(K)$ be the constant spinor associated with the momentum vector K, the index s taking the values $s = 1, 2, 3, 4$ corresponding to the two possible spin and two possible energy states of an electron. The spinor $u_s(K)$ is required to be a bounded continuous function of K, and to satisfy, for each value of K, the conditions

$$\overline{u}_{s\mu}(K)\, u_{s'\mu}(K) = \delta_{ss'}, \tag{15}$$

$$\sum_{s=1}^{4} \overline{u}_{s\mu}(K)\, u_{s\nu}(K) = \delta_{\mu\nu}, \tag{16}$$

$$-(\alpha_i K_i + m\alpha_0)\, u_s(K) = E_s(K)\, u_s(K). \tag{17}$$

The function $E_s(K)$ is equal to $+(K^2+m^2)^{\frac{1}{2}}$ when $S = 1, 2$, and $-(K^2+m^2)^{\frac{1}{2}}$ when $s = 3, 4$. Heitler ((4), p. 107) gives details of the determination of $u_s(K)$.

LEMMA 2. *For any $\phi \in R_a$, expressed in the form $\phi = \phi_\mu(x) \otimes \rho_\mu$, then*

$$\phi = \text{l.i.m.}\,(2\pi)^{-\frac{3}{2}} \int \sum_{s=1}^{4} u_s(K)\, a_s(K) \exp(iKx)\, dK, \tag{18}$$

where l.i.m. *means limit in the mean with respect to the integral over E_3. Further*

$$a_s(K) = \text{l.i.m.}\,(2\pi)^{-\frac{3}{2}} \int \sum_{\mu} \overline{u}_{s\mu}(K)\, \phi_\mu(x) \exp(-iKx)\, dx, \tag{19}$$

where $a_s(K) \in L_2(E_3)$.

This lemma is proved immediately by means of Plancherel's theorem. This theorem proves that there exists, for each μ, a function $\psi_\mu(x) \in L_2(E_3)$ for which

$$\phi_\mu(x) = \text{l.i.m.}\,(2\pi)^{-\frac{3}{2}} \int \psi_\mu(K) \exp(iKx)\, dK$$

and

$$\psi_\mu(K) = \text{l.i.m.}\,(2\pi)^{-\frac{3}{2}} \int \phi_\mu(x) \exp(-iKx)\, dx.$$

https://doi.org/10.1017/S0305004100031807 Published online by Cambridge University Press

For each K, s, the function $a_s(K)$ is introduced to satisfy the equation

$$\psi(K) = \sum_{s=1}^{4} a_s(K)\, u_s(K),$$

so that

$$a_s(K) = \psi_\mu(K)\overline{u}_{s\mu}(K).$$

Then $a_s(K) \in L_2(E_3)$, from the properties of $u_s(K)$. Hence

$$\phi_\mu(x) = \text{l.i.m.}\,(2\pi)^{-\frac{3}{2}} \int \sum_{s=1}^{4} u_{s\mu}(K)\, a_s(K) \exp(iKx)\, dK,$$

where

$$a_s(K) = \text{l.i.m.}\,(2\pi)^{-\frac{3}{2}} \int \overline{u}_{s\mu}(K)\, \phi_\mu(x) \exp(-iKx)\, dx.$$

Let J be the linear transformation on R_a defined in terms of the Fourier expansion of equation (18) by

$$J\phi = \text{l.i.m.}\,(2\pi)^{-\frac{3}{2}} \int \sum_{s=1}^{4} \overline{a}_s(K)\, u_s(K) \exp(iKx)\, dK. \tag{20}$$

DEFINITION 1. *A conjugation T on R_a is a linear transformation on R_a with domain R_a, which satisfies the conditions*

$$T^2 \equiv I, \quad (Tf, Tg) = \overline{(f,g)} = (g,f).$$

for any $f, g \in R_a$.

Stone ((5), p. 357) discusses conjugations in more detail.

LEMMA 3. *J is a conjugation which commutes with H and $I_\pm$.*

For $J^2 \equiv I$, and the domain of J is R_a. For any $\phi = \phi_\mu \otimes \rho_\mu$, $\psi = \psi_\mu \otimes \rho_\mu$ in R_a, let

$$\phi = \text{l.i.m.}\,(2\pi)^{-\frac{3}{2}} \int \sum_{s} u_s(K)\, a_s(K) \exp(iKx)\, dK, \tag{21}$$

$$\psi = \text{l.i.m.}\,(2\pi)^{-\frac{3}{2}} \int \sum_{s} u_s(K)\, b_s(K) \exp(iKx)\, dK. \tag{22}$$

The application of Plancherel's theorem to equations (21), (22), together with the conditions (15), (16), (17) on $u_s(K)$, gives

$$(J\phi, J\psi) = \left(\sum_{s=1}^{4} \overline{a}_s u_s, \sum_{t=1}^{4} \overline{b}_t u_t\right) = \sum_{s,t,\mu} \int \overline{a}_s(K)\, b_t(K)\, u_{s\mu}(K)\, \overline{u}_{t\mu}(K)\, dK$$

$$= \sum_{s} \int \overline{a}_s(K)\, b_s(K)\, dK = \sum_{s,\mu,\nu} \int \overline{f}_\mu(K)\, g_\nu(K)\, \overline{u}_{s\nu}(K)\, u_{s\mu}(K)\, dK = (g,f) = (\psi, \phi),$$

where f_μ, g_μ are the Fourier transforms of ϕ_μ, ψ_μ. Thus J is a conjugation on R_a.

Let ϕ be any element in the domain of H. If ϕ is represented by the Fourier transform in equation (18), then $H\phi$ has the Fourier transform

$$H\phi = \text{l.i.m.}\,(2\pi)^{-\frac{3}{2}} \sum_{s} \int E_s(K)\, a_s(K)\, u_s(K) \exp(iKx)\, dK. \tag{23}$$

https://doi.org/10.1017/S0305004100031807 Published online by Cambridge University Press

The interchange of differentiation and integration required to obtain the right-hand side of equation (23) is permitted since the right-hand side is an element of $L_2(E_3)$, and since if $f(x)$, $\partial f/\partial x_j$ are both in $L_2(E_3)$, then

$$f(x) = \text{l.i.m.}\,(2\pi)^{-\frac{3}{2}}\int g(K)\exp(iKx)\,dK$$

and

$$\partial f/\partial x_j = \text{l.i.m.}\,(2\pi)^{-\frac{3}{2}}\int iK_j g(K)\exp(iKx)\,dK.$$

Then

$$JH\phi = \text{l.i.m.}\,(2\pi)^{-\frac{3}{2}}\sum_s\int E_s(K)\,\bar{a}_s(K)\,u_s(K)\exp(iKx)\,dK = HJ\phi,$$

and so $JH = HJ$. Similarly, $JI_\pm = I_\pm J$. This concludes the proof of the lemma.

Another result that will be required is

LEMMA 4. *Let $\phi(x)\in L_2(E_3)$ be such that $\phi(x)\otimes\rho_\mu$ is in the domain of H for each value of μ. Then $HJ(\phi\otimes\rho_\mu) = H_{\mu\nu}J(\phi\otimes\rho_\nu)$, where $H_{\mu\nu} = (i\alpha_j\partial/\partial x_j - m\alpha_0)_{\mu\nu}$ is a densely defined linear transformation on $L_2(E_3)$.*

The result of Lemma 2 shows that

$$\begin{aligned}H(\phi\otimes\rho_\mu) &= H_{\nu\mu}(\phi\otimes\rho_\nu)\\ &= -\text{l.i.m.}\,(2\pi)^{-\frac{3}{2}}\sum_s\int u_s(K)\,(\alpha_jK_j + m\alpha_0)_{\nu\mu}a_{s,\nu}(K)\exp(iKx)\,dK,\end{aligned}\tag{24}$$

where $a_{s,\nu}(K) = \text{l.i.m.}\,(2\pi)^{-\frac{3}{2}}\int\bar{u}_{s\nu}(K)\,\phi(x)\exp(-iKx)\,dx$. The interchange of integration and differentiation is permitted, for the same reason as in equation (23). Then, since $\overline{(\alpha_\lambda)_{\mu\nu}} = (\alpha_\lambda)_{\nu\mu}\,(\lambda = 0, 1, 2, 3)$,

$$\begin{aligned}HJ(\phi\otimes\rho_\mu) &= -\text{l.i.m.}\,(2\pi)^{-\frac{3}{2}}\sum_s\int u_s(K)\,\bar{a}_{s,\nu}(K)\,(\alpha_jK_j + m\alpha_0)_{\mu\nu}\exp(iKx)\,dK\\ &= H_{\mu\nu}J(\phi\otimes\rho_\mu).\end{aligned}$$

The electron-positron field operator $\Phi_\mu(\phi)$, associated with any $\phi(x)\in L_2(E_3)$, is defined to be

$$\Phi_\mu(\phi) = \omega(JI_-\phi\otimes\rho_\mu) + \omega^*(I_+\phi\otimes\rho_\mu).\tag{25}$$

Then $\Phi_\mu(\phi)$ and the adjoint operator $\Phi^*_\mu(\phi)$ are densely defined closed linear transformations on $\mathscr{A}$. There is an evident resemblance between $\Phi_\mu(\phi)$ and the operator $\Psi_\mu(x)$ defined in (4), p. 115, equation (1 *a*). This resemblance will be analysed more closely in the next section. Before this is done, the time-dependence of the operator $\Phi_\mu(\phi)$ will be investigated.

The time-dependent field operator is

$$\Phi_\mu(\phi, t) = \exp(it\Omega(|H|))\,\Phi_\mu(\phi)\exp(-it\Omega(|H|)).$$

Then

$$\begin{aligned}i\partial/\partial t\,\Phi_\mu(\phi,t) &= \exp(it\Omega(|H|))\,[\Phi_\mu(\phi), \Omega(|H|)]^\sim\exp(-it\Omega(|H|))\\ &= -\omega(\exp(it|H|)\,|H|\,JI_-\phi\otimes\rho_\mu) + \omega^*(\exp(it|H|)\,|H|\,I_+\phi\otimes\rho_\mu).\end{aligned}\tag{26}$$

From the results of Lemmas 1, 3 and 4,

$$i\partial/\partial t\,\Phi_\mu(\phi,t) = H_{\mu\nu}\omega(\exp(it|H|)\,JI_-\phi\otimes\rho_\nu) + H'_{\mu\nu}\omega^*(\exp(it|H|)\,I_+\phi\otimes\rho_\nu),\tag{27}$$

https://doi.org/10.1017/S0305004100031807 Published online by Cambridge University Press

where $$H'_{\mu\nu} = -(i\alpha_j \partial/\partial x_j + m\alpha_0)_{\mu\nu} \quad \text{and} \quad \partial/\partial x\,\omega(\phi) = \omega(\partial\phi/\partial x),$$

$$\partial/\partial x\,\omega^*(\phi) = \omega^*(\partial\phi/\partial x).$$

It is necessary to alter the definition of the field operator $\Phi_\mu(\phi, t)$ slightly, so that $\Phi_\mu(\phi, t)$ satisfies the Dirac equation. Let y be any point in E_3, and let ϕ_y be the function obtained from ϕ by translation through the vector y. Then $\phi_y(x) = \phi(x-y)$. The field operator $\Phi_y(\phi_y, t)$, which will be written $\Phi_\mu(\phi, y, t)$ to show the dependence on y separately from that on ϕ, can now easily be shown to satisfy the Dirac equation

$$i\partial/\partial t\,\Phi_\mu(\phi, y, t) = (i\alpha_j \partial/\partial y_j - m\alpha_0)_{\mu\nu}\,\Phi_\nu(\phi, y, t). \tag{28}$$

This follows immediately from equation (27), using the results

$$\partial/\partial x_j \phi(x-y) = -\partial/\partial y_j \phi(x-y), \quad \partial/\partial x_j \phi(x+y) = \partial/\partial y_j \phi(x+y),$$

and the following

LEMMA 5. *For any* $\phi \in L_2(E_3)$, $J(\phi_y(x) \otimes \rho_\mu) = \psi_\nu(x+y) \otimes \rho_\nu$ *for some functions* $\psi_\nu \in L_2(E_3)$, *and for any* $y \in E_3$.

For from Lemma 1,

$$J(\phi_y(x) \otimes \rho_\mu) = \text{l.i.m.}\,(2\pi)^{-\frac{3}{2}} \int u_s(K)\,\bar{a}_s(K) \exp(iKx)\,dK,$$

where $$a_s(K) = \text{l.i.m.}\,(2\pi)^{-\frac{3}{2}} \int \bar{u}_{s\mu}(K)\,\phi(x-y) \exp(-iKx)\,dx$$

$$= \text{l.i.m.}\,(2\pi)^{-\frac{3}{2}} \exp(-iKy) \int \bar{u}_{s\mu}(K)\,\phi(t) \exp(-iKt)\,dt,$$

so that $$J(\phi_y(x) \otimes \rho_\mu) = \text{l.i.m.}\,(2\pi)^{-\frac{3}{2}} \int u_s(K) \exp(iK(x+y))\,\bar{b}_s(K)\,dK \in R_a,$$

where $$b_s(K) = \text{l.i.m.}\,(2\pi)^{-\frac{3}{2}} \int \bar{u}_{s\mu}(K)\,\phi(t) \exp(-iKt)\,dt \in L_2(E_3).$$

Thus $J(\phi_y(x) \otimes \rho_\mu) = \tau_{-y} J(\phi \otimes \rho_\mu)$, where $\tau_y \phi(x) = \phi(x-y)$ and τ is the space translation operator.

Then from equation (27) and Lemma 5,

$$i\partial/\partial t\,\Phi_\mu(\phi, y, t) = H_{\mu\nu}\omega(\exp(i \mid H \mid t)\,\tau_{-y} J I_- \phi \otimes \rho_\mu) + H'_{\mu\nu}\omega^*(\exp(i \mid H \mid t)\,I_+ \tau_y \phi \otimes \rho_\nu)$$
$$= -(\alpha_j \partial/\partial(iy_j) + m\alpha_0)_{\mu\nu}\,\Phi_\nu(\phi, y, t).$$

The formalism can now be developed in close analogy with that of intuitive quantum field theory. The total momentum operator P_j is evidently to be defined as

$$P_j = \Omega(I_+ p_j \otimes 1) - \Omega(I_- p_j \otimes 1),$$

where $p_j = -i\partial/\partial x_j$. Similarly, the total spin operator S_j is to be defined as

$$S_j = \Omega(I_+ 1 \otimes (\tfrac{1}{2}\sigma_j)) - \Omega(I_- 1 \otimes (\tfrac{1}{2}\sigma_j)).$$

The total current-vector operator J_l is defined to be

$$J_l = \Omega(I_+ 1 \otimes \alpha_l) - \Omega(I_- 1 \otimes \alpha_l).$$

https://doi.org/10.1017/S0305004100031807 Published online by Cambridge University Press

The total angular momentum operator M_{ij} is defined to be

$$M_{ij} = \Omega(I_+(x_i p_j - x_j p_i) \otimes 1) - \Omega(I_-(x_i p_j - x_j p_i) \otimes 1).$$

All these operators are densely defined self-adjoint transformations on $\mathscr{A}$. Since $\Omega(|H|)$ commutes with all these operators, then the corresponding observables are constants of the motion.

4. *Relativistic invariance.* It is necessary to ensure that the present formalism has the correct relativistic invariance properties. The relativistic invariance properties of the formalism for the electromagnetic field have been discussed by Cook. In this discussion, a difficulty arises which will now be considered. Let G be any Lorentz transformation which does not reverse time, and which has the equation $x'_\mu = a_{\nu\mu} x_\nu$, where G takes undashed into dashed coordinates, or in matrix form $x' = A_{\text{tr}} x$. G induces the transformation $A_c = A^{-1}$ on the base ρ_μ of the photon spin-space C_s so that $\rho'_\mu = A_{c\mu\nu}\rho_\nu$. Hence if $\phi = \phi_\mu \otimes \rho_\mu \in R_s$, the space of single-photon wave functions, then under G,

$$\phi = \phi'_\mu(x', t') \otimes \rho'_\mu = \phi'_\mu(x', t') \otimes A_{c\mu\nu}\rho_\nu = \phi_\nu(x, t) \otimes \rho_\nu, \tag{29}$$

so that $\phi'_\mu = A_{\mu\nu}\phi_\nu$. Hence the quantities $(\phi_1, \phi_2, \phi_3, \phi_4)$ transform as the contravariant components of a four-vector under G, as is required.

Under G the electromagnetic potential operators transform as

$$A'_\mu(\phi) = Q(\phi \otimes \rho'_\mu) = (A^{-1})_{\mu\nu} A_\nu(\phi). \tag{30}$$

Equation (30) shows that the four quantities $A_\mu(\phi)$ transform as the covariant components of a four-vector. These potentials are time-independent operators, since the Schrödinger picture is being used for the system. For any operator A associated with the system in the Schrödinger picture we may define the operator $\partial_t A$ by the equation $\partial_t A = i[\Omega(H), A]_-$. Then the expectation value of $\partial_t A$, in any state of the system, is equal to the rate of change of the expectation value of A in that state, which makes evident the relation between ∂_t and $\partial/\partial t$. In (1) it was shown that the equations

$$(\partial_t^2 - \nabla^2) A_\mu(\phi) = 0,$$

$$\partial_t A_4(\phi) - \partial/\partial x_i A_i(\phi) = 0,$$

are then satisfied by the electromagnetic potentials. The usual argument for the relativistic invariance of these equations, using the transformation properties obtained for the electromagnetic potentials, does not seem to apply here, since

$$(\partial_t, \partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$$

is not a four-vector under Lorentz transformations. Indeed, since we are keeping $\phi \in L_2(E_3)$ invariant then we would expect to keep $\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3$ invariant. On the other hand, since $\Omega(H)$ should transform as the time component of a four-vector then ∂_t should transform similarly.

We may alternatively consider the photon-field system in the Heisenberg picture. The electromagnetic potentials will now be time-dependent; they are written $A_\mu(\phi, t)$.

https://doi.org/10.1017/S0305004100031807 Published online by Cambridge University Press

Whereas in the Schrödinger picture it was not possible to obtain relativistically invariant field equations, we find now that it is not meaningful even to consider the transformation properties of the potentials themselves. For the time t enters into the potentials in a non-relativistic manner, completely separated from (x_1, x_2, x_3), so that it does not seem possible to define a transformed time t' at which the transformed potential may be defined in terms of $A_\mu(\phi, t)$. This difficulty would seem to arise from the appearance of the pair (t, ϕ) in the rigorous formalism in place of (t, x_1, x_2, x_3) in the intuitive formalism. The difficulty arising in the discussion of relativistic invariance of the field equations, in the Schrödinger picture, seems to occur for this same reason. The above difficulties arise in a similar manner when the relativistic transformation properties of the electron field operators $\Psi_\mu(\phi, t)$ are considered.

In § 3 we considered the operators $\Psi_\mu(\phi, x, t)$, and we may consider also the operators $A_\mu(\phi, x, t)$. This change in definition alters the discussion of relativistic invariance. This discussion will not be given here. In fact it will be shown elsewhere that the rigorous formalism cannot possess all the relativistic transformation properties of intuitive field theory unless we take $\phi = \delta_{(0)}$. This corresponds to taking the pair $(t, \delta_{(x)})$ in place of (t, ϕ), or equivalently to taking the vector (t, x).

5. *Relation to quantum field theory.* It is the purpose of this section to discuss the relation of the rigorous formalism developed in (1) and the present paper to that of intuitive quantum field theory.

Consider first the potential operators of the electromagnetic field, which were defined in (1), p. 243. For any $\phi \in L_2(E_3)$ in the domain of $H^{-\frac{1}{2}}$ and H, let ϕ have the Fourier transform $\psi \in L_2(E_3)$, so that

$$H^{-\frac{1}{2}}\phi = \text{l.i.m.}\,(2\pi)^{-\frac{3}{2}}\int \exp(iKx)\,K^{-\frac{1}{2}}\psi(K)\,dK.$$

Then
$$\omega_s(H^{-\frac{1}{2}}\phi \otimes \rho_\mu) = (2\pi)^{-\frac{3}{2}}\omega_s\left(\int K^{-\frac{1}{2}}\psi(K)\exp(iKx)\,dK \otimes \rho_\mu\right),$$

where the integration over K is to be taken as a limit in the mean in $L_2(E_3)$. It is possible to write this expression formally as $(2\pi)^{-\frac{3}{2}}\int K^{-\frac{1}{2}}\psi(K)\,\omega_s(\exp(iKx)\otimes\rho_\mu)\,dK$, but since $\exp(iKx)\otimes\rho_\mu$ does not belong to R_s, then $a_{K\mu} = \omega_s(\exp(iKx)\otimes\rho^\mu)$ is not a creation operator on S. Similarly, it is possible to write $\omega_s^*(H^{-\frac{1}{2}}\phi\otimes\rho_\mu)$ heuristically as

$$(2\pi)^{-\frac{3}{2}}\int K^{-\frac{1}{2}}\overline{\psi}(K)\,\omega_s^*(\exp(iKx)\otimes\rho_\mu)\,dK,$$

though this expression is not really an operator on S. Then the following heuristic expressions for the potential operators are obtained

$$\left.\begin{aligned} A_j(\phi, y) &= (2\pi)^{-\frac{3}{2}}2^{-\frac{1}{2}}\int dK\,.\,K^{-\frac{1}{2}}[\psi(K)\,a_{Kj}\exp(iKy) + \overline{\psi}(K)\,a_{Kj}^*\exp(-iKy)],\\ \Phi(\phi, y) &= (2\pi)^{-\frac{3}{2}}2^{-\frac{1}{2}}\int dK\,.\,K^{-\frac{1}{2}}[\psi(K)\,a_{K4}\exp(iKy) - \overline{\psi}(K)\,a_{K4}^*\exp(-iKy)],\end{aligned}\right\} \quad (31)$$

https://doi.org/10.1017/S0305004100031807 Published online by Cambridge University Press

where the commutation brackets of the $a_{K\mu}$, $a^*_{K\mu}$ are formally

$$[a^*_{K\mu}, a_{K'\nu}]_- = [\omega^*_s(\exp(iKx)\otimes\rho_\mu), \omega_s(\exp(iK'x)\otimes\rho_\nu)]_- = (2\pi)^3\delta_{\mu\nu}\delta(K-K')I. \quad (32)$$

On comparison of the expansions of equation (31) with those arising in intuitive quantum field theory (for example (4), p. 98, equation (44)) it is easily seen that only if $\psi = 1$ do $A_j(\phi, y)$, $\Phi(\phi, y)$ reduce to the potentials given in the above reference. In other words, it is necessary to take $\phi(x) = \delta_{(0)}$ (which is the Dirac measure associated with the origin) to make the potentials defined above reduce to those of intuitive quantum field theory.

A similar analysis may be applied to the electron-positron field operators defined in the last section. Then formally

$$\Phi_\mu(\phi, y, t) = (2\pi)^{-\frac{3}{2}}\int dK\Big[\sum_{s=1,2}\int dz\, a_{sK}u_{s\mu}(K)\overline{\phi}(z)\exp(-iE_{sK}t+ik(z+y)) + \sum_{s=3,4}\int b^*_{sK}\exp(-iE_{sK}t+iK(y-z))\overline{\phi}(z)u_{s\mu}(K)\,dz\Big], \quad (33)$$

where $a_{sK} = \omega^*_A(\exp(iKx)u_s)$, $(s = 1, 2)$; $b^*_{sK} = \omega_A(\exp(iKx)u_s)$, $(s = 3, 4)$. The non-zero anti-commutation brackets of the a_{sK}, b_{sK}, etc., are

$$\left.\begin{aligned}[a_{sK}, a^*_{s'K'}]_+ &= (2\pi)^3\delta_{ss'}\delta(K-K')I,\\ [b_{sK}, b^*_{s'K'}]_+ &= (2\pi)^3\delta_{ss'}\delta(K-K')I,\end{aligned}\right\} \quad (34)$$

while all the other anti-commutation brackets are zero. It is necessary to choose $\Phi(z) = \delta_{(0)}$ to make equation (33) agree with the expression for $\Psi_\mu(x, t)$ in (4), p. 115, equation $(1a)$.

When E_3 is replaced by torus space, an analysis similar to the above gives very similar results. The only difference is that the integration over K must be replaced by summation over a discrete denumerable set (K_n) of frequencies. The operators a_{sK_n}, b_{sK_n} and $a_{K_n\mu}$ are now rigorously defined as densely defined closed linear transformations on $\mathscr{A}$ and S respectively. It is still necessary to take $\overline{\phi}(z) = \delta_{(0)}$ to reduce the rigorous formalism to that of intuitive quantum field theory.

The commutation brackets of the electromagnetic potential operators are, for $\phi_1, \phi_2 \in D_{H^{-\frac{1}{2}}}$,

$$\begin{aligned}[A_\mu(\phi_1, y_1, t_1,), A_\nu(\phi_2, y_2, t_2)]^\sim_- &= (\tfrac{1}{2})g_{\mu\nu}[(H^{-\frac{1}{2}}\exp(iH(t_2-t_1))\tau_{y_2-y_1}\phi_2, H^{-\frac{1}{2}}\phi_1)\\ &\quad -(H^{-\frac{1}{2}}\exp(iH(t_1-t_2))\tau_{y_1-y_2}\phi_1, H^{-\frac{1}{2}}\phi_2)]\\ &= ig_{\mu\nu}D(\phi_1, \phi_2, y_1-y_2, t_1-t_2), \end{aligned} \quad (35)$$

where $g_{\mu\nu} = +1$ if $\mu = \nu = 1, 2, 3$, $g_{\mu\nu} = -1$ if $\mu = \nu = 4$, and $g_{\mu\nu} = 0$ for all other values of μ, ν. Let ψ_1, ψ_2 be the Fourier transforms of ϕ_1, ϕ_2. Then

$$D(\phi_1, \phi_2, y, t) = (2\pi)^{-3}\int d^3K(2iK)^{-1}[\overline{\psi}_1(K)\psi_2(K)\exp(iKy-iKt) - \psi_1(K)\overline{\psi}_2(K)\exp(-iKy+iKt)]. \quad (36)$$

$D(\phi_1, \phi_2, y, t)$ is closely related to the invariant D-function and reduces to this function when $\psi_1 = \psi_2 = 1$, as is to be expected from the discussion at the beginning of this

https://doi.org/10.1017/S0305004100031807 Published online by Cambridge University Press

section. $D(\phi_1, \phi_2, y, t)$ is a finite real-valued function of y and t, for each pair ϕ_1, $\phi_2 \in L_2(E_3)$. When ϕ_1, ϕ_2 are real-valued functions in $L_2(E_3)$, then

$$D(\phi_1, \phi_2, y, 0) = 0, \quad \partial/\partial t\, D(\phi_1, \phi_2, y, t)\,|_{t=0} = (\tau_{-y}\phi_1, \phi_2).$$

The anti-commutation brackets of the electron field operators are

$$\begin{aligned}[\Phi^*_\mu(\phi_1, y_1, t_1), \Phi_\nu(\phi_2, y_2, t_2)]_+ &= S_{\mu\nu}(\phi_1, \phi_2, y_1 - y_2, t_1 - t_2) \\ &= [(\exp(iHt_1)\, I_-\tau_{y_1}\phi_1 \otimes \rho_\mu, \exp(iHt_2)\, I_-\tau_{y_2}\phi_2 \otimes \rho_\nu) \\ &\quad + (\exp(iHt_1)\, I_+\tau_{y_1}\phi_1 \otimes \rho_\mu, \exp(iHt_2)\, I_+\tau_{y_2}\phi_2 \otimes \rho_\nu)]\, I. \end{aligned} \tag{37}$$

Using the representation of Lemma 2, and the results of Lemmas 3, 4, then

$$\begin{aligned} &S_{\mu\nu}(\phi_1, \phi_2, y_1 - y_2, t_1 - t_2) \\ &= (2\pi)^{-3} \sum_{\substack{s,s'=1,2 \\ s,s'=3,4}} \int dK \exp(i(E_{sK}t_1 - E_{s'K}t_2) - iK(y_1 - y_2)) \sum_\sigma u_{s\sigma}(K)\overline{u_{s'\sigma}}(K)\, a^{(1)}_{s\mu}(K)\, \overline{a^{(2)}_{s'\nu}}(K), \end{aligned} \tag{38}$$

where $a^{(i)}_{s\mu}(K) = \overline{u}_{s\mu}(K)\, \psi_i(K), i = 1, 2$. By equation (15),

$$S_{\mu\nu}(\phi_1, \phi_2, y, t) = \sum_{s=1}^{4} \int \exp(i(E_{sK}t - Ky))\, \overline{u}_{s\mu}(K)\, u_{s\nu}(K)\, \psi_1(K)\, \overline{\psi}_2(K)\, dK. \tag{39}$$

When $\psi_1 = \psi_2 = 1$, the right-hand side of equation (39) reduces to that of (4), p. 119, equation (16), except for a factor α_0. More generally it is only possible to reduce $S_{\mu\nu}(\phi_1, \phi_2, y, t)$ to a differential operator acting on $D(\phi_1, \phi_2, y, t)$ when ϕ_1, ϕ_2 are real valued functions. In this case the arguments of (4), p. 119, may be used to obtain

$$S_{\mu\nu}(\phi_1, \phi_2, y, t) = (\delta_{\mu\nu}\partial/\partial t - iH_{\mu\nu})\, D(\phi_1, \phi_2, y, t),$$

where $H_{\mu\nu}$ is to be regarded as a differential operator, acting on $D(\phi_1, \phi_2, y, t)$ regarded as a function of y. $S_{\mu\nu}(\phi_1, \phi_2, y, t)$ is a finite-valued function of y, t, for each pair ϕ_1, $\phi_2 \in L_2(E_3)$. The remaining anti-commutation brackets of the electron field operators are

$$[\Phi_\mu(\phi_1, y_1, t_1), \Phi_\nu(\phi_2, y_2, t_2)]_+ = [\Phi^*_\mu(\phi_1, y_1, t_1), \Phi^*_\nu(\phi_2, y_2, t_2)]_+ = 0.$$

These considerations show that it is not possible to formulate, in a rigorous way, intuitive quantum field theory in terms of Hilbert space theory. What has been developed in the preceding sections is a mathematically rigorous model which reduces exactly to intuitive quantum field theory when the δ-function is introduced in the manner described in this section. It would thus seem necessary to use a mathematical formalism other than Hilbert space theory to achieve a rigorous formulation for intuitive field theory. This will not be attempted here.

6. *Application of the rigorous formalism.* One might inquire if it is possible to apply the formalism to problems involving electrons and positrons. It is first necessary to define an interaction energy between the electromagnetic field and the electron-positron field. The interaction energy in field theory is usually taken to be

$$-\sum_{\mu=1}^{4} g_{\mu\mu} \int_{E_3} \Phi^*(\delta_{(y)}, t)\, \alpha_\mu\, \Phi(\delta_{(y)}, t)\, A_\mu(\delta_{(y)}, t)\, dy.$$

https://doi.org/10.1017/S0305004100031807 Published online by Cambridge University Press

By analogy it would seem that for the present formalism the interaction energy should be

$$-\sum_{\mu=1}^{4} g_{\mu\mu} \int_M \Phi^*(\phi, t)\, \alpha_\mu \Phi(\phi, t)\, A_\mu(\phi, t)\, dm(\phi),$$

where m is a measure on a suitable sub-space M of $L_2(E_3)$. A heuristic condition on m would be that when M is taken as the set of all δ-measures, then

$$\int_{y \in E_3} f(\delta_{(y)})\, dm(\delta_{(y)}) = \int_{E_3} F(y)\, dy,$$

where

$$F(y) = f(\delta_{(y)}).$$

An immediate difficulty in determining m is that there are no non-trivial translation invariant measures on $L_2(E_3)$, so that it is not possible to give a simple extension of Lebesgue integration on E_3 to an integration procedure over $L_2(E_3)$. Further, though it is possible to set up a measure m satisfying the heuristic condition above (this has been done), it does not seem possible to prevent the occurrence of terms having infinite integrals with respect to m when the S-matrix for such an interaction energy is developed by perturbation methods. That such an interaction expression has meaning as an operator except for the most trivial cases for m is extremely doubtful. It must be concluded that it is extremely difficult to develop the Hilbert space formalism for quantum field theory in any useful manner.

REFERENCES

(1) Cook, J. M. *Trans. Amer. Math. Soc.* 74 (1953), 222.
(2) Dirac, P. A. M. *The principles of quantum mechanics*, 3rd ed. (Oxford, 1947).
(3) Gupta, S. N. *Proc. Phys. Soc.* 63 (1950), 681.
(4) Heitler, W. *The quantum theory of radiation*, 3rd ed. (Oxford, 1953).
(5) Stone, M. H. *Linear transformations in Hilbert space* (Colloq. Publ. Amer. Math. Soc. 15, 1932).
(6) Valatin, J. G. *J. Phys. Radium*, 12 (1951), 131.
(7) Weyl, H. *The theory of groups and quantum mechanics* (New York, 1931).

Christ's College
Cambridge

https://doi.org/10.1017/S0305004100031807 Published online by Cambridge University Press