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Kinetic Alfvén solitons in a low-beta plasma

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Kinetic Alfvén solitons with hot electrons and finite electron inertia in a low-beta ($\beta = 8\pi n_0 T/B_G^2$, the ratio of the kinetic to the magnetic pressure) plasma is studied analytically, with the ion motion being considered dominant through the polarization drift. Both compressive and rarefactive kinetic Alfvén solitons are found to exist within a definite range of k_z (the direction of propagation of the kinetic Alfvén solitary waves with respect to the direction of the magnetic field) for each pair of assigned values of β and M (Mach number). Unlike in previous theoretical investigations, β appears as an explicit parameter for the kinetic Alfvén solitons in this case. In addition, consideration of the electron pressure gradient is found to suppress the speed of both the Alfvén solitons considerably for A ($= 2QM^2/\beta k_z^2$, with Q the electron-to-ion mass ratio) less than unity.

1. Introduction

The enhanced level of particle transport over the classical values in a thermal equilibrium plasma arises from the low-frequency and long-wavelength modes that convect the plasma across the confining magnetic field as observed by Taylor and McNamara (1970). Okuda (1973) *et al.* and Montgomery (1972) *et al.* observed that an electromagnetic mode exists in a thermal equilibrium plasma through $\mathbf{E} \times \mathbf{B}$ drifts by means of plasma convection. It has been established that electrostatic and electromagnetic modes of different frequencies exist in a thermal equilibrium plasma. The modes below the ion cyclotron frequency and above the magnetic drift frequency are known as shear Alfvén waves, whereas high-frequency and short-wavelength modes are known as fast magnetosonic Alfvén waves. The kinetic effect modifies the properties of shear Alfvén wave solutions from the magnetohydrodynamic (MHD) solutions to kinetic Alfvén-wave solutions. The departure from the MHD model leads to the existence of shear or kinetic Alfvén waves, as reported by Stefant (1978) and Lashmore-Devies and May (1976).

Several workers have observed that kinetic Alfvén waves can propagate in a plasma with $\beta > m_e/m_i$ (the electron-to-ion mass ratio). A low-β plasma supports low-frequency kinetic Alfvén waves and electrostatic ion-cyclotron waves. Kinetic Alfvén waves are found to have a two-dimensional structure, with electrostatic and magnetic components with phase velocity (parallel to the external magnetic field $B_0 \hat{\mathbf{z}}$) much smaller than the electron thermal velocity. Furthermore, kinetic Alfvén waves are observed to propagate everywhere uniformly in a magnetized hot plasma, as reported by Stix (1981). He found

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that for the waves to propagate across the density gradient from the antenna to the resonance region, either $\beta_e > (m_e/m_i)^{1/2}$ or $\beta_i > (m_e/m_i)^{1/2}$ must hold. If neither of these conditions is satisfied, kinetic Alfvén waves, described by the cold plasma equations, form a surface mode after reflection from the resonance region. However, for hydromagnetic Alfvén waves, $\omega < \Omega_i$, and so the velocity component of an Alfvén soliton must be less than the ion cyclotron frequency to allow the neglect of resonances.

Solitary-wave solutions for kinetic Alfvén waves were investigated by Hasegawa and Mima (1976) and Yu and Shukla (1978) in the case of a plasma with $1 \gg \beta \gg m_e/m_i$. An exact supersonic solitary-wave solution with density hump was shown to exist by Hasegawa and Mima when the electron pressure dominates over the electron inertia that establishes equilibrium in the direction of the magnetic field. Kalita and Kalita (1986) showed the existence of both super- and subsonic rarefactive Alfvén solitons in a low-β ($\ll m_e/m_i$) plasma, using the cold electron fluid equation and taking account of ion inertia and the displacement current. The existence of a new kind of super-Alfvénic soliton was established by Shukla *et al.* (1982) for the case where ion inertia and displacement current are neglected.

The nonlinear behaviour of azimuthally symmetric Alfvén waves for modulational instability was studied by Kumar and Srivastava (1991) using the nonlinear Schrödinger equation. Buti (1991) showed that parallel propagation of large-amplitude Alfvén waves depending on β is governed by the modified derivative nonlinear Schrödinger (MDNLS) equation. Large-amplitude Alfvén waves propagating along the magnetic field in an inhomogeneous plasma were shown by Buti (1992) to be governed by the derivative nonlinear Schrödinger (DNLS) equation. Kinetic effects, such as the finite-Larmor-radius effect that increases the order of equations and leads to particle resonances on the ideal Alfvén-wave continuum, were investigated by Akerstedt (1990) and shown to result in singular perturbations.

In this paper, we investigate exact nonlinear localized Alfvén solitary-wave solutions in a low-β plasma with hot electrons and finite electron inertia. In Sec. 2, we outline the method for normalizing the basic set of equations governing the plasma. In Sec. 3, the energy integral used to investigate solitary-wave solutions is derived. The conditions for the existence of Alfvén solitons are derived in Sec. 4. In Sec. 5 we deal with the solutions for solitary Alfvén waves in the small-amplitude limit. Finally, we analyse the results of our investigation in Sec. 6.

2. Dynamics of motion

We consider a low-β plasma with electron pressure gradient, finite electron inertia and current density assuming ion motion to be dominant through polarization drift. Consequently, not all the nonlinear terms are incorporated in the analysis. Also, the highly magnetized electrons of negligible inertia are assumed to move only in the direction of the magnetic field $B_0\,\hat{\mathbf{z}}$, where $\hat{\mathbf{z}}$ is the unit vector along the z axis. We use the fact that when the ion Larmor radius remains finite, the charge separation that develops between the ions and electrons generates a significant longitudinal electric (electrostatic) field, which may be much larger than the transverse (inductive) electric field connected with

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the kinetic Alfvén waves. Because of the low-β ($\ll 1$) assumption for high magnetic pressure, the electric field can be written in terms of two potentials ϕ and ψ (Kadomtsev 1965):

$$E_x = -\frac{\partial \phi}{\partial x}, \qquad E_z = -\frac{\partial \psi}{\partial z}.$$

For the weak Larmor-radius effect, in the fluid description of the plasma it is assumed that the scaling of the variables is such that the frequency $\omega \ll \Omega_i$ (the gyrofrequency), the phase velocity $\omega/k \ll v_{\text{the}}$ (electron thermal velocity) and the wavenumber $k \ll 1/\rho_i$ (where ρ_i is the ion gyroradius). Hence the fluid equations governing the dynamics of such a plasma in the (x, z) plane, after normalizing the density by the equilibrium plasma density n_0, distances by the Debye length λ_D, the potentials by T_e/e, velocity by the Alfvén velocity v_A and time by the ion gyroperiod Ω_i^{-1}, is given by

$$\frac{\partial n_e}{\partial t} + \frac{\partial}{\partial z}(n_e v_{ez}) = 0, \tag{1}$$

$$\frac{\partial v_{ez}}{\partial t} + v_{ez}\frac{\partial v_{ez}}{\partial z} = \frac{\beta}{2Q}\left(\frac{\partial \psi}{\partial z} - \frac{1}{n_e}\frac{\partial n_e}{\partial z}\right), \tag{2}$$

$$\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x}(n_i v_{ix}) + \frac{\partial}{\partial z}(n_i v_{iz}) = 0, \tag{3}$$

$$v_{ix} = -\frac{\beta}{2}\frac{\partial^2 \phi}{\partial t\, \partial x}, \tag{4}$$

$$\frac{\partial v_{iz}}{\partial t} + v_{ix}\frac{\partial v_{iz}}{\partial x} + v_{iz}\frac{\partial v_{iz}}{\partial z} = -\frac{\beta}{2}\frac{\partial \psi}{\partial z}, \tag{5}$$

$$\frac{\partial^4(\phi - \psi)}{\partial x^2\, \partial z^2} = -\frac{\partial^4 \psi}{\partial z^2\, \partial t^2}\frac{v_A^2}{c^2} + \frac{2}{\beta}\left[\frac{\partial^2 n_e}{\partial t^2} - \frac{\partial^2 n_i}{\partial t^2} - \frac{\partial^2 (n_i v_{ix})}{\partial x\, \partial t}\right], \tag{6}$$

where

$$Q = \frac{m_e}{m_i}, \qquad \lambda_D = \left(\frac{T_e}{4\pi n_0 e^2}\right)^{1/2},$$

$$v_A = \left(\frac{B_G}{4\pi n_0 m_i}\right)^{1/2}, \qquad \beta = \frac{8\pi n_0 T}{B_G^2},$$

Ω_i is the ion gyrofrequency and T_e is the electron temperature. Subscripts i and e stand for ions and electrons respectively. In (4), the ion motion is considered to be dominanted by the polarization drift. The linear dispersion relation describing kinetic Alfvén waves in this high-electron-frequency plasma is

$$\omega^2 = k_z^2\left[v_A^2\left(1 + \frac{v_A^2}{c^2}\right) + c^2\lambda_D^2\right]\frac{\lambda^2}{\lambda^2 + 1} \qquad \left(\lambda = \frac{\omega_{pe}}{c}\right).$$

This admits (from the dispersion relation) the formation of kinetic Alfvén waves in such a plasma.

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3. Derivation of the energy integral

It is necessary to obtain a reduced model equation in which nonlinearity and dispersion produce small deviations of the wave speed from that of an underlying MHD wave. Therefore we assume that the wave is propagating obliquely to the external magnetic field and depends on the quantity

$$\eta = k_x x + k_z z - Mt, \tag{7}$$

where

$$k_x^2 + k_z^2 = 1,$$

$$M = \frac{\text{wave velocity}}{\text{Alfvén velocity}} = \frac{v}{v_A}.$$

Using the new coordinate η moving with the wave in a quasi-neutral plasma, we get from (1)–(5), after integration and simplification,

$$v_{ez} = \frac{M}{k_z}\left(1-\frac{1}{n}\right) = \frac{k_x}{k_z}v_{ix} + v_{iz},$$

$$v_{ix} = \frac{1}{k_x}\left[\frac{\beta k_z^2}{2M}(1-n) + M(1+Q)\left(1-\frac{1}{n}\right)\right],$$

$$v_{iz} = -\left[\frac{\beta k_z}{2M}(1-n) + \frac{QM}{k_z}\left(1-\frac{1}{n}\right)\right],$$

$$\frac{\partial^2\psi}{\partial\eta^2} = \frac{\partial}{\partial\eta}\left[\frac{1}{n}\frac{\partial n}{\partial\eta}\left(1-\frac{2QM^2}{\beta k_z^2 n^2}\right)\right],$$

$$\frac{\partial^2\phi}{\partial\eta^2} = \frac{1}{k_x^2}\left[\frac{k_z^2}{M^2}(1-n) + \frac{2(1+Q)}{\beta}\left(1-\frac{1}{n}\right)\right].$$

The eliminating v_{ix} from (6) in terms of the new coordinate η (after integrating twice) gives

$$k_x^2 k_z^2 \frac{\partial^2(\phi-\psi)}{\partial\eta^2} = -k_z^2\frac{M^2 v_A^2}{c^2}\frac{\partial^2\phi}{\partial\eta^2} + k_z^2 n(1-n) - \frac{2M^2}{\beta}(1+Q)(1-n). \tag{8}$$

In deducing these equations we have applied the boundary conditions

$$v_{ix} = v_{iz} = v_{ez} = 0, \qquad \phi = \psi = \frac{\partial n}{\partial\eta} = 0 \quad \text{at } n = 1 \quad \text{as } |\eta| \to \infty$$

and the charge-neutrality relation $n_e = n_i = n$.

Putting the values of $\partial^2\psi/\partial\eta^2$ and $\partial^2\phi/\partial\eta^2$ into (8) and integrating twice, we get

$$\frac{d}{d\eta}\left[\left(1-\frac{2QM^2}{\beta k_z^2 n^2}\right)\frac{1}{n}\frac{dn}{d\eta}\right] = -\frac{c^2}{M^2 v_A^2 - c^2 k_x^2}\left[\frac{k_z^2}{M^2}(1-n) + \frac{2(1+Q)}{\beta}\left(1-\frac{1}{n}\right) - n(1-n) + \frac{2M^2(1+Q)}{\beta k_z^2}(1-n)\right]. \tag{9}$$

Multiplying both sides of (9) by

$$\left(1-\frac{2QM^2}{\beta k_z^2 n^2}\right)\frac{1}{n}\frac{dn}{d\eta}$$

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and integrating once, we get the energy integral for classical particles in a potential well as

$$\frac{1}{2}\left(\frac{dn}{d\eta}\right)^2+\Psi(n,M,k_z,v_A,\beta)=0, \tag{10}$$

where the Sagdeev potential $\Psi(n)$, a function of density n with variable parameters M, k_z, V_A and β, is given by

$$\begin{aligned}\Psi(n,M,k_z,v_A,\beta)=\frac{c^2n^6}{(c^2k_x^2-M^2v_A^2)(n^2-A)^2}\Bigg[&\left(-\frac{k_z^2}{M^2}R-S+A\right)\log n\\&+\left(\frac{k_z^2}{M^2}R+1\right)(n-1)+\left(\frac{Ak_z^2}{M^2}R-S+A\right)\left(\frac{1}{n}-1\right)\\&-\left(\frac{Ak_z^2}{2M^2}R+\tfrac{1}{2}AS\right)\left(\frac{1}{n^2}-1\right)+\frac{AS}{3}\left(\frac{1}{n^3}-1\right)-\frac{n^2-1}{2}\Bigg],\end{aligned} \tag{11}$$

with

$$A=\frac{2QM^2}{\beta k_z^2},\qquad R=\frac{2M^4(1+Q)}{\beta k_z^4}+1,$$

$$S=\frac{2(1+Q)}{\beta}.$$

We put

$$P=\frac{c^2}{c^2k_x^2-M^2v_A^2}.$$

For a maximum variation N of the density n, the nonlinear dispersion equation is given by $\Psi(N)=0$; that is,

$$\begin{aligned}&\left(-\frac{k_z^2}{M^2}R-S+A\right)\log N+\left(\frac{k_z^2}{M^2}R+1\right)(N-1)\\&\quad+\left(\frac{Ak_z^2}{M^2}R-S+A\right)\left(\frac{1}{N}-1\right)-\left(\frac{Ak_z^2}{2M^2}R+\tfrac{1}{2}AS\right)\left(\frac{1}{N^2}-1\right)\\&\quad+\frac{AS}{3}\left(\frac{1}{N^3}-1\right)-\tfrac{1}{2}(N^2-1)=0.\end{aligned} \tag{12}$$

This determines the amplitude N of the solitary Alfvén waves for the given set of parameters.

4. Conditions for the existence of solitary Alfvén waves

To find the conditions for the existence of solitary waves, it is necessary to study the behaviour of the potential $\Psi(n)$ of (11) near $n=1$ and $n=N$. For solitary-wave solutions, the required conditions are

$$\Psi(1)=\Psi(N)=\left(\frac{d\Psi}{dn}\right)_{n=1}=0,$$

$$\Psi(n)<0,$$

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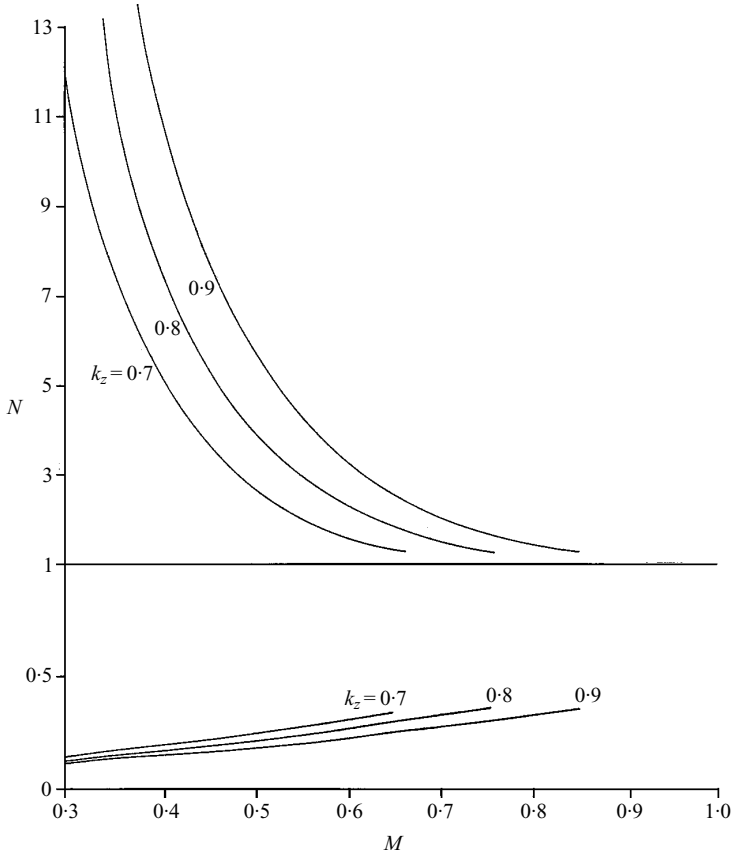


Figure 1. Amplitude of Alfvén solitons as a function of Mach number M for different values of k_z and fixed $\beta = 0.005$ ($A < 1$ and $M < k_z$).

between $n = 1$ and $n = N$ (amplitude). Expanding in Taylor series near $n = 1$ and $n = N$, we get, for $n \approx 1$,

$$\Psi(n) = -\frac{P(n-1)^2}{2(1-A)}\left[(1+S)-\frac{k_z^2}{M^2}R\right],$$

and, for $n \approx N$,

$$\Psi(n) = -\frac{PN^3(N-1)(n-N)}{N^2-A}\left[\left(N+\frac{S}{N}\right)-\frac{k_z^2}{M^2}R\right].$$

Hence for $\Psi(n)$ to be negative between $n = 1$ and $n = N$, it is necessary to have the following conditions: for, $M < k_z$ ($P > 0$),

$$N+\frac{S}{N} < \frac{k_z^2}{M^2}R < 1+S \quad \text{when } A < 1 < N, \tag{13a}$$

$$N+\frac{S}{N} > 1+S > \frac{k_z^2}{M^2}R \quad \text{when } A < N < 1 \text{ but } N^2 < A; \tag{13b}$$

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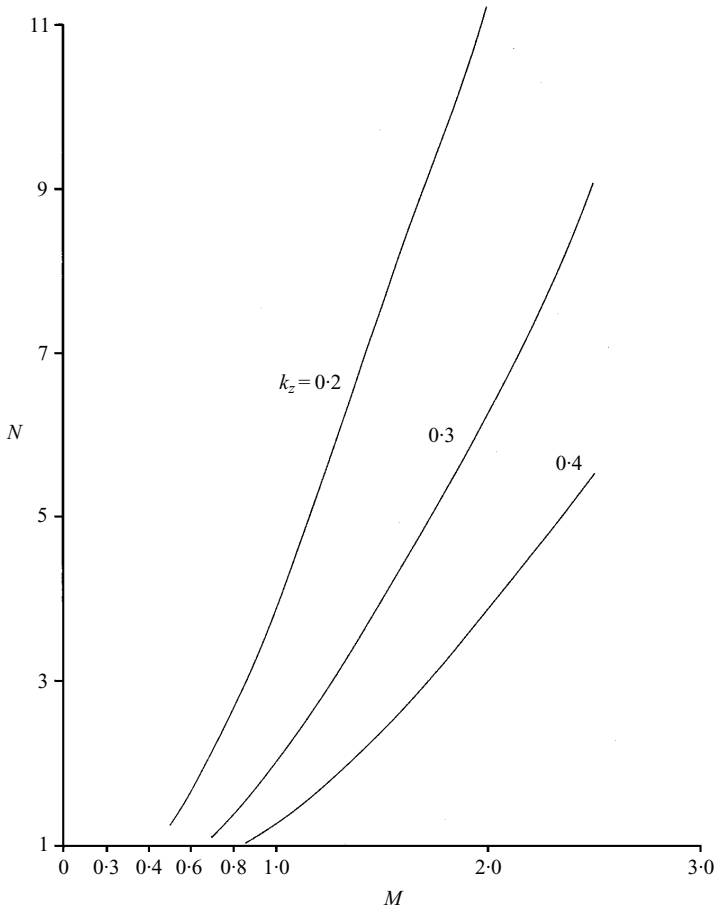


Figure 2. Amplitude of Alfvén solitons as a function of Mach number M for different values of k_z and fixed $\beta = 0.005$ ($A > 1$ and $M > k_z$).

for $M > k_z$ ($P > 0$),

$$N + \frac{S}{N} < 1 + S < \frac{k_z^2}{M^2} R \quad \text{when } A > N > 1 \text{ but } N^2 > A, \tag{14a}$$

$$1 + S < \frac{k_z^2}{M^2} R < N + \frac{S}{N} \quad \text{when } N < 1 < A. \tag{14b}$$

The amplitude N of the solitary waves is determined by assigning different values of β, k_z, $v_A/c = 0.01$ and M in the nonlinear dispersion relation (12). Substituting these values of the parameters and allowing n to vary up to N in (11), the maximum depth d of the Sagdeev potential $\Psi(n)$ is obtained. The width Δ of the soliton can then be determined from

$$\Delta = \frac{N}{d^{1/2}}. \tag{15}$$

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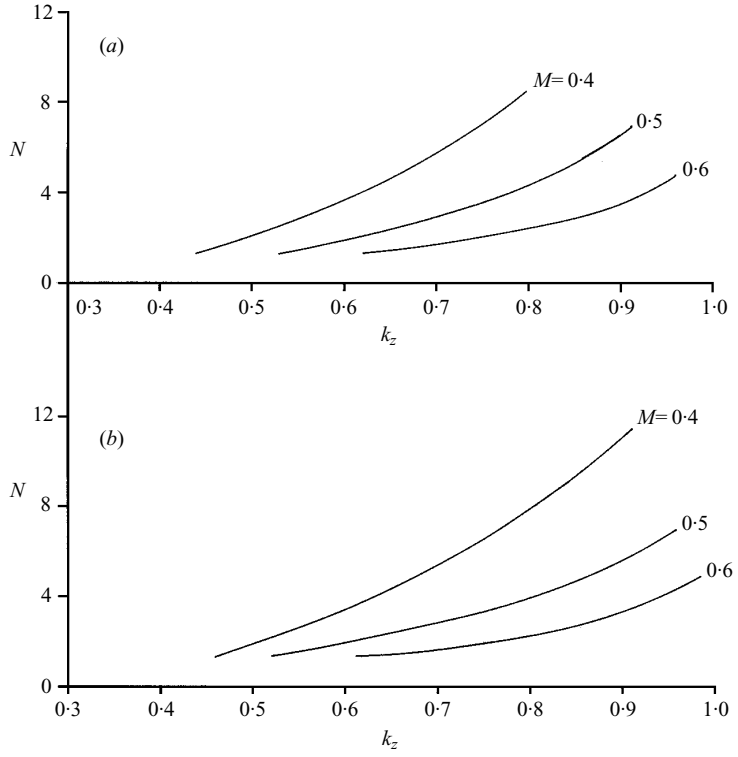


Figure 3. Amplitude of Alfvén solitons versus k_z for different values of the Mach number M: (*a*) $\beta = 0.05$; (*b*) $\beta = 0.005$. ($A < 1$ and $M < k_z$.)

5. Small-amplitude limit

Assuming $n-1$ to be very small, so that $n-1 = \delta n$ $(\delta n \ll 1), \Psi(n)$ can be expressed near $n = 1$ as

$$\psi(n \approx 1) = \Delta_1 B_1 \delta n^2 \left(1 + \frac{2C_1}{3\Delta_1}\delta n\right), \tag{16}$$

where

$$\Delta_1 = (1-A)\left(1 + S - \frac{k_z^2}{M^2}R\right),$$

$$B_1 = \frac{P}{2(1-A)^2},$$

$$C_1 = \left[3 - 9A + \frac{k_z^2}{M^2}R(1-3A) + 2A(1-S)\right].$$

The energy integral for the classical particle is

$$\frac{1}{2}\left(\frac{dn}{d\eta}\right)^2 = B_1 \Delta_1 \delta n^2 \left(1 + \frac{2C_1}{3\Delta_1}\delta n\right).$$

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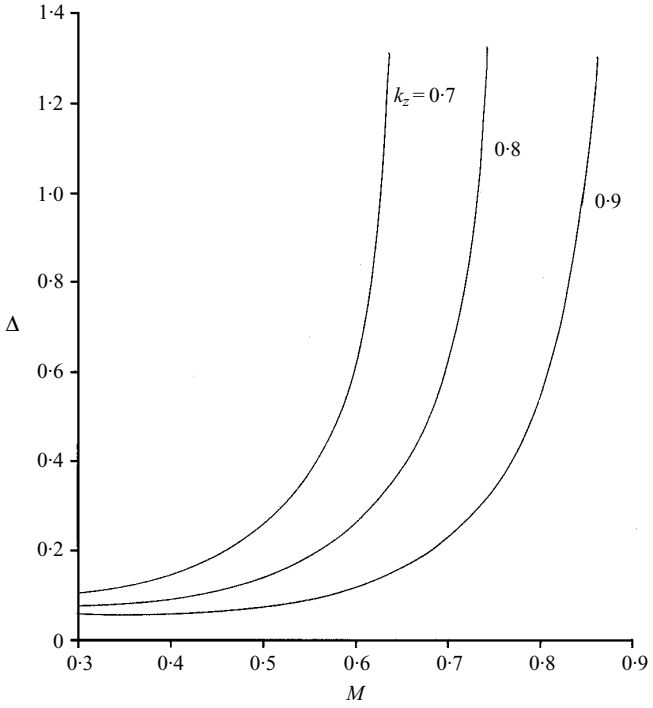


Figure 4. Width of Alfvén solitons versus Mach number M for different values of k_z and fixed $\beta = 0.005$ ($A < 1$, $N > 1$ and $M < k_z$).

After integration, this gives the soliton solution as

$$\delta n = -\frac{3\Delta_1}{2C_1}\operatorname{sech}^2[(\tfrac{1}{2}B_1\Delta_1)^{1/2}\eta]. \tag{17}$$

The analytical solutions in the small-amplitude limit are obtained from (17):

$$k_z v_{ez} = M\delta n,$$

$$k_x v_{ix} = M\left(1+Q-\frac{\beta k_z^2}{2M^2}\right)\delta n,$$

$$k_z v_{iz} = -M\left(Q-\frac{\beta k_z^2}{2M^2}\right)\delta n,$$

$$\tfrac{1}{2}\beta k_x^2\frac{\partial^2\phi}{\partial\eta^2} = \left(1+Q-\frac{\beta k_z^2}{2M^2}\right)\delta n,$$

$$\tfrac{1}{2}\beta k_z^2\frac{\partial^2\psi}{\partial\eta^2} = \left[-\tfrac{2}{3}QM^2\frac{\partial^2}{\partial\eta^2}(1-3\delta n)\right].$$

The soliton solution (17) is identical to the stationary solution of the KdV equation, which has width

$$\Delta' = (\tfrac{1}{2}B_1\Delta_1)^{-1/2}.$$

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6. Discussion

In the low-β plasma under consideration, both compressive and rarefactive Alfvén solitons with interesting properties have been found to exist for $A = 2QM^2/\beta k_z^2 <$ or > 1, depending on β, M and k_z. The amplitude of the compressive ($N > 1$) Alfvén solitons uniformly decreases as $M(< k_z)$ increases for $A < 1$ and a fixed value of $\beta = 0.005$ (Fig. 1). Figure 1 also reveals that smaller-amplitude compressive Alfvén solitons are faster than high-amplitude Alfvén solitons. On the other hand, the amplitude of the corresponding rarefactive ($N < 1$) Alfvén solitons increase very slowly (lower curves of Fig. 1) with M. In contrast, the amplitude of compressive Alfvén solitons for $A > 1$ and fixed $\beta = 0.005$ shows a rapid increase (Fig. 2) with $M\,(> k_z)$ in all directions for $k_z = 0.2$, 0.3 and 0.4. In this case, the corresponding lower limits on M allowing compressive Alfvén solitons increase ($M > k_z$) as k_z increases, i.e. as the direction of propagation approaches the direction of the magnetic field. Numerical investigations show that there exist rarefactive ($N < 1$) Alfvén solitons of very small amplitude and negligible width Δ for the same value of $\beta = 0.005$ and $A > 1$. Figure 3 demonstrates the uniform increase in amplitude of compressive Alfvén solitons with k_z for both values of $\beta = 0.05$ (a) and 0.005 (b) and fixed M. It is found that the region of existence of compressive Alfvén solitons departs from the direction of the magnetic field as the speed of the soliton decreases ($M < k_z$). The amplitudes are found to be slightly higher for $\beta = 0.05$ than for $\beta = 0.005$. Figure 4 shows that the growth of the width Δ with M remains almost constant in the lower range of M for different values of k_z and $A < 1$. Finally, consideration of the electron pressure gradient in the plasma is found to considerably suppress the speed ($M < k_z$) of both types of Alfvén solitons compared with those (which are almost everywhere supersonic) established by Kalita and Kalita (1986), and here again the region of existence is shifted away from the direction of the magnetic field.

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References

Akerstedt, O. 1990 *J. Plasma Phys.* **44**, 507.
Buti, B. 1991 *Geophys. Res. Lett.* **18**, 709.
Buti, B. 1992 *J. Plasma Phys.* **47**, 39.
Hasegawa, A. & Mima, K. 1976 *Phys. Rev. Lett.* **37**, 690.
Kadomtsev, B. B. 1965 *Plasma Turbulence*, p. 82. Academic Press, New York.
Kalita, M. K. & Kalita, B. C. 1986 *J. Plasma Phys.* **35**, 267.
Kumar, N. & Srivastava, K. M. 1991 *J. Plasma Phys.* **45**, 89.
Lashmore-Devies, C. N. & May, R. M. 1976 *Phys. Fluids* **15**, 1616.
Montgomery, D., Liu, C. S. & Vahala, G. 1972 *Phys. Fluids* **15**, 815.
Okuda, H. & Dawson, J. M. 1973 *Phys. Fluids* **16**, 408.
Shukla, P. K., Rahman, H. U. & Sarma, R. P. 1982 *J. Plasma Phys.* **28**, 125.
Stefant, R. A. 1978 *Phys. Fluids* **13**, 440.

https://doi.org/10.1017/S0022377896004849 Published online by Cambridge University Press

Stix, T. H. 1981 *Heating in Toroidal Plasmas. Proceedings of 2nd Joint Grenoble–Varenna International Symposium, 980*, Vol. 2, p. 631. Commission of the European Communities, Brussels.

Taylor, J. B. & McNamara, B. 1970 *Phys. Fluids* **14**, 492.

Yu, M. Y. & Shukla, P. K. 1978 *Phys. Fluids* **21**, 1457.

https://doi.org/10.1017/S0022377896004849 Published online by Cambridge University Press