Math. Proc. Camb. Phil. Soc. (2009), **146**, 435 © 2008 *Cambridge Philosophical Society*
doi:10.1017/S0305004108001709 *Printed in the United Kingdom*
First published online 18 December 2008

Harmonic G-structures

BY J. C. GONZÁLEZ–DÁVILA AND F. MARTÍN CABRERA
Department of Fundamental Mathematics, University of La Laguna, 38200 *La Laguna, Tenerife, Spain.*
e-mail: jcgonza@ull.es, fmartin@ull.es

(*Received* 12 *December* 2007*; Revised* 28 *April* 2008)

Abstract

For closed and connected subgroups G of $SO(n)$, we study the energy functional on the space of G-structures of a (compact) Riemannian manifold $(M, \langle\cdot,\cdot\rangle)$, where G-structures are considered as sections of the quotient bundle $\mathcal{SO}(M)/G$. We deduce the corresponding first and second variation formulae and the characterising conditions for critical points by means of tools closely related to the study of G-structures. In this direction, we show the rôle in the energy functional played by the intrinsic torsion of the G-structure. Moreover, we analyse the particular case $G = U(n)$ for $2n$-dimensional manifolds. This leads to the study of harmonic almost Hermitian manifolds and harmonic maps from M into $\mathcal{SO}(M)/U(n)$.

1. *Introduction*

The energy of a map between Riemannian manifolds is a functional which has been widely studied [**3**, **4**, **22**]. Critical points for the energy functional are called *harmonic maps* and have been characterised by Eells and Sampson [**5**] as maps with vanishing *tension field*.

For a Riemannian manifold $(M, \langle\cdot,\cdot\rangle)$, we denote by $(T_1M, \langle\cdot,\cdot\rangle^S)$ its unit tangent bundle equipped with the Sasaki metric $\langle\cdot,\cdot\rangle^S$ (see [**17**]). Looking at unit vectors fields as maps $M \to T_1M$, if M is compact and oriented, one can consider the energy functional as defined on the set $\mathfrak{X}_1(M)$ of unit vector fields. Critical points for this functional give rise to the notion of *harmonic unit vector field*. The condition characterising harmonic unit vector fields has been obtained by Wiegmink [**25**] (see also Wood [**27**]). This has also been extended in a natural way to sections of sphere bundles (see [**9**, **19**]) and to oriented distributions, considered as sections of the corresponding Grassmann bundle [**8**].

In [**28**], for principal G-bundles $Q \to M$ over a Riemannian manifold $(M, \langle\cdot,\cdot\rangle)$, Wood considers global sections $\sigma: M \to Q/H$ of the quotient bundle $\pi: Q/H \to M$, where H is a closed subgroup of G such that G/H is reductive. Note that such global sections are in one-to-one correspondence with the H-reductions of the G-bundle $Q \to M$. Likewise, a connection on $Q \to M$ and a G-invariant metric on G/H are fixed. Thus, Q/H can be equipped in a natural way with a metric $\langle\cdot,\cdot\rangle_{Q/H}$, defined by using the metrics on M and G/H. For such a metric on Q/H, the submersion $\pi: Q/H \to M$ is Riemannian and has totally geodesic fibres. In such conditions, harmonic sections are characterised as those with vanishing vertical tension field. This situation arises when the Riemannian manifold M is

https://doi.org/10.1017/S0305004108001709 Published online by Cambridge University Press

equipped with some additional geometric structure, viewed as reduction of the structure group of the orthonormal frame bundle.

In this paper, we consider the particular situation for G-structures defined on an oriented Riemannian n-manifold $(M, \langle\cdot,\cdot\rangle)$, where G is a closed and connected subgroup of $SO(n)$. The manifold M is said to be equipped with a G-structure if its oriented orthogonal frame bundle $\mathcal{SO}(M)$ admits a reduction $\mathcal{G}(M)$ to the subgroup G. Moreover, if $\mathcal{SO}(M)/G = \mathcal{SO}(M) \times_G SO(n)/G$ is the quotient bundle under the action of G on $\mathcal{SO}(M)$, the existence of a G-structure on M is equivalent to the existence of a global section $\sigma: M \to \mathcal{SO}(M)/G$. In this paper we analyse the energy functional defined on the space of sections $\Gamma^\infty(\mathcal{SO}(M)/G)$ of the quotient bundle. We show the central rôle played by ξ^G, intrinsic torsion of the G-structure, in the energy functional (Theorem 3·3). Furthermore, the first variation formula is deduced (Theorem 3·6). Then, we show several equivalent characterising conditions of critical points for the energy functional on the space of G-structures defined on $(M, \langle\cdot,\cdot\rangle)$ (Theorem 3·7). This gives rise to the notion of *harmonic G-structure* for general Riemannian manifolds, not necessarily compact and oriented. It is worthwhile to note that harmonic G-structures are not necessarily critical for the energy functional on all maps from $(M, \langle\cdot,\cdot\rangle)$ to $(\mathcal{SO}(M)/G, \langle\cdot,\cdot\rangle_{\mathcal{SO}(M)/G})$. They are harmonic maps when the corresponding harmonic G-structures satisfy a condition involving the curvature of the Riemannian manifold. Additionally, we deduce the second variation formula (Theorem 3·8).

We point out that because the intrinsic torsion of the G-structures is involved in all the results mentioned above, this makes it possible to go further in the study of relations between harmonicity and G-structures. This will be illustrated in Section 4, where we focus attention on the study of harmonic almost Hermitian structures initiated by Wood in [**26**, **28**]. Thus, we study harmonicity of almost Hermitian structures by using the tools developed in Section 3, recovering Wood's results and proving some additional ones. In Theorem 4·1, several equivalent characterising conditions for harmonic almost Hermitian structures are shown. The relation of harmonicity with Gray–Hervella's classes of almost Hermitian structures is studied in Theorem 4·5. Note that those results characterise harmonic almost Hermitian structures by means of conditions on the Riemannian curvature. Concretely, in terms of the particular Ricci tensor Ric^* determined by the almost Hermitian structure. Finally, we point out that Theorem 4·5, in some sense, generalises the results proved by Bor et al. [**2**] (see Theorem 4·9). In fact, note that the results in [**2**] are stated for conformally flat manifolds, i.e., the Weyl curvature tensor vanishes.

We then focus attention on the study of harmonicity of almost Hermitian structures as maps. Results in that area were already obtained by Wood [**28**]. We complete such results by using tools present here.

For completeness, we briefly give a detailed and self-contained explanation for nearly Kähler manifolds. Thus, we recover results already known, some of which were originally proved by Gray and others by Wood. However, we will display alternative proofs in terms of the intrinsic torsion. Additionally, a result by Kirichenco [**14**] is also shown, asserting that, for nearly Kähler manifolds, the intrinsic torsion is parallel with respect to the minimal connection.

2. *Preliminaries*

First we recall some notions relative to G-structures, where G is a subgroup of the linear group $GL(n, \mathbb{R})$. The Lie algebra of G will be denoted by $\mathfrak{g}$. An n-dimensional manifold M is

https://doi.org/10.1017/S0305004108001709 Published online by Cambridge University Press

equipped with a G-structure if its frame bundle admits a reduction $\mathcal{G}(M)$ to the subgroup G. Moreover, if $(M, \langle\cdot,\cdot\rangle)$ is an n-dimensional oriented Riemannian manifold, we can consider the principal $SO(n)$-bundle $\pi_{SO(n)}\colon \mathcal{SO}(M) \to M$ of the oriented orthonormal frames with respect to the metric $\langle\cdot,\cdot\rangle$. A G-structure on $(M, \langle\cdot,\cdot\rangle)$ is a reduction $\mathcal{G}(M) \subset \mathcal{SO}(M)$ to a subgroup G of $SO(n)$.

In what follows, we always assume that G is closed and connected. Then, the quotient space $SO(n)/G$ is a homogeneous manifold and becomes a normal homogeneous Riemannian manifold with the metric induced by the bi-invariant metric on $SO(n)$ defined by $\langle X, Y\rangle = -\operatorname{trace} XY$. This last metric coincides with the natural extension of the usual Euclidean product $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^n$ to $\mathrm{End}(\mathbb{R}^n)$. Let $\mathcal{SO}(M)/G$ be the orbit space under the action of G on $\mathcal{SO}(M)$ on the right as subgroup of $SO(n)$. Then the G-orbit map $\pi_G\colon \mathcal{SO}(M) \to \mathcal{SO}(M)/G$ is a principal G-bundle and we have $\pi_{SO(n)} = \pi \circ \pi_G$, where $\pi\colon \mathcal{SO}(M)/G \to M$ is a fibre bundle with fibre $SO(n)/G$, which is naturally isomorphic to the associated bundle $\mathcal{SO}(M) \times_{SO(n)} SO(n)/G$. The map $\sigma\colon M \to \mathcal{SO}(M)/G$ given by $\sigma(m) = \pi_G(p)$, for all $p \in \mathcal{G}(M)$ with $\pi_{SO(n)}(p) = m$, is well-defined because π_G is constant on each fiber of the reduced bundle. It is a smooth section and we have $\mathcal{G}(M) = \pi_G^{-1}(\sigma(M))$. Hence, there is a one-to-one correspondence between the total of G-structures and the manifold $\Gamma^\infty(\mathcal{SO}(M)/G)$ of all global sections of $\mathcal{SO}(M)/G$. In what follows we shall also denote by σ the G-structure determined by the section σ.

If $u_1 = (1, 0, \ldots, 0), \ldots, u_n = (0, \ldots, 0, 1)$ is the canonical orthonormal frame on $\mathbb{R}^n$, then an oriented frame $p \in \mathcal{SO}(M)$ can be viewed as an isomorphism $p\colon \mathbb{R}^n \to \mathrm{T}_{\pi_{SO(n)}(p)}M$ such that $\{p(u_1), \ldots, p(u_n)\}$ is a positive oriented basis of $T_{\pi_{SO(n)}(p)}M$. From now on, we will make reiterated use of the *musical isomorphisms* $\flat\colon \mathrm{T}M \to \mathrm{T}^*M$ and $\sharp\colon \mathrm{T}^*M \to \mathrm{T}M$, induced by the metric $\langle\cdot,\cdot\rangle$ on M, respectively defined by $X^\flat = \langle X, \cdot\rangle$ and $\langle\theta^\sharp, \cdot\rangle = \theta$.

In the presence of a G-structure determined by a section $\sigma\colon M \to \mathcal{SO}(M)/G$, a frame $p \in \mathcal{SO}(M)$ is said to be an *adapted frame* to the G-structure, if $p \in \sigma \circ \pi_{SO(n)}(p)$ or, equivalently, if $p \in \mathcal{G}(M) \subseteq \mathcal{SO}(M)$. Note also that, in a first instance, the bundle of endomorphisms $\mathrm{End}(\mathrm{T}M)$ on the fibers in the tangent bundle $\mathrm{T}M$ coincides with the associated vector bundle $\mathcal{SO}(M) \times_{SO(n)} \mathrm{End}(\mathbb{R}^n)$, where $SO(n)$ acts on $\mathrm{End}(\mathbb{R}^n)$ in the usual way $(g \cdot \varphi)(x) = g\varphi(g^{-1}x) = (\mathrm{Ad}_{SO(n)}(g)\varphi)(x)$. Thus, it is identified

$$\varphi_m = a_{ji}\, p(u_i)^\flat \otimes p(u_j) \cong [(p, a_{ji}\, u_i^\flat \otimes u_j)], \tag{2·1}$$

where $m \in M$ and $p \in \pi_{SO(n)}^{-1}(m)$ and the summation convention is used. Such a convention will be followed in the sequel. When there is a risk of confusion the sum will be written in detail.

In this context, we also have a reduced subbundle $\mathcal{G}(M)$. So that we can do the identification $\mathrm{End}(\mathrm{T}M) = \mathcal{G}(M) \times_G \mathrm{End}(\mathbb{R}^n)$ because any φ_m can be identified with an element in $\mathcal{G}(M) \times_G \mathrm{End}(\mathbb{R}^n)$ as in (2·1), but in this case $p \in \sigma(m)$ must hold.

Now we restrict our attention to the subbundle $\mathfrak{so}(M)$ of $\mathrm{End}(\mathrm{T}M)$ of skew-symmetric endomorphisms φ_m, for all $m \in M$, i.e., $\langle\varphi_m X, Y\rangle = -\langle\varphi_m Y, X\rangle$. Note that this subbundle $\mathfrak{so}(M)$ is expressed as $\mathfrak{so}(M) = \mathcal{SO}(M) \times_{SO(n)} \mathfrak{so}(n) = \mathcal{G}(M) \times_G \mathfrak{so}(n)$. The corresponding matrices (a_{ij}) for $\mathfrak{so}(M)$, given by (2·1), are such that $a_{ij} = -a_{ji}$. Furthermore, because $\mathfrak{so}(n)$ is decomposed into the G-modules $\mathfrak{g}$ and the orthogonal complement $\mathfrak{m}$ on $\mathfrak{so}(n)$ with respect to the inner product $\langle\cdot,\cdot\rangle$, the bundle $\mathfrak{so}(M)$ is also decomposed into $\mathfrak{so}(M) = \mathfrak{g}_\sigma(M) \oplus \mathfrak{m}_\sigma(M)$, where $\mathfrak{g}_\sigma(M) = \mathcal{G}(M) \times_G \mathfrak{g}$ and $\mathfrak{m}_\sigma(M) = \mathcal{G}(M) \times_G \mathfrak{m}$. The matrices (a_{ij}) in (2·1) corresponding to $\mathfrak{g}_\sigma(M)$ and $\mathfrak{m}_\sigma(M)$ are such that they are in $\mathfrak{g}$ and

https://doi.org/10.1017/S0305004108001709 Published online by Cambridge University Press

$\mathfrak{m}$, respectively. The subindex σ in $\mathfrak{g}_\sigma(M)$ and $\mathfrak{m}_\sigma(M)$ points out that these bundles are determined by the G-structure σ. From now on, we will merely write $\mathfrak{g}_\sigma$ and $\mathfrak{m}_\sigma$.

Under the conditions fixed above, if M is equipped with a G-structure, then there exists a G-connection $\widetilde{\nabla}$ defined on M. Doing the difference $\widetilde{\xi}_X = \widetilde{\nabla}_X - \nabla_X$, where ∇ is the Levi–Civita connection of $\langle\cdot,\cdot\rangle$, a tensor $\widetilde{\xi}_X \in \mathfrak{so}(M)$ is obtained. Because ∇ is torsion-free, $\widetilde{\xi}$ is an alternative way of giving the torsion of $\widetilde{\nabla}$. In fact, if $\widetilde{T}$ is the usual torsion tensor of $\widetilde{\nabla}$ given by $\widetilde{T}(X,Y) = \widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X - [X,Y]$, then it is satisfied

$$\begin{aligned} &\widetilde{T}(X,Y) = \widetilde{\xi}_X Y - \widetilde{\xi}_Y X, \\ &2\langle \widetilde{\xi}_X Y, Z\rangle = \langle \widetilde{T}(X,Y), Z\rangle - \langle \widetilde{T}(Y,Z), X\rangle + \langle \widetilde{T}(Z,X), Y\rangle. \end{aligned} \tag{2·2}$$

Decomposing $\widetilde{\xi}_X = (\widetilde{\xi}_X)_{\mathfrak{g}_\sigma} + (\widetilde{\xi}_X)_{\mathfrak{m}_\sigma}$, $(\widetilde{\xi}_X)_{\mathfrak{g}_\sigma} \in \mathfrak{g}_\sigma$ and $(\widetilde{\xi}_X)_{\mathfrak{m}_\sigma} \in \mathfrak{m}_\sigma$, a new G-connection ∇^G, defined by $\nabla^G_X = \widetilde{\nabla}_X - (\widetilde{\xi}_X)_{\mathfrak{g}_\sigma}$, can be considered. Because the difference between two G-connections must be in $\mathfrak{g}_\sigma$, ∇^G is the unique G-connection on M such that its torsion satisfies the condition $\xi^G_X = (\widetilde{\xi}_X)_{\mathfrak{m}_\sigma} = \nabla^G_X - \nabla_X \in \mathfrak{m}_\sigma$. ∇^G is called the *minimal connection* and ξ^G is referred to as the *intrinsic torsion* of the G-structure σ [**6**, **18**]. A natural way of classifying G-structures arises by decomposing the space $\mathrm{T}^*M \otimes \mathfrak{m}_\sigma$ of possible intrinsic torsions into irreducible G-modules. If $\xi^G = 0$, the G-structure is usually referred to as an *integrable* G-structure. In such a case, the Riemannian holonomy group of M is contained in G.

Associated to the metric connections ∇ and ∇^G there are connection one-forms ω and ω^G defined on $\mathcal{SO}(M)$ and $\mathcal{G}(M)$ with values in $\mathfrak{so}(n)$ and $\mathfrak{g}$, respectively. Note that the projection $\mathcal{G}(M) \to M$ of the reduced bundle is $\pi_{SO(n)}$ restricted to $\mathcal{G}(M)$. Therefore, if $\wp = \{e_1, \dots e_n\}\colon U \to \mathcal{G}(M)$ is a local frame field adapted to the G-structure, then

$$\left\langle \xi^G_X e_i, e_j \right\rangle_m = \left\langle \nabla^G_X e_i, e_j \right\rangle_m - \langle \nabla_X e_i, e_j\rangle_m = \omega^G_{\wp(m)}(\wp_{*m} X)_{ji} - \omega_{\wp(m)}(\wp_{*m} X)_{ji}.$$

Since the matrices $(\langle \xi^G_X e_i, e_j\rangle_m) \in \mathfrak{m}$ and $(\omega^G_{\wp(m)}(\wp_{*m} X)_{ji}) \in \mathfrak{g}$, the following identities for matrices are obtained

$$((\omega_{\wp(m)}(\wp_{*m} X)_{ji})_{\mathfrak{g}} = \left(\omega^G_{\wp(m)}(\wp_{*m} X)_{ji}\right), \qquad (\omega_{\wp(m)}(\wp_{*m} X)_{ji})_{\mathfrak{m}} = -\left(\left\langle \xi^G_X e_i, e_j \right\rangle_m\right).$$

Therefore, the intrinsic torsion is expressed as

$$\xi^G_X = -(\omega(\wp_* X)_{ji})_{\mathfrak{m}}\, e_i^\flat \otimes e_j, \tag{2·3}$$

where $\wp = \{e_1, \dots e_n\}$ is a local frame field adapted to the G-structure.

Finally, we need to point out that, along the present paper, we will consider the natural extension of the metric $\langle\cdot,\cdot\rangle$ to (r,s)-tensor fields on M. Such an extension is defined by

$$\langle \Psi, \Phi\rangle = \Psi^{i_1\dots i_r}_{j_1\dots j_s} \Phi^{i_1\dots i_r}_{j_1\dots j_s}, \tag{2·4}$$

where $\Psi^{i_1\dots i_r}_{j_1\dots j_s}$ and $\Phi^{i_1\dots i_r}_{j_1\dots j_s}$ are the components of Ψ and Φ with respect to an orthonormal local frame.

3. *Characterising harmonic G-structures via the intrinsic torsion*

Now we consider the bundle $\pi_G\colon \mathcal{SO}(M) \to \mathcal{SO}(M)/G$. Because we have $\mathrm{T}\mathcal{SO}(M) = \ker \pi_{SO(n)*} \oplus \ker \omega$, the tangent bundle of $\mathcal{SO}(M)/G$ is decomposed into $\mathrm{T}\mathcal{SO}(M)/G = \mathcal{V} \oplus \mathcal{H}$, where $\mathcal{V} = \pi_{G*}(\ker \pi_{SO(n)*})$ and $\mathcal{H} = \pi_{G*}(\ker \omega)$. Then the *vertical* and *horizontal* distributions $\mathcal{V}$ and $\mathcal{H}$ are such that $\pi_* \mathcal{V} = 0$ and $\pi_* \mathcal{H} = \mathrm{T}M$.

https://doi.org/10.1017/S0305004108001709 Published online by Cambridge University Press

Moreover, we consider the pullback or induced bundle $\pi^*\mathfrak{so}(M)$ of $\mathfrak{so}(M)$ by π, that is, the vector bundle over $\mathcal{SO}(M)/G$ consisting of those pairs (pG, φ_m), where $\pi(pG) = m$ and $\varphi_m \in \mathfrak{so}(M)_m$. Alternatively, $\pi^*\mathfrak{so}(M)$ is also described as the associated bundle $\mathcal{SO}(M) \times_G \mathfrak{so}(n)$ to π_G. Then $\pi^*\mathfrak{so}(M)$ is decomposed into $\pi^*\mathfrak{so}(M) = \mathfrak{g}_{\mathcal{SO}(M)} \oplus \mathfrak{m}_{\mathcal{SO}(M)}$, where $\mathfrak{g}_{\mathcal{SO}(M)} = \mathcal{SO}(M) \times_G \mathfrak{g}$ and $\mathfrak{m}_{\mathcal{SO}(M)} = \mathcal{SO}(M) \times_G \mathfrak{m}$. A metric on each fiber of $\pi^*\mathfrak{so}(M)$ is defined by

$$\langle (pG, \varphi_m), (pG, \psi_m) \rangle = \langle \varphi_m, \psi_m \rangle,$$

where $\langle \cdot, \cdot \rangle$ in the right side is the extension to $(1, 1)$-tensors of the metric on M given by (2·4). With respect to this metric, the decomposition $\pi^*\mathfrak{so}(M) = \mathfrak{g}_{\mathcal{SO}(M)} \oplus \mathfrak{m}_{\mathcal{SO}(M)}$ is orthogonal.

Additionally, we have a covariant derivative ∇ on $\pi^*\mathfrak{so}(M)$ induced by the Levi-Civita connection associated to the metric $\langle \cdot, \cdot \rangle$ on M and given by

$$(\nabla_A \tilde{\varphi})_{pG} = \left(pG, \frac{\nabla}{ds}_{|s=0} \mathrm{pr}_2^{\pi} \tilde{\varphi}_{\tilde{\gamma}(s)} \right), \tag{3·1}$$

for all $A \in \mathfrak{X}(\mathcal{SO}(M)/G) = \Gamma^{\infty}(\mathrm{T}\mathcal{SO}(M)/G)$ and $\tilde{\varphi} \in \Gamma^{\infty}(\pi^*\mathfrak{so}(M))$, where $s \to \tilde{\gamma}(s)$ is a curve in $\mathcal{SO}(M)/G$ such that $\tilde{\gamma}(0) = pG$ and $\tilde{\gamma}'(0) = A_{pG}$ and pr_2^{π} is the projection $\mathrm{pr}_2^{\pi}(pG, \varphi_m) = \varphi_m$ on $\mathfrak{so}(M)$. Note that, in the right side, the covariant derivative is along the curve $\gamma(s) = \pi \circ \tilde{\gamma}(s)$.

There is a canonical isomorphism between $\mathcal{V}$ and the bundle $\mathfrak{m}_{\mathcal{SO}(M)}$. For describing such an isomorphism, let us firstly say that the elements in $\mathfrak{m}_{\mathcal{SO}(M)}$ can be seen as pairs (pG, φ_m) such that if φ_m is expressed with respect to p as in (2·1), then $(a_{ji}) \in \mathfrak{m}$. Now, let us describe the mentioned canonical isomorphism $\phi_{|\mathcal{V}_{pG}}: \mathcal{V}_{pG} \to (\mathfrak{m}_{\mathcal{SO}(M)})_{pG}$. For all $a \in \mathfrak{m}$, we have the fundamental vector field a^* on $\mathcal{SO}(M)$ given by

$$a_p^* = \frac{d}{dt}_{|t=0} p \cdot \exp ta \in \ker \pi_{SO(n)*p} \subseteq \mathrm{T}_p\mathcal{SO}(M).$$

Any vector in $\mathcal{V}_{pG}$ is given by $\pi_{G*p}(a_p^*)$, for some $a = (a_{ji}) \in \mathfrak{m}$. The isomorphism $\phi_{|\mathcal{V}_{pG}}$ is defined by

$$\phi_{|\mathcal{V}_{pG}}(\pi_{G*p}(a_p^*)) = (pG, a_{ji}\, p(u_i)^{\flat} \otimes p(u_j)).$$

Next the map $\phi_{|\mathcal{V}}: \mathcal{V} \to \mathfrak{m}_{\mathcal{SO}(M)}$ is extended to $\phi: \mathrm{T}\mathcal{SO}(M)/G \to \mathfrak{m}_{\mathcal{SO}(M)}$ by saying that $\phi(A) = 0$, for all $A \in \mathcal{H}$, and $\phi(V) = \phi_{|\mathcal{V}}(V)$, for all $V \in \mathcal{V}$. This is used to define a metric $\langle \cdot, \cdot \rangle_{\mathcal{SO}(M)/G}$ on $\mathcal{SO}(M)/G$ by

$$\langle A, B \rangle_{\mathcal{SO}(M)/G} = \langle \pi_* A, \pi_* B \rangle + \langle \phi(A), \phi(B) \rangle. \tag{3·2}$$

For this metric, the projection $\pi: \mathcal{SO}(M)/G \to M$ is a Riemannian submersion with totally geodesic fibres (see [**24**] and [**1**, page 249]). That is, if $\mathsf{v}: \mathrm{T}\mathcal{SO}(M)/G \to \mathcal{V}$ and $\mathsf{h}: \mathrm{T}\mathcal{SO}(M)/G \to \mathcal{H}$ are respectively the vertical and horizontal projections and ∇^q is the Levi–Civita connection of $\langle \cdot, \cdot \rangle_{\mathcal{SO}(M)/G}$, then $\nabla^q_V W = \mathsf{v}\nabla^q_V W$ and $\nabla^q_V H = \mathsf{h}\nabla^q_V H$, for all $H \in \Gamma^{\infty}(\mathcal{H})$ and $V, W \in \Gamma^{\infty}(\mathcal{V})$.

Because $\mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{m}$ is a reductive decomposition, that is, it satisfies $Ad_{SO(n)}(G)\mathfrak{m} \subseteq \mathfrak{m}$, the component $\omega_{\mathfrak{g}}$ in $\mathfrak{g}$ of the connection-form ω is a connection-form for the bundle $\pi_G: \mathcal{SO}(M) \to \mathcal{SO}(M)/G$ which is referred to as *canonical connection*. This connection provides a covariant derivative ∇^c on $\mathfrak{m}_{\mathcal{SO}(M)}$, with respect to which the fibre metric is holonomy invariant. The Levi–Civita connection ∇^q is related with ∇^c on $\mathfrak{m}_{\mathcal{SO}(M)}$ via the projection of the $\mathfrak{m}$-component of the curvature form Ω of the Levi–Civita connection ∇ of

https://doi.org/10.1017/S0305004108001709 Published online by Cambridge University Press

M. Thus, the two-form Φ on $\mathcal{SO}(M)/G$ is considered, with values in $\mathfrak{m}_{\mathcal{SO}(M)}$ and defined by

$$\Phi(A, B) = \phi\pi_{G*}\Omega(\tilde{A}, \tilde{B})^*_{\mathfrak{m}} = \phi\pi_{G*}d\omega(\tilde{A}, \tilde{B})^*_{\mathfrak{m}} + \phi\pi_{G*}[\omega(\tilde{A}), \omega(\tilde{B})]^*_{\mathfrak{m}},$$

where $\tilde{A}, \tilde{B} \in T\mathcal{SO}(M)$ such that $\pi_{G*}\tilde{A} = A$, $\pi_{G*}\tilde{B} = B$. Therefore, if we consider on $\mathcal{SO}(M)/G$ the vertical vectors U and V and the horizontal vectors H and K, then

$$\Phi(U, V) = 0, \quad \Phi(U, H) = 0, \quad \Phi(H, K) = \phi\pi_{G*}\Omega(\tilde{H}, \tilde{K})^*_{\mathfrak{m}} = \phi\pi_{G*}d\omega(\tilde{H}, \tilde{K})^*_{\mathfrak{m}}.$$

Next, we recall some useful facts proved in [**28**, corollary 2·4 and proposition 2·7].

LEMMA 3·1 ([**28**]). *We have:*
(i) $\nabla^c_A\tilde{V} = \nabla_A\tilde{V} - [\phi A, \tilde{V}]$;
(ii) $\phi(\nabla^q_A B) - \nabla^c_A\phi B = \frac{1}{2}\{[\phi A, \phi B]_{\mathfrak{m}} - \Phi(A, B)\}$;
for all $A, B \in \mathfrak{X}(\mathcal{SO}(M)/G)$ *and* $\tilde{V} \in \Gamma^\infty(\mathfrak{m}_{\mathcal{SO}(M)})$.

From this we obtain

$$\phi\big(\nabla^q_A V\big) = \nabla^c_A\phi\, V + \frac{1}{2}[\phi A, \phi V]_{\mathfrak{m}} = \nabla_A\phi\, V - \frac{1}{2}[\phi A, \phi V]_{\mathfrak{m}} - [\phi A, \phi V]_{\mathfrak{g}}, \tag{3·3}$$

for all $A \in \mathfrak{X}(\mathcal{SO}(M)/G)$ and $V \in \Gamma^\infty(\mathcal{V})$.

Remark 3·2. (1) The Lie bracket on $\pi^*\mathfrak{so}(M)$ is defined by

$$[(pG, \varphi_m), (pG, \psi_m)] = (pG, [\varphi_m, \psi_m]) = (pG, \varphi_m \circ \psi_m - \psi_m \circ \varphi_m).$$

(2) Given a G-structure $\sigma\colon M \to \mathcal{SO}(M)/G$, the bundle $\sigma^*\pi^*\mathfrak{so}(M)$ is identified with $\mathfrak{so}(M)$ by the bijection map $\mathrm{pr}^\pi_2 \circ \mathrm{pr}^\sigma_2\colon (m, (\sigma(m), \varphi_m)) \mapsto \varphi_m$ and likewise, $\sigma^*\mathfrak{g}_{\mathcal{SO}(M)} \cong \mathfrak{g}_\sigma$ and $\sigma^*\mathfrak{m}_{\mathcal{SO}(M)} \cong \mathfrak{m}_\sigma$. With respect to sections, if $\varphi \in \Gamma^\infty(\mathfrak{so}(M))$ then $pG \to (pG, \varphi_{\pi(pG)})$ belongs to $\Gamma^\infty(\pi^*\mathfrak{so}(M))$ and conversely, if $\tilde{\varphi} \in \Gamma^\infty(\mathfrak{m}_{\mathcal{SO}(M)})$ (respectively, $\tilde{\varphi} \in \Gamma^\infty(\mathfrak{g}_{\mathcal{SO}(M)})$), then $m \to \mathrm{pr}^\pi_2\tilde{\varphi}_{\sigma(m)}$ is in $\Gamma^\infty(\mathfrak{m}_\sigma)$ (respectively, in $\Gamma^\infty(\mathfrak{g}_\sigma)$).

Now, we consider the set of all possible G-structures on a closed and oriented Riemannian manifold $(M, \langle\cdot,\cdot\rangle)$. As has been already mentioned, this set is identified with the manifold $\Gamma^\infty(\mathcal{SO}(M)/G)$ of all global sections $\sigma\colon M \to \mathcal{SO}(M)/G$. Then the *energy* of the G-structure is defined as the energy of the corresponding section σ, given by the integral

$$\mathcal{E}(\sigma) = \frac{1}{2}\int_M \|\sigma_*\|^2\, dv, \tag{3·4}$$

where $\|\sigma_*\|^2$ is the norm of the differential σ_* of σ with respect to the metrics $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_{\mathcal{SO}(M)/G}$, and dv denotes the volume form on $(M, \langle\cdot,\cdot\rangle)$. On the domain of a local orthonormal frame field $\{e_1, \ldots, e_n\}$ on M, $\|\sigma_*\|^2$ can be expressed as $\|\sigma_*\|^2 = \langle\sigma_* e_i, \sigma_* e_i\rangle_{\mathcal{SO}(M)/G}$. Furthermore, from (3·4) and using (3·2), it is obtained that the energy $\mathcal{E}(\sigma)$ of σ is given by

$$\mathcal{E}(\sigma) = \frac{n}{2}\mathrm{Vol}(M) + \frac{1}{2}\int_M \|\phi\,\sigma_*\|^2\, dv.$$

The relevant part of this formula, $B(\sigma) = \frac{1}{2}\int_M \|\phi\,\sigma_*\|^2 dv$, will be called the *total bending* of the G-structure. Because we will show that $\phi\,\sigma_* = -\xi^G$, the total bending provides a measure of how the G-structure σ fails to be integrable. Here, we are doing the identification $\sigma^*\mathfrak{m}_{\mathcal{SO}(M)} \cong \mathfrak{m}_\sigma$ pointed out in Remark 3·2.

https://doi.org/10.1017/S0305004108001709 Published online by Cambridge University Press

THEOREM 3·3. *If σ is a global section of $\mathcal{SO}(M)/G$ then $\phi\,\sigma_* = -\xi^G$, where ξ^G is the intrinsic torsion of the G-structure determined by σ, and the total bending of the G-structure σ is given by*

$$B(\sigma) = \frac{1}{2}\int_M \|\xi^G\|^2\, dv.$$

Proof. For $X \in \mathrm{T}_m M$, we will compute $\phi\,\sigma_* X$. If $\wp = \{e_1, \ldots, e_n\}: U \to \mathcal{G}(M)$ is a local frame field adapted to the G-structure σ with $m \in U$, then $\pi_{SO(n)} \circ \wp = Id_U$ and taking $\pi_{G|\mathcal{G}(M)} = \sigma \circ \pi_{SO(n)}$ into account, we have $\sigma_* = \pi_{G*} \circ \wp_*$. Therefore, we get

$$\begin{aligned}\phi(\sigma_* X) &= \phi(\mathsf{v}(\sigma_* X)) = \phi(\mathsf{v}(\pi_{G*}\wp_* X)) = \phi\big(\pi_{G*}(\omega(\wp_* X)_{ji} u_i^\flat \otimes u_j)^*\big)\\ &= \phi\big((\pi_{G*}(\omega(\wp_* X)_{ji})_{\mathfrak{m}} u_i^\flat \otimes u_j)^*\big) = \big(\sigma(m), (\omega(\wp_* X)_{ji})_{\mathfrak{m}} e_i^\flat \otimes e_j\big).\end{aligned}$$

Thus, from (2·3), we have

$$\phi\sigma_* X = \big(\sigma(m), -\xi^G_X\big)$$

and

$$\|\phi\sigma_*\|^2 = \langle \phi\,\sigma_*(e_i)), \phi\,\sigma_*(e_i)\rangle = \big\langle \big(\sigma, \xi^G_{e_i}\big), \big(\sigma, \xi^G_{e_i}\big)\big\rangle = \big\langle \xi^G_{e_i}, \xi^G_{e_i}\big\rangle = \|\xi^G\|^2.$$

Now, the theorem follows using the above identification $\sigma^*\mathfrak{m}_{\mathcal{SO}(M)} \cong \mathfrak{m}_\sigma$.

Some immediate consequences of last Theorem, most of them already proved in [**28**], are given in the following corollary.

COROLLARY 3·4. *The following conditions are equivalent*:

(i) *$\sigma_* X$ is horizontal, for all $X \in \mathrm{T}M$;*
(ii) *σ is an integrable G-structure, i.e., $\xi^G = 0$, or ∇^G is torsion-free*;
(iii) *σ is an isometric immersion*;
(iv) *∇ can be reduced to a G-connection.*

Next, we determine the Euler–Lagrange equation or the critical point condition for the energy functional $\mathcal{E}$ on closed and oriented Riemannian manifolds. If we consider a smooth variation $\sigma_t \in \Gamma^\infty(\mathcal{SO}(M)/G)$ of $\sigma = \sigma_0$, then the corresponding *variation field* $m \to \varphi(m) = (d/dt)_{|t=0}\sigma_t(m)$ is a section of the pullback bundle $\sigma^*\mathcal{V}$ over M. Thus, the tangent space $\mathrm{T}_\sigma\Gamma^\infty(\mathcal{SO}(M)/G)$ is identified with the space $\Gamma^\infty(\sigma^*\mathcal{V})$ of global sections of $\sigma^*\mathcal{V}$ [**22**]. Because ϕ determines also an identification $\sigma^*\mathcal{V} \cong \mathfrak{m}_\sigma$ by the bijection

$$(m, \pi_{G*\sigma(m)} a^*_{\sigma(m)}) \longmapsto \varphi_m = a_{ji}\; p(u_i)^\flat \otimes p(u_j),$$

where $a = (a_{ij}) \in \mathfrak{m}$ and $p \in \mathcal{G}(M)$ with $\pi_{SO(n)}(p) = m$, we can identify the tangent space $\mathrm{T}_\sigma\Gamma^\infty(\mathcal{SO}(M)/G)$ with $\Gamma^\infty(\mathfrak{m}_\sigma)$.

In following results, we will consider the coderivative $d^*\xi^G$ of the intrinsic torsion ξ^G, which is defined by

$$(d^*\xi^G)_m(X) = -(\nabla_{e_i}\xi^G)_{e_i} X,$$

where $\{e_1, \ldots, e_n\}$ is any orthonormal frame on $m \in M$. In a first instance, $d^*\xi^G$ is a global section of $\mathfrak{so}(M) = \mathfrak{g}_\sigma \oplus \mathfrak{m}_\sigma$.

LEMMA 3·5. *The coderivative $d^*\xi^G$ is a global section of $\mathfrak{m}_\sigma$ and is given by*

$$d^*\xi^G = -\big(\nabla^G_{e_i}\xi^G\big)_{e_i} - \xi^G_{\xi^G_{e_i} e_i}. \tag{3·5}$$

https://doi.org/10.1017/S0305004108001709 Published online by Cambridge University Press

Proof. Because $\nabla^G = \nabla + \xi^G$, it follows that $d^*\xi^G = -(\nabla^G_{e_i}\xi^G)_{e_i} + (\xi^G_{e_i}\xi^G)_{e_i}$. But one can check that $(\xi^G_{e_i}\xi)_{e_i} = -\xi^G_{\xi^G_{e_i}e_i}$. Thus, (3·5) is obtained. It is obvious that $\xi^G_{\xi^G_{e_i}e_i}$ is in $\mathfrak{m}_\sigma$. Since ∇^G is a G-connection, ∇^G preserves the G-type of a tensor. Therefore, from $\xi^G_X \in \mathfrak{m}_\sigma$, it follows that $(\nabla^G_{e_i}\xi^G)_{e_i} \in \mathfrak{m}_\sigma$.

THEOREM 3·6 (The first variation formula). *If $(M, \langle\cdot,\cdot\rangle)$ is a closed and oriented Riemannian manifold and σ a global section of $\mathcal{SO}(M)/G$, then, for all $\varphi \in \Gamma^\infty(\mathfrak{m}_\sigma) \cong \mathrm{T}_\sigma\Gamma^\infty(\mathcal{SO}(M)/G)$, we have*

$$d\mathcal{E}_\sigma(\varphi) = -\int_M \langle \xi^G, \nabla\varphi\rangle\, dv = -\int_M \langle d^*\xi^G, \varphi\rangle\, dv,$$

where ξ^G is the intrinsic torsion of σ.

Proof. We will also denote by φ the section in $\Gamma^\infty(\sigma^*\mathcal{V})$ which is identified with $\varphi \in \Gamma^\infty(\mathfrak{m}_\sigma)$, i.e., $\mathrm{pr}_2^\pi\phi\varphi = \varphi$. If $I_{\varepsilon_1} =]-\varepsilon_1, \varepsilon_1[\to \Gamma^\infty(\mathcal{SO}(M)/G)$, $t \to \sigma_t$, is a curve such that $\sigma_0 = \sigma$, and $(\sigma_t)'(0) = \varphi$, then we obtain

$$\begin{aligned} d\mathcal{E}_\sigma(\varphi) = \frac{d}{dt}_{|t=0}\mathcal{E}(\sigma_t) &= \frac{1}{2}\int_M \frac{d}{dt}_{|t=0}\langle \mathsf{v}\sigma_{t*}, \mathsf{v}\sigma_{t*}\rangle_{\mathcal{SO}(M)/G}\, dv \\ &= \int_M \left\langle \mathsf{v}\sigma_*, \frac{\nabla^q}{dt}_{|t=0}\mathsf{v}\sigma_{t*}\right\rangle_{\mathcal{SO}(M)/G} dv. \end{aligned}$$

Now, since π has totally geodesic fibres and the tangent vector $(\sigma_t(m))'_{t=0} = \varphi(m)$ of the curve $t \to \sigma_t(m)$ is vertical, it follows

$$\left\langle \mathsf{v}\sigma_*, \frac{\nabla^q}{dt}_{|t=0}\mathsf{v}\sigma_{t*}\right\rangle_{\mathcal{SO}(M)/G} = \left\langle \mathsf{v}\sigma_*, \frac{\nabla^q}{dt}_{|t=0}\sigma_{t*}\right\rangle_{\mathcal{SO}(M)/G}.$$

Next, if $I_{\varepsilon_2} =]-\varepsilon_2, \varepsilon_2[\to M$, $s \to \gamma(s)$, is a curve such that $\gamma(0) = m$ and $\gamma'(0) = X$ and we consider the smooth map $I_{\varepsilon_1} \times I_{\varepsilon_2} \to \mathcal{SO}(M)/G$ defined by $(t, s) \to \sigma_t(\gamma(s))$, then we obtain

$$\frac{\nabla^q}{dt}_{|t=0}(\sigma_{t*m}X) = \frac{\nabla^q}{\partial t}_{|t=0}\frac{\partial}{\partial s}_{|s=0}(\sigma_t(\gamma(s))) = \frac{\nabla^q}{\partial s}_{|s=0}\frac{\partial}{\partial t}_{|t=0}(\sigma_t(\gamma(s))) = \frac{\nabla^q}{ds}_{|s=0}\varphi(\gamma(s)).$$

Therefore,

$$\begin{aligned} \left\langle \mathsf{v}\sigma_*X, \frac{\nabla^q}{dt}_{|t=0}\sigma_{t*}X\right\rangle_{\mathcal{SO}(M)/G} &= \left\langle \mathsf{v}\sigma_*X, \frac{\nabla^q}{ds}_{|s=0}\varphi(\gamma(s))\right\rangle_{\mathcal{SO}(M)/G} \\ &= \left\langle \phi\sigma_*X, \phi\frac{\nabla^q}{ds}_{|s=0}\varphi(\gamma(s))\right\rangle. \end{aligned}$$

Hence, using (3·3), we get

$$\begin{aligned} \left\langle \mathsf{v}\sigma_*X, \frac{\nabla^q}{dt}_{|t=0}\sigma_{t*}X\right\rangle_{\mathcal{SO}(M)/G} &= \left\langle \phi\sigma_*X, \frac{\nabla}{ds}_{|s=0}\phi\varphi(\gamma(s)) - \frac{1}{2}[\phi\sigma_*X, \phi\varphi]_{\mathfrak{m}}\right\rangle \\ &= \left\langle \phi\sigma_*X, \frac{\nabla}{ds}_{|s=0}\phi\varphi(\gamma(s))\right\rangle, \end{aligned}$$

where we have used that $SO(n)/G$ is a normal homogeneous Riemannian manifold and $(\nabla/ds)_{|s=0}\phi\varphi(\gamma(s))$ means the covariant derivative along the curve $s \to \sigma(\gamma(s))$. Finally,

https://doi.org/10.1017/S0305004108001709 Published online by Cambridge University Press

since by (3·1) we have

$$\frac{\nabla}{ds}_{|s=0}\phi\varphi(\gamma(s)) = \left(\sigma(\gamma(0)), \frac{\nabla}{ds}_{|s=0}\mathrm{pr}_2^{\pi}\phi\varphi(\gamma(s))\right),$$

then we obtain

$$\left\langle \mathrm{v}\,\sigma_* X, \frac{\nabla^q}{dt}_{|t=0}\sigma_{t*}X\right\rangle_{\mathcal{SO}(M)/G} = \left\langle \mathrm{pr}_2^{\pi}\phi\sigma_* X, \frac{\nabla}{ds}_{|s=0}\mathrm{pr}_2^{\pi}\phi\varphi(\gamma(s))\right\rangle$$
$$= -\langle \xi_X^G, \nabla_X \mathrm{pr}_2^{\pi}\phi\varphi\rangle.$$

From this, and taking into account that $\mathrm{pr}_2^{\pi}\phi\varphi = \varphi$, we will get the required identity

$$d\mathcal{E}_\sigma(\varphi) = -\int_M \langle \xi^G, \nabla\varphi\rangle\, dv. \tag{3·6}$$

On the other hand, we have the equality

$$\langle \xi^G, \nabla\varphi\rangle = \mathrm{div}(\xi^G)^{\mathrm{t}}\varphi + \langle d^*\xi^G, \varphi\rangle,$$

where t means the *transpose* operator which is applied to any section $\Psi \in \Gamma^\infty(\mathrm{T}^*M \otimes \mathfrak{so}(M))$ and defined by $\Psi^{\mathrm{t}}\colon \mathfrak{so}(M) \to \mathfrak{X}(M)$, $\langle \Psi^{\mathrm{t}}\varphi, X\rangle = \langle \Psi_X, \varphi\rangle$. Using the last identity in (3·6), we will finally have the other expression for $d\mathcal{E}_\sigma(\varphi)$ required in the Theorem.

THEOREM 3·7. *Under the same assumptions as in Theorem* 3·6*, the following conditions are equivalent*:

(i) σ *is a critical point for the energy functional on* $\Gamma^\infty(\mathcal{SO}(M)/G)$;
(ii) $d^*\xi^G = 0$;
(iii) $(\nabla^G_{e_i}\xi^G)_{e_i} = -\xi^G_{\xi^G_{e_i}e_i}$;
(iv) *If* T^G *is the torsion of the minimal connection* ∇^G*, then*
  (a) $\langle (\nabla_{e_i}T^G)(X, Y), e_i\rangle = 0$*, for all* $X, Y \in \mathfrak{X}(M)$*, and*
  (b) d^*T^G *is a skew-symmetric endomorphism, i.e.,* $d^*T^G \in \mathfrak{so}(M)$.

Proof. An immediate consequence of Theorem 3·6 and Lemma 3·5 is that conditions (i) and (ii) are equivalent. The equivalence with (iii) follows from (3·5). Finally, the equivalence with the conditions in (iv) is a direct consequence of the identity

$$2\langle (\nabla_X\xi^G)_Y Z, U\rangle = \langle Y, (\nabla_X T^G)(Z, W)\rangle - \langle Z, (\nabla_X T^G)(W, Y)\rangle + \langle W, (\nabla_X T^G)(Y, Z)\rangle.$$

For general Riemannian manifolds $(M, \langle\cdot,\cdot\rangle)$, not necessarily closed and oriented, we will say that a G-structure σ is *harmonic*, if it satisfies $d^*\xi^G = 0$.

THEOREM 3·8 (The second variation formula). *With the same assumptions as in Theorem* 3·6*, if* σ *is a harmonic G-structure, then the Hessian form* $(\mathrm{Hess}\ \mathcal{E})_\sigma$ *on* $\Gamma^\infty(\mathfrak{m}_\sigma) \cong T_\sigma\Gamma^\infty(\mathcal{SO}(M)/G)$ *is given by*

$$(\mathrm{Hess}\ \mathcal{E})_\sigma\varphi = \int_M \left(\|\nabla\varphi\|^2 - \frac{1}{2}\|[\xi^G, \varphi]_{\mathfrak{m}_\sigma}\|^2 + \langle \nabla\varphi, 2[\xi^G, \varphi] - [\xi^G, \varphi]_{\mathfrak{m}_\sigma}\rangle\right) dv.$$

In particular, if $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{g}$ *or equivalently* $SO(n)/G$ *is locally symmetric, then*

$$(\mathrm{Hess}\ \mathcal{E})_\sigma\varphi = \int_M \left(\|\nabla\varphi\|^2 - 2\|[\xi^G, \varphi]\|^2\right) dv.$$

https://doi.org/10.1017/S0305004108001709 Published online by Cambridge University Press

Proof. From results contained in the proof of Theorem 3·6, relative to the first variation formula, we have

$$(\mathrm{Hess}\,\mathcal{E})_\sigma\varphi = \frac{d^2}{dt^2}_{|t=0}\mathcal{E}(\sigma_t) = \int_M \frac{d}{dt}_{|t=0}\left\langle \mathsf{v}\,\sigma_{t*}, \frac{\nabla^q}{dt}_{|t=t}\sigma_{t*}\right\rangle_{\mathcal{SO}(M)/G} dv.$$

But using the same arguments as in the mentioned proof, we will get

$$\frac{d}{dt}_{|t=0}\left\langle \mathsf{v}\,\sigma_{t*}X, \frac{\nabla^q}{dt}_{|t=t}\sigma_{t*}X\right\rangle_{\mathcal{SO}(M)/G} = \frac{d}{dt}_{|t=0}\left\langle \mathsf{v}\,\sigma_{t*}X, \frac{\nabla^q}{ds}_{|s=0}\varphi_t(\gamma(s))\right\rangle_{\mathcal{SO}(M)/G},$$

where $s\to\gamma(s)$ is a curve in M such that $\gamma(0)=m$ and $\gamma'(0)=X$, $(\sigma_t)'_{t=t_0}=\varphi_{t_0}$, and ∇^q/ds is the covariant derivative along the curve $s\to\sigma_t(\gamma(s))$. From the last identity, using that the fibers are totally geodesic, it follows

$$\begin{aligned}\frac{d}{dt}_{|t=0}\left\langle \mathsf{v}\,\sigma_{t*}X, \frac{\nabla^q}{dt}_{|t=t}\sigma_{t*}X\right\rangle_{\mathcal{SO}(M)/G} &= \left\|\mathsf{v}\frac{\nabla^q}{ds}_{|s=0}\varphi(\gamma(s))\right\|^2_{\mathcal{SO}(M)/G}\\ &\quad+\left\langle \mathsf{v}\,\sigma_*X, \frac{\nabla^q}{dt}_{|t=0}\frac{\nabla^q}{ds}_{|s=0}\varphi_t(\gamma(s))\right\rangle_{\mathcal{SO}(M)/G}.\end{aligned}$$

Now, from (3·3), the first term is expressed as

$$\left\|\mathsf{v}\frac{\nabla^q}{ds}_{|s=0}\varphi(\gamma(s))\right\|^2_{\mathcal{SO}(M)/G} = \left\|\phi\frac{\nabla^q}{ds}_{|s=0}\varphi(\gamma(s))\right\|^2 = \left\|\nabla_X\varphi+\frac{1}{2}\left[\xi^G_X,\varphi\right]_{\mathfrak{m}_\sigma}+\left[\xi^G_X,\varphi\right]_{\mathfrak{g}_\sigma}\right\|^2 \tag{3·7}$$

and the second term can be given by

$$\begin{aligned}\left\langle \mathsf{v}\,\sigma_*X, \frac{\nabla^q}{dt}_{|t=0}\frac{\nabla^q}{ds}_{|s=0}\varphi_t(\gamma(s))\right\rangle_{\mathcal{SO}(M)/G} &= \left\langle \mathsf{v}\,\sigma_*X, \frac{\nabla^q}{ds}_{|s=0}\frac{\nabla^q}{dt}_{|t=0}\varphi_t(\gamma(s))\right\rangle_{\mathcal{SO}(M)/G}\\ &\quad+\langle R^q(\varphi(m),\mathsf{v}\sigma_*X)\varphi(m),\mathsf{v}\,\sigma_*X\rangle_{\mathcal{SO}(M)/G},\end{aligned}$$

where $R^q(A,B)=\nabla^q_{[A,B]}-[\nabla^q_A,\nabla^q_B]$ is the Riemannian curvature tensor of $\langle\cdot,\cdot\rangle_{\mathcal{SO}(M)/G}$ and we have used that π has totally geodesic fibres. On the one hand, by using similar arguments as in the proof of Theorem 3·6, we get

$$\begin{aligned}\left\langle \mathsf{v}\,\sigma_*X, \frac{\nabla^q}{ds}_{|s=0}\frac{\nabla^q}{dt}_{|t=0}\varphi_t(\gamma(s))\right\rangle_{\mathcal{SO}(M)/G} &= \left\langle \phi\,\sigma_*X, \phi\frac{\nabla^q}{ds}_{|s=0}(\sigma_t)''_{t=0}(\gamma(s))\right\rangle\\ &= -\left\langle \xi^G_X, \nabla_X\mathrm{pr}_2^\pi(\sigma_t)''_{t=0}(m)\right\rangle.\end{aligned} \tag{3·8}$$

Additionally, since σ is harmonic, $d^*\xi^G=0$, we have the identity

$$\left\langle \xi^G, \nabla\mathrm{pr}_2^\pi(\sigma_t)''_{t=0}(m)\right\rangle = \mathrm{div}(\xi^G)^{\mathrm{t}}\mathrm{pr}_2^\pi(\sigma_t)''_{t=0}(m). \tag{3·9}$$

On the other hand, in order to compute $\langle R^q(\varphi(m),\mathsf{v}\sigma_*X)\varphi(m),\mathsf{v}\,\sigma_*X\rangle_{\mathcal{SO}(M)/G}$, note that the $\mathsf{v}\nabla^q_\varphi\psi$ is a well defined connection on the fibres of π. In our case, $\mathsf{v}\nabla^q_\varphi\psi=\nabla^q_\varphi\psi$ and the corresponding Riemannian curvature tensor R^{v} satisfies $R^{\mathsf{v}}(\varphi,\psi_1)\psi_2=R^q(\varphi,\psi_1)\psi_2$. Therefore,

$$\langle R^q(\varphi(m),\mathsf{v}\sigma_*X)\varphi(m),\mathsf{v}\,\sigma_*X\rangle_{\mathcal{SO}(M)/G} = \langle R^{\mathsf{v}}(\varphi(m),\mathsf{v}\sigma_*X)\varphi(m),\mathsf{v}\,\sigma_*X\rangle_{\mathcal{SO}(M)/G}.$$

https://doi.org/10.1017/S0305004108001709 Published online by Cambridge University Press

Now, using (3·3), we get

$$\phi\mathsf{v}\nabla^q_{\varphi(m)}\mathsf{v}\nabla^q_{\mathsf{v}\sigma_*X}\varphi = \frac{1}{4}[\phi\varphi,[\phi\sigma_*X,\phi\varphi]_{\mathfrak{m}}]_{\mathfrak{m}} + \frac{1}{2}[\phi\varphi,[\phi\sigma_*X,\phi\varphi]_{\mathfrak{g}}]_{\mathfrak{m}} + \frac{1}{2}[\phi\varphi,[\phi\sigma_*X,\phi\varphi]_{\mathfrak{m}}]_{\mathfrak{g}} + [\phi\varphi,[\phi\sigma_*X,\phi\varphi]_{\mathfrak{g}}]_{\mathfrak{g}},$$

$$\phi\mathsf{v}\nabla^q_{\mathsf{v}\sigma_*X}\mathsf{v}\nabla^q_{\varphi(m)}\varphi = 0,$$

$$\phi\mathsf{v}\nabla^q_{[\varphi,\sigma_*X]}\varphi = -\frac{1}{2}[\phi\varphi,[\phi\sigma_*X,\phi\varphi]_{\mathfrak{m}}]_{\mathfrak{m}} - \frac{1}{2}[\phi\varphi,[\phi\sigma_*X,\phi\varphi]_{\mathfrak{g}}]_{\mathfrak{m}} - [\phi\varphi,[\phi\sigma_*X,\phi\varphi]_{\mathfrak{m}}]_{\mathfrak{g}} - [\phi\varphi,[\phi\sigma_*X,\phi\varphi]_{\mathfrak{g}}]_{\mathfrak{g}}.$$

From these identities and because of the bi-invariance of the metric induced by the inner product $\langle\cdot,\cdot\rangle$ on $\mathcal{SO}(n)$, it is not hard to deduce

$$\begin{aligned}\langle R^{\mathsf{v}}(\varphi(m),\mathsf{v}\sigma_*X)\varphi(m),\mathsf{v}\,\sigma_*X\rangle_{\mathcal{SO}(M)/G} &= -\frac{3}{4}\big\|\big[\xi^G_X,\varphi\big]_{\mathfrak{m}_\sigma}\big\|^2 - \big\langle\big[\varphi,\big[\xi^G_X,\varphi\big]_{\mathfrak{g}_\sigma}\big]_{\mathfrak{m}_\sigma},\xi^G_X\big\rangle \\ &= -\frac{3}{4}\big\|\big[\xi^G_X,\varphi\big]_{\mathfrak{m}_\sigma}\big\|^2 - \big\|\big[\xi^G_X,\varphi\big]_{\mathfrak{g}_\sigma}\big\|^2. \qquad (3\cdot10)\end{aligned}$$

The required formula for the second variation follows from (3·7), (3·8), (3·9) and (3·10). For the last part of the theorem, we use that $\nabla\varphi = \nabla^G\varphi - [\xi,\varphi]$ and $\nabla^G\varphi \in \Gamma^\infty(\mathfrak{m}_\sigma)$.

To study harmonicity of G-structures as maps, we need to consider $\nabla\sigma_*$, where $(\nabla_X\sigma_*)(Y) = \nabla^q_X\sigma_*Y - \sigma_*(\nabla_XY)$, for all $X, Y \in \mathfrak{X}(M)$. Here as before, ∇^q also denotes the induced connection on $\sigma^*T\mathcal{SO}(M)/G$.

LEMMA 3·9. *If $R(X,Y) = \nabla_{[X,Y]} - [\nabla_X,\nabla_Y]$ is the Riemannian curvature tensor of $(M,\langle\cdot,\cdot\rangle)$ and σ is a G-structure on M, then*

$$\sigma^*\Phi(X,Y) = -(\nabla_X\xi^G)_Y + (\nabla_Y\xi^G)_X - 2\big[\xi^G_X,\xi^G_Y\big] + \big[\xi^G_X,\xi^G_Y\big]_{\mathfrak{m}_\sigma} = -R(X,Y)_{\mathfrak{m}_\sigma}.$$

Proof. If $\wp\colon U \to \mathcal{G}(M)$ is a local section of the reduced bundle $\mathcal{G}(M) \subseteq \mathcal{SO}(M)$, then

$$\Phi_{\sigma(m)}(\sigma_*X,\sigma_*Y) = \phi\pi_{G*\wp(m)}\Omega(\wp_*X,\wp_*Y)^*_{\mathfrak{m}} = (\sigma(m),-R(X,Y)_{\mathfrak{m}_\sigma}) = -R(X,Y)_{\mathfrak{m}_\sigma}$$

(see [**15**, proposition 4·5]). Now, if we use $\nabla^G = \nabla + \xi^G$ in the expression for R, it is not hard to see that

$$R(X,Y) = R^G(X,Y) + \big(\nabla^G_X\xi^G\big)_Y - \big(\nabla^G_Y\xi^G\big)_X + \xi^G_{\xi^G_XY} - \xi^G_{\xi^G_YX} - \big[\xi^G_X,\xi^G_Y\big],$$

where $R^G(X,Y) = \nabla^G_{[X,Y]} - [\nabla^G_X,\nabla^G_Y]$. Finally, since $R^G \in \Lambda^2T^*M\otimes\mathfrak{g}_\sigma$, $\xi^G \in T^*M\otimes\mathfrak{m}_\sigma$ and ∇^G is a G-connection, we get

$$\begin{aligned}R(X,Y)_{\mathfrak{m}_\sigma} &= \big(\nabla^G_X\xi^G\big)_Y - \big(\nabla^G_Y\xi^G\big)_X + \xi^G_{\xi^G_XY} - \xi^G_{\xi^G_YX} - \big[\xi^G_X,\xi^G_Y\big]_{\mathfrak{m}_\sigma} \\ &= (\nabla_X\xi^G)_Y - \big(\nabla_Y\xi^G\big)_X + 2\big[\xi^G_X,\xi^G_Y\big] - \big[\xi^G_X,\xi^G_Y\big]_{\mathfrak{m}_\sigma}.\end{aligned}$$

From all this, the Lemma follows. Finally, note also that $R(X,Y)_{\mathfrak{g}_\sigma} = R^G(X,Y) - [\xi^G_X,\xi^G_Y]_{\mathfrak{g}_\sigma}$.

If $\sigma^*\Phi = 0$, the G-structure σ is called a *flat G-structure*. By the final remark in the proof of the last Lemma, this notion is characterised by $R(X,Y) = R^G(X,Y) - [\xi^G_X,\xi^G_Y]_{\mathfrak{g}_\sigma} \in S^2\mathfrak{g}_\sigma$. Therefore, the intrinsic torsion of a flat G-structure has not contributions in the G-components of R orthogonal to $S^2\mathfrak{g}_\sigma$. Thus, R is in the space of algebraic curvature tensors for manifolds with integrable G-structure.

https://doi.org/10.1017/S0305004108001709 Published online by Cambridge University Press

Now we have the tools to show some results (Theorem 3·10, Theorem 3·11 and Theorem 3·12) which are versions of Wood's results given in [**28**], expressed in terms of the intrinsic torsion ξ^G and the Riemannian curvature tensor R. But we firstly recall that a G-structure σ is said to be *totally geodesic*, if $\nabla\sigma_* = 0$. In such a situation, $\sigma(M)$ is a totally geodesic submanifold of $\mathcal{SO}(M)/G$. Weaker conditions can be considered by saying that a G-structure σ is *vertically geodesic* (resp., *horizontally geodesic*), if the vertical component (resp., *horizontal component*) of $\nabla\sigma_*$ vanishes. In these situations, σ sends geodesics to paths with horizontal (resp., vertical) acceleration.

THEOREM 3·10. *If σ is a G-structure on $(M, \langle\cdot,\cdot\rangle)$, then*:

(a) $\phi(\nabla_X\sigma_*)Y = -(1/2)((\nabla_X\xi^G)_Y + (\nabla_Y\xi^G)_X)$. *Therefore, σ is vertically geodesic if and only if $(\nabla_X\xi^G)_X = 0$. In particular, if σ is vertically geodesic, then σ is a harmonic G-structure*;

(b) $2\langle\pi_*(\nabla_X\sigma_*)Y, Z\rangle = \langle\xi^G_X, R(Y,Z)\rangle + \langle\xi^G_Y, R(X,Z)\rangle$. *Therefore, σ is horizontally geodesic if and only if $\langle\xi^G_X, R(Y,Z)\rangle$ is a skew-symmetric three-form. In particular, if σ is a flat G-structure, then σ is horizontally geodesic.*

Proof. For (a). Using Lemma 3·1, we have

$$\begin{aligned}\phi(\nabla_X\sigma_*)Y &= \nabla^c_{\sigma_*X}\phi\sigma_*Y + \frac{1}{2}\{[\phi\sigma_*X, \phi\sigma_*Y]_{\mathfrak{m}} - \Phi(\sigma_*X, \sigma_*Y)\} - \phi\sigma_*(\nabla_X Y)\\ &= \nabla_{\sigma_*X}\phi\,\sigma_*Y - [\phi\sigma_*X, \phi\sigma_*Y] + \frac{1}{2}[\phi\sigma_*X, \phi\sigma_*Y]_{\mathfrak{m}}\\ &\quad -\phi\sigma_*(\nabla_X Y) - \frac{1}{2}\Phi(\sigma_*X, \sigma_*Y).\end{aligned}$$

Now, taking $\phi\sigma_* = -\xi^G$ into account and using Lemma 3·9, the required identity in (a) follows.

For (b). In [**28**, theorem 3.4 (ii)], it is proved that

$$2\langle\pi_*(\nabla_X\sigma_*)Y, Z\rangle = \langle\phi\sigma_*X, \Phi(\sigma_*Y, \sigma_*Z)\rangle + \langle\phi\sigma_*Y, \Phi(\sigma_*X, \sigma_*Z)\rangle.$$

Since $\phi\sigma_* = -\xi^G$ and $\Phi(\sigma_*Y, \sigma_*Z\rangle = -R(Y,Z)_{\mathfrak{m}_\sigma}$, (b) follows.

Next, we compute the respective vertical and horizontal components of the *tension field* $\tau(\sigma) = (\nabla^q_{e_i}\sigma_*)(e_i)$ used in variational problems [**22**]. Given a G-structure σ on a closed Riemannian manifold $(M, \langle\cdot,\cdot\rangle)$, the map $(M, \langle\cdot,\cdot\rangle) \mapsto (\mathcal{SO}(M)/G, \langle\cdot,\cdot\rangle_{\mathcal{SO}(M)/G})$ is harmonic, i.e., σ is a critical point for the energy functional on $\mathcal{C}^\infty(M, \mathcal{SO}(M)/G)$, if and only if $\tau(\sigma)$ vanishes. Because variational vector fields of smooth variations of σ through sections belong to $\Gamma^\infty(\sigma^*\mathcal{V})$, it follows that harmonic sections are characterised by the vanishing of the vertical component of $\tau(\sigma)$. By Theorem 3·10(a), it follows $\phi\tau(\sigma) = -(\nabla_{e_i}\xi^G)_{e_i} = d^*\xi^G$ which coincides with the above exposed relative to harmonic G-structures. Since, by Theorem 3·10(b), the horizontal component of $\tau(\sigma)$ is determined by the horizontal lift of the vector field metrically equivalent to the one-form $\langle\xi^G_{e_i}, R(e_i,\cdot)\rangle$, then next result follows.

THEOREM 3·11. *A G-structure σ on a closed and oriented Riemannian manifold $(M, \langle\cdot,\cdot\rangle)$ is a harmonic map if and only if σ is a harmonic G-structure such that $\langle\xi^G_{e_i}, R(e_i,\cdot)\rangle = 0$. Therefore, if σ is flat, then σ is a harmonic map if and only if σ is a harmonic G-structure.*

Such a G-structure σ is said to determine a *harmonic map*, even when M is possibly non-compact or non-orientable and if $\mathsf{v}(\nabla.\mathsf{v}\sigma_*)\cdot = 0$, the G-structure σ is called *super-flat*.

https://doi.org/10.1017/S0305004108001709 Published online by Cambridge University Press

THEOREM 3·12. *We have*

$$\phi(\nabla_X \mathsf{v}\sigma_*)(Y) = -\frac{1}{2}((\nabla_X \xi^G)_Y + (\nabla_Y \xi^G)_X + R(X, Y)_{\mathfrak{m}_\sigma}).$$

Therefore, σ is super-flat if and only if σ is flat and totally geodesic. In particular, an integrable G-structure is super-flat.

Proof. Using Lemma 3·1, we have

$$\phi(\nabla_X \mathsf{v}\sigma_*)Y = \phi(\nabla_X \sigma_*)Y - \phi\nabla^q_X \mathsf{h}\sigma_* Y = \phi(\nabla_X \sigma_*)Y + \frac{1}{2}\Phi(\sigma_* X, \sigma_* Y).$$

Then, the identity follows using Lemma 3·9 and Theorem 3·10. Finally, note that if σ is super-flat, then the vanishing of the symmetric part for X and Y of $\phi(\nabla_X \mathsf{v}\sigma_*)Y$ implies that σ is vertically geodesic. Meanwhile, the vanishing of the skew-symmetric part for X and Y implies that σ is flat.

Relevant types of diverse G-structures are characterised by saying that its intrinsic torsion ξ^G is metrically equivalent to a skew-symmetric three-form, that is, $\xi^G_X Y = -\xi^G_Y X$. Now we will show some facts satisfied by such G-structures.

PROPOSITION 3·13. *For a G-structure σ such that $\xi^G_X Y = -\xi^G_Y X$, we have*:

(i) *if $[\xi^G_X, \xi^G_Y] \in \mathfrak{g}_\sigma$, for all $X, Y \in \mathfrak{X}(M)$, then $\langle R(X, Y)_{\mathfrak{m}_\sigma} X, Y\rangle = 2\langle \xi^G_X Y, \xi^G_X Y\rangle$. Therefore, σ is integrable if and only if σ is flat if and only if σ is super-flat;*

(ii) *if σ is a harmonic G-structure, then σ is also a harmonic map.*

Proof. (i) Because the condition $\xi^G_X Y = -\xi^G_Y X$ implies that $(\nabla_X \xi^G)_Y Z = -(\nabla_X \xi^G)_Z Y$ and $(\nabla^G_X \xi^G)_Y Z = -(\nabla^G_X \xi^G)_Z Y$, we will get the required identity in (i) by using the expression for $R(X, Y)_{\mathfrak{m}_\sigma}$ contained in Lemma 3·9.

(ii) Applying the first Bianchi's identity, we have

$$\left\langle \xi^G_{e_i}, R(e_i, X)\right\rangle = \frac{1}{3}\left\langle \xi^G_{e_i} e_j, e_k\right\rangle(\langle R(e_j, e_k)e_i, X\rangle + \langle R(e_k, e_i)e_j, X\rangle + \langle R(e_i, e_j)e_k, X\rangle) = 0.$$

In next Section, we will study harmonicity of almost Hermitian structures. Such structures are examples of G-structures defined by means of one or several (r, s)-tensor fields Ψ which are stabilised under the action of G, i.e., $g\cdot\Psi = \Psi$, for all $g \in G$. Moreover, it will be possible to characterise the harmonicity of such G-structures by conditions on those tensors Ψ. The *connection Laplacian* (or *rough Laplacian*) [**15**] $\nabla^*\nabla\Psi$ will play a relevant rôle in such conditions. We recall that

$$\nabla^*\nabla\Psi = -(\nabla^2\Psi)_{e_i, e_i},$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal frame field and $(\nabla^2\Psi)_{X,Y} = \nabla_X(\nabla_Y\Psi) - \nabla_{\nabla_X Y}\Psi$. Next Lemma provides an expression for $\nabla^*\nabla\Psi$ in terms of ∇^G and ξ^G which will be useful in the sequel.

LEMMA 3·14. *Let $(M, \langle\cdot,\cdot\rangle)$ be an oriented Riemannian n-manifold equipped with a G-structure, where the Lie group $G \subseteq SO(n)$ is closed and connected. If Ψ is an (r, s)-tensor field on M which is stabilised under the action of G, then*

$$\nabla^*\nabla\Psi = \left(\nabla^G_{e_i}\xi^G\right)_{e_i}\Psi + \xi^G_{\xi^G_{e_i} e_i}\Psi - \xi^G_{e_i}\left(\xi^G_{e_i}\Psi\right). \tag{3·11}$$

Moreover, if the G-structure is harmonic, then $\nabla^*\nabla\Psi = -\xi^G_{e_i}(\xi^G_{e_i}\Psi)$.

https://doi.org/10.1017/S0305004108001709 Published online by Cambridge University Press

Proof. Since ∇^G is a G-connection and Ψ is stabilised under the action of G, then $\nabla^G\Psi = 0$. Taking $\nabla^G = \nabla + \xi^G$ into account, this implies that $\nabla\Psi = -\xi^G\Psi$. Therefore,

$$(\nabla^2\Psi)_{X,Y} = -\nabla_X\big(\xi^G_Y\Psi\big) + \xi^G_{\nabla_X Y}\Psi = -\nabla^G_X\big(\xi^G_Y\Psi\big) + \xi^G_X\big(\xi^G_Y\Psi\big) + \xi^G_{\nabla_X Y}\Psi.$$

Because of the presence of the metric $\langle\cdot,\cdot\rangle$, any (r,s)-tensor field on M is metrically equivalent to a $(0, r+s)$-tensor field. Therefore, we have only to make the proof for covariant tensors fields. Thus, we can assume that Ψ is a $(0,s)$-tensor field on M. By a straightforward computation we get

$$\begin{aligned}\nabla^G_X\big(\xi^G_Y\Psi\big)(Z_1,\dots,Z_s) = &-\sum_{i=1}^{s} X\big(\Psi(Z_1,\dots,\xi^G_Y Z_i,\dots,Z_s)\big)\\ &+\sum_{i=1}^{s}\Psi\big(Z_1,\dots,\xi^G_Y\nabla^G_X Z_i,\dots,Z_s\big)\\ &+\sum_{\substack{i,j=1\\ i\neq j}}^{s}\Psi\big(Z_1,\dots,\xi^G_Y Z_i,\dots,\nabla^G_X Z_j,\dots,Z_s\big).\end{aligned}$$

Now using $\nabla^G\Psi = 0$, we have

$$\begin{aligned}\sum_{i=1}^{s} X\big(\Psi\big(Z_1,\dots,\xi^G_Y Z_i,\dots,Z_s\big)\big) = &\sum_{i=1}^{s}\Psi\big(Z_1,\dots,\nabla^G_X\xi^G_Y Z_i,\dots,Z_s\big)\\ &+\sum_{\substack{i,j=1\\ i\neq j}}^{s}\Psi\big(Z_1,\dots,\xi^G_Y Z_i,\dots,\nabla^G_X Z_j,\dots,Z_s\big).\end{aligned}$$

Taking this identity into account in the expression for $\nabla^G_X(\xi^G_Y\Psi)$, we will obtain $\nabla^G_X(\xi^G_Y\Psi) = (\nabla^G_X\xi^G)_Y\Psi + \xi^G_{\nabla^G_X Y}\Psi$. Finally, by using this last identity in the second covariant derivative displayed above, (3·11) follows.

4. *Harmonic almost Hermitian structures*

An almost Hermitian manifold is a $2n$-dimensional Riemannian manifold $(M,\langle\cdot,\cdot\rangle)$ equipped with an almost complex structure J compatible with the metric, that is, $J^2 = -\mathrm{Id}$ and $\langle JX, JY\rangle = \langle X, Y\rangle$, for all vector fields X, Y. Associated to the almost Hermitian structure, the two-form $\omega = \langle\cdot, J\cdot\rangle$, called the *Kähler form*, is usually considered. Using ω, M can be oriented by fixing a constant multiple of $\omega^n = \omega\wedge\overset{(n)}{\dots}\wedge\omega$ as volume form. Likewise, the presence of an almost Hermitian structure is equivalent to say that M is equipped with a $U(n)$-structure. It is well known that $U(n)$ is a closed and connected subgroup of $SO(2n)$ and $SO(2n)/U(n)$ is reductive; in fact, it is a Riemannian symmetric space. Moreover, we have the decomposition into $U(n)$-modules $\mathfrak{so}(M) = \mathfrak{u}(n)(M)\oplus\mathfrak{m}(M)$. We will omit the subindex σ used in previous Sections. Also, as in references, we shall simply denote $\mathfrak{u}(n)(M)$ and $\mathfrak{m}(M)$ by $\mathfrak{u}(n)$ and $\mathfrak{u}(n)^\perp$. The bundle $\mathfrak{u}(n)$ (resp., $\mathfrak{u}(n)^\perp$) consists of those skew-symmetric endomorphisms A on tangent vectors such that $AJ = JA$ (resp., $AJ = -JA$). The identification $b_A(\cdot,\cdot) = \langle A\cdot,\cdot\rangle$ implies $\Lambda^2\mathrm{T}^*M \cong \mathfrak{so}(M)$. Therefore, $\Lambda^2\mathrm{T}^*M = \mathfrak{u}(n)\oplus\mathfrak{u}(n)^\perp$, where in this case $\mathfrak{u}(n)$ (resp., $\mathfrak{u}(n)^\perp$) consists of those two-forms on M which are Hermitian (resp., anti-Hermitian), i.e., $b(J\cdot, J\cdot) = b(\cdot,\cdot)$ (resp., $b(J\cdot, J\cdot) = -b(\cdot,\cdot)$).

https://doi.org/10.1017/S0305004108001709 Published online by Cambridge University Press

The minimal $U(n)$-connection is given by $\nabla^{U(n)} = \nabla + \xi^{U(n)}$, with

$$\xi^{U(n)}_X Y = -\frac{1}{2} J\,(\nabla_X J)\,Y \tag{4·1}$$

(see [**7**]). Moreover, $\xi^{U(n)} \in \mathrm{T}^*M \otimes \mathfrak{u}(n)^\perp$ is equivalent to the condition

$$\xi^{U(n)} J + J\xi^{U(n)} = 0.$$

Since $U(n)$ stabilises the Kähler form ω, it follows that $\nabla^{U(n)}\omega = 0$. Taking this into account, $\xi^{U(n)} \in \mathrm{T}^*M \otimes \mathfrak{u}(n)^\perp$ implies $\nabla\omega = -\xi^{U(n)}\omega \in \mathrm{T}^*M \otimes \mathfrak{u}(n)^\perp$. Thus, one can identify the $U(n)$-components of $\xi^{U(n)}$ with the $U(n)$-components of $\nabla\omega$:

(i) if $n = 1$, $\xi^{U(1)} \in \mathrm{T}^*M \otimes \mathfrak{u}(1)^\perp = \{0\}$;
(ii) if $n = 2$, $\xi^{U(2)} \in \mathrm{T}^*M \otimes \mathfrak{u}(2)^\perp = \mathcal{W}_2 \oplus \mathcal{W}_4$;
(iii) if $n \geqslant 3$, $\xi^{U(n)} \in \mathrm{T}^*M \otimes \mathfrak{u}(n)^\perp = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$.

Here the $\mathcal{W}_i$ are the irreducible $U(n)$-modules given by Gray and Hervella in [**13**]. In the following, we will merely write $\xi = \xi^{U(n)}$ and $\xi_{(i)}$ will denote the component in $\mathcal{W}_i$ of the intrinsic torsion ξ. For one-forms θ, we will set $J\theta(X) = -\theta(JX)$, for all $X \in \mathfrak{X}(M)$. The one-form $Jd^*\omega$ is a constant multiple of the Lee one-form which determines the $\mathcal{W}_4$-part of the intrinsic torsion ξ [**13**]. Moreover, from (4·1), we will have $2\langle \xi_X Y, Z\rangle = -(\nabla_X\omega)(Y, JZ)$. Now, using this last identity, it is obtained that the vector field $\xi_{e_i} e_i$ which takes part in the harmonicity criteria (see Theorem 3·7) is given by $2\xi_{e_i} e_i = -J(d^*\omega)^\sharp$.

THEOREM 4·1. *For an almost Hermitian $2n$-manifold $(M, \langle\cdot,\cdot\rangle, J)$ with Kähler form ω, we have that the following conditions are equivalent*:

(i) *the almost Hermitian structure is harmonic*;
(ii) $[J, \nabla^*\nabla J] = 0$, *where* $[\cdot,\cdot]$ *denotes the commutator bracket for endomorphisms;*
(iii) $\nabla^*\nabla\omega$ *is a Hermitian two-form*;
(iv) $\nabla^*\nabla\omega(X, Y) = -4\omega(\xi_{e_i} X, \xi_{e_i} Y)$, *for all* $X, Y \in \mathfrak{X}(M)$.

Remark 4·2. Condition (ii) represents the Euler–Lagrange equations given in [**26**] for the harmonic almost Hermitian structure determined by J.

Proof. Using Theorem 3·7, Lemma 3·14 and $\xi J = -J\xi$, it follows that (i) implies (iv) and (iv) implies

$$\big(\big(\nabla^{U(n)}_{e_i}\xi\big)_{e_i} + \xi_{\xi_{e_i} e_i}\big)\omega = 0.$$

But note that the map $A \to -\omega(A\cdot,\cdot) - \omega(\cdot, A\cdot)$ from $\mathfrak{u}(n)^\perp \subseteq \mathfrak{so}(2n)$ to $\mathfrak{u}(n)^\perp \subseteq \Lambda^2\mathrm{T}^*M$ is a $U(n)$-isomorphism. Therefore, $(\nabla^{U(n)}_{e_i}\xi)_{e_i} + \xi_{\xi_{e_i} e_i} = 0$.

Taking into account that $(\nabla^{U(n)}_{e_i}\xi)_{e_i}\omega$, $\xi^{U(n)}_{\xi_{e_i} e_i}\omega$ belong to $\mathfrak{u}(n)^\perp$, the equivalence between (iii) and (iv) is an immediate consequence of Lemma 3·14 and $\xi J = -J\xi$.

Because we have $(\nabla_X\omega)(Y, Z) = \langle Y, (\nabla_X J)Z\rangle$, it follows that

$$(\nabla^*\nabla\omega)(X, Y) = \langle X, (\nabla^*\nabla J)Y\rangle.$$

This implies the equivalence between (ii) and (iii).

Tricerri and Vanhecke [**21**] gave a complete decomposition of the Riemannian curvature tensor R of an almost Hermitian manifold M into irreducible $U(n)$-components. These divide naturally into two groups, one forming the space $\mathcal{K} = \mathcal{K}(\mathfrak{u}(n))$ of algebraic curvature tensors for a Kähler manifold (characterised by $\xi = 0$), and the other, $\mathcal{K}^\perp$, being its orthogonal complement. Additionally, Falcitelli et al. [**7**] showed that the components of R

https://doi.org/10.1017/S0305004108001709 Published online by Cambridge University Press

in $\mathcal{K}^{\perp}$ are linearly determined by the covariant derivative $\nabla\xi$. By using the minimal $U(n)$-connection $\nabla^{U(n)}$ of M, Falcitelli et al. display some tables which show whether or not the tensors $\nabla^{U(n)}\xi_{(i)}$ and $\xi_{(i)} \odot \xi_{(j)}$ contribute to the components of R in $\mathcal{K}^{\perp}$. Some variations of such tables have been given in [**16**]. Explanations for these variations are based in Equation (4·3) given below. All of this has provided a unified approach to many of the curvature results obtained by Gray [**12**].

To study some components of R, it is necessary to consider the usual Ricci curvature tensor Ric, associated to the metric structure, and another tensor Ric^*, called the *$*$-Ricci curvature tensor*, associated to the almost Hermitian structure and defined by $\mathrm{Ric}^*(X, Y) = \langle R_{X,e_i} JY, Je_i\rangle$.

In general, Ric^* is not symmetric. However, because $\mathrm{Ric}^*(JX, JY) = \mathrm{Ric}^*(Y, X)$, it can be claimed that its Hermitian part coincides with its symmetric part $\mathrm{Ric}^*_{\mathrm{s}}$, and its anti-Hermitian part is equal to its skew-symmetric part $\mathrm{Ric}^*_{\mathrm{alt}}$. Under the action of $U(n)$, Ric^* is decomposed into $\mathrm{Ric}^* = \mathrm{Ric}^*_{\mathrm{s}} + \mathrm{Ric}^*_{\mathrm{alt}}$, where $\mathrm{Ric}^*_{\mathrm{s}} \in \mathbb{R}\langle\cdot,\cdot\rangle \oplus \mathfrak{su}(n)_s \subseteq S^2\mathrm{T}^*M$ and $\mathrm{Ric}^*_{\mathrm{alt}} \in \mathfrak{u}(n)^{\perp} \subseteq \Lambda^2\mathrm{T}^*M$ [**21**]. Because in the present work the tensor $\mathrm{Ric}^*_{\mathrm{alt}}$ will play a special rôle, we recall the following result.

LEMMA 4·3 ([**16**]). *If M is an almost Hermitian $2n$-manifold with minimal $U(n)$-connection $\nabla^{U(n)} = \nabla + \xi$, then the skew-symmetric part $\mathrm{Ric}^*_{\mathrm{alt}}$ of the $*$-Ricci tensor is given by*

$$\mathrm{Ric}^*_{\mathrm{alt}}(X, Y) = -\langle \xi_{J\xi_{e_i}e_i} JX, Y\rangle + \big\langle \big(\nabla^{U(n)}_{e_i}\xi\big)_{Je_i} JX, Y\big\rangle. \tag{4·2}$$

From the fact $d^2\omega = 0$, writing $d^2\omega$ by means of $\nabla^{U(n)}$ and ξ, the identity

$$\begin{aligned} 0 = {} & 3\big\langle \big(\nabla^{U(n)}_{e_i}\xi_{(1)}\big)_{e_i} X, Y\big\rangle - \big\langle \big(\nabla^{U(n)}_{e_i}\xi_{(3)}\big)_{e_i} X, Y\big\rangle + (n-2)\big\langle \big(\nabla^{U(n)}_{e_i}\xi_{(4)}\big)_{e_i} X, Y\big\rangle \\ & + \big\langle \xi_{(3)X}e_i, \xi_{(1)e_i}Y\big\rangle - \big\langle \xi_{(3)Y}e_i, \xi_{(1)e_i}X\big\rangle + \big\langle \xi_{(3)X}e_i, \xi_{(2)e_i}Y\big\rangle - \big\langle \xi_{(3)Y}e_i, \xi_{(2)e_i}X\big\rangle \\ & - \frac{n-5}{n-1}\big\langle \xi_{(1)\xi_{(4)e_i}e_i} X, Y\big\rangle - \frac{n-2}{n-1}\big\langle \xi_{(2)\xi_{(4)e_i}e_i} X, Y\big\rangle + \big\langle \xi_{(3)\xi_{(4)e_i}e_i} X, Y\big\rangle \end{aligned} \tag{4·3}$$

was deduced in [**16**]. Here, we will make use of (4·3) below. Likewise, we need to point out that $\xi_{(4)\xi_{e_i}e_i} = 0$. In fact, this directly follows from the expression for $\xi_{(4)}$ [**13**] given by

$$\langle \xi_{(4)X}Y, JZ\rangle = -\frac{1}{4(n-1)}\{X^{\flat} \wedge d^*\omega(Y, Z) - JX^{\flat} \wedge Jd^*\omega(Y, Z)\}. \tag{4·4}$$

Some results proved in [**26**] are recovered in Theorem 4·5 below which is completed with other additional results. In proving those results next Lemma will be useful.

LEMMA 4·4. *For an almost Hermitian $2n$-manifold $(M, \langle\cdot,\cdot\rangle, J)$, we have*

$$\begin{aligned} 2(n-1)\big\langle \big(\nabla^{U(n)}_{e_i}\xi_{(4)}\big)_{e_i} X, Y\big\rangle = {} & d\big(\xi^{\flat}_{e_i}e_i\big)(X, Y) - d\big(\xi^{\flat}_{e_i}e_i\big)(JX, JY) \\ & - 4\big\langle \xi_{(1)\xi_{e_i}e_i} X, Y\big\rangle + 2\big\langle \xi_{(2)\xi_{e_i}e_i} X, Y\big\rangle. \end{aligned}$$

Proof. From the expression (4·4) we have

$$2(n-1)\xi_{(4)X} = X^{\flat} \otimes \xi_{e_i}e_i - \xi^{\flat}_{e_i}e_i \otimes X - JX^{\flat} \otimes J\xi_{e_i}e_i + J\xi^{\flat}_{e_i}e_i \otimes JX. \tag{4·5}$$

Now, fixing a local orthonormal frame field $\{e_1, \ldots, e_{2n}\}$ such that $(\nabla_{e_i}e_j)_m = 0$, for a given

https://doi.org/10.1017/S0305004108001709 Published online by Cambridge University Press

$m \in M$, we will compute $((\nabla_{e_i}\xi_{(4)})_{e_i}X)_m$. In fact, by a straightforward computation we obtain

$$2(n-1)\langle(\nabla_{e_i}\xi_{(4)})_{e_i}X, Y\rangle = d(\xi^{\flat}_{e_i}e_i)(X,Y) - d(\xi^{\flat}_{e_i}e_i)(JX,JY) + 2\langle\xi_{JX}JY - \xi_{JY}JX, \xi_{e_i}e_i\rangle.$$

Then, taking the properties of $\xi_{(i)}$ given in [**13**] into account, we will get

$$\langle\xi_{(4)X}Y - \xi_{(4)Y}X, \xi_{e_i}e_i\rangle = 0.$$

Thus, we obtain the identity

$$2(n-1)\langle(\nabla_{e_i}\xi_{(4)})_{e_i}X, Y\rangle = d(\xi^{\flat}_{e_i}e_i)(X,Y) - d(\xi^{\flat}_{e_i}e_i)(JX,JY) - 4\langle\xi_{(1)\xi_{e_i}e_i}X, Y\rangle + 2\langle\xi_{(2)\xi_{e_i}e_i}X, Y\rangle + 2\langle\xi_{(3)X}Y - \xi_{(3)Y}X, \xi_{e_i}e_i\rangle.$$

Finally, it is not hard to show

$$2(n-1)\langle(\xi_{e_i}\xi_{(4)})_{e_i}X, Y\rangle = -2\langle\xi_{(3)X}Y - \xi_{(3)Y}X, \xi_{e_i}e_i\rangle.$$

From the last two identities, the required identity in the Lemma follows.

THEOREM 4·5. *For an almost Hermitian $2n$-manifold $(M, \langle\cdot,\cdot\rangle, J)$, we have*:

(i) *if M is of type $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4$, then the almost Hermitian structure is harmonic if and only if*

$$(n-1)\mathrm{Ric}^*_{\mathrm{alt}}(X,Y) = d(\xi^{\flat}_{e_i}e_i)(X,Y) - d(\xi^{\flat}_{e_i}e_i)(JX,JY) + 2(n-3)\langle\xi_{(1)\xi_{e_i}e_i}X, Y\rangle + 2n\langle\xi_{(2)\xi_{e_i}e_i}X, Y\rangle;$$

(ii) *if M is quasi-Kähler $(\mathcal{W}_1 \oplus \mathcal{W}_2)$, then the almost Hermitian structure is harmonic if and only if* $\mathrm{Ric}^*_{\mathrm{alt}} = 0$;

(iii) *if $n > 2$ and M is locally conformal almost Kähler $(\mathcal{W}_2 \oplus \mathcal{W}_4)$, then the almost Hermitian structure is harmonic if and only if*

$$(n-1)\mathrm{Ric}^*_{\mathrm{alt}}(X,Y) = 2n\langle\xi_{\xi_{e_i}e_i}X, Y\rangle,$$

for all $X, Y \in \mathfrak{X}(M)$;

(iv) *if M is of type $\mathcal{W}_1 \oplus \mathcal{W}_4$ and $n \neq 2$, then the almost Hermitian structure is harmonic if and only if*

$$(n-1)(n-5)\mathrm{Ric}^*_{\mathrm{alt}}(X,Y) = 2(n+1)(n-3)\langle\xi_{\xi_{e_i}e_i}X, Y\rangle,$$

for all $X, Y \in \mathfrak{X}(M)$;

(v) *if M is Hermitian $(\mathcal{W}_3 \oplus \mathcal{W}_4)$, then the almost Hermitian structure is harmonic if and only if*

$$\mathrm{Ric}^*_{\mathrm{alt}}(X,Y) = -2\langle\xi_{\xi_{e_i}e_i}X, Y\rangle.$$

In particular:

(i)* *a nearly Kähler structure $(\mathcal{W}_1)$ is a harmonic map*;

(ii)* *if the exterior derivative of the Lee form is Hermitian (in particular, if it is closed), a Hermitian structure is harmonic if and only if* $\mathrm{Ric}^*_{\mathrm{alt}} = 0$;

(iii)* *a balanced Hermitian structure $(\mathcal{W}_3)$ is a harmonic almost Hermitian structure*;

(iv)* *if $n > 2$, a locally conformal Kähler structure $(\mathcal{W}_4)$ is a harmonic almost Hermitian structure. In such a case, the Lee form is closed and, therefore,* $\mathrm{Ric}^*_{\mathrm{alt}} = 0$.

https://doi.org/10.1017/S0305004108001709 Published online by Cambridge University Press

Proof. (i) By Lemma 4·3, using the properties of $\xi_{(i)}$ given in [**13**], we have

$$\mathrm{Ric}^*_{\mathrm{alt}}(X,Y) = \langle \xi_{(1)\xi_{e_i}e_i}X, Y\rangle + \langle \xi_{(2)\xi_{e_i}e_i}X, Y\rangle - \langle (\nabla^{U(n)}_{e_i}\xi_{(1)})_{e_i}X, Y\rangle$$
$$- \langle (\nabla^{U(n)}_{e_i}\xi_{(2)})_{e_i}X, Y\rangle + \langle (\nabla^{U(n)}_{e_i}\xi_{(4)})_{e_i}X, Y\rangle.$$

Now, by Theorem 3·7 and Lemma 4·4, (i) follows. In particular, if the structure is nearly Kähler, by Equation (4·3), we have $(\nabla^{U(n)}_{e_i}\xi)_{e_i} = 0$. Thus, we get $\mathrm{Ric}^*_{\mathrm{alt}} = 0$. Finally, by Proposition 3·13 (ii), (i)* follows.

Parts (ii) and (iii) are immediate consequences of (i). We recall that, if $n > 2$ and the structure is locally conformal almost Kähler, then the Lee one-form is closed. Moreover, by Lemma 4·4, $\langle (\nabla^{U(n)}_{e_i}\xi)_{e_i}X, Y\rangle = 0$. Since we will also have $\langle \xi_{(4)\xi_{e_i}e_i}X, Y\rangle = 0$, then (iv)* follows.

(iv) Because the structure is of type $\mathcal{W}_1 \oplus \mathcal{W}_4$, (4·3) and (4·2) are respectively given by

$$0 = 3\langle (\nabla^{U(n)}_{e_i}\xi_{(1)})_{e_i}X, Y\rangle + (n-2)\langle (\nabla^{U(n)}_{e_i}\xi_{(4)})_{e_i}X, Y\rangle - \frac{n-5}{n-1}\langle \xi_{(1)\xi_{(4)e_i}e_i}X, Y\rangle, \quad (4{\cdot}6)$$

$$\mathrm{Ric}^*_{\mathrm{alt}}(X,Y) = \langle \xi_{(1)\xi_{(4)e_i}e_i}X, Y\rangle - \langle (\nabla^{U(n)}_{e_i}\xi_{(1)})_{e_i}X, Y\rangle + \langle (\nabla^{U(n)}_{e_i}\xi_{(4)})_{e_i}X, Y\rangle. \quad (4{\cdot}7)$$

Likewise, the characterising condition for harmonic almost Hermitian structures given in Theorem 3·7 is expressed by

$$-\langle \xi_{(1)\xi_{(4)e_i}e_i}X, Y\rangle = \langle (\nabla^{U(n)}_{e_i}\xi_{(1)})_{e_i}X, Y\rangle + \langle (\nabla^{U(n)}_{e_i}\xi_{(4)})_{e_i}X, Y\rangle. \quad (4{\cdot}8)$$

Now, for $n \geqslant 3$, it is straightforward to check that (4·6), (4·7) and (4·8) imply the expression for $\mathrm{Ric}^*_{\mathrm{alt}}$ required in (iv).

Reciprocally, it is also direct to see that such an expression for $\mathrm{Ric}^*_{\mathrm{alt}}$, (4·6) and (4·7) imply (4·8). Therefore, the almost Hermitian structure is harmonic.

(v) The intrinsic torsion ξ for Hermitian structures is such that $\xi_{JX}JY = \xi_X Y$ [**13**]. Therefore, the required identity in (v) is an immediate consequence of Theorem 3·7 and Lemma 4·3.

(ii)* By Lemma 4·4, if the exterior derivative of the Lee form is Hermitian, then $(\nabla^{U(n)}_{e_i}\xi_{(4)})_{e_i} = 0$ in this case. But we also have $(\nabla^{U(n)}_{e_i}\xi_{(3)})_{e_i} = \xi_{(3)\xi_{e_i}e_i}$ by (4·3). Therefore, the assertion is a consequence of (v).

(iii)* Now, we have $\xi_{e_i}e_i = 0$. Moreover, (4·3) implies $(\nabla^{U(n)}_{e_i}\xi_{(3)})_{e_i} = 0$.

Example 4·6. It is well known that a 3-symmetric space $(M, \langle\cdot,\cdot\rangle)$ admits a canonical almost complex structure J compatible with $\langle\cdot,\cdot\rangle$ and $(M, \langle\cdot,\cdot\rangle, J)$ becomes a quasi-Kähler manifold. Further, the intrinsic torsion $\xi = -(1/2)J(\nabla J)$ of the corresponding $U(n)$-structure is a homogeneous structure (see for example [**20**]). Hence, ξ is $\nabla^{U(n)}$-parallel and then we get $\mathrm{Ric}^*_{\mathrm{alt}} = 0$. Then, from Theorem 4·5(ii), we can conclude that *the canonical almost Hermitian structure of a* 3*-symmetric space is harmonic*.

If we write $\langle \xi_{(1)X}Y, Z\rangle = \Psi_\xi(X, Y, Z)$, then Ψ_ξ is a skew-symmetric three-form such that $\Psi_\xi(JX, JY, Z) = -\Psi_\xi(X, Y, Z)$ [**13**]. For $n \geqslant 3$, if we have a harmonic almost Hermitian structure of type $\mathcal{W}_1 \oplus \mathcal{W}_4$, then it follows, using Theorem 4·1(iv) and (4·4), that the connection Laplacian of ω is given by

$$\nabla^*\nabla\omega(X,Y) = 4\langle X \lrcorner \Psi_\xi, JY \lrcorner \Psi_\xi\rangle + \frac{1}{4(n-1)^2}d^*\omega \wedge Jd^*\omega(X,Y).$$

https://doi.org/10.1017/S0305004108001709 Published online by Cambridge University Press

Note that, in general, the right-hand side of this equality is not collinear with ω. In particular, if $n = 3$, we obtain

$$\nabla^*\nabla\omega = \frac{\|\Psi_\xi\|^2}{36}\omega + \frac{1}{16}d^*\omega \wedge Jd^*\omega.$$

A harmonic section σ into a sphere bundle of a Riemannian vector bundle is characterised by the condition $\nabla^*\nabla\sigma = (\|\nabla\sigma\|^2/\|\sigma\|^2)\sigma$ or, equivalently, $\nabla^*\nabla\sigma$ is collinear with σ (see [**9**, **19**]). From the previous paragraphs, the first part of next result is immediate.

PROPOSITION 4·7. *In dimension six, the nearly Kähler structures are the only harmonic almost Hermitian structures of type $\mathcal{W}_1 \oplus \mathcal{W}_4$, such that ω is also a harmonic section into a sphere bundle in $\Lambda^2 T^*M$. In dimension four, if the structure is locally conformal Kähler and the Lee form is closed, then ω is a harmonic section into a sphere bundle in $\Lambda^2 T^*M$.*

Proof. Let M be a locally conformal Kähler four-manifold. In order to compute $(\nabla^*\nabla\omega)_m$, for $m \in M$, we will consider a local orthonormal frame field $\{e_1, \ldots, e_4\}$ such that $(\nabla_{e_i} e_j)_m = 0$. Thus, because in this case, $\nabla_X\omega = X^\flat \wedge (\theta^\sharp \lrcorner \omega) - \theta \wedge (X \lrcorner \omega)$, where $\theta = \frac{1}{2}Jd^*\omega = -\xi_{e_i} e_i$ [**13**], we have

$$(\nabla^*\nabla\omega)_m = -e_i^\flat \wedge (\theta^\sharp \lrcorner (\nabla_{e_i}\omega)) - e_i^\flat \wedge ((\nabla_{e_i}\theta)^\sharp \lrcorner \omega) + \nabla_{e_i}\theta \wedge (e_i \lrcorner \omega) - \theta \wedge d^*\omega.$$

Now, using the expression for $\nabla\omega$ and the identities $e_i \wedge (e_i \lrcorner \omega) = 2\omega$ and $e_i^\flat \wedge \theta \wedge (\theta^\sharp \lrcorner (e_i \lrcorner \omega)) = \theta \wedge (\theta^\sharp \lrcorner \omega)$, we obtain

$$-e_i^\flat \wedge (\theta^\sharp \lrcorner (\nabla_{e_i}\omega)) = -2\theta \wedge (\theta^\sharp \lrcorner \omega) + 2\|\theta\|^2\omega.$$

Moreover, because θ is closed, we have $(\nabla_X\theta)(Y) = (\nabla_Y\theta)(X)$ and it is not hard to see

$$e_i^\flat \wedge ((\nabla_{e_i}\theta)^\sharp \lrcorner \omega) = \nabla_{e_i}\theta \wedge (e_i \lrcorner \omega).$$

Finally, from all of this and $d^*\omega = -2\theta^\sharp \lrcorner \omega$, we get $\nabla^*\nabla\omega = 2\|\theta\|^2\omega$.

Remark 4·8. For nearly Kähler connected six-manifolds which are not Kähler, if 5α denotes the Einstein constant and using [**16**, equation (3.10)], we have

$$\nabla^*\nabla\omega(X, Y) = 4\langle \xi_{e_i} X, \xi_{e_i} JY\rangle = 4\alpha\, \omega(X, Y).$$

Therefore, $\|\Psi_\xi\|^2 = 144\alpha$.

On the other hand, for locally conformal Kähler four-manifolds with closed Lee form, we have $\nabla^*\nabla\omega = 2\|\theta\|^2\omega$. Therefore, $(1/16)\|\nabla\omega\|^2 = (1/2)\|\theta\|^2 = (1/2)\|\xi_{e_i} e_i\|^2 = (1/8)\|Jd^*\omega\|^2$ that, in general, is not constant.

In [**2**], Bor et al. have obtained some results on the energy of almost Hermitian structures defined on certain compact Riemannian manifolds. Concretely, they prove the following

THEOREM 4·9 ([**2**]). *Let $(M^{2n}, \langle\cdot,\cdot\rangle)$ be a compact Riemannian manifold such that*:

(i) *$n \geqslant 3$ and $(M, \langle\cdot,\cdot\rangle)$ is conformally flat*; *or*

(ii) *$n = 2$ and $(M, \langle\cdot,\cdot\rangle)$ is anti-self-dual.*

Then an orthogonal almost complex structure J on M is an energy minimiser in each one of the following three cases:

(i) *$n = 3$ and J is of type $\mathcal{W}_1 \oplus \mathcal{W}_4$*;

(ii) *$n = 2$ or $n \geqslant 4$ and J is of type $\mathcal{W}_4$*;

(iii) *n arbitrary and J is of type $\mathcal{W}_2$.*

https://doi.org/10.1017/S0305004108001709 Published online by Cambridge University Press

Because $\mathrm{Ric}^*_{\mathrm{alt}}$ determines certain $U(n)$-component, $n \geqslant 2$, of the Weyl curvature tensor W on almost Hermitian manifolds (see [**7**, **16**]), then we have $\mathrm{Ric}^*_{\mathrm{alt}} = 0$ for almost Hermitian $2n$-manifolds which are locally conformal flat. In particular, for $n = 2$, if we consider the action $SO(4)$ determined by the volume form given by $Vol = (1/2)\omega \wedge \omega$, the Weyl curvature tensor is decomposed into two components, that is, $W = W^+ + W^-$. If $W^+ = 0$ ($W^- = 0$), the manifold is called *anti-self-dual* (*self-dual*). More details can be found in [**7**, **18**]. Since $\mathrm{Ric}^*_{\mathrm{alt}}$ determines certain $U(2)$-component of W^+, if the manifold is anti-self-dual, then we will also have $\mathrm{Ric}^*_{\mathrm{alt}} = 0$. Therefore, it follows that the results presented here are agree with Theorem 4·9.

Now, we focus attention on harmonicity of almost Hermitian structures as maps. Results in that direction were already obtained in [**28**]. We will complete such results by using tools here presented. In next Lemma, s^* will denote the $*$*scalar curvarture* defined by $s^* = \mathrm{Ric}^*(e_i, e_i)$. If $\mathrm{Ric}^*(X, Y) = \frac{1}{2n} s^* \langle X, Y \rangle$, then the almost Hermitian manifold is said to be *weakly* $*$*Einstein*. If s^* is constant, a weakly-$*$Einstein manifold is called $*$*Einstein*.

In Riemannian geometry, $2d^*\mathrm{Ric} + ds = 0$ is satisfied, where s is the scalar curvature. In general, the *analogue in almost Hermitian geometry is not true. In fact, this is clarified by the following two results.

LEMMA 4·10. *For almost Hermitian manifolds, we have*

$$2d^*\mathrm{Ric}^{*t}(X) + ds^*(X) = 2\langle R(e_i, X), \xi_{Je_i} J \rangle - 4\mathrm{Ric}^*(X, \xi_{e_i} e_i) + 4\big\langle \mathrm{Ric}^*, \xi^\flat_X \big\rangle,$$

where $\mathrm{Ric}^{*t}(X, Y) = \mathrm{Ric}^*(Y, X)$ *and* $\xi^\flat_X(Y, Z) = \langle \xi_X Y, Z \rangle$. *In particular, if the manifold is weakly* $*$*Einstein, then*

$$\frac{n-1}{n} ds^*(X) = 2\big\langle R(e_i, X), \xi_{Je_i} J \big\rangle - 2s^* \big\langle \xi_{e_i} e_i, X \big\rangle.$$

Proof. Note that $\mathrm{Ric}^{*t}(X, Y) = \frac{1}{2}\langle R(e_i, Je_i)Y, JX \rangle$. Then, we get

$$\begin{aligned}
d^*\mathrm{Ric}^{*t}(X) &= -\big(\nabla_{e_j} \mathrm{Ric}^{*t}\big)(e_j, X) \\
&= -\frac{1}{2} e_j \langle R(e_i, Je_i)X, Je_j \rangle + \frac{1}{2}\langle R(e_i, Je_i)\nabla_{e_j} X, Je_j \rangle \\
&\quad + \frac{1}{2}\langle R(e_i, Je_i)X, J\nabla_{e_j} e_j \rangle \\
&= -\frac{1}{2}\langle (\nabla_{e_j} R)(e_i, Je_i)X, Je_j \rangle - \langle R(\nabla_{e_j} e_i, Je_i)X, Je_j \rangle \\
&\quad - \frac{1}{2}\langle R(e_i, Je_i)X, (\nabla_{e_j} J)e_j \rangle.
\end{aligned}$$

Now, by symmetry properties of R and $\xi = -(1/2)J(\nabla J)$, it follows that

$$\begin{aligned}
d^*\mathrm{Ric}^{*t}(X) = &-\frac{1}{2}\langle (\nabla_{e_j} R)(X, Je_j)e_i, Je_i \rangle + \langle R(X, e_j)e_i, \nabla_{Je_j} Je_i) \rangle \\
&- \langle R(e_i, Je_i)X, J\xi_{e_j} e_j \rangle.
\end{aligned}$$

Using the second Bianchi identity and taking into account that

$$\big\langle R(X, e_j)e_i, \nabla_{Je_j} Je_i) \big\rangle = \big\langle R(X, e_j)e_i, \nabla^{U(n)}_{Je_j} Je_i \big) \big\rangle - \big\langle R(X, e_j)e_i, \xi_{Je_j} Je_i \big) \big\rangle,$$

we get

$$d^*\mathrm{Ric}^{*t}(X) = -\frac{1}{4}\langle (\nabla_X R)(e_j, Je_j)e_i, Je_i \rangle - \langle R(X, e_j), \xi_{Je_j} J \rangle - 2\mathrm{Ric}^*(X, \xi_{e_j} e_j).$$

https://doi.org/10.1017/S0305004108001709 Published online by Cambridge University Press

Note that

$$\left\langle R(X, e_j)e_i, \nabla^{U(n)}_{Je_j} Je_i \right\rangle = \langle R(X, e_j)e_i, e_k\rangle \left\langle \nabla^{U(n)}_{Je_j} Je_i, e_k \right\rangle = 0,$$

because it is a scalar product of a skew-symmetric matrix by a Hermitian symmetric matrix.

Finally, it is obtained

$$\begin{aligned} 2d^*\mathrm{Ric}^{*t}(X) = &-\frac{1}{2}\langle (\nabla_X R)(e_j, Je_j)e_i, Je_i\rangle - 2\langle R(X, e_j), \xi_{Je_j} J\rangle \\ &- 4\,\mathrm{Ric}^*(X, \xi_{e_j} e_j). \end{aligned} \tag{4·9}$$

In a second instance, $ds^*(X) = (1/2)X\langle R(e_i, Je_i)e_j, Je_j\rangle$. Hence, we get

$$ds^*(X) = \frac{1}{2}\langle (\nabla_X R)(e_i, Je_i)e_j, Je_j\rangle + 2\langle R(e_i, Je_i)e_j, \nabla_X Je_j\rangle.$$

But we also have that

$$\begin{aligned} \langle R(e_i, Je_i)e_j, \nabla_X Je_j\rangle &= \langle R(e_i, Je_i)e_j, e_k\rangle \left\langle \nabla^{U(n)}_X Je_j, e_k \right\rangle - \langle R(e_i, Je_i)e_j, e_k\rangle \langle \xi_X Je_j, e_k\rangle \\ &= \langle R(e_i, Je_i)e_j, J\xi_X e_j\rangle = 2\,\mathrm{Ric}^*(e_i, \xi_X e_i) = 2\langle \mathrm{Ric}^*, \xi_X\rangle. \end{aligned}$$

Thus, it follows that

$$ds^*(X) = \frac{1}{2}\langle (\nabla_X R)(e_i, Je_i)e_j, Je_j\rangle + 4\langle \mathrm{Ric}^*, \xi_X\rangle. \tag{4·10}$$

From (4·9) and (4·10), the required identity is obtained.

THEOREM 4·11. *For an almost Hermitian $2n$-manifold $(M, \langle\cdot,\cdot\rangle, J)$, we have*:

(i) *if M is of type $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4$, then the almost Hermitian structure is a harmonic map if and only if the almost Hermitian structure is harmonic and*

$$\begin{aligned} (n-1)d^*\mathrm{Ric}^{*t}(X) + \frac{n-1}{2}ds^*(X) = &\,\mathrm{Ric}(X, \xi_{e_i} e_i) - (2n-1)\mathrm{Ric}^*(X, \xi_{e_i} e_i) \\ &+ 2(n-1)\langle \mathrm{Ric}^*, \xi^\flat_X\rangle, \end{aligned}$$

for all $X \in \mathfrak{X}(M)$;

(ii) *if M is quasi-Kähler $(\mathcal{W}_1 \oplus \mathcal{W}_2)$, then the almost Hermitian structure is a harmonic map if and only if* Ric^* *is symmetric and $2d^*\mathrm{Ric}^* + ds^* = 0$. In particular, if the quasi-Kähler manifold is weakly-$*$Einstein, then the almost Hermitian structure is a harmonic map if and only if s^* is constant*;

(iii) *if M is Hermitian $(\mathcal{W}_3 \oplus \mathcal{W}_4)$, then the almost Hermitian structure is a harmonic map if and only if* $\mathrm{Ric}^*_{\mathrm{alt}} = -2\xi^\flat_{\xi_{e_i} e_i}$ *and*

$$2d^*\mathrm{Ric}^{*t}(X) + ds^*(X) + 4\,\mathrm{Ric}^*(X, \xi_{e_j} e_j) - 4\langle \mathrm{Ric}^*, \xi^\flat_X\rangle = 0,$$

for all $X \in \mathfrak{X}(M)$. In particular:

(a)* *if the exterior derivative of the Lee form is Hermitian (in particular, if it is closed), then the Hermitian structure is a harmonic map if and only if* $\mathrm{Ric}^*_{\mathrm{alt}} = 0$ *and* $2d^*\mathrm{Ric}^* + ds^* + 4\xi_{e_i} e_i \lrcorner \mathrm{Ric}^* = 0$;

(b)* *if* Ric^* *is symmetric, then the Hermitian structure is a harmonic map if and only if $\xi_{\xi_{e_i} e_i} = 0$ and $2d^*\mathrm{Ric}^* + ds^* + 4\xi_{e_i} e_i \lrcorner \mathrm{Ric}^* = 0$. In particular, if the manifold is weakly-$*$Einstein, then the Hermitian structure is a harmonic map if and only if $\xi_{\xi_{e_i} e_i} = 0$ and $(n-1)ds^* + 2s^*\xi^\flat_{e_i} e_i = 0$;*

https://doi.org/10.1017/S0305004108001709 Published online by Cambridge University Press

(c)* *if the manifold is balanced Hermitian* ($\mathcal{W}_3$)*, then the almost Hermitian structure is a harmonic map if and only if* $2d^*\mathrm{Ric}^* + ds^* = 0$*. Furthermore, if the balanced Hermitian manifold is weakly-*$*$*Einstein, the almost Hermitian structure is a harmonic map if and only if* s^* *is constant*;

(d)* *if the manifold is locally conformal Kähler* ($\mathcal{W}_4$) *with closed Lee form, then the almost Hermitian structure is a harmonic map if and only if* $2d^*\mathrm{Ric}^* + ds^* + 4\xi_{e_i e_i} \lrcorner \mathrm{Ric}^* = 0$ *if and only if, for all* $X \in \mathfrak{X}(M)$, $(\mathrm{Ric} - \mathrm{Ric}^*)(X, \xi_{e_i e_i}) = 0$.

Proof. All results contained in the Theorem are immediate consequences of Theorem 4·5, Lemma 4·10, and the following consequence of the expression for $\xi_{(4)}$ given in (4·5)

$$(n-1)\langle \xi_{(4)e_i}, R_{(e_i,X)} \rangle = (\mathrm{Ric} - \mathrm{Ric}^*)(X, \xi_{e_i e_i}).$$

Example 4·12. Hopf manifolds are diffeomorphic to $S^1 \times S^{2n-1}$ and admit a locally conformal Kähler structure with parallel Lee form $(1/2(n-1))\xi^\flat_{e_i e_i}$ [**23**]. Furthermore, $\xi_{e_i e_i}$ is nowhere zero and tangent to S^1. The metric on $S^1 \times S^{2n-1}$ is the product metric of constant multiples of the metrics on S^1 and S^{2n-1} induced by the respective Euclidean metrics on $\mathbb{R}^2$ and $\mathbb{R}^{2n}$. The set $\mathcal{L}(S^{2n-1})$ will consist of those vector fields on $S^1 \times S^{2n-1}$ which are lifts of vector fields on S^{2n-1}. The Riemannian curvature tensor R satisfies

$$\langle R(X,Y)Z_1, Z_2\rangle = k(\langle X, Z_1\rangle\langle Y, Z_2\rangle - \langle X, Z_2\rangle\langle Y, Z_1\rangle), \qquad R(X, \xi_{e_i e_i}) = 0,$$

for all $X, Y, Z_1, Z_2 \in \mathcal{L}(S^{2n-1})$, where k is a constant. Therefore, $\langle R(e_i, \xi_{e_i e_i}), \xi_{e_i}\rangle = 0$. Moreover, using the expression given by (4·5), for all $X \in \mathcal{L}(S^{2n-1})$, we have

$$\langle R(e_i, X), \xi_{e_i}\rangle = 2k\langle \xi_{e_i e_i}, X\rangle = 0.$$

Additionally, it can be checked that

$$\frac{n-1}{k}\langle \xi_X, R(Y,Z)\rangle = -JX^\flat \wedge J\xi^\flat_{e_i e_i}(Y,Z),$$

for all X, Y, Z orthogonal to $\xi_{e_i e_i}$. Therefore, the almost Hermitian structure is not horizontally geodesic. As a consequence, it is also not a flat structure.

Finally, again using the expression (4·5) and the fact that $\xi_{e_i e_i}$ is parallel, it is obtained

$$2(n-1)^2(\nabla_X \xi)_X = J\xi^\flat_{e_i e_i}(X)\,(X^\flat \otimes J\xi_{e_i e_i} e_i - J\xi^\flat_{e_i e_i} e_i \otimes X + JX^\flat \otimes \xi_{e_i e_i} e_i - \xi^\flat_{e_i e_i} e_i \otimes JX).$$

Note that this expression does not vanish for all X.

In conclusion, *the locally conformal Kähler structure on* $S^1 \times S^{2n-1}$ *is a harmonic map which is neither horizontally geodesic, nor vertically geodesic.*

Example 4·13. In general, locally conformal Kähler structures are not harmonic maps. In fact, one can consider the Kähler structure on $\mathbb{R}^{2n}$ determined by the Euclidean metric $\langle\cdot,\cdot\rangle$ and the standard almost complex structure J. If we do a conformal change of metric using a function f on $\mathbb{R}^{2n}$, the new metric $\langle\cdot,\cdot\rangle_o = e^f\langle\cdot,\cdot\rangle$ and J determine a new almost Hermitian structure which is locally conformal Kähler. The Lee form for the new structure is df and the Riemannian curvature tensor is given by

$$\begin{aligned} -2e^{-f}\langle R_o(X,Y)Z, W\rangle_o = {} & L(X,Z)\langle Y,W\rangle + L(Y,W)\langle X,Z\rangle \\ & - L(X,W)\langle Y,Z\rangle - L(Y,Z)\langle X,W\rangle \\ & + \frac{\|df\|^2}{2}\{\langle X,Z\rangle\langle Y,W\rangle - \langle Y,Z\rangle\langle X,W\rangle\}, \end{aligned}$$

https://doi.org/10.1017/S0305004108001709 Published online by Cambridge University Press

where $L(X, Y) = (\nabla_X df)(Y) - (1/2)df(X)df(Y)$ and ∇ is the Levi–Civita connection associated to $\langle \cdot, \cdot \rangle$ (see [**21**]). If ξ_o denotes the intrinsic torsion of the structure $(J, \langle \cdot, \cdot \rangle_o)$, a straightforward computation shows that

$$16e^f \langle R_o(e_{o\,i}, X), \xi_{oe_{o\,i}} \rangle_o = -\frac{2n-3}{2} d(\|df\|^2)(X) + d^*(df)df(X) + (\nabla_{JX} df)(J \operatorname{grad} f),$$

where $\{e_{o\,1}, \ldots, e_{o\,2n}\}$ is an orthonormal basis for vectors with respect to $\langle \cdot, \cdot \rangle_o$ and the terms in the right-hand side, the norm $\| \cdot \|$, grad, etc., are considered with respect to the Euclidean metric $\langle \cdot, \cdot \rangle$. Therefore, it is not hard to find functions f such that $\langle R_o(e_{o\,i}, X), \xi_{oe_{o\,i}} \rangle_o \neq 0$. For instance, if $f = \sin x_1$, then $\langle R_o(e_{o\,i}, X), \xi_{oe_{o\,i}} \rangle_o = ((n-1)/8)\, e^{-\sin x_1} \sin x_1 \cos x_1\, dx_1$.

If we take the function f such that $(x^i \circ f)(x) = (x^i \circ f)(x + 2\pi)$, $i = 1, \ldots, 2n$, then $\langle \cdot, \cdot \rangle_o$ determines a Riemannian metric on the torus $T^{2n} = S^1 \times \ldots \times S^1$ and the natural projection of $\mathbb{R}^{2n}$ on T^{2n} becomes a local isometry. Hence, we also get *locally conformal Kähler structures which are not harmonic maps on the torus* $(T^{2n}, \langle \cdot, \cdot \rangle_o)$.

4·1. *Nearly Kähler manifolds*

For completeness, we will give here a detailed and self-contained explanation of the situation for nearly Kähler manifolds. Thus, we will recover results already known originally proved, some of them, by Gray and, others, by Wood. However, we will display alternative proofs in terms of the intrinsic torsion ξ. Additionally, it is also shown that, for nearly Kähler manifolds, ξ is parallel with respect to the minimal connection $\nabla^{U(n)}$, i.e., $\nabla^{U(n)}\xi = 0$. This last result is originally due to Kirichenko [**14**].

The intrinsic torsion ξ of a nearly Kähler manifold is characterised by the condition $\xi_X Y = -\xi_Y X$. Because this property is preserved by the action of $O(2n)$, then we also have $(\nabla_X \xi)_Y Z = -(\nabla_X \xi)_Z Y$ and $(\nabla^{U(n)}_X \xi)_Y Z = -(\nabla^{U(n)}_X \xi)_Z Y$. Moreover, with respect to the almost complex structure J, $\xi_{JX} JY = -\xi_X Y$ is also satisfied. Therefore, $(\nabla^{U(n)}_X \xi)_{JY} JZ = -(\nabla^{U(n)}_X \xi)_Y Z$.

For nearly Kähler manifolds, Gray [**10**] showed that the following identities are satisfied

$$\langle R(X, Y)X, Y \rangle - \langle R(X, Y)JX, JY \rangle = 4\|\xi_X Y\|^2, \tag{4·11}$$

$$\langle R(JX, JY)JZ, JW \rangle = \langle R(X, Y)Z, W \rangle. \tag{4·12}$$

In fact, since $\langle (\nabla_X \xi)_Y X, Y \rangle = 0 = \langle (\nabla_Y \xi)_X X, Y \rangle$, it is immediate that

$$\langle R(X, Y)_{\mathfrak{u}(n)^\perp} X, Y \rangle = \frac{1}{2} (\langle R(X, Y)X, Y \rangle - \langle R(X, Y)JX, JY \rangle) = 2\langle [\xi_X, \xi_Y]X, Y \rangle.$$

From this, (4·11) follows. Also (4·11) follows from Proposition 3·13, because $[\mathfrak{u}(n)^\perp, \mathfrak{u}(n)^\perp] \subseteq \mathfrak{u}(n)$.

For (4·12). Using (4·11), it is not hard to prove $\langle R(JX, JY)JX, JY \rangle = \langle R(X, Y)X, Y \rangle$. Then, by linearizing, we will have (4·12).

THEOREM 4·14. *Nearly Kähler structures are vertically geodesic harmonic maps. Moreover, for nearly Kähler manifolds, we have*

$$\langle R(X, Y)Z, W \rangle - \langle R(X, Y)JZ, JW \rangle = 4\langle \xi_X Y, \xi_Z W \rangle, \tag{4·13}$$

$$\nabla^{U(n)}_X \xi = 0. \tag{4·14}$$

In particular, if the nearly Kähler structure is flat, then it is Kähler.

Remark 4·15. Equation (4·13) is due to Gray [**11**]. On the other hand, Wood proved in [**28**] that nearly Kähler structures are vertically geodesic harmonic maps.

https://doi.org/10.1017/S0305004108001709 Published online by Cambridge University Press

Proof. Since $\xi_X Y = -\xi_Y X$ and $\nabla\xi = \nabla^{U(n)}\xi - \xi\xi$, it is direct to show that

$$\begin{aligned}\langle R(X,Y)_{\mathfrak{u}(n)^\perp} Z, W\rangle &= \frac{1}{2}\left(\langle R(X,Y)Z, W\rangle - \langle R(X,Y)JZ, JW\rangle\right)\\ &= \left\langle \left(\nabla^{U(n)}_X \xi\right)_Y Z, W\right\rangle - \left\langle \left(\nabla^{U(n)}_Y \xi\right)_X Z, W\right\rangle + 2\langle \xi_X Y, \xi_Z W\rangle. \quad (4{\cdot}15)\end{aligned}$$

Now, we consider the map $\mathsf{s}\colon \Lambda^2 T^*M \otimes \Lambda^2 T^*M \to S^2(\Lambda^2 T^*M)$ defined by $\mathsf{s}(a\otimes b) = a\otimes b + b\otimes a$ and the map $\mathsf{b}\colon S^2(\Lambda^2 T^*M) \to S^2(\Lambda^2 T^*M)$ defined by

$$\mathsf{b}(\Upsilon)(X,Y,Z,W) = 2\Upsilon(X,Y,Z,W) - \Upsilon(Z,X,Y,W) - \Upsilon(Y,Z,X,W).$$

Applying the composition $\mathsf{b}\circ\mathsf{s}$ to both sides of Equation (4·15) and, then, making use of (4·12) and the first Bianchi identity, we obtain

$$\begin{aligned}&3\langle R(X,Y)Z, W\rangle - 2\langle R(X,Y)JZ, JW\rangle + \langle R(Z,X)JY, JW\rangle + \langle R(Y,Z)JX, JW\rangle\\ &\quad = 8\langle \xi_Z W, \xi_X Y\rangle + 4\langle \xi_Y W, \xi_X Z\rangle - 4\langle \xi_X W, \xi_Y Z\rangle. \quad (4{\cdot}16)\end{aligned}$$

Note that we have also taken $(\nabla^{U(n)}_X \xi)_Y Z = -(\nabla^{U(n)}_X \xi)_Z Y$ into account. Now, if we replace Z and W by JZ and JW, subtract the result from (4·16) and use $\xi_{JX} JY = -\xi_X Y$, then we get

$$\begin{aligned}&5\langle R(X,Y)Z, W\rangle - 5\langle R(X,Y)JZ, JW\rangle - \langle R(X,JY)Z, JW\rangle\\ &\quad -\langle R(X,JY)JZ, W\rangle = 16\langle \xi_X Y, \xi_Z W\rangle. \quad (4{\cdot}17)\end{aligned}$$

Finally, replacing in (4·17) Y and Z by JY and JZ, multiplying by $1/5$ the resulting equation and adding the final result to (4·17), the required identity (4·13) is obtained.

Now, from the following identity

$$\begin{aligned}\frac{1}{2}\left(\langle R(X,Y)X, Z\rangle - \langle R(X,Y)JX, JZ\rangle\right) &= \langle (\nabla_X\xi)_Y X, Z\rangle - \langle (\nabla_Y\xi)_X X, Z\rangle\\ &\quad + 2\langle [\xi_X, \xi_Y]X, Z\rangle,\end{aligned}$$

using (4·13), we get $(\nabla_X\xi)_X = 0$. Hence the nearly Kähler structure is vertically geodesic.

For (4·14). Since

$$\begin{aligned}\langle R(X,Y)Z, W\rangle - \langle R(X,Y)JZ, JW\rangle &= 2\left\langle \left(\nabla^{U(n)}_X \xi\right)_Y Z, W\right\rangle - 2\left\langle \left(\nabla^{U(n)}_Y \xi\right)_X Z, W\right\rangle\\ &\quad + 4\langle \xi_X Y, \xi_Z W\rangle\\ &= 4\langle \xi_X Y, \xi_Z W\rangle,\end{aligned}$$

we have $(\nabla^{U(n)}_X \xi)_Y = (\nabla^{U(n)}_Y \xi)_X$. Moreover, from the identity

$$\langle (\nabla_X\xi)_Y Z, W\rangle = \left\langle \left(\nabla^{U(n)}_X \xi\right)_Y Z, W\right\rangle + \langle \xi_Z W, \xi_X Y\rangle - \langle \xi_X \xi_Y Z, W\rangle + \langle \xi_Y \xi_X Z, W\rangle,$$

taking $(\nabla_X\xi)_X = 0$ into account, it follows $(\nabla^{U(n)}_X \xi)_X = 0$. Therefore, $(\nabla^{U(n)}_X \xi)_Y = (\nabla^{U(n)}_Y \xi)_X = -(\nabla^{U(n)}_X \xi)_Y = 0$.

The final remark of the Theorem follows from Proposition 3·13 (i).

Acknowledgments. The authors are supported by grants from MEC (Spain), projects MTM2004-06015-C02-01 and MTM2007-66375.

REFERENCES

[**1**] A. L. BESSE. *Einstein manifolds*. Ergebnisse der Mathematik und ihrer Grenzgebiete 3. Folge **10** (Springer, 1987).

https://doi.org/10.1017/S0305004108001709 Published online by Cambridge University Press

[**2**] G. BOR, L. HERNÁNDEZ LAMONEDA and M. SALVAI. Orthogonal almost-complex structures of minimal energy on conformally, and half-conformally flat manifolds. *Geom. Dedicata* **127** (2007), 75–85. arXiv:math.DG/0609511

[**3**] J. EELLS and L. LEMAIRE. A report on harmonic maps. *Bull. London Math. Soc.* **10** (1978), 1–68.

[**4**] J. EELLS and L. LEMAIRE. Another report on harmonic maps. *Bull. London Math. Soc.* **20** (1988), 385–524.

[**5**] J. EELLS and J. H. SAMPSON. Harmonic mappings of Riemannian manifolds. *Amer. J. Math.* **86** (1964), 109–160.

[**6**] R. CLEYTON and A. F. SWANN. Einstein metrics via intrinsic or parallel torsion. *Math. Z.* **247** no. 3 (2004), 513–528.

[**7**] M. FALCITELLI, A. FARINOLA and S. M. SALAMON. Almost-Hermitian geometry. *Differential Geom. Appl.* **4** (1994), 259–282.

[**8**] O. GIL-MEDRANO, J. C. GONZÁLEZ–DÁVILA and L. VANHECKE. Harmonicity and minimality of oriented distributions. *Israel J. Math.* **143** (2004), 253–279.

[**9**] J. C. GONZÁLEZ–DÁVILA, F. MARTÍN CABRERA and M. SALVAI. Harmonicity of sections of sphere bundles. *Math. Z.* to appear. arXiv:math.DG/0711.3703

[**10**] A. GRAY. Almost complex submanifolds of the six sphere. *Proc. Amer. Math. Soc.* **20** (1970), 277–279.

[**11**] A. GRAY. Nearly Kähler manifolds. *J. Diff. Geom.* **4** (1976), 283–309.

[**12**] A. GRAY. Curvature identities for Hermitian and almost Hermitian manifolds. *Tôhoku Math. J.* (2) **28** no. 4 (1976), 601–612.

[**13**] A. GRAY. and L. M. HERVELLA. The sixteen classes of almost Hermitian manifolds and their linear invariants. *Ann. Mat. Pura Appl.* (4) **123** (1980), 35–58.

[**14**] V. F. KIRICHENKO. K-spaces of maximal rank. *Mat. Zametki* **22** (1977), 465–476.

[**15**] B. LAWSON and M. L. MICHELSOHN. *Spin Geometry.* (Princeton University Press, 1989).

[**16**] F. MARTÍN CABRERA and A. SWANN. Curvature of special almost Hermitian manifolds. *Pacific J. Math.* **228** (2006), 165–184. arXiv:math.DG/0501062

[**17**] T. SAKAI. *Riemannian Geometry.* Transl. Math. Mon. **149** (Amer. Math. Soc., 1996).

[**18**] S. SALAMON. *Riemannian Geometry and Holonomy Groups.* Pitman Research Notes in Math. Series, **201** (Longman, 1989).

[**19**] M. SALVAI. On the energy of sections of trivializable sphere bundles. *Rendiconti del Seminario Matematico dell'Università e Politecnico di Torino* **60** (2002), 147–155.

[**20**] T. SATO. Riemannian 3-symmetric spaces and homogeneous K-spaces. *Mem. Fac. of Technology Kanazawa Univ.* **12**(2) (1979), 137–143.

[**21**] F. TRICERRI and L. VANHECKE. Curvature tensors on almost Hermitian manifolds. *Trans. Amer. Math. Soc.* **267** (1981), 365–398.

[**22**] H. URAKAWA. *Calculus of variations and harmonic maps.* Transl. Math. Mon. **132** (Amer. Math. Soc., 1993).

[**23**] I. VAISMAN. Locally conformal Kähler manifolds with parallel Lee form. *Rend. Mat.* (6) **12** no. 2 (1979), 263–284.

[**24**] J. VILMS. Totally geodesic maps. *J. Diff. Geom.* **4** (1970), 73–79.

[**25**] G. WIEGMINK. Total bending of vector fields on Riemannian manifolds. *Math. Ann.* **303** (1995), 325–344.

[**26**] C. M. WOOD. Harmonic almost-complex structures. *Comp. Math.* **99** (1995), 183–212.

[**27**] C. M. WOOD. On the energy of a unit vector field. *Geom. Dedicata* **64** (1997), 319–330.

[**28**] C. M. WOOD. Harmonic sections of homogeneous fibre bundles. *Differential Geom. Appl.* **19** (2003), 193–210.

https://doi.org/10.1017/S0305004108001709 Published online by Cambridge University Press