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Genus two stable maps, local equations and modular resolutions

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Abstract

We provide a geometric construction of a sequence of modular blowups of the Artin stack parameterizing pre-stable pairs consisting of a genus 2 nodal curve and a smooth divisor. The resulting stack locally diagonalizes the tautological derived objects associated with the moduli of stable maps from genus 2 curves to projective space. As a consequence, the singularities of the main component of the moduli space of stable maps are resolved, and the entire space admits only normal crossing singularities. Our approach is expected to generalize to higher genera.

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1. Introduction

Moduli problems are of central importance in algebraic geometry. Among them, the moduli spaces $\overline{M}_{g,k}(\mathbb{P}^n, d)$ of genus g degree d stable maps into projective space with k marked points are particularly important. For example, the Gromov–Witten theory is an intersection theory on these fundamental moduli spaces (introduced and constructed in [Kon95, Ale94, FP96]).

This article focuses on the deep geometry of the moduli space $\overline{M}_{g,k}(\mathbb{P}^n, d)$. By Murphy's law of Vakil ([Vak06]), the moduli space $\overline{M}_{g,k}(\mathbb{P}^n, d)$ is arbitrarily singular: every singularity of finite type over $\mathbb{Z}$ appears for some n, g, and d. The resolution of singularity is arguably one of the hardest problems in algebraic geometry ([Hir64a, Hir64b, dJ96, ATW20, McQ20], etc.). Several moduli problems offer platforms for geometric solutions, and $\overline{M}_{g,k}(\mathbb{P}^n, d)$ is one of these (and this is *the pivotal motivation* for one of us to study the moduli $\overline{M}_{g,k}(\mathbb{P}^n, d)$). A canonical resolution of $\overline{M}_{g,k}(\mathbb{P}^n, d)$ is also desirable for applications computing Gromov–Witten (GW) invariants, even though these invariants can be introduced without using any resolution ([BF97] and [LT98]). When the genus g is 0, the moduli space $\overline{M}_{0,k}(\mathbb{P}^n, d)$ is smooth. For $g = 1$, the moduli space $\overline{M}_{1,k}(\mathbb{P}^n, d)$ is singular and the resolution of the main component of $\overline{M}_{1,k}(\mathbb{P}^n, d)$

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was constructed by Vakil and Zinger in [VZ08]. This was followed by an algebraic approach to the entire moduli space $\overline{M}_1(\mathbb{P}^n, d)$ (without marked points) by Hu and Li [HL10].

Our current article begins to treat the more difficult cases of general genera. We start with the moduli space $\overline{M}_2(\mathbb{P}^n, d)$; for conciseness and without loss of generality in terms of singularity, we still work with curves without marked points. The goal of this paper is to *geometrically* describe a canonical sequence of blowups of $\overline{M}_2(\mathbb{P}^n, d)$ such that the resulting stack admits only normal crossing singularities and each of its irreducible components is smooth. This is the best that one can hope for as far as smoothness is concerned.

In previous works [HL11, HN21], the first and third named authors proved that the blowup stacks constructed in both [HL10] and the present article admit modular interpretations: they parameterize certain geometric objects. This perspective has the advantage of being more amenable to generalization to higher genus.

Although not a primary motivation of this series of works, the smooth blowup of $\overline{M}_2(\mathbb{P}^n, d)$ obtained in this paper can be used, for instance, to prove the 'hyperplane' property of the GW invariants of quintic Calabi–Yau threefolds conjectured in [LZ09] (along with other possible calculations). Further, the method provided in this paper may be helpful for the resolution of other singular moduli used in the recent papers describing progress in mirror symmetry.

To state our main results, consider the smooth Artin stack $\mathfrak{M}_2^{\mathrm{div}}$ (written as $\mathfrak{D}_2$ in [HN21]) consisting of pre-stable pairs (C, D) where C are connected genus 2 nodal curves and $D \subset C$ are simple divisors away from the nodes of C. A pair (C, D) is pre-stable if any smooth rational component of C missing the divisor D has at least three nodes.

To resolve $\overline{M}_2(\mathbb{P}^n, d)$, we construct a sequence of smooth blowups of $\mathfrak{M}_2^{\mathrm{div}}$, then pull back this sequence to $\overline{M}_2(\mathbb{P}^n, d)$. The process would be more straightforward if there existed a natural global morphism from $\overline{M}_2(\mathbb{P}^n, d)$ to $\mathfrak{M}_2^{\mathrm{div}}$. Fortunately, although such a global morphism is not available, a rich supply of local morphisms suffices for our purpose.

For every $[C, u] \in \overline{M}_2(\mathbb{P}^n, d)$, we have $(C, u^{-1}(H)) \in \mathfrak{M}_2^{\mathrm{div}}$ for a generic hyperplane $H \subset \mathbb{P}^n$. Let $\check{\mathbb{P}}^n = \mathrm{Gr}(n, n+1)$ be the space of the hyperplanes of $\mathbb{P}^n$. Then, for every neighborhood $U \subset \overline{M}_2(\mathbb{P}^n, d)$ of $[C, u]$, we set

$$\mathbf{H}_U := \{H \in \check{\mathbb{P}}^n \,:\, (C', (u')^{-1}(H)) \in \mathfrak{M}_2^{\mathrm{div}} \,\forall\, [C', u'] \in U\}, \tag{1.1}$$

which is nonempty as long as U is small. Each $H \in \mathbf{H}_U$ gives rise to a morphism

$$f_{U,H} : U \longrightarrow \mathfrak{M}_2^{\mathrm{div}}, \quad [C, u] \mapsto (C, u^{-1}(H)). \tag{1.2}$$

We denote by

$$\pi : \mathcal{X} \longrightarrow \overline{M}_2(\mathbb{P}^n, d), \quad \mathfrak{f} : \mathcal{X} \longrightarrow \mathbb{P}^n \tag{1.3}$$

the universal family of the moduli $\overline{M}_2(\mathbb{P}^n, d)$. The main technical goal of this article is to locally diagonalize the derived objects $\mathbf{R}\pi_*\mathfrak{f}^*\mathscr{O}_{\mathbb{P}^n}(k)$, $k \geq 1$, in the sense of Definition 2.7.

Theorem 1.1. *There exist a sequence of blowups $\widetilde{\mathfrak{M}}_2^{\mathrm{div}} \longrightarrow \mathfrak{M}_2^{\mathrm{div}}$, a Deligne–Mumford stack $\widetilde{M}_2(\mathbb{P}^n, d)$, and a morphism $\widetilde{M}_2(\mathbb{P}^n, d) \longrightarrow \overline{M}_2(\mathbb{P}^n, d)$ such that for every small open $U \subset \overline{M}_2(\mathbb{P}^n, d)$ and every $H \in \mathbf{H}_U$, there exists a commutative diagram*

$$\begin{array}{ccc} U \times_{f_{U,H}, \mathfrak{M}_2^{\mathrm{div}}} \widetilde{\mathfrak{M}}_2^{\mathrm{div}} & \xrightarrow{\sim} & U \times_{\overline{M}_2(\mathbb{P}^n, d)} \widetilde{M}_2(\mathbb{P}^n, d) \\ & \searrow \quad \swarrow & \\ & U & \end{array}$$

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where the horizontal arrow is an isomorphism (of stacks over U), and the other two arrows are natural morphisms.

Moreover, the pullback of the derived object $\mathbf{R}\pi_*\mathfrak{f}^*\mathscr{O}_{\mathbb{P}^n}(1)$ to $\widetilde{M}_2(\mathbb{P}^n,d)$ becomes locally diagonalizable. For every integer $k>1$, the pullback of the derived object $\mathbf{R}\pi_*\mathfrak{f}^*\mathscr{O}_{\mathbb{P}^n}(k)$ to the main component $\widetilde{M}_2(\mathbb{P}^n,d)^{\mathrm{mc}}$ of $\widetilde{M}_2(\mathbb{P}^n,d)$ is also locally diagonalizable.

Here, the main components

$$\overline{M}_2(\mathbb{P}^n,d)^{\mathrm{mc}}\subset\overline{M}_2(\mathbb{P}^n,d)\quad\text{and}\quad\widetilde{M}_2(\mathbb{P}^n,d)^{\mathrm{mc}}\subset\widetilde{M}_2(\mathbb{P}^n,d)$$

are, respectively, the irreducible components whose general points are stable maps with smooth domain curves.

The stack $\widetilde{M}_2(\mathbb{P}^n,d)$ is obtained by gluing $U\times_{f_{U,H};\mathfrak{M}_2^{\mathrm{div}}}\widetilde{\mathfrak{M}}_2^{\mathrm{div}}$ for all small open U and $H\in\mathbf{H}_U$. For the precise construction of $\widetilde{M}_2(\mathbb{P}^n,d)$, see Corollary 5.3.

The case of Theorem 1.1 when $k=1$ leads to the following conclusion.

Corollary 1.2. *$\widetilde{M}_2(\mathbb{P}^n,d)$ has smooth irreducible components and, at worst, normal crossing singularities. For $d>2$, the main component is of the expected dimension $d(n+1)-n+3$.*

Theorem 1.1 and Corollary 1.2 are proved in Section 5; see Corollary 5.3, Proposition 5.5, and Theorem 5.4. Theorem 1.1 also gives rise to the following statement on the desingularization of the sheaves $\pi_*\mathfrak{f}^*\mathscr{O}_{\mathbb{P}^n}(k)$. This is proved in Corollary 5.6.

Corollary 1.3. *Let $(\widetilde{\pi}^{\mathrm{mc}},\widetilde{\mathfrak{f}}^{\mathrm{mc}})$ be the pullback to $\widetilde{M}_2(\mathbb{P}^n,d)^{\mathrm{mc}}$ of the universal family $(\pi,\mathfrak{f})$. Then the direct image sheaf $(\widetilde{\pi}^{\mathrm{mc}})_*(\widetilde{\mathfrak{f}}^{\mathrm{mc}})^*\mathscr{O}_{\mathbb{P}^n}(k)$ is locally free for all $k\geq 1$. It is of rank $kd-1$ if $d>2$.*

As a by-product of the construction of $\widetilde{\mathfrak{M}}_2^{\mathrm{div}}$, in Proposition 5.7 we construct a Deligne–Mumford stack $\widetilde{M}_2(\mathbb{P}^n,d;k)$ for every integer $k>1$ in a manner similar to $\widetilde{M}_2(\mathbb{P}^n,d)$. The stacks $\widetilde{M}_2(\mathbb{P}^n,d;k)$, although they are less smooth than $\widetilde{M}_2(\mathbb{P}^n,d)$, behave more nicely as far as the local diagonalizability of the pullback of the derived object $\mathbf{R}\pi_*\mathfrak{f}^*\mathscr{O}_{\mathbb{P}^n}(k)$ is concerned. The following statement is a recapitulation of Proposition 5.7.

Theorem 1.4. *For every integer $k>1$, the pullback of $\mathbf{R}\pi_*\mathfrak{f}^*\mathscr{O}_{\mathbb{P}^n}(k)$ to $\widetilde{M}_2(\mathbb{P}^n,d;k)$ becomes locally diagonalizable. Moreover, for every irreducible component N of $\widetilde{M}_2(\mathbb{P}^n,d;k)$, with $(\widetilde{\pi}_N,\widetilde{\mathfrak{f}}_N)$ denoting the pullback to N of the universal family (1.3), the direct image sheaf $(\widetilde{\pi}_N)_*(\widetilde{\mathfrak{f}}_N)^*\mathscr{O}_{\mathbb{P}^n}(k)$ is locally free. It is of rank $kd-1$ if $d>2$ and the general points of N have smooth domain curves.*

Our approach to $\widetilde{\mathfrak{M}}_2^{\mathrm{div}}/\mathfrak{M}_2^{\mathrm{div}}$ improves the technique of [HL10] and combines with the idea of the derived modular. To explain this, notice that the stack $\mathfrak{M}_2^{\mathrm{div}}$ is equipped with a universal curve $\rho:\mathcal{C}\to\mathfrak{M}_2^{\mathrm{div}}$ as well as a universal divisor $\mathcal{D}\subset\mathcal{C}$. We study the parallel diagonalization problem of the derived object $\rho_*\mathscr{O}_{\mathcal{C}}(\mathcal{D})$. Locally over a smooth chart $\mathcal{V}$ of $\mathfrak{M}_2^{\mathrm{div}}$, the object $\rho_*\mathscr{O}_{\mathcal{C}}(\mathcal{D})|_{\mathcal{V}}$ can be represented by a two-term structural homomorphism

$$\varphi:\mathscr{O}_{\mathcal{V}}^{\oplus m+1}\longrightarrow\mathscr{O}_{\mathcal{V}}^{\oplus 2},\tag{1.4}$$

where m is the degree of $\mathcal{D}$. *Throughout this article, m can be any positive integer; it corresponds to the case of $\pi_*\mathfrak{f}^*\mathscr{O}_{\mathbb{P}^n}(k)$ when $m=kd$.*

We provide a detailed analysis of this homomorphism in terms of local modular parameters in Section 2 (and later in Section 4). The analysis of φ, summarized in Proposition 2.22, guides

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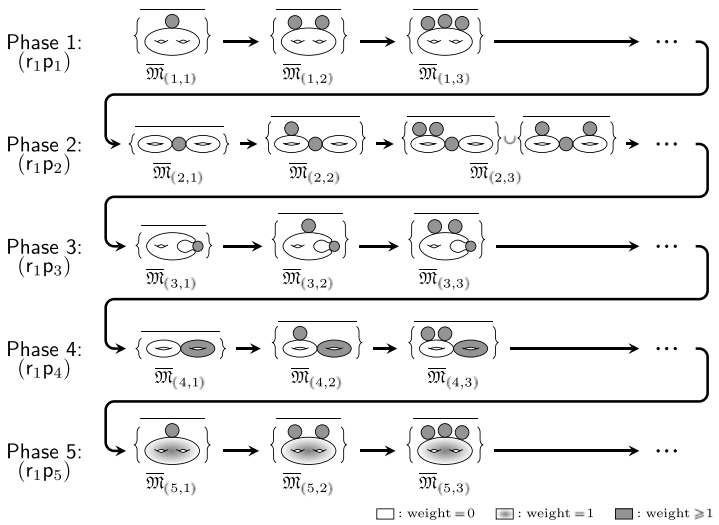


FIGURE 1. The first round (r_1) of the modular blowups.

our construction of the modular blowups in Section 3. The goal is to find global *smooth* blowups of $\mathfrak{M}_2^{\mathrm{div}}$ such that all the pullback of (1.4) can be diagonalized everywhere, locally. As proved in Section 5, this implies Theorem 1.1 as well as Corollaries 1.2 and 1.3. The procedure involves three **rounds**, the first and the last of which contain several **phases**. When the degree d is fixed, each round or phase consists of finitely many **steps**. For conciseness, we hereafter write

$$\mathsf{r}_i, \quad \mathsf{p}_j, \quad \mathsf{s}_\ell \tag{1.5}$$

respectively, for the ith round, the jth phase, and the ℓth step, whenever applicable.

We point out here that, in order to locally diagonalize $\mathbf{R}\pi_*\mathfrak{f}^*\mathscr{O}_{\mathbb{P}^n}(k)$ with $k \geq 2$, $(\mathsf{r}_1\mathsf{p}_5)$ and $(\mathsf{r}_3\mathsf{p}_4)$ can be omitted; to locally diagonalize $\mathbf{R}\pi_*\mathfrak{f}^*\mathscr{O}_{\mathbb{P}^n}(k)$ with $k \geq 3$, $(\mathsf{r}_3\mathsf{p}_3)$ can be omitted as well.

Many sequences of our blowups are performed in the Artin stack $\mathfrak{M}_2^{\mathrm{wt}}$ of stable weighted curves of genus 2 (i.e. genus 2 nodal curves whose irreducible components are decorated by weights in $\mathbb{Z}_{\geq 0}$, satisfying a certain stability condition; cf. Section 2.1), which comes equipped with the morphisms

$$\begin{aligned} &\mathfrak{M}_2^{\mathrm{div}} \longrightarrow \mathfrak{M}_2^{\mathrm{wt}}, \quad (C, D) \mapsto (C, c_1(D)), \\ &\varrho : \overline{M}_2(\mathbb{P}^n, d) \longrightarrow \mathfrak{M}_2^{\mathrm{wt}}, \quad [C, u] \mapsto (C, c_1(u^*\mathscr{O}_{\mathbb{P}^n}(1))), \end{aligned} \tag{1.6}$$

where, for every small open $U \subset \overline{M}_2(\mathbb{P}^n, d)$ and $H \in \mathbf{H}_U$, the composition of the natural inclusion $U \to \overline{M}_2(\mathbb{P}^n, d)$ and the latter morphism factors through (1.2) and the former morphism. Each blowup $\widetilde{\mathfrak{M}}' \to \mathfrak{M}_2^{\mathrm{wt}}$ determines the blowup $\mathfrak{M}_2^{\mathrm{div}} \times_{\mathfrak{M}_2^{\mathrm{wt}}} \widetilde{\mathfrak{M}}'$ of $\mathfrak{M}_2^{\mathrm{div}}$.

1.1 *The first round of blowups* (r_1)

We first blow up $\mathfrak{M}_2^{\mathrm{wt}}$ successively along the proper transforms of the substacks that are illustrated in Figure 1 and precisely described in Section 3.1. This round consists of five phases of sequential blowups: the phases are classified by the types of the **cores** of curves (i.e. the smallest subcurve of genus 2; cf. Definition 2.1) in the corresponding centers as well as the distribution of

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weights (see Figure 1). Within each given phase, the blowup centers are classified by the number of rational tails attached to the cores (also see Figure 1).

Here, we comment that in $(\mathsf{r}_1\mathsf{p}_5)$, we treat the case when the core of a curve has weight 1, even though this does not correspond to any point on the main component $\overline{M}_2(\mathbb{P}^n,d)^{\mathrm{mc}}$. The reason is twofold. First, the structural homomorphism (1.4) is not diagonalized near the corresponding points of $\mathfrak{M}_2^{\mathrm{div}}$. Since we have reduced the problem of resolving $\overline{M}_2(\mathbb{P}^n,d)$ to locally diagonalizing (1.4), we shall not specify which points of $\mathfrak{M}_2^{\mathrm{div}}$ miss the image of $\overline{M}_2(\mathbb{P}^n,d)$ or its main component under (1.2). Second, the underlying weighted curves of the blowup center of $(\mathsf{r}_1\mathsf{p}_5)$ correspond to some boundary components of $\overline{M}_2(\mathbb{P}^n,d)$. As presented in Corollary 1.2, we aim to resolve the entire moduli so that all the components are smooth and meet transversely.

When (r_1) terminates, we denote the final stack by $\widetilde{\mathfrak{M}}^{\mathsf{r}_1}$. Every point of $\mathfrak{M}_2^{\mathrm{wt}}$ has a neighborhood $\mathcal{V}$ such that the pullback $\widetilde{\varphi}^{\mathsf{r}_1}$ of the structural homomorphism φ of (1.4) to $\widetilde{\mathcal{V}}^{\mathsf{r}_1}=\mathcal{V}\times_{\mathfrak{M}_2^{\mathrm{wt}}}\widetilde{\mathfrak{M}}^{\mathsf{r}_1}$ has one row that contains an element that divides all other entries of $\widetilde{\varphi}^{\mathsf{r}_1}$ in that row (in such a case, we say that this row is locally 'diagonalized'). However, $\widetilde{\varphi}^{\mathsf{r}_1}|_{\widetilde{\mathcal{V}}^{\mathsf{r}_1}}$ may *not* be locally diagonalizable (cf. Definition 2.6) because of the existence of (1) some distinctive directions in the exceptional divisors obtained in (r_1); and/or (2) the Weierstrass and conjugate points on the smallest subcurves of genus 2.

The second round of blowups, (r_2), solves the former case; the third round of blowups, (r_3), solves the latter.

1.2 *The second round of blowups* (r_2)

After the first round is done, one of the two rows, say, the first row, of the structural homomorphism φ can be locally 'diagonalized', as explained above. We then turn our attention to the second row. In the second row, on any affine chart, some terms of the proper transform of φ may acquire exceptional parameters. Note that locally, the blowup centers in the first round are always some intersections of the (local) divisors corresponding to the loci where nodes are not smoothed (we may call them nodal divisors). The blowup centers in the second round are similar in the sense that, locally, they are always some intersections of the proper transforms of the exceptional divisors created in $(\mathsf{r}_1\mathsf{p}_1)$ as well as the proper transforms of nodal divisors. In particular, the blowup centers $X_{i,j}$ in the second round are always contained in the exceptional divisors of $(\mathsf{r}_1\mathsf{p}_1)$, as illustrated in Figure 2. However, locally, the process in (r_2) is parallel to that in (r_1); the difference is just that some exceptional parameters in (r_2) will take the roles of node-smoothing parameters in (r_1). In this narrow sense, we may say that the second round of blowups is in essence totally analogous to the first round. The blowup centers $X_{i,j}$ in (r_2) are defined in Section 3.2, specifically in (3.17).

On $\widetilde{\mathfrak{M}}^{\mathsf{r}_1}$, let $\mathcal{E}_k$ be the proper transform of the exceptional divisor created in the kth step of $(\mathsf{r}_1\mathsf{p}_1)$, which may meet the proper transforms of some nodal divisors. Locally, each connected component of $X_{k,k'}$ is the intersection of $\mathcal{E}_k$ with (the proper transforms of) some $k'-1$ many nodal divisors. Then

$$\widetilde{\mathfrak{M}}^{\mathsf{r}_2}\longrightarrow\widetilde{\mathfrak{M}}^{\mathsf{r}_1}$$

decomposes as the sequential blowups along the proper transforms of $\{X_{k,k'}\}$, with respect to the lexicographic order on (k,k') so that the order on k' is the usual order of natural numbers, whereas that on k is the reverse order of natural numbers; cf. Figure 2.

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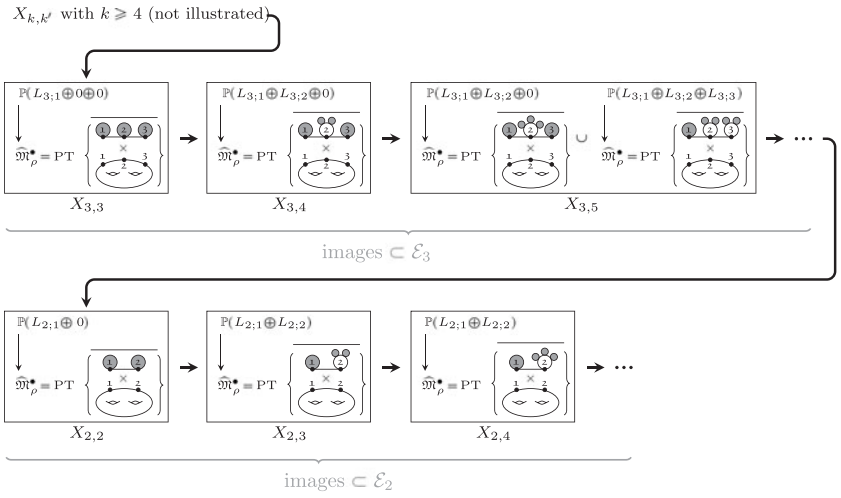


FIGURE 2. The second round (r_2) of the modular blowups.

After the second round, (r_2), the only obstacle to the pullback of φ being locally diagonalizable is the existence of the Weierstrass and conjugate points. These will be dealt with in the next (i.e. the last) round of blowups.

1.3 *The third round of blowups* (r_3)

There are four phases of sequential blowups in this round.

The blowup centers in the first phase $(\mathsf{r}_3\mathsf{p}_1)$ locally lie in the proper transforms of intersections of exceptional divisors of $(\mathsf{r}_1\mathsf{p}_1)$ and some nodal divisors. Each of such intersections, after (r_2), gives rise to two distinctive directions in the corresponding exceptional divisor of $(\mathsf{r}_1\mathsf{p}_1)$. For the underlying curve of a general point of the blowup center of any step in $(\mathsf{r}_3\mathsf{p}_1)$, its core is of weight zero, and there are two rational tails attached to conjugate points of the core that correspond to the two aforementioned distinctive directions.

The blowup centers in the second phase $(\mathsf{r}_3\mathsf{p}_2)$ locally lie in the proper transforms of intersections of exceptional divisors of (r_2) and some nodal divisors. Each of such intersections gives rise to a distinctive direction in the corresponding exceptional divisor of (r_2). For the underlying curve of a general point of the blowup center of any step in $(\mathsf{r}_3\mathsf{p}_2)$, its core is of weight zero, and there is a rational tail attached to a Weierstrass point of the core that corresponds to the aforementioned distinctive direction.

All the previous blowups are performed over the base stack $\mathfrak{M}_2^{\mathrm{wt}}$. The remaining blowups are to be performed over the base stack $\mathfrak{M}_2^{\mathrm{div}}$.

The general points of the blowup centers in the third phase $(\mathsf{r}_3\mathsf{p}_3)$ are related to the pairs (C, D) such that the core F of each C is smooth and satisfies the condition that $D \cap F$ consists of two conjugate points. More precisely, let $\mathcal{H}_k$ $(k \geq 0)$ be the closed substack of $\mathfrak{M}_2^{\mathrm{div}}$ whose general points are the pairs (C, D) such that there are k rational tails attached to the smooth core F, satisfying the condition that $D \cap F$ consists of two conjugate points. We blow up the stack $\mathfrak{M}_2^{\mathrm{div}} \times_{\mathfrak{M}_2^{\mathrm{wt}}} \widetilde{\mathfrak{M}}^{\mathsf{r}_3\mathsf{p}_2}$ along the proper transforms of $\mathcal{H}_0 \times_{\mathfrak{M}_2^{\mathrm{wt}}} \widetilde{\mathfrak{M}}^{\mathsf{r}_3\mathsf{p}_2}, \ldots, \mathcal{H}_k \times_{\mathfrak{M}_2^{\mathrm{wt}}} \widetilde{\mathfrak{M}}^{\mathsf{r}_3\mathsf{p}_2}, \ldots$ and obtain $\widetilde{\mathfrak{M}}^{\mathsf{r}_3\mathsf{p}_3}$.

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The blowup centers in the fourth phase $(\mathsf{r}_3\mathsf{p}_4)$ locally lie in the proper transforms of intersections of exceptional divisors of $(\mathsf{r}_1\mathsf{p}_5)$ and some nodal divisors. Each of such intersections gives rise to a distinctive direction in the corresponding exceptional divisor of $(\mathsf{r}_1\mathsf{p}_5)$. For the underlying point (C, D) of a general point of the blowup center of any step in $(\mathsf{r}_3\mathsf{p}_4)$, its core F contains exactly one point of D, and this point is conjugate to a point of the core where a rational tail is attached; this tail corresponds to the distinctive direction of the exceptional divisor mentioned above. We remark that $(\mathsf{r}_3\mathsf{p}_4)$ solely treats the boundary components of $\overline{M}_2(\mathbb{P}^n, d)$, just as in $(\mathsf{r}_1\mathsf{p}_5)$.

A set of examples that illustrate the above blowups is given in Section 3.6.

The key fact to verify is that the pullback of φ becomes locally diagonalizable when (r_3) terminates; see Proposition 4.1. To this end, it is crucial to understand how each round or phase of (the globally defined) modular blowups affects the moduli space *locally*, so as to study the local behavior of the structural homomorphism φ in Section 4.

Once Proposition 4.1 is established in Section 4, we apply the techniques in [HL10] and [HN21] to obtain the local structure of $\widetilde{M}_2(\mathbb{P}^n, d)$ in Section 5.

Throughout the paper, we work over $\mathbb{C}$.

1.4 Related works

Using logarithmic geometry and Gorenstein curves, Battistella and Carocci in [BCG+18], construct a desingularization of the main component of $\overline{M}_2(\mathbb{P}^n, d)$. Related work on desingularizations of sheaves and reduced GW invariants can be found in [RMM+26] by Rabano, Mann, Manolache, and Picciotto. In [LLO24], Lee, Li, and Oh construct the analogous desingularization and prove the quantum Lefschetz property for the moduli genus 2 stable quasimaps.

2. Local defining equations of $\overline{M}_2(\mathbb{P}^n, d)$

In this section, we study in detail the local defining equations of $\overline{M}_2(\mathbb{P}^n, d)$. Our resolution process is guided by these local defining equations. Some crucial examples will be provided to illustrate the simple ideas before introducing the heavy machinery.

As explained in [HL10], the local structure of $\overline{M}_2(\mathbb{P}^n, d)$, or indeed of $\overline{M}_g(\mathbb{P}^n, d)$ for any g, is governed by the derived object $\mathbf{R}\pi_*\mathfrak{f}^*\mathscr{O}_{\mathbb{P}^n}(1)$. Our aim of this section is twofold: (1) to describe $\mathbf{R}\pi_*\mathfrak{f}^*\mathscr{O}_{\mathbb{P}^n}(k)$ as explicitly as possible; and (2) to explain how to locally diagonalize (in the sense of Definition 2.7) the object $\mathbf{R}\pi_*\mathfrak{f}^*\mathscr{O}_{\mathbb{P}^n}(k)$, the $k=1$ case of which will imply a desingularization of $\overline{M}_2(\mathbb{P}^n, d)$, as desired.

The local defining equations of $\overline{M}_2(\mathbb{P}^n, d)$ are summarized in Proposition 2.22.

2.1 Basic setup

For the aforesaid purpose, we consider two stacks throughout this paper: $\mathfrak{M}_2^{\mathrm{wt}}$ and $\mathfrak{M}_2^{\mathrm{div}}$.

First, let $\mathfrak{M}_2^{\mathrm{wt}}$ be the Artin stack of stable pairs $(C, \mathbf{w})$ of genus 2 nodal curves C with non-negative weights $\mathbf{w} \in H^2(C, \mathbb{Z})$, meaning that $\mathbf{w}(\Sigma) \geq 0$ for all irreducible components Σ of C. Here $(C, \mathbf{w})$ is said to be **stable** if every rational irreducible component of weight 0 contains at least three nodal points; cf. [HL10, Section 2.1]. The stability requirement guarantees that each connected component of $\mathfrak{M}_2^{\mathrm{wt}}$, determined by the total weight on the curve, is of finite type.

Next, let $\mathfrak{M}_2^{\mathrm{div}}$ be the Artin stack of stable pairs (C, D) where C are genus 2 nodal curves and $D \subset C$ are simple and effective divisors on C away from nodes. Here (C, D) is said to

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be pre-stable if every rational irreducible component disjoint from D contains at least three nodal points; cf. [HL10, Section 2.3]. The stability requirement guarantees that each connected component of $\mathfrak{M}_2^{\mathrm{div}}$, determined by the degree of D, is also of finite type.

Between the above two stacks, there is a morphism

$$\mathfrak{M}_2^{\mathrm{div}} \longrightarrow \mathfrak{M}_2^{\mathrm{wt}}, \quad (C, D) \mapsto (C, c_1(D)). \tag{2.1}$$

We point out that this is a smooth morphism.

In studying $\overline{M}_2(\mathbb{P}^n, d)$ (along with $\pi_* \mathfrak{f}^* \mathscr{O}_{\mathbb{P}^n}(k)$), we cover it by sufficiently small étale opens $\{U \to \overline{M}_2(\mathbb{P}^n, d)\}$ such that for each $U \in \{U\}$, the set $\mathbf{H}_U$ as in (1.1) is nonempty. We pick an affine étale $\{\mathcal{V} \to \mathfrak{M}_2^{\mathrm{div}}\}$ such that for each $U \in \{U\}$ and $H \in \mathbf{H}_U$, the morphism

$$f_{U,H} : U \longrightarrow \mathfrak{M}_2^{\mathrm{div}}, \quad [C, u] \mapsto (C, u^{-1}(H))$$

introduced in (1.2) factors through $U \to \mathcal{V}$ for some $\mathcal{V} \in \{\mathcal{V}\}$. This way, we cover $\overline{M}_2(\mathbb{P}^n, d)$ by the charts $\{U/\mathcal{V}\}$.

Using the (local) morphism (1.2), it suffices to study a parallel problem on $\mathfrak{M}_2^{\mathrm{div}}$. That is, for a fixed $(C, D) \in \mathfrak{M}_2^{\mathrm{div}}$ lying over $(C, \mathbf{w}) \in \mathfrak{M}_2^{\mathrm{wt}}$ with $\deg D = m$ for some positive integer m, we pick an affine smooth chart $\mathcal{V} \to \mathfrak{M}_2^{\mathrm{div}}$ containing (C, D). Here, m can be any positive integer; it corresponds to the case of $\mathbf{R}\pi_* \mathfrak{f}^* \mathscr{O}_{\mathbb{P}^n}(k)$ when $m = dk$.

Let $\rho : \mathcal{C} \to \mathcal{V}$ and $\mathcal{D} \subset \mathcal{C}$ be the universal family on $\mathcal{V}$. We find disjoint sections $\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}$ of $\mathcal{C}/\mathcal{V}$, disjoint from $\mathcal{D}$, called the auxiliary divisors. We list this package as

$$\rho : \mathcal{C} \to \mathcal{V} \,(\to \mathfrak{M}_2^{\mathrm{div}}), \quad \mathcal{D} \subset \mathcal{C}; \quad \text{plus } \mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2,\ \mathcal{B} \subset \mathcal{C}. \tag{2.2}$$

Let

$$\mathscr{L} = \mathscr{O}_{\mathcal{C}}(\mathcal{D}), \quad \mathscr{M} = \mathscr{O}_{\mathcal{C}}(\mathcal{A} - \mathcal{B}), \quad \text{and} \quad \mathscr{O}_{\mathcal{A}}(\mathcal{A}) = \mathscr{O}_{\mathcal{V}}(\mathcal{A}) \otimes_{\mathscr{O}_{\mathcal{C}}} \mathscr{O}_{\mathcal{A}}.$$

Note that $\mathscr{L}(\mathcal{A} - \mathcal{B}) = \mathscr{M}(\mathcal{D})$. We consider the two-term complex

$$\rho_* \mathscr{L}(\mathcal{A}) \longrightarrow \rho_* \mathscr{O}_{\mathcal{A}}(\mathcal{A}) \tag{2.3}$$

via the evaluation homomorphism. These data will be fixed throughout the paper.

DEFINITION 2.1. Let C be a nodal curve. We call a node q (respectively, an irreducible component C') of C separating if $C \backslash q$ (respectively, $C \backslash C'$) is disconnected. We call other nodes (respectively, irreducible components) of C non-separating. A curve C is said to be separable (respectively, inseparable) if it contains at least one separating node (respectively, contains no separating node). A genus 1 inseparable component of C is a subcurve of C that is itself an inseparable curve of (arithmetic) genus 1. If C is of positive (arithmetic) genus g, we call the smallest connected genus g subcurve F of C the core of C. Removing F from C results in disjoint connected trees of rational curves, called the tails of C. Each tail is attached to the core F at a separating node, called a pivotal node.

We remark that it is possible that a non-separating node of C is a separating node of a subcurve of C, and hence for a genus 1 inseparable component C' of C, its nodes must be non-separating in C'. It may also be worth pointing out here that an inseparable F can be reducible: an example is a smooth rational curve attached to two distinct points of a smooth genus 1 curve.

ASSUMPTION 2.2. *Possibly after an étale base change and shrinking $\mathcal{V}$, we assume:*

(A) *$a_i = \mathcal{A}_i \cap C$ and $b = \mathcal{B} \cap C$ lie in the core F of C and are in general position; if F is separable, then a_1 and a_2 lie on different genus 1 inseparable components of F;*

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(B) $\mathcal{D} = \mathcal{D}_1 + \cdots + \mathcal{D}_m$ *is a union of* m *disjoint sections* $\mathcal{D}_i$ *of* $\mathcal{C} \to \mathcal{V}$*; and*

(C) *assumptions* (A) *and* (B) *hold for all fibers of* $\mathcal{C} \to \mathcal{V}$.

We emphasize that the general position requirement in Assumption (A) implies that none of a_1, a_2, and b are Weierstrass points in the sense of Definition 2.12, and that any two of these three points are not conjugate in the sense of Definition 2.11. In particular, a_1, a_2, and b cannot be on any non-separating bridge (should one exist; see Definition 2.9 for terminology), because by Proposition 2.19(4) (in Section 2.5), any point on a non-separating bridge is a Weierstrass point.

The lemma below follows from direct verification.

Lemma 2.3. *Under Assumption 2.2, we have* $R^1\rho_*\mathscr{L}(\mathcal{A}) = 0$, $\rho_*\mathscr{L}(\mathcal{A})$ *is locally free, and* $\mathbf{R}\rho_*\mathscr{L}$ *is quasi-isomorphic to the complex* (2.3).

2.2 The first reduction

The standard inclusion $\mathscr{O}_\mathcal{C} \subset \mathscr{O}_\mathcal{C}(\mathcal{D})$ determines the obvious section $1 \in \Gamma(\rho_*\mathscr{L})$. For other sections, let $\mathscr{M} = \mathscr{O}_\mathcal{C}(\mathcal{A} - \mathcal{B})$ and consider the inclusion

$$\mathscr{M}(\mathcal{D}_i) = \mathscr{O}_\mathcal{C}(\mathcal{D}_i + \mathcal{A} - \mathcal{B}) \xrightarrow{\subset} \mathscr{M}(\mathcal{D}) = \mathscr{O}_\mathcal{C}(\mathcal{D} + \mathcal{A} - \mathcal{B}),$$

and the induced inclusions

$$\eta_i : \rho_*\mathscr{M}(\mathcal{D}_i) \xrightarrow{\subset} \rho_*\mathscr{M}(\mathcal{D}).$$

Because of Assumption 2.2, the above terms are locally free. By the Riemann–Roch Theorem, $\rho_*\mathscr{M}(\mathcal{D}_i)$ is invertible and $\rho_*\mathscr{M}(\mathcal{D})$ has rank m. Let

$$\varphi : \rho_*\mathscr{M}(\mathcal{D}) \longrightarrow \rho_*\mathscr{O}_\mathcal{A}(\mathcal{A}) \quad \text{and} \quad \varphi_i : \rho_*\mathscr{M}(\mathcal{D}_i) \longrightarrow \rho_*\mathscr{O}_\mathcal{A}(\mathcal{A}) \tag{2.4}$$

be the evaluation homomorphisms; then $\varphi_i = \varphi \circ \eta_i$. We denote their sum by

$$(\eta_i) : \oplus_{i=1}^m \rho_*\mathscr{M}(\mathcal{D}_i) \longrightarrow \rho_*\mathscr{M}(\mathcal{D}),$$
$$(\varphi_i) : \oplus_{i=1}^m \rho_*\mathscr{M}(\mathcal{D}_i) \longrightarrow \rho_*\mathscr{O}_\mathcal{A}(\mathcal{A}).$$

Lemma 2.4. *We have:* (1). $\rho_*\mathscr{L} \cong \mathscr{O}_\mathcal{V} \oplus \rho_*\mathscr{L}(-\mathcal{B})$; (2). $\rho_*\mathscr{L}(-\mathcal{B}) \cong \ker\varphi$; (3). (η_i) *is an isomorphism; and* (4). $(\varphi_i) = \varphi \circ (\eta_i)$.

Consequently,

$$\rho_*\mathscr{L} \cong \mathscr{O}_\mathcal{V} \oplus \ker\{(\varphi_i)\}.$$

Proof. The proofs of (1) and (2) are the same as for the genus 1 case, [HL10, Lemma 4.10]; we omit the details. The proof of (3) is also similar. We now give details on it. Since both $\rho_*\mathscr{M}(\mathcal{D}_i)$ and $\rho_*\mathscr{M}(\mathcal{D})$ are locally free, we only need to show that for any closed $z \in \mathcal{V}$, $\oplus_{i=1}^m \rho_*\mathscr{M}(\mathcal{D}_i)|_z \to \rho_*\mathscr{M}(\mathcal{D})|_z$ is an isomorphism. By base change, it suffices to show that the tautological homomorphism

$$\bigoplus_{i=1}^m \eta_i(z) : \bigoplus_{i=1}^m H^0(\mathscr{O}_{\mathcal{C}_z}(\mathcal{D}_i + \mathcal{A} - \mathcal{B})) \longrightarrow H^0(\mathscr{O}_{\mathcal{C}_z}(\mathcal{D} + \mathcal{A} - \mathcal{B}))$$

is an isomorphism. Because both sides are of the same dimension, it suffices to show that it is injective. For this, we look at the composite of $\oplus\eta_i(z)$ with the restriction

$$\phi_j(z) : H^0(\mathcal{C}_z, \mathscr{O}_{\mathcal{C}_z}(\mathcal{D} + \mathcal{A} - \mathcal{B})) \longrightarrow H^0(\mathcal{C}_z \cap \mathcal{D}_j, \mathscr{O}_{\mathcal{C}_z \cap \mathcal{D}_j}(\mathcal{D} + \mathcal{A} - \mathcal{B})).$$

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For any i, when $i=j$, $\phi_j(z)\circ\eta_i(z)$ is an isomorphism for all the points $z\in\mathcal{V}$, and when $i\neq j$, $\phi_j(z)\circ\eta_i(z)=0$ holds for a general point $z\in\mathcal{V}$ and hence also holds for all the points. Thus, if $(\oplus_{i=1}^m\eta_i(z))(\oplus v_i)=\oplus_{i=1}^m\eta_i(z)(v_i)=0$ where $v_i\in H^0(\mathscr{O}_{\mathcal{C}_z}(\mathcal{D}_i+\mathcal{A}-\mathcal{B}))$, then $\oplus_{i=1}^m\phi_j(z)\circ\eta_i(z)(v_i)=\phi_j(z)\circ\eta_j(z)(v_j)=0$. Hence, $v_j=0$ for all $1\leq j\leq m$. This shows that $\oplus_{i=1}^m\eta_i(z)$ is injective. Item (4) is a tautology. This proves the lemma. □

This proves that the homomorphism φ in (2.4) is equivalent to the homomorphism (φ_i), via the isomorphism (η_i).

We will call either φ or (φ_i) a structural homomorphism. Note that each φ_i has the form

$$\varphi_i=\varphi_i^1\oplus\varphi_i^2:\rho_*\mathscr{M}(\mathcal{D}_i)\longrightarrow\rho_*\mathscr{O}_{\mathcal{A}_1}(\mathcal{A}_1)\oplus\rho_*\mathscr{O}_{\mathcal{A}_2}(\mathcal{A}_2). \tag{2.5}$$

These homomorphisms will be our focus in this section.

2.3 Local open immersions

To obtain local equations of $\overline{M}_2(\mathbb{P}^n,d)$, we consider the case of $\pi_*\mathfrak{f}^*\mathscr{O}_{\mathbb{P}^n}(1)$, and let $m=d$. Then, as in Section 2.1, we cover $\overline{M}_2(\mathbb{P}^n,d)$ by charts $\{U/\mathcal{V}\}$. Let $\mathscr{E}_{\mathcal{V}}$ be the total space of the vector bundle $\rho_*\mathscr{L}(\mathcal{A})^{\oplus n}$ and let

$$p:\mathscr{E}_{\mathcal{V}}\to\mathcal{V}$$

be the projection. We also set

$$\rho_*(\mathscr{L}(\mathcal{A})^{\oplus n}|_{\mathcal{A}})=\rho_*(\mathscr{L}(\mathcal{A})^{\oplus n}|_{\mathcal{A}_1})\oplus\rho_*(\mathscr{L}(\mathcal{A})^{\oplus n}|_{\mathcal{A}_2}).$$

Then the tautological restriction homomorphism

$$\text{rest}:\rho_*\mathscr{L}(\mathcal{A})^{\oplus n}\longrightarrow\rho_*(\mathscr{L}(\mathcal{A})^{\oplus n}|_{\mathcal{A}}) \tag{2.6}$$

lifts to a section

$$\Phi\in\Gamma(\mathscr{E}_{\mathcal{V}},\ p^*\rho_*(\mathscr{L}(\mathcal{A})^{\oplus n}|_{\mathcal{A}})). \tag{2.7}$$

By choosing a suitable trivialization, the homomorphism 'rest' of (2.6) can be identified with the homomorphism

$$(0\oplus\varphi)^{\oplus n}:(\mathscr{O}_{\mathcal{V}}\oplus\rho_*\mathscr{M}(\mathcal{D}))^{\oplus n}\longrightarrow(\rho_*\mathscr{O}_{\mathcal{A}}(\mathcal{A}))^{\oplus n}, \tag{2.8}$$

where φ is the structural homomorphism in (2.4). Then, the section Φ of (2.7) is the lift of $(0\oplus\varphi)^{\oplus n}$ in (2.8).

Theorem 2.5. *Let $\mathcal{U}=\mathcal{V}\times_{\mathfrak{M}_2^{\mathrm{div}};f_{U,H}}U$. Then there is a canonical open immersion*

$$\mathcal{U}\longrightarrow(\Phi=0)\subset\mathscr{E}_{\mathcal{V}}. \tag{2.9}$$

Proof. This theorem is the natural extension of [HL10, Theorem 2.17]; the proof is also parallel. □

In what follows, we will focus on the structures of the objects

$$\varphi:\rho_*\mathscr{M}(\mathcal{D})\longrightarrow\rho_*\mathscr{O}_{\mathcal{A}}(\mathcal{A}).$$

We begin with the notion of locally diagonalizable derived objects, as introduced in [HL10]. This is key to this article.

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2.4 Locally diagonalizable homomorphisms

Let $\varphi : \mathscr{E}_1 \to \mathscr{E}_2$ be a homomorphism of locally free sheaves of $\mathscr{O}_U$-modules on a Deligne-Mumford (DM) stack U.

Definition 2.6. We say φ is diagonalizable if there are integers r, l_1 and $l_2 \in \mathbb{Z}_{\geq 0}$, $p_i \in \Gamma(\mathscr{O}_U)$ such that the ideals $(p_i) \supset (p_{i+1})$, and isomorphisms $\mathscr{E}_i \cong \mathscr{O}_U^{\oplus r} \oplus \mathscr{O}_U^{\oplus l_i}$ such that

$$\varphi = \operatorname{diag}[p_1, \ldots, p_r] \oplus 0 : \mathscr{O}_U^{\oplus r} \oplus \mathscr{O}_U^{\oplus l_1} \longrightarrow \mathscr{O}_U^{\oplus r} \oplus \mathscr{O}_U^{\oplus l_2},$$

where $\operatorname{diag}[p_1, \ldots, p_r] : \mathscr{O}_U^{\oplus r} \to \mathscr{O}_U^{\oplus r}$ is the diagonal homomorphism and $0 : \mathscr{O}_U^{\oplus l_1} \to \mathscr{O}_U^{\oplus l_2}$ is the zero homomorphism. We say φ is locally diagonalizable if there is an étale cover U_α of U such that φ is diagonalizable over U_α.

Definition 2.7. Let $\mathbf{E}$ be a perfect derived object on a DM stack M. We say $\mathbf{E}$ is locally diagonalizable if there is an étale cover $\{U\}$ of M such that $\mathbf{E}|_U$ is quasi-isomorphic to $[\varphi : \mathscr{E}_1 \to \mathscr{E}_2]$, where $\mathscr{E}_1$ and $\mathscr{E}_2$ are locally free sheaves and φ is diagonalizable.

Using [HL10, Proposition 3.2] (the universality of diagonalization), one can check directly that the definition does not depend on the local representation of the object $\mathbf{E}$. Here it is worth pointing out that in Definition 2.6, if the requirement $(p_{i+1}) \subset (p_i)$ is removed, then the 'diagonalization' thus defined is not a property of the derived object but merely a property of a presentation of the object $\mathbf{E}$.

The following result is a conclusion of [HL10, Corollary 4.4]. It is needed to deduce Corollary 1.3 from Theorem 1.1 in Section 1.

Proposition 2.8 [HL10]. *Let M and $\widetilde{M}$ be two DM stacks and $\mathbf{E}$ be a perfect derived object on M. Assume there is a birational morphism $\psi : \widetilde{M} \to M$ such that the pullback of $\mathbf{E}$ is locally diagonalizable in the sense of Definition 2.6. Then, for any irreducible component N of $\widetilde{M}$ endowed with the reduced stack structure, $\mathscr{H}^0(L\psi^*\mathbf{E}|_N)$ is locally free.*

As local diagonalization is functorial, a local diagonalization of $\mathbf{R}\rho_*\mathscr{O}_{\mathcal{C}}(\mathcal{D})$ will imply a resolution of $\overline{M}_2(\mathbb{P}^n, d)$. The remainder of Section 2 is devoted to a detailed analysis of the local structures of the object $\mathbf{R}\rho_*\mathscr{O}_{\mathcal{C}}(\mathcal{D})$.

2.5 Notation on nodal curves and the initial forms of structural homomorphisms

By the deformation theory of nodal curves, for each node $q \in C$ there is a regular function $\zeta_q \in \Gamma(\mathscr{O}_{\mathcal{V}})$ such that $\{\zeta_q = 0\}$ is the locus where the node q is not smoothed; the divisor $(\zeta_q = 0)$ is smooth. We will refer to ζ_q as a modular parameter or a node-smoothing parameter.

To proceed, we follow the notation and terminology of nodal curves in Definition 2.1. Let C be a nodal curve of genus 2. We denote by $N(C)$ the set of the nodes of C. A node $q \in N(C)$ is called a core node if it is a node (i.e. a singular point) of the core F of C; in particular, any pivotal node cannot be a core node.

We say a separating node q separates (or lies between) two smooth points δ and δ' of C if δ and δ' lie in different connected components of $C \backslash \{q\}$. If we replace δ' by a connected subcurve C' of C, we say q lies between δ and C' if δ' and $C' \backslash \{q\}$ lie in different connected components of $C \backslash \{q\}$. We denote by

$$N_{[\delta,\delta']} \quad \text{(respectively } N_{[\delta,C']}\text{)}$$

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the set of the *separating* nodes lying between δ and δ' (respectively, C'). In particular, for $C' = F$, we write

$$N_{[\delta]} := N_{[\delta,F]}.$$

Below, we introduce an important type of rational subcurve of the core F of C.

DEFINITION 2.9. If F is a union of two connected subcurves, B of arithmetic genus 0 and F_1 of arithmetic genus 1, such that B and F_1 do not share any common irreducible component, then we call B a **non-separating bridge** of C. In other words, B is a non-separating bridge if there exist two non-separating core nodes $\mathfrak{p}$ and $\mathfrak{q}$ such that

$$F = \mathrm{B} \cup F_1, \quad \mathrm{B} \cap F_1 = \{\mathfrak{p}, \mathfrak{q}\}, \quad p_a(\mathrm{B}) = 0, \quad p_a(F_1) = 1,$$

where p_a denotes the arithmetic genus.

If F is a union of three connected subcurves, B of arithmetic genus 0, and F_1 and F_2 both of arithmetic genus 1, such that any two of B, F_1 and F_2 do not share any common irreducible component, then we call B a **separating bridge** of C. In other words, B is a separating bridge if there exist two separating core nodes $\mathfrak{p}$ and $\mathfrak{q}$ such that

$$F = \mathrm{B} \cup F_1 \cup F_2, \quad \mathrm{B} \cap F_1 = \{\mathfrak{p}\}, \quad \mathrm{B} \cap F_2 = \{\mathfrak{q}\}, \quad p_a(\mathrm{B}) = 0, \quad p_a(F_1) = p_a(F_2) = 1$$

(which implies $F_1 \cap F_2 = \emptyset$ because $p_a(F) = 2$).

A bridge, non-separating or separating, is **maximal** if it is not a strict subset of any other bridge.

When we need to, we denote a bridge B by $\mathrm{B}[\mathfrak{p}, \mathfrak{q}]$ to emphasize the two nodes.

It is straightforward to show that all the core nodes meeting a non-separating (respectively, separating) bridge $\mathrm{B}[\mathfrak{p}, \mathfrak{q}]$, including $\mathfrak{p}$ and $\mathfrak{q}$, are non-separating (respectively, separating). By the topology of nodal curves, a genus 2 nodal curve C may contain at most three non-separating bridges, or at most one separating bridge. Indeed, if C contains three non-separating bridges, then it cannot have any separating bridge; if C contains a separating bridge, then it may contain at most two non-separating bridges.

For example, the shaded component of the leftmost picture in the third row (respectively, the second row) of Figure 1 is a non-separating bridge (respectively, a separating bridge).

If there exists a non-separating bridge $B = \mathrm{B}[\mathfrak{p}, \mathfrak{q}]$ on F, then for every smooth point $\delta \in B$, we define

$$N_{[\delta,\mathfrak{p}]} \quad (\text{respectively } N_{[\delta,\mathfrak{q}]}) \tag{2.10}$$

to be set of all the core nodes of B between δ and $\mathfrak{p}$ (respectively, $\mathfrak{q}$), including $\mathfrak{p}$ (respectively, $\mathfrak{q}$). We emphasize that both $N_{[\delta,\mathfrak{p}]}$ and $N_{[\delta,\mathfrak{q}]}$ are nonempty as long as B exists, because $\mathfrak{p} \in N_{[\delta,\mathfrak{p}]}$ and $\mathfrak{q} \in N_{[\delta,\mathfrak{q}]}$.

If T is a tail and $\delta, \delta' \in T$, then among all the nodes of $N_{[\delta]} \cap N_{[\delta']}$, i.e. all the common nodes lying between δ and F as well as between δ' and F, we denote by $\delta \wedge \delta'$ the one that is the farthest from F. In other words, $\delta \wedge \delta'$ is the node satisfying

$$N_{[\delta \wedge \delta']} = N_{[\delta]} \cap N_{[\delta']}; \tag{2.11}$$

in particular, $\delta \wedge \delta'$ cannot be any core node. If δ and δ' are on different tails, or at least one of them is on the core, we define $N_{[\delta \wedge \delta']} := \emptyset$, which is in accordance with (2.11). However, $\delta \wedge \delta'$ is not defined in these cases.

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Next, consider the family $\mathcal{C}\to\mathcal{V}$ with $C=\mathcal{C}\times_{\mathcal{V}}0$ the central fiber mentioned before. For $1\le i\le m$ and $1\le s\le 2$, let

$$\delta_i=\mathcal{D}_i\cap C\quad\text{and}\quad a_s=\mathcal{A}_s\cap C.$$

For smooth points $\delta,\delta'\in C$, we define

$$\zeta_{[\delta,\delta']}=\prod_{q\in N_{[\delta,\delta']}}\zeta_q\quad\text{and}\quad\zeta_{[\delta]}=\prod_{q\in N_{[\delta]}}\zeta_q. \tag{2.12}$$

If there exists a non-separating bridge $B=\mathrm{B}[\mathfrak{p},\mathfrak{q}]$ on F, then for every smooth point $\delta\in B$, we set

$$\zeta_{[\delta,\mathfrak{p}]}=\prod_{q\in N_{[\delta,\mathfrak{p}]}}\zeta_q\quad\text{and}\quad\zeta_{[\delta,\mathfrak{q}]}=\prod_{q\in N_{[\delta,\mathfrak{q}]}}\zeta_q. \tag{2.13}$$

In the case that $N_{[\delta,\delta']}$ (respectively, $N_{[\delta]}$, $N_{[\delta,\mathfrak{p}]}$, $N_{[\delta,\mathfrak{p}]}$) is empty, we set $\zeta_{[\delta,y]}$ (respectively, $\zeta_{[\delta]}$, $\zeta_{[\delta,\mathfrak{p}]}$, $\zeta_{[\delta,\mathfrak{q}]}$) to be 1. Similarly, if $\delta\wedge\delta'$ does not exist (i.e. $N_{[\delta\wedge\delta']}=\emptyset$), we set $\zeta_{[\delta\wedge\delta']}=1$.

The above notation on nodal curves is illustrated in Figure 3.

We fix the trivializations

$$\rho_*\mathscr{M}(\mathcal{D}_i)\cong\mathscr{O}_{\mathcal{V}},\quad\rho_*\mathscr{O}_{\mathcal{A}_s}(\mathcal{A}_s)\cong\mathscr{O}_{\mathcal{V}};\quad 1\le i\le m,\quad 1\le s\le 2, \tag{2.14}$$

and the induced trivialization $\rho_*\mathscr{O}_{\mathcal{A}}(\mathcal{A})\cong\mathscr{O}_{\mathcal{V}}^{\oplus 2}$, throughout this section, unless stated otherwise.

Proposition 2.10. *After fixing (2.14), the evaluation homomorphism*

$$\varphi_{si}:\rho_*\mathscr{M}(\mathcal{D}_i)\longrightarrow\rho_*\mathscr{O}_{\mathcal{A}_s}(\mathcal{A}_s) \tag{2.15}$$

is given by the multiplication

$$\varphi_{si}=c_{si}\cdot\zeta_{[\delta_i,a_s]}\ :\mathscr{O}_{\mathcal{V}}\longrightarrow\mathscr{O}_{\mathcal{V}},\quad c_{si}\in\Gamma(\mathscr{O}_{\mathcal{V}}^*).$$

Proof. The proof is parallel to that of [HKK+03, Proposition 4.13] and is thus omitted. □

By abuse of notation, we still denote by φ the $2\times m$ matrix $(c_{si}\cdot\zeta_{[\delta_i,a_s]})$ with entries as in (2.15) and still call it the structural homomorphism when the context is clear.

2.6 Conjugate and Weierstrass loci

In the genus 1 case, the analogue of Proposition 2.10 contains all the necessary data for locally diagonalizing the structural homomorphism in the sense of Definition 2.6. In the genus 2 case, however, Proposition 2.10 alone is not sufficient for studying the 2×2 minors of φ, which is crucial in diagonalizing φ. To this end, we need to first study the loci of $\mathfrak{M}_2^{\mathrm{div}}$ consisting of (C,D) when some points of D are in the special positions of C. For conciseness, in this subsection we deal with the loci of $\bigsqcup_{\ell>0}\overline{\mathcal{M}}_{2,\ell}$ rather than $\mathfrak{M}_2^{\mathrm{div}}$, because *locally* neither the order of the marked points nor the subtle difference in the stability conditions of the two stacks affects the local equations of the conjugate or Weierstrass loci; see Remark 2.21 for more detail.

Definition 2.11. Two distinct points δ_1 and δ_2 on a smooth genus 2 curve C are said to be conjugate if

$$h^0(C,\mathscr{O}_C(\delta_1+\delta_2))=2\quad(\text{or equivalently, } h^0(C,\omega_C(-\delta_1-\delta_2))=1).$$

We denote by $\mathcal{K}$ the locus of smooth genus 2 curves with a pair of conjugate marked points and by $\overline{\mathcal{K}}$ the closure of $\mathcal{K}$ in $\overline{\mathcal{M}}_{2,2}$. For every two-marked stable curve $(C,\delta_1,\delta_2)\in\overline{\mathcal{K}}$, we say the marked points δ_1 and δ_2 are conjugate.

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DEFINITION 2.12. A point δ on a smooth genus 2 curve C is called a **Weierstrass** point if

$$h^0(C, \mathscr{O}_C(2\delta)) = 2, \quad \text{(or equivalently, } h^0(C, \omega_C(-2\delta)) = 1\text{)}.$$

We denote by $\mathcal{W} \subset \mathcal{M}_{2,1}$ the locus of smooth genus 2 curves with one marked Weierstrass point and by $\overline{\mathcal{W}}$ the closure of $\mathcal{W}$ in $\overline{\mathcal{M}}_{2,1}$. For every one-marked stable curve $(C, \delta) \in \overline{\mathcal{W}}$, we call the marked point δ a **Weierstrass point**.

When more marked points are involved, we define conjugacy and Weierstrassness via pullbacks with respect to the forgetful morphisms.

Concretely, for $1 \le i, j \le \ell$ with $i \ne j$, we set $\overline{\mathcal{K}}_{\ell;i,j} \subset \overline{\mathcal{M}}_{2,\ell}$ to be the pullback of $\overline{\mathcal{K}}$ in $\overline{\mathcal{M}}_{2,\ell}$ via the forgetful morphism that forgets all but the ith and jth marked points. If $x = (C, \delta_1, \dots, \delta_\ell) \in \overline{\mathcal{K}}_{\ell;i,j}$, we say that δ_i and δ_j are **conjugate**.

Similarly, for $1 \le i \le \ell$, let $\overline{\mathcal{W}}_{\ell;i} \subset \overline{\mathcal{M}}_{2,\ell}$ be the pullback of $\overline{\mathcal{W}}$ in $\overline{\mathcal{M}}_{2,\ell}$ via the forgetful morphism that forgets all but the ith marked point. If $x = (C, \delta_1, \dots, \delta_\ell) \in \overline{\mathcal{W}}_{\ell;i}$, we say that δ_i is **Weierstrass**.

The main goal of Section 2.6 is to study the local equations of $\overline{\mathcal{K}}_{\ell;i,j}$ and $\overline{\mathcal{W}}_{\ell;i}$. We begin with $\overline{\mathcal{K}}$ and $\overline{\mathcal{W}}$.

2.6.1 *Local equations of $\overline{\mathcal{K}}$ and $\overline{\mathcal{W}}$.* Recall that the Hodge bundle $\Omega\mathcal{M}_{2,\ell}$ is a rank 2 vector bundle over $\mathcal{M}_{2,\ell}$ whose fiber over every $(C, \delta_1, \dots, \delta_\ell) \in \mathcal{M}_{2,\ell}$ consists of holomorphic Abelian differentials η on C. Following [BC23], $\overline{\mathcal{K}}$ and $\overline{\mathcal{W}}$ can be interpreted as the *incidence variety compactifications* of certain strata of the projectivized Hodge bundles $\mathbb{P}\Omega\mathcal{M}_{2,2}$ and $\mathbb{P}\Omega\mathcal{M}_{2,1}$, respectively.

More precisely, let $\Omega\mathcal{M}_{2,2}(1,1)$ be the stratum of $\Omega\mathcal{M}_{2,2}$ consisting of all $(C, \delta_1, \delta_2, \eta)$ satisfying the condition that η has simple zeros at δ_1 and δ_2. Similarly, let $\Omega\mathcal{M}_{2,1}(2)$ be the stratum of $\Omega\mathcal{M}_{2,1}$ consisting of all (C, δ_1, η) satisfying the condition that η has a zero of order 2 at δ_1. Then, by forgetting the differentials, it is straightforward that $\mathbb{P}\Omega\mathcal{M}_{2,2}(1,1)$ and $\mathbb{P}\Omega\mathcal{M}_{2,1}(2)$ are isomorphic to $\mathcal{K}$ and $\mathcal{W}$, respectively. Let $\mathbb{P}\Omega\overline{\mathcal{M}}^{\mathrm{inc}}_{2,2}(1,1)$ and $\mathbb{P}\Omega\overline{\mathcal{M}}^{\mathrm{inc}}_{2,1}(2)$ be the incidence variety compactifications of $\mathbb{P}\Omega\mathcal{M}_{2,2}(1,1)$ and $\mathbb{P}\Omega\mathcal{M}_{2,1}(2)$, respectively, in the sense of [BC23, Theorem 1.3]. Then, by [BC23, Corollary 1.4], the projections by forgetting the differentials induce surjections

$$\mathbb{P}\Omega\overline{\mathcal{M}}^{\mathrm{inc}}_{2,2}(1,1) \longrightarrow \overline{\mathcal{K}} \quad \text{and} \quad \mathbb{P}\Omega\overline{\mathcal{M}}^{\mathrm{inc}}_{2,1}(2) \longrightarrow \overline{\mathcal{W}}, \tag{2.16}$$

respectively. These surjections lead to the following lemmas on the possible locations of conjugate and Weierstrass points on a genus 2 stable curve with marked points.

For convenience, for every point $\delta \in C$, hereafter we write

$$\langle \delta \rangle := \begin{cases} \delta, & \text{if } \delta \text{ is on the core of } C, \\ \text{the pivotal node of the tail containing } \delta, & \text{if } \delta \text{ is on a tail of } C; \end{cases} \tag{2.17}$$

see Figure 3 for an illustration.

LEMMA 2.13. *Given $(C, \delta) \in \overline{\mathcal{M}}_{2,1}$, if δ is on a non-separating bridge, then $(C, \delta) \in \overline{\mathcal{W}}$; if δ is on a separating bridge, then $(C, \delta) \notin \overline{\mathcal{W}}$.*

Proof. The statements follow directly from the proof of [BC23, Proposition 3.12], where the boundary components of $\mathbb{P}\Omega\overline{\mathcal{M}}^{\mathrm{inc}}_{2,1}(2)$ are described explicitly. □

LEMMA 2.14. *Given $(C, \delta_1, \delta_2) \in \overline{\mathcal{M}}_{2,2}$, we have the following:*

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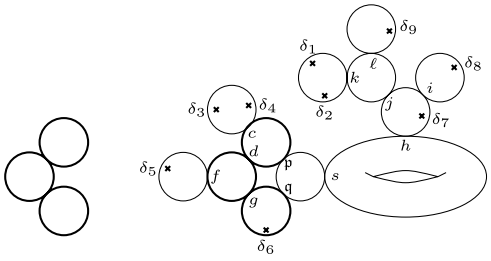


FIGURE 3. An example of the notation on nodal curves.

- *Assume δ_1 is on a tail, which implies δ_2 is on the same tail by stability. Then $(C, \delta_1, \delta_2) \in \overline{\mathcal{K}}$ if and only if $\langle\delta_1\rangle$ $(= \langle\delta_2\rangle)$ is a Weierstrass point of the core F of C.*
- *Assume δ_1 is on the core F, which implies $\delta_2 \in F$ by stability.*
  - *Assume in addition that δ_1 is on a maximal non-separating bridge* B. *Then $(C, \delta_1, \delta_2) \in \overline{\mathcal{K}}$ if and only if $\delta_2 \in$ B.*
  - *Assume in addition that δ_1 is not on any non-separating bridge.*

    ∗*If δ_1 is a Weierstrass point, then $(C, \delta_1, \delta_2) \notin \overline{\mathcal{K}}$.*

    ∗*If δ_1 is not a Weierstrass point, then there exists a unique smooth point $\delta_1^\vee \in C$ such that $(C, \delta_1, \delta_2) \in \overline{\mathcal{K}}$ if and only if $\delta_2 = \delta_1^\vee$. Moreover, if δ_1 is on a genus 1 inseparable component C' of C, then $\delta_2 \in C'$ as well; if δ_1 is on a separating bridge (of F), then δ_1 and δ_2 are on the same smooth rational subcurve.*

Proof. Consider the first case when δ_1 and δ_2 are on a tail R, which implies that R is smooth. In this case, we further assume that F is smooth; the sub-case when F is singular follows from the closedness of $\overline{\mathcal{K}}$ and $\overline{\mathcal{W}}$. By (2.16), we see that $(C, \delta_1, \delta_2) \in \overline{\mathcal{K}}$ if and only if there exists a pointed stable differential of type $(1, 1)$ over (C, δ_1, δ_2) satisfying the conditions in [BC23, Theorem 1.3]. Notice that on R, to scaling, there uniquely exists a (non-zero) meromorphic differential with simple zeros at δ_1 and δ_2, as well as a pole at $q := \langle\delta_1\rangle$ whose order and residue are, respectively, -4 and 0. Analyzing the five conditions of [BC23, Definitions 1.1 & 1.2], we conclude that $(C, \delta_1, \delta_2) \in \overline{\mathcal{K}}$ if and only if there exists a (non-zero) holomorphic differential on F with a zero of order 2 at q (because F is assumed to be smooth), i.e. $h^0(F, K_F - 2q) = 1$, which is equivalent to $(F, q) \in \overline{\mathcal{W}}$.

Consider the second case when $\delta_1 \in F$ is a on a maximal non-separating bridge $\mathrm{B} = \mathrm{B}[\mathfrak{p}, \mathfrak{q}]$. Let $\widehat{T}$ be the (connected) genus 1 subcurve of F that meets B at $\mathfrak{p}$ and $\mathfrak{q}$.

- If $\delta_2 \in$ B, to show $(C, \delta_1, \delta_2) \in \overline{\mathcal{K}}$, we only need to consider the situation when B and $\widehat{T}$ are smooth (because of the closedness of $\overline{\mathcal{K}}$); see Graph (a) of Figure 4 for illustration. In this situation, notice that there exists a (non-zero) meromorphic differential on B with simple zeros at δ_1 and δ_2 as well as poles of order 2 at $\mathfrak{p}$ and $\mathfrak{q}$, whose total residue on B is 0. Also notice that every (non-zero) holomorphic differential on $\widehat{T}$ has no zeros. Then, by [BC23, Theorem 1.3], we have $(C, \delta_1, \delta_2) \in \overline{\mathcal{K}}$.
- If $\delta_2 \notin$ B, let T' be the (unique) genus 1 inseparable subcurve of $\widehat{T}$. Notice that every element of $H^0(C, \omega_C(-\delta_1))$, which is of rank 1, vanishes identically on $C \backslash T'$. In addition, observe that for every level graph $\overline{\Gamma}$ (in the sense of [BC23, Section 1.5]) whose maxima

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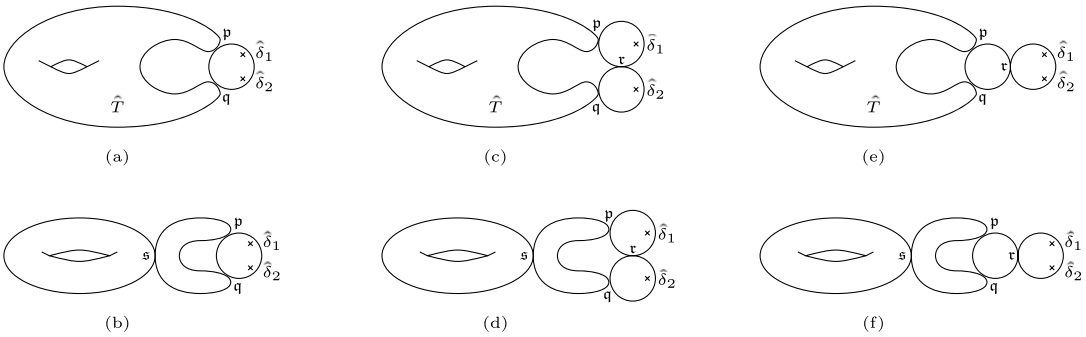


FIGURE 4. $(\widehat{C}, \widehat{\delta}_1, \widehat{\delta}_2) \in \overline{\mathcal{K}}$ whose marked points are on a non-separating bridge.

are the irreducible components of T', every twisted differential η compatible with $\overline{\Gamma}$ (in the sense of [BC23, Definitions 1.1 & 1.2]) that has a simple zero at δ_1 must have another simple zero on B. This implies $\eta(\delta_2) \neq 0$, so by [BC23, Theorem 1.3], we have $(C, \delta_1, \delta_2) \notin \overline{\mathcal{K}}$.

The statements when $\delta_1 \in F$ is not on any non-separating bridge can be proved by applying [BC23, Theorem 1.3] analogously (on a case-by-case basis). We omit further details. □

It is well known that $\mathcal{W}$ is a smooth Cartier divisor of $\mathcal{M}_{2,1}$; see [Cuk89, Proposition 3.2.1], for example. In addition, the restriction to $\mathcal{K}$ of the forgetful morphism $\mathcal{M}_{2,2} \longrightarrow \mathcal{M}_{2,1}$ that forgets the second marked point is an isomorphism to $\mathcal{M}_{2,1}\backslash\mathcal{W}$, so $\mathcal{K}$ is a smooth Cartier divisor of $\mathcal{M}_{2,2}$. Below, we show that the same result holds for $\overline{\mathcal{K}} \subset \overline{\mathcal{M}}_{2,2}$ and $\overline{\mathcal{W}} \subset \overline{\mathcal{M}}_{2,1}$ as well.

PROPOSITION 2.15. *$\overline{\mathcal{K}}$ is a smooth Cartier divisor of $\overline{\mathcal{M}}_{2,2}$.*

Proof. Let $Z \subset \overline{\mathcal{K}}$ be the substack consisting of $(C, \delta_1, \delta_2) \in \overline{\mathcal{K}}$ so that there exists a non-separating bridge of (the core of) C containing $\langle\delta_1\rangle$ and $\langle\delta_2\rangle$. Similarly, let $Z' \subset \overline{\mathcal{W}}$ be the substack consisting of $(C, \delta_1) \in \overline{\mathcal{W}}$ so that δ_1 is on a non-separating bridge. Then, by Lemmas 2.13 and 2.14,

$$Z \simeq \overline{\mathcal{M}}_{1,2} \times \overline{\mathcal{M}}_{0,4} \quad \text{and} \quad Z' \simeq \overline{\mathcal{M}}_{1,2} \times \overline{\mathcal{M}}_{0,3},$$

so both Z and Z' are closed.

Graph (a) of Figure 4 shows a general point of Z. When one of the marked points approaches a node or the other marked point, we obtain Graph (b) or Graph (c) of Figure 4. In addition, the genus 1 subcurve $\widehat{T}$ in the first row of Figure 4 may be nodal; in particular, $\widehat{T}$ may even contain a separating node, as shown in the second row of Figure 4.

By Lemma 2.14, the restriction to $\overline{\mathcal{K}}$ of the forgetful morphism $\overline{\mathcal{M}}_{2,2} \longrightarrow \overline{\mathcal{M}}_{2,1}$ that forgets the second marked point, still denoted by

$$\widehat{\mathrm{pr}} : \overline{\mathcal{K}} \longrightarrow \overline{\mathcal{M}}_{2,1},$$

restricts to an isomorphism of the open substacks

$$\overline{\mathcal{K}}\backslash Z \longrightarrow \overline{\mathcal{M}}_{2,1}\backslash Z'.$$

Hence $\overline{\mathcal{K}}\backslash Z$ is smooth and of codimension 1 in $\overline{\mathcal{M}}_{2,2}\backslash Z$. Therefore, to establish Proposition 2.15, it suffices to fix an arbitrary point

$$\widehat{x} = (\widehat{C}, \widehat{\delta}_1, \widehat{\delta}_2) \in Z$$

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as well as a small affine smooth chart $\mathcal{V} \longrightarrow \overline{\mathcal{M}}_{2,2}$ of x, and show that $\overline{\mathcal{K}} \cap \mathcal{V}$ is a smooth Cartier divisor of $\mathcal{V}$.

Below, we prove this by analyzing $\kappa \in \Gamma(\mathscr{O}_{\mathcal{V}})$ with

$$\overline{\mathcal{K}} \cap \mathcal{V} = \{\kappa = 0\}. \tag{2.18}$$

To this end, we denote by $\mathrm{B}[\mathfrak{p}, \mathfrak{q}]$ the maximal non-separating bridge of $\widehat{C}$ containing $\langle \widehat{\delta}_1 \rangle$ and $\langle \widehat{\delta}_2 \rangle$, and by $\widehat{T}$ the genus 1 connected subcurve of $\widehat{C}$ meeting $\mathrm{B}[\mathfrak{p}, \mathfrak{q}]$ at $\mathfrak{p}$ and $\mathfrak{q}$; see Figure 4. Observe that

$$Z \cap \mathcal{V} = \{\zeta_{\mathfrak{p}} = \zeta_{\mathfrak{q}} = 0\}.$$

This implies that $\kappa = 0$ whenever $\zeta_{\mathfrak{p}} = \zeta_{\mathfrak{q}} = 0$, so there exist $f, g \in \Gamma(\mathscr{O}_{\mathcal{V}})$ such that

$$\kappa = f\zeta_{\mathfrak{p}} + g\zeta_{\mathfrak{q}}. \tag{2.19}$$

We aim to show that both f and g are invertible. By symmetry, we only need to show that either of them is invertible. The proof of this is divided into six cases as per Figure 4.

Case (a). In this case, we assume that $\widehat{x}$ is given by Graph (a) of Figure 4, where $\widehat{T}$ may have (up to two) non-separating nodes. Recall that the Hodge bundle $\Omega\mathcal{M}_{2,2}$ extends over $\overline{\mathcal{M}}_{2,2}$ as a rank 2 vector bundle $\Omega\overline{\mathcal{M}}_{2,2}$, whose fiber over every $(C, \delta_1, \delta_2) \in \overline{\mathcal{M}}_{2,2}$ parameterizes meromorphic differentials η of C that have, at worst, simple poles at the nodes of C with opposite residues on the two branches of each node (i.e. the fiber is $H^0(C, \omega_C)$).

Since the chart $\mathcal{V}$ containing $\widehat{x}$ is assumed to be small, the marked points are not on a tail or a separating bridge for any point of $\mathcal{V}$. Therefore, for every point $(C, \delta_1, \delta_2) \in \mathcal{V}$, $\{\eta \in H^0(C, \omega_C) : \eta|_{\delta_1} = 0\}$ is always of rank 1. In other words,

$$\mathfrak{L}_1/\mathcal{V} := \{(C, \delta_1, \delta_2, \eta) \in \Omega\overline{\mathcal{M}}_{2,2}/\mathcal{V} : \eta|_{\delta_1} = 0\}$$

is a line bundle. We choose a nowhere vanishing section $\mathfrak{h}_1$ of $\mathfrak{L}_1/\mathcal{V}$. In addition, let $L_2/\overline{\mathcal{M}}_{2,2}$ be the universal tangent line bundle at the second marked point. We choose a nowhere vanishing section h_2 of $L_2/\mathcal{V}$. They together give rise to a regular function $\gamma \in \Gamma(\mathscr{O}_{\mathcal{V}})$ as follows:

$$\gamma(C, \delta_1, \delta_2) = \mathfrak{h}_1(C, \delta_1, \delta_2)|_{\delta_2} \otimes h_2(C, \delta_1, \delta_2) \quad \forall\, (C, \delta_1, \delta_2) \in \mathcal{V}. \tag{2.20}$$

Here, $T^*_{\delta_2} C \otimes T_{\delta_2} C$ is canonically identified with $\mathbb{C}$.

With κ as in (2.19), we claim that

$$\gamma/\kappa \in \Gamma(\mathscr{O}_{\mathcal{V}}). \tag{2.21}$$

Indeed, for every

$$x = (C, \delta_1, \delta_2) \in \overline{\mathcal{K}} \cap (\mathcal{V} \backslash \{\zeta_{\mathfrak{p}} \cdot \zeta_{\mathfrak{q}} = 0\}),$$

by Definition 2.11, we have $\mathfrak{h}_1(x)|_{\delta_2} = 0$, so γ/κ is regular on $\mathcal{V} \backslash \{\zeta_{\mathfrak{p}} \cdot \zeta_{\mathfrak{q}} = 0\}$. Moreover, shrinking $\mathcal{V}$ if necessary, we have

$$\overline{\mathcal{K}} \cap \mathcal{V} \cap (\{\zeta_{\mathfrak{p}} \neq 0, \zeta_{\mathfrak{q}} = 0\} \cup \{\zeta_{\mathfrak{q}} \neq 0, \zeta_{\mathfrak{p}} = 0\}) = \emptyset,$$

for otherwise, without loss of generality we suppose that there exists a sequence of elements of $\overline{\mathcal{K}} \cap \{\zeta_{\mathfrak{p}} \neq 0,\ \zeta_{\mathfrak{q}} = 0\}$ approaching to $\widehat{x}$. Then, by [BC23, Theorem 1.3], $H^0(\widehat{T}, \omega_{\widehat{T}})$ must have a nonzero element vanishing at $\mathfrak{p}$. As this is impossible, we see that γ/κ is regular on the complement of the codimension 2 substack $Z \cap \mathcal{V}$, and hence on the entire $\mathcal{V}$ by Hartogs' Lemma.

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Next, we construct a morphism of a small disk $D \longrightarrow \mathcal{V}$ whose image contains $\widehat{x}$, and analyze the vanishing order of the pullback of γ at the center of D. More precisely, we write

$$\widehat{\eta}^{+} := \mathfrak{h}_1(\widehat{x}) \quad (\in H^0(\widehat{C}, \omega_{\widehat{C}}) \backslash \{0\}).$$

Then $\widehat{\eta}^{+}$ vanishes identically 0 on $\mathrm{B}[\mathfrak{p}, \mathfrak{q}]$, which in Case (a) is a smooth rational subcurve of C, denoted by $\widehat{R}$. On the genus 1 inseparable component $\widehat{T}$, the restriction of $\widehat{\eta}^{+}$ is holomorphic, hence does not have any zero. Next, we choose another

$$\widehat{\xi} \in H^0(\widehat{C}, \omega_{\widehat{C}}) \quad \text{s.t.} \quad H^0(\widehat{C}, \omega_{\widehat{C}}) = \mathrm{span}\{\widehat{\eta}^{+}, \widehat{\xi}\}.$$

Then, $\widehat{\xi}$ does not have any zero on $\widehat{R}$. Moreover, $\widehat{\xi}$ has simple poles at $\mathfrak{p}$ and $\mathfrak{q}$, such that on both $\widehat{T}$ and $\widehat{R}$, the residues of $\widehat{\xi}$ sum up to 0. Finally, there exists a unique

$$\widehat{\eta}^{-} \in H^0(\widehat{R}, K_{\widehat{R}} + 2\mathfrak{p}) \quad \text{s.t.} \quad (\widehat{\eta}^{-} + \widehat{\xi})|_{\widehat{\delta}_1} = 0.$$

Notice that $(\widehat{\eta}^{-} + \widehat{\xi})|_{\widehat{R}}$ does not have any other zero on $\widehat{R}$, because its only poles are $\mathfrak{p}$ (of order 2) and $\mathfrak{q}$ (of order 1). Consequently, we have

$$(\widehat{\eta}^{-} + \widehat{\xi})|_{\widehat{\delta}_2} \neq 0. \tag{2.22}$$

Next, we smooth out $\mathfrak{p}$ while keeping the node $\mathfrak{q}$, and apply [BC23, Theorem 4.5 (Higher-order Plumbing)]. Concretely, we take D to be a small disk. By taking

$$\eta_t^{+} = \widehat{\eta}^{+} + t\widehat{\xi}, \quad \eta_t^{-} = t(\widehat{\eta}^{-} + \widehat{\xi}), \quad t \in D,$$

in [BC23, Theorem 4.5], we obtain a family of stable curves with two marked points

$$x_t = (C_t, \delta_{1;t}, \delta_{2;t}), \quad t \in D$$

satisfying:

- *(the central fiber):* $x_0 = \widehat{x}$ $(= (\widehat{C}, \widehat{\delta}_1, \widehat{\delta}_2))$;
- *(the nodes):* $\zeta_{\mathfrak{p}}(x_t) \neq 0$ for all $t \in D\backslash\{0\}$, and $\zeta_{\mathfrak{q}}(x_t) = 0$ for all $t \in D$; and
- *(the vanishing order of $\gamma(x_t)$):* there exists $u \in \Gamma(\mathscr{O}_D^*)$ such that for every $t \in D$,

$$\gamma(x_t) = t \cdot u(t) \cdot ((\widehat{\eta}^{-} + \widehat{\xi})|_{\widehat{\delta}_2} \otimes h_2(\widehat{x})).$$

Thus, on the one hand, by (2.22), $\gamma(x_t)/t$ is invertible on D. On the other hand, shrinking D if necessary, we see there exist unique integers $\widehat{n} \geq 1$ and $\widehat{m} \geq 0$ such that $\zeta_{\mathfrak{p}}(x_t)/t^{\widehat{n}}$ and $f(x_t)/t^{\widehat{m}}$ are both invertible on D, where f is as in (2.19). Taking $\zeta_{\mathfrak{q}}(x_t) = 0$ for all $t \in D$ into consideration, we conclude from (2.19) and (2.21) that

$$\gamma(x_t)/t^{\widehat{n}+\widehat{m}} = (\gamma(x_t)/\kappa(x_t)) \cdot (f(x_t)/t^{\widehat{m}}) \cdot (\zeta_{\mathfrak{p}}(x_t)/t^{\widehat{n}}) \ \in \ \Gamma(\mathscr{O}_D).$$

Therefore, $\widehat{n} = 1$ and $\widehat{m} = 0$. In particular, we have $f(\widehat{x}) \neq 0$, i.e. f is invertible on the small chart $\mathcal{V}$. This completes the proof in Case (a).

Case (b). In this case, we assume that $\widehat{x}$ is given by Graph (b) of Figure 4, where $\widehat{T}$ is separable and may have (at most) one non-separating node. The stability of $\widehat{x}$ implies that the separating node of $\widehat{T}$ is unique, and we denote this by $\mathfrak{s}$. Let $\widehat{T}'$ and $\widehat{T}''$ be the two genus 1 inseparable components of $\widehat{C}$ satisfying

$$\widehat{T}' \cap \widehat{T}'' = \{\mathfrak{s}\} \quad \text{and} \quad \widehat{\delta}_1, \widehat{\delta}_2 \in \widehat{T}''.$$

Then $\widehat{T}''$ consists of two smooth rational subcurves $\widehat{R}$ and $\widehat{R}_{\mathfrak{s}}$ such that

$$\widehat{R} = \mathrm{B}[\mathfrak{p}, \mathfrak{q}] \quad \text{and} \quad \mathfrak{s} \in \widehat{R}_{\mathfrak{s}}.$$

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The construction of the line bundle $\mathfrak{L}_1/\mathcal{V}$ of Case (a) is still valid in Case (b), so we define $\gamma \in \Gamma(\mathscr{O}_{\mathcal{V}})$ in the same way as in (2.20). Notice that for every $x = (C, \delta_1, \delta_2) \in \{\zeta_{\mathfrak{s}} = 0\}$, $\mathfrak{h}_1(x)$ vanishes identically on the genus 1 inseparable component of C that contains δ_1 and δ_2, and hence γ vanishes along $\{\zeta_{\mathfrak{s}} = 0\}$, i.e.

$$\gamma/\zeta_{\mathfrak{s}} \in \Gamma(\mathscr{O}_{\mathcal{V}}).$$

Moreover, with κ as in (2.19), we claim that

$$\gamma/(\zeta_{\mathfrak{s}} \cdot \kappa) \in \Gamma(\mathscr{O}_{\mathcal{V}}). \tag{2.23}$$

Indeed, Case (a) implies that $\gamma/(\zeta_{\mathfrak{s}}\kappa)$ is regular on $\mathcal{V}\backslash\{\zeta_{\mathfrak{s}} = 0\}$. Moreover, for a general point $y \in \{\zeta_{\mathfrak{s}} = 0\}$, the two marked points of y are not conjugate, by Lemma 2.14, hence $\kappa(y) \neq 0$, which means that $\gamma/(\zeta_{\mathfrak{s}}\kappa)$ is also regular near every general point of $\{\zeta_{\mathfrak{s}} = 0\}$. By Hartogs' Lemma, we obtain (2.23).

Next, we choose the meromorphic differentials $\widehat{\eta}^+$, $\widehat{\xi}$, and $\widehat{\eta}^-$ in the same way as in Case (a). In particular, (2.22) still holds. In Case (b), $\widehat{\eta}^+$ vanishes identically on $\widehat{R}_{\mathfrak{s}}$. We choose

$$\widehat{\eta}_{\mathfrak{s}} \in H^0(\widehat{R}_{\mathfrak{s}}, K_{\widehat{R}_{\mathfrak{s}}} + 2\mathfrak{s})\backslash\{0\},$$

which is unique up to scaling.

Next, we smooth out $\mathfrak{s}$ and $\mathfrak{p}$ while keeping the node $\mathfrak{q}$, and apply the Higher-order Plumbing theorem of [BC23] as in Case (a). Let D be a small disk. In [BC23, Theorem 4.5], we take

$$\eta_t^+ = \widehat{\eta}^+, \quad \eta_t^- = t(\widehat{\eta}_{\mathfrak{s}} + t\widehat{\xi}), \quad t \in D$$

for the node $\mathfrak{s}$, and

$$\eta_t^+ = t(\widehat{\eta}_{\mathfrak{s}} + t\widehat{\xi}), \quad \eta_t^- = t^2(\widehat{\eta}^- + \widehat{\xi}), \quad t \in D$$

for the node $\mathfrak{p}$. Then, [BC23, Theorem 4.5] gives rise to a family of stable curves with two marked points

$$x_t = (C_t, \delta_{1;t}, \delta_{2;t}), \quad t \in D$$

over a small disk D, satisfying:

- *(the central fiber):* $x_0 = \widehat{x}$ $(= (\widehat{C}, \widehat{\delta}_1, \widehat{\delta}_2))$;
- *(the nodes):* $\zeta_{\mathfrak{s}}(x_t) \neq 0$ and $\zeta_{\mathfrak{p}}(x_t) \neq 0$ for all $t \in D\backslash\{0\}$, and $\zeta_{\mathfrak{q}}(x_t) = 0$ for all $t \in D$; and
- *(the vanishing order of $\gamma(x_t)$):* there exists $u \in \Gamma(\mathscr{O}_D^*)$ such that for every $t \in D$,

$$\gamma(x_t) = t^2 \cdot u(t) \cdot ((\widehat{\eta}^- + \widehat{\xi})|_{\widehat{\delta}_2} \otimes h_2(\widehat{x})).$$

Thus, on the one hand, by (2.22), $\gamma(x_t)/t^2$ is invertible on D. On the other hand, shrinking D if necessary, we see there exist unique integers $\widehat{k} \geq 1$, $\widehat{n} \geq 1$, and $\widehat{m} \geq 0$ such that $\zeta_{\mathfrak{s}}(x_t)/t^{\widehat{k}}$, $\zeta_{\mathfrak{p}}(x_t)/t^{\widehat{n}}$, and $f(x_t)/t^{\widehat{m}}$ are all invertible on D, where f is as in (2.19). Taking into consideration that $\zeta_{\mathfrak{q}}(x_t) = 0$ for all $t \in D$, we conclude from (2.19) and (2.23) that

$$\gamma(x_t)/t^{\widehat{k}+\widehat{n}+\widehat{m}} = (\gamma(x_t)/(\zeta_{\mathfrak{s}}(x_t)\kappa(x_t))) \cdot (\zeta_{\mathfrak{s}}(x_t)/t^{\widehat{k}}) \cdot (f(x_t)/t^{\widehat{m}}) \cdot (\zeta_{\mathfrak{p}}(x_t)/t^{\widehat{n}}) \in \Gamma(\mathscr{O}_D).$$

Therefore, $\widehat{k} = \widehat{n} = 1$ and $\widehat{m} = 0$. As in Case (a), this implies that f is invertible on the small chart $\mathcal{V}$ and completes the proof of Case (b).

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Cases (c) and (d). In Case (c) (respectively, Case (d)), we assume that $\widehat{x}$ is given by Graph (c) (respectively, Graph (d)) of Figure 4, where $\widehat{T}$ may contain non-separating nodes (respectively, $\widehat{T}$ is separable and may have a non-separating node). The proof of Case (c) (respectively, Case (d)) is indeed parallel to that of Case (a) (respectively, Case (b)), except that $\mathrm{B}[\mathfrak{p},\mathfrak{q}]$ consists of two rational subcurves $\widehat{R}_1$ and $\widehat{R}_2$ respectively containing $\widehat{\delta}_1$ and $\widehat{\delta}_2$. Without loss of generality we assume $\mathfrak{p}\in\widehat{R}_1$ and $\mathfrak{q}\in\widehat{R}_2$.

In either case, we copy the construction of the line bundle $\mathfrak{L}_1$ and the 1-forms $\widehat{\eta}^+$ and $\widehat{\xi}$ verbatim, and define $\widehat{\eta}^-$ likewise on $\widehat{R}_1$. In addition, in Case (d), we choose $\widehat{\eta}_{\mathfrak{s}}$ on $\widehat{R}_{\mathfrak{s}}$ as in Case (b). Then, in Case (c) (respectively, Case (d)), we still have (2.22), and hence [BC23, Theorem 4.5] still gives rise to a family $\{x_t\}\subset\mathcal{V}$ satisfying the same properties as in Case (a) (respectively, Case (b)). Hence the argument in Case (a) (respectively, Case (b)) still applies to Case (c) (respectively, Case (d)).

Case (e). In this case, we assume that $\widehat{x}$ is given by Graph (e) of Figure 4, where $\widehat{T}$ may have non-separating nodes. In particular, $\widehat{\delta}_1$ and $\widehat{\delta}_2$ are on a smooth tail $\widehat{R}_{\mathfrak{r}}$ that attaches to $\widehat{R}:=\mathrm{B}[\mathfrak{p},\mathfrak{q}]$ at the node $\mathfrak{r}$.

On the small chart $\mathcal{V}$ containing $\widehat{x}$,

$$\mathfrak{L}_1'/\mathcal{V}:=\{(C,\delta_1,\delta_2,\eta)\in\Omega\overline{\mathcal{M}}_{2,2}|_{\mathcal{V}}:\ \eta|_{\langle\delta_1\rangle}=0\}$$

is a line bundle, on which we choose a nowhere vanishing section $\mathfrak{h}_1'$. Let $L_2/\mathcal{V}$ still be the universal tangent line bundle at the second marked point, on which we choose a nowhere vanishing section h_2.

Consider the regular function $\gamma'\in\Gamma(\mathscr{O}_{\mathcal{V}})$ given by

$$\gamma'(C,\delta_1,\delta_2)=\mathfrak{h}_1'(C,\delta_1,\delta_2)|_{\delta_2}\otimes h_2(C,\delta_1,\delta_2)\quad\forall\,(C,\delta_1,\delta_2)\in\mathcal{V}.$$

First, we claim that

$$\gamma:=\gamma'/\zeta_{\mathfrak{r}}^2\in\Gamma(\mathscr{O}_{\mathcal{V}}),\tag{2.24}$$

where $\zeta_{\mathfrak{r}}$ is a node-smoothing parameter corresponding to the node $\mathfrak{r}$. To see this, we denote by $\Delta_{12}\subset\overline{\mathcal{M}}_{2,2}$ the (divisorial) boundary component of $\overline{\mathcal{M}}_{2,2}$ when δ_1 and δ_2 are on the same tail. Locally, we have $\Delta_{12}\cap\mathcal{V}=\{\zeta_{\mathfrak{r}}=0\}$. Note that

$$L_2\simeq\widehat{\mathrm{pr}}_2^{\,*}L\otimes\mathscr{O}(-\Delta_{12}),$$

where $\widehat{\mathrm{pr}}_2:\overline{\mathcal{M}}_{2,2}\longrightarrow\overline{\mathcal{M}}_{2,1}$ denotes the forgetful morphism that forgets the *first* marked point, and $L/\overline{\mathcal{M}}_{2,1}$ is the universal tangent line bundle at the unique marked point. Taking into consideration the fact that $\mathfrak{h}_1'(C,\delta_1,\delta_2)|_{\langle\delta_2\rangle}$ vanishes on $\Delta_{12}\cap\mathcal{V}$ (which follows immediately from the definition of $\mathfrak{L}_1'$), we see that the vanishing order of γ' along $\Delta_{12}\cap\mathcal{V}$ is at least 2, and thus we obtain (2.24).

Next, we claim that this defined γ vanishes on $\overline{\mathcal{K}}$; in other words, with κ as in (2.19), we have

$$\gamma'/(\zeta_{\mathfrak{r}}^2\cdot\kappa)\in\Gamma(\mathscr{O}_{\mathcal{V}}).\tag{2.25}$$

To see this, first observe that Case (a) implies that γ/κ is regular on $\mathcal{V}\backslash\Delta_{12}$. In addition, for a general point $y\in\Delta_{12}\cap\mathcal{V}$, the node $\mathfrak{r}$ is not a Weierstrass point on the core of F. Hence, by Lemma 2.14, $\kappa(y)\neq0$, which means that γ/κ is regular near every general point of $\Delta_{12}\cap\mathcal{V}$. Once again by Hartogs' Lemma, we obtain (2.25).

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Parallel to Case (a), we set

$$\widehat{\eta}^{+} := \mathfrak{h}_1'(\widehat{x}) \quad (\in H^0(\widehat{C}, \omega_{\widehat{C}})\backslash\{0\})$$

and take $\widehat{\xi}$ and $\widehat{\eta}^{-}$ in the same way as in Case (a). In addition, we take $\widehat{\eta}_{\mathfrak{r}}$ to be a (unique up to scaling) nonzero meromorphic differential on $\widehat{R}_{\mathfrak{r}}$ whose only pole and zero are an order 3 pole at $\mathfrak{r}$ and a simple zero at $\widehat{\delta}_1$, respectively. In particular,

$$\widehat{\eta}_{\mathfrak{r}}|_{\widehat{\delta}_2} \neq 0. \tag{2.26}$$

Next, we smooth out $\mathfrak{p}$ and $\mathfrak{r}$ while keeping the node $\mathfrak{q}$, and apply the Higher-order Plumbing theorem of [BC23] again. Let D be a small disk. In [BC23, Theorem 4.5], we take

$$\eta_t^{+} = \widehat{\eta}^{+} + t\widehat{\xi}, \quad \eta_t^{-} = t(\widehat{\eta}^{-} + \widehat{\xi}), \quad t \in D$$

for the node $\mathfrak{p}$, and

$$\eta_t^{+} = t(\widehat{\eta}^{-} + \widehat{\xi}), \quad \eta_t^{-} = t^3\widehat{\eta}_{\mathfrak{r}}, \quad t \in D$$

for the node $\mathfrak{r}$. Then, [BC23, Theorem 4.5] gives rise to a family of stable curves with two marked points

$$x_t = (C_t, \delta_{1;t}, \delta_{2;t}), \quad t \in D$$

over a small disk D, satisfying:

- *(the central fiber):* $x_0 = \widehat{x}$ $(= (\widehat{C}, \widehat{\delta}_1, \widehat{\delta}_2))$;
- *(the nodes):* $\zeta_{\mathfrak{p}}(x_t) \neq 0$ and $\zeta_{\mathfrak{r}}(x_t) \neq 0$ for all $t \in D\backslash\{0\}$, and $\zeta_{\mathfrak{q}}(x_t) = 0$ for all $t \in D$; and
- *(the vanishing order of $\gamma'(x_t)$):* there exists $u \in \Gamma(\mathscr{O}_D^*)$ such that for every $t \in D$,

$$\gamma'(x_t) = t^3 \cdot u(t) \cdot (\widehat{\eta}_{\mathfrak{r}}|_{\widehat{\delta}_2} \otimes h_2(\widehat{x})).$$

Thus, on the one hand, by (2.26), $\gamma'(x_t)/t^3$ is invertible on D. On the other hand, shrinking D if necessary, we see there exist unique integers $\widehat{n} \geq 1$, $\widehat{k} \geq 1$, and $\widehat{m} \geq 0$ such that $\zeta_{\mathfrak{p}}(x_t)/t^{\widehat{n}}$, $\zeta_{\mathfrak{r}}(x_t)/t^{\widehat{k}}$, and $f(x_t)/t^{\widehat{m}}$ are all invertible on D, where f is as in (2.19). Taking into consideration that $\zeta_{\mathfrak{q}}(x_t) = 0$ for all $t \in D$, we conclude from (2.19) and (2.25) that

$$\gamma'(x_t)/t^{2\widehat{k}+\widehat{n}+\widehat{m}} = (\gamma'(x_t)/(\zeta_{\mathfrak{r}}^2(x_t)\kappa(x_t))) \cdot (\zeta_{\mathfrak{r}}(x_t)/t^{\widehat{k}})^2 \cdot (f(x_t)/t^{\widehat{m}}) \cdot (\zeta_{\mathfrak{p}}(x_t)/t^{\widehat{n}}) \in \Gamma(\mathscr{O}_D).$$

Therefore, $\widehat{k} = \widehat{n} = 1$ and $\widehat{m} = 0$. As in Case (a), this implies that f is invertible on the small chart $\mathcal{V}$ and completes the proof of Case (e).

Case (f). In this case, we assume that $\widehat{x}$ is given by Graph (f) of Figure 4, where $\widehat{T}$ is separable and may have a non-separating node.

The proof of this case simply combines the arguments used in Cases (b) and (e). Specifically, we construct the line bundle $\mathfrak{L}_1'/\mathcal{V}$ and the regular function γ' as in Case (e), and observe that

$$\gamma'/(\zeta_{\mathfrak{r}}^2 \cdot \zeta_{\mathfrak{s}} \cdot \kappa) \in \Gamma(\mathscr{O}_{\mathcal{V}}),$$

where $\mathfrak{s}$ denotes the separating node of the core and $\mathfrak{r}$ denotes the pivotal node where the tail attaches to the core. Then, we choose the meromorphic differentials $\widehat{\eta}^{+}$, $\widehat{\xi}$, $\widehat{\eta}^{-}$ and $\widehat{\eta}_{\mathfrak{r}}$ as in Case (e), as well as $\widehat{\eta}_{\mathfrak{s}}$ as in Case (b). Once again, we smooth out $\mathfrak{s}$, $\mathfrak{p}$ and $\mathfrak{r}$ while keeping the node $\mathfrak{q}$, and apply [BC23, Theorem 4.5] to obtain a family $\{x_t\} \subset \mathcal{V}$ with $x_0 = \widehat{x}$ such that $\gamma'(x_t)/t^4$ is invertible, which implies that $f(x_t)$ is invertible. We omit further details. □

Corollary 2.16. *$\overline{\mathcal{W}}$ is a smooth Cartier divisor of $\overline{\mathcal{M}}_{2,1}$.*

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Proof. Observe that the first statement of Lemma 2.14 implies an inclusion

$$\overline{\mathcal{W}} \simeq \overline{\mathcal{W}} \times \overline{\mathcal{M}}_{0,3} \hookrightarrow \overline{\mathcal{K}}$$

by identifying the unique marked point of $\overline{\mathcal{W}}$ with the first marked point of $\overline{\mathcal{M}}_{0,3}$. Given $x = (C, \delta_1) \in \overline{\mathcal{W}}$, we denote by $\widehat{x} = (\widehat{C}, \widehat{\delta}_1, \widehat{\delta}_2)$ its image in $\overline{\mathcal{K}}$, and by $\mathfrak{r}$ the node where the unique tail attaches to the core $\widehat{F}$. On a small affine smooth chart $\mathcal{V} \longrightarrow \overline{\mathcal{M}}_{2,2}$ containing $\widehat{x}$, the image of $\overline{\mathcal{W}}$ under the above inclusion can be written as

$$\{\zeta_{\mathfrak{r}} = \kappa = 0\}, \tag{2.27}$$

where κ is the local parameter satisfying $\overline{\mathcal{K}} \cap \mathcal{V} = \{\kappa = 0\}$.

Let $Z \subset \overline{\mathcal{K}}$ be the closed substack of $\overline{\mathcal{K}}$ as in the beginning of the proof of Proposition 2.15.

If $\widehat{x} \in \overline{\mathcal{K}} \backslash Z$, then $\mathfrak{r}$ is a Weierstrass point of $\widehat{F}$ away from any separating or non-separating bridge, and hence is a branch point of the hyperelliptic admissible cover. Therefore, at $\widehat{x}$, the normal direction to $\{\kappa = 0\}$ is given by changing the position of $\mathfrak{r}$ on $\widehat{F}$, which is transverse to the smoothing of the nodes of $\widehat{C}$. Hence $\{\kappa\} \sqcup \{\zeta_q : q \in N(\widehat{C})\}$ forms a subset of a system of local parameters on $\mathcal{V}$. This implies that (2.27) is smooth on $\mathcal{V}$.

If $\widehat{x} \in Z$, let $\mathfrak{p}$ and $\mathfrak{q}$ be the core nodes of $\widehat{C}$ determining the (maximal) non-separating bridge containing $\mathfrak{r}$. By (2.19), $\kappa = f\zeta_{\mathfrak{p}} + g\zeta_{\mathfrak{q}}$ for some units f and g on $\mathcal{V}$. Since the node-smoothing parameters $\zeta_{\mathfrak{p}}$, $\zeta_{\mathfrak{q}}$ and $\zeta_{\mathfrak{r}}$ form a subset of a system of local parameters on $\mathcal{V}$, we conclude that (2.27) is smooth on $\mathcal{V}$. □

2.6.2 *Local equations of $\overline{\mathcal{K}}_{\ell;i,j}$ and $\overline{\mathcal{W}}_{\ell;i}$.* We are ready to present the local equations of $\overline{\mathcal{K}}_{\ell;i,j}$ and and $\overline{\mathcal{W}}_{\ell;i}$.

PROPOSITION 2.17. *Let $1 \le i, j \le \ell$ with $i \ne j$, $x = (C, \delta_1, \ldots, \delta_\ell) \in \overline{\mathcal{M}}_{2,\ell}$, and $\mathcal{V} \longrightarrow \overline{\mathcal{M}}_{2,\ell}$ be a sufficiently small smooth chart containing x. Let $\{\zeta_q : q \in N(C)\}$ be a set of node-smoothing parameters (corresponding to all the nodes of C) on $\mathcal{V}$ as in Section 2.5, and $\langle \delta_i \rangle \in F$ be as in (2.17) for all i. Then we have the following statements.*

(i) There exists $\kappa \in \Gamma(\mathscr{O}_{\mathcal{V}})$ such that

$$\overline{\mathcal{K}}_{\ell;i,j} \cap \mathcal{V} = (\kappa = 0).$$

In other words, $\overline{\mathcal{K}}_{\ell;i,j}$ is a Cartier divisor of $\overline{\mathcal{M}}_{2,\ell}$.

(ii) If $x \in \overline{\mathcal{K}}_{\ell;i,j}$ and $\langle \delta_i \rangle$ and $\langle \delta_j \rangle$ do not lie on any non-separating bridge, then $\overline{\mathcal{K}}_{\ell;i,j}$ is smooth at x. In this case, $\{\kappa\} \sqcup \{\zeta_q : q \in N(C)\}$ forms a subset of a system of local parameters on $\mathcal{V}$.

(iii) If $\langle \delta_i \rangle$ lies on a maximal non-separating bridge $\mathrm{B}[\mathfrak{p}, \mathfrak{q}]$, then $x \in \overline{\mathcal{K}}_{\ell;i,j}$ if and only if $\langle \delta_j \rangle$ lies on $\mathrm{B}[\mathfrak{p}, \mathfrak{q}]$. Moreover, if $\langle \delta_i \rangle, \langle \delta_j \rangle \in \mathrm{B}[\mathfrak{p}, \mathfrak{q}]$, then by taking two arbitrary smooth points $z_{\langle \delta_i \rangle}, z_{\langle \delta_j \rangle} \in \mathrm{B}[\mathfrak{p}, \mathfrak{q}]$ that are sufficiently close to $\langle \delta_i \rangle$ and $\langle \delta_j \rangle$, respectively, there then exist $f, g \in \Gamma(\mathscr{O}_{\mathcal{V}}^*)$ such that

$$\kappa = f \cdot \prod_{q \in N_{[z_{\langle \delta_i \rangle}, \mathfrak{p}]} \cap N_{[z_{\langle \delta_j \rangle}, \mathfrak{p}]}} \zeta_q \;+\; g \cdot \prod_{q \in N_{[z_{\langle \delta_i \rangle}, \mathfrak{q}]} \cap N_{[z_{\langle \delta_j \rangle}, \mathfrak{q}]}} \zeta_q.$$

We remark that in the above identity, the indices of the products of the ζ_q only involve the core nodes on $\mathrm{B}[\mathfrak{p}, \mathfrak{q}]$; see (2.10) for the definition and Figure 3 for an illustration. We further point out that the reason for using $z_{\langle \delta_i \rangle}$ instead of $\langle \delta_i \rangle$ in $N_{[z_{\langle \delta_i \rangle}, \mathfrak{p}]}$ and $N_{[z_{\langle \delta_i \rangle}, \mathfrak{q}]}$ is to make it clear that $\zeta_{\langle \delta_i \rangle}$ does not divide either of the products when $\langle \delta_i \rangle$ is a pivotal node.

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When $\langle\delta_i\rangle$ and $\langle\delta_j\rangle$ lie on a maximal non-separating bridge $\mathrm{B}[\mathfrak{p},\mathfrak{q}]$, Proposition 2.17 implies that $\overline{\mathcal{K}}_{\ell;i,j}$ is singular unless there exists a smooth rational subcurve of $\mathrm{B}[\mathfrak{p},\mathfrak{q}]$ containing at least one of $\langle\delta_i\rangle$ and $\langle\delta_j\rangle$, as well as at least one of $\mathfrak{p}$ and $\mathfrak{q}$. For instance, assume the maximal non-separating bridge $\mathrm{B}[\mathfrak{p},\mathfrak{q}]$ consists of one smooth rational subcurve. Then there exist $f,g\in\Gamma(\mathscr{O}_{\mathcal{V}}^*)$ such that

$$\overline{\mathcal{K}}_{\ell;i,j}\cap\mathcal{V}=\{f\zeta_{\mathfrak{p}}+g\zeta_{\mathfrak{q}}=0\},$$

which implies that $\overline{\mathcal{K}}_{\ell;i,j}$ is smooth at x. Consider another example. Assume that $\mathrm{B}[\mathfrak{p},\mathfrak{q}]$ consists of three smooth rational subcurves R_a, R_b and R_c with $\mathfrak{p}\in R_a$, $\mathfrak{q}\in R_c$, and $\langle\delta_i\rangle,\langle\delta_j\rangle\in R_b$. We write $q_a=R_a\cap R_b$ and $q_b=R_b\cap R_c$. Then, there exist $f,g\in\Gamma(\mathscr{O}_{\mathcal{V}}^*)$ such that

$$\overline{\mathcal{K}}_{\ell;i,j}\cap\mathcal{V}=\{f\zeta_{q_a}\zeta_{\mathfrak{p}}+g\zeta_{q_b}\zeta_{\mathfrak{q}}=0\}.$$

Hence, in this case, $\overline{\mathcal{K}}_{\ell;i,j}$ is singular at x.

To prove Proposition 2.17, we need the following local property of the forgetful maps. Consider the forgetful map $\widehat{\mathrm{pr}}:\overline{\mathcal{M}}_{2,\ell}\longrightarrow\overline{\mathcal{M}}_{2,\ell-1}$ that forgets the last marked point. Given $\widehat{x}=(\widehat{C},\widehat{\delta}_1,\dots,\widehat{\delta}_\ell)$, we write $\widehat{\mathrm{pr}}\,(\widehat{x})=(\widehat{C}',\widehat{\delta}'_1,\dots,\widehat{\delta}'_{\ell-1})$. Since $\widehat{\mathrm{pr}}$ can be identified with the universal curve $\overline{\mathcal{C}}_{2,\ell-1}\longrightarrow\overline{\mathcal{M}}_{2,\ell-1}$, the ℓ-marked curve $\widehat{x}$ can be identified with a pair $(\widehat{\mathrm{pr}}\,(\widehat{x}),\widehat{\delta}'_\ell)$, where $\widehat{\delta}'_\ell$ can be either a smooth point or a node of $\widehat{C}'$. We take smooth charts $\widehat{\mathcal{U}}\longrightarrow\overline{\mathcal{M}}_{2,\ell}$ and $\widehat{\mathcal{V}}\longrightarrow\overline{\mathcal{M}}_{2,\ell-1}$ centered at $\widehat{x}$ and $\widehat{\mathrm{pr}}\,(\widehat{x})$, respectively.

Lemma 2.18. *Shrinking $\widehat{\mathcal{U}}$ and $\widehat{\mathcal{V}}$ if necessary, we have the following.*

- *If $\widehat{\delta}'_\ell$ is a smooth point of $\widehat{C}'$, every subset S of a system of local parameters on $\widehat{\mathcal{V}}$ pulls back to a subset of a system of local parameters on $\widehat{\mathcal{U}}$ via $\widehat{\mathrm{pr}}$.*
- *If $\widehat{\delta}'_\ell$ is a node q of $\widehat{C}'$, then $\widehat{\delta}_\ell$ is the marked point on a smooth rational component of $\widehat{C}$ that contains exactly two nodal points q_1 and q_2. With ζ_q denoting the node-smoothing parameter of the node q, if $\mathsf{S}\sqcup\{\zeta_q\}$ is a subset of a system of local parameters on $\widehat{\mathcal{V}}$, then $\widetilde{\mathsf{S}}\sqcup\{\zeta_{q_1},\zeta_{q_2}\}$ forms a subset of a system of local parameters of $\widehat{\mathcal{U}}$, where $\widetilde{\mathsf{S}}$ is the pullback of S via $\widehat{\mathrm{pr}}$. In addition, ζ_q pulls back to $\zeta_{q_1}\zeta_{q_2}$ up to a unit.*

The statements of Lemma 2.18 are standard in the deformation theory of nodal curves with marked points, and hence the proof is omitted.

Proof of Proposition 2.17. Part (i) holds because $\overline{\mathcal{K}}$ is a Cartier divisor of $\overline{\mathcal{M}}_{2,2}$ by Proposition 2.15.

To verify part (ii), we denote by $\overline{x}=(\overline{C},\overline{\delta}_i,\overline{\delta}_j)\in\overline{\mathcal{K}}$ the image of x by forgetting all but the ith and jth marked points. Let $\overline{\mathcal{V}}\subset\overline{\mathcal{M}}_{2,1}$ be a small affine smooth chart centered at $\overline{x}$. By the smoothness of $\overline{\mathcal{K}}$ (proved in Proposition 2.15), there exists a local parameter $\overline{\kappa}$ on $\overline{\mathcal{V}}$ such that

$$\overline{\mathcal{K}}\cap\overline{\mathcal{V}}=\{\overline{\kappa}=0\}.$$

If $\overline{\delta}_i$ (hence $\overline{\delta}_j$) is on the core and not on any non-separating bridge, then by Lemma 2.14, for every $(C',\delta'_1,\delta'_2)\in\{\overline{\kappa}=0\}$, the location of δ'_2 on C' is determined by δ'_1. Therefore, at $\overline{x}$, the normal direction to $\{\overline{\kappa}=0\}$ is given by changing the position of $\overline{\delta}_j$ on $\overline{C}$, which is transverse to the smoothing of the nodes of $\overline{C}$. Consequently, $\{\overline{\kappa}\}\sqcup\{\zeta_q:q\in N(\overline{C})\}$ forms a subset of a system of local parameters on $\overline{\mathcal{V}}$. Applying Lemma 2.18 inductively, we see that $\overline{\kappa}$ pulls back to a local parameter κ on $\mathcal{V}$, satisfying

$$\overline{\mathcal{K}}_{\ell;i,j}\cap\mathcal{V}=\{\kappa=0\}.$$

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In addition, Lemma 2.18 implies that $\{\kappa\} \sqcup \{\zeta_q : q \in N(C)\}$ is a subset of a system of local parameters on $\mathcal{V}$.

If $\overline{\delta}_i$ and $\overline{\delta}_j$ are on the same tail R that meets the core $\overline{F}$ of $\overline{C}$ at a pivotal node $\mathfrak{r}$, and $\mathfrak{r}$ is not on any non-separating bridge, then, as shown in the proof of Corollary 2.16, $\{\overline{\kappa}\} \sqcup \{\zeta_q : q \in N(\overline{C})\}$ still forms a subset of a system of local parameters on $\overline{\mathcal{V}}$, and hence the argument from the previous paragraph applies to this case verbatim.

To verify part (iii), we denote by $\overline{x} = (\overline{C}, \overline{\delta}_i, \overline{\delta}_j) \in \overline{\mathcal{M}}_{2,2}$ the image of x by forgetting all but the ith and jth marked points. Let $\overline{\mathcal{V}} \subset \overline{\mathcal{M}}_{2,1}$ be a small affine smooth chart centered at $\overline{x}$. Assume that $\langle \overline{\delta}_i \rangle$ is on a maximal non-separating bridge $\mathrm{B}[\overline{\mathfrak{p}}, \overline{\mathfrak{q}}]$ of $\overline{C}$. Then by Lemmas 2.14 and 2.13, we see that $\overline{x} \in \overline{\mathcal{K}}$ if and only if $\langle \overline{\delta}_j \rangle \in \mathrm{B}[\overline{\mathfrak{p}}, \overline{\mathfrak{q}}]$. Moreover, let $\zeta_{\overline{\mathfrak{p}}}$ and $\zeta_{\overline{\mathfrak{q}}}$ be node-smoothing parameters on $\overline{\mathcal{V}}$ corresponding to $\overline{\mathfrak{p}}$ and $\overline{\mathfrak{q}}$, respectively. Then, by the proof of Proposition 2.15 (cf. (2.19)), there exist invertible functions f and g on $\overline{\mathcal{V}}$ such that

$$\overline{\kappa} = f\zeta_{\overline{\mathfrak{p}}} + g\zeta_{\overline{\mathfrak{q}}}.$$

Applying Lemma 2.18 inductively, we see that $\zeta_{\overline{\mathfrak{p}}}$ and $\zeta_{\overline{\mathfrak{q}}}$ pull back to

$$\prod_{q \in N_{[z_{\langle \delta_i \rangle}, \mathfrak{p}]} \cap N_{[z_{\langle \delta_j \rangle}, \mathfrak{p}]}} \zeta_q \quad \text{and} \quad \prod_{q \in N_{[z_{\langle \delta_i \rangle}, \mathfrak{q}]} \cap N_{[z_{\langle \delta_j \rangle}, \mathfrak{q}]}} \zeta_q,$$

respectively, each up to a unit. This completes part (iii). □

PROPOSITION 2.19. *Let $1 \le i \le \ell$, $x = (C, \delta_1, \dots, \delta_\ell) \in \overline{\mathcal{M}}_{2,\ell}$, and $\mathcal{V} \longrightarrow \overline{\mathcal{M}}_{2,\ell}$ be a sufficiently small smooth chart containing x. Let $\{\zeta_q : q \in N(C)\}$ be a set of node-smoothing parameters (corresponding to all the nodes of C) on $\mathcal{V}$ as in Section 2.5. Then we have the following.*

(1) *There exists $\kappa \in \Gamma(\mathscr{O}_{\mathcal{V}})$ such that*

$$\overline{\mathcal{W}}_{\ell;i} \cap \mathcal{V} = (\kappa = 0).$$

In other words, $\overline{\mathcal{W}}_{\ell;i}$ is a Cartier divisor of $\overline{\mathcal{M}}_{2,\ell}$.

(2) *If δ_i is on a separating bridge of the core of C, then $x \notin \overline{\mathcal{W}}_{\ell;i}$. In other words, a Weierstrass marked point cannot be on any separating bridge (of the core).*

(3) *Assume that $x \in \overline{\mathcal{W}}_{\ell;i}$, and $\langle \delta_i \rangle$ as in (2.17) does not lie on any non-separating bridge. Then, $\overline{\mathcal{W}}_{\ell;i}$ is smooth at x. In this case, $\{\kappa\} \sqcup \{\zeta_q : q \in N(C)\}$ forms a subset of a system of local parameters on $\mathcal{V}$.*

(4) *If $\langle \delta_i \rangle$ lies on a maximal non-separating bridge $\mathrm{B}[\mathfrak{p}, \mathfrak{q}]$, then $x \in \overline{\mathcal{W}}_{\ell;i}$. In this case, with $z_{\langle \delta_i \rangle}$ denoting a smooth point on $\mathrm{B}[\mathfrak{p}, \mathfrak{q}]$ that is sufficiently close to $\langle \delta_i \rangle$, there exist $f, g \in \Gamma(\mathscr{O}_{\mathcal{V}}^*)$ such that*

$$\kappa = f\zeta_{[z_{\langle \delta_i \rangle}, \mathfrak{p}]} + g\zeta_{[z_{\langle \delta_i \rangle}, \mathfrak{q}]}.$$

Proof. Part (2) follows from Lemma 2.13. The proof of the remaining statements of Proposition 2.19 is parallel to that of Proposition 2.17, and hence is omitted. □

We conclude this subsection with a relation between the $\overline{\mathcal{W}}_{\ell;i}$ and the $\overline{\mathcal{K}}_{\ell;i,j}$.

LEMMA 2.20. *Let $1 \le i, j \le \ell$, $i \ne j$, and $x = (C, \delta_1, \dots, \delta_\ell) \in \overline{\mathcal{M}}_{2,\ell}$. If δ_i, δ_j are on the same (rational) tail, then*

$$x \in \overline{\mathcal{K}}_{\ell;i,j} \iff x \in \overline{\mathcal{W}}_{\ell;i} \iff x \in \overline{\mathcal{W}}_{\ell;j}.$$

Proof. The statements follow from the first statement of Lemma 2.14 via pullback. □

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Remark 2.21. As mentioned at the beginning of this subsection, all the statements for $\overline{\mathcal{W}}_{\ell;i}$ and $\overline{\mathcal{K}}_{\ell;i,j}$ can be applied to $\mathfrak{M}_2^{\mathrm{div}}$ verbatim.

More precisely, a point $(C, D) \in \mathfrak{M}_2^{\mathrm{div}}$ is allowed to have smooth (tail) rational subcurves that each contain exactly one nodal point of C and exactly one element of D and hence may not be stable curves with marked points. By writing $D = \{\delta_1, \dots, \delta_m\}$ (recall that D is simple and effective), without loss of generality we assume that there exists $\ell \le m$ such that each δ_i, $1 \le i \le \ell$, is on a smooth rational subcurve R_i that contains exactly one nodal point q_i of C and satisfies $D \cap R_i = \{\delta_i\}$. We denote by $(\overline{C}, \overline{D} = \{\overline{\delta}_1, \dots, \overline{\delta}_m\})$ the image of (C, D) in $\overline{\mathcal{M}}_{2,m}$ after applying stabilization. Then the aforementioned subcurves R_i are all contracted in $(\overline{C}, \overline{D})$.

For distinct $1 \le j, k \le m$, we say that δ_j is Weierstrass (respectively, δ_j and δ_k are conjugate) if $\overline{\delta}_j$ is too (respectively, $\overline{\delta}_j$ and $\overline{\delta}_k$ are too). It is straightforward that Propositions 2.17 and 2.19 and Lemma 2.20 still hold for $\mathfrak{M}_2^{\mathrm{div}}$; cf. the proof of [HN21, Lemma 4.2] for details.

In the remainder of this paper, when a property for the Weierstrass or conjugate loci on $\mathfrak{M}_2^{\mathrm{div}}$ is presented (e.g. in Corollary 2.31 or Proposition 2.22 (SH_3)), it suffices to prove the same property on $\overline{\mathcal{M}}_{2,m}$.

2.7 Structural homomorphisms: general results

At this stage, we are ready to present the general results on the structural homomorphism

$$\varphi\colon \rho_*\mathscr{M}(\mathcal{D})\ (=\rho_*\mathscr{O}_{\mathcal{C}}(\mathcal{D}+\mathcal{A}-\mathcal{B})) \longrightarrow \rho_*\mathscr{O}_{\mathcal{A}}(\mathcal{A})$$

in Proposition 2.22. This key proposition, along with Proposition 2.17, contains everything we need for diagonalizing φ (considered as a $2 \times m$ matrix): its entries, 2×2 minors, and the relations between them.

We continue with the notation from previous subsections of Section 2. We still fix an arbitrary point of $\mathfrak{M}_2^{\mathrm{div}}$ and an affine smooth chart containing it:

$$z_0 = (C,\ D = \delta_1 + \cdots + \delta_m) \in \mathcal{V} \subset \mathfrak{M}_2^{\mathrm{div}}.$$

We always assume that $\mathcal{V}$ is sufficiently small. Recall that the zero locus of $\kappa_{ij} \in \Gamma(\mathscr{O}_{\mathcal{V}})$ consists of $z \in \mathcal{V}$ with $\mathcal{D}_i(z)$ and $\mathcal{D}_j(z)$ being a pair of conjugate points on $\mathcal{C}_z$. Also recall that $\delta_i \wedge \delta_j$ refers to the node in (2.12), which intuitively is a node between the core F of C and both δ_i and δ_j, and is the 'farthest away' from the core among all such nodes. Finally, for each δ_i, recall that $\langle \delta_i \rangle$ is the 'projection' of δ_i onto the core F of C, as in (2.17).

Proposition 2.22. *With notation as above, we have the following.*

(SH_1) *Under a suitable trivialization of $\rho_*\mathscr{M}(\mathcal{D})$ and $\rho_*\mathscr{O}_{\mathcal{A}}(\mathcal{A})$, φ takes the form*

$$\varphi = \begin{bmatrix} c_{11}\zeta_{[\delta_1,a_1]} & c_{12}\zeta_{[\delta_2,a_1]} & \cdots & c_{1i}\zeta_{[\delta_i,a_1]} & \cdots \\ c_{21}\zeta_{[\delta_1,a_2]} & c_{22}\zeta_{[\delta_2,a_2]} & \cdots & c_{2i}\zeta_{[\delta_i,a_2]} & \cdots \end{bmatrix}.$$

(SH_2) *For any distinct $1 \le i, j \le m$, there exists $u \in \Gamma(\mathscr{O}_{\mathcal{V}}^*)$ such that the determinant $\theta_{ij} := \det \begin{bmatrix} c_{1i} & c_{1j} \\ c_{2i} & c_{2j} \end{bmatrix}$ satisfies*

$$\theta_{ij} = u \cdot \kappa_{ij} \cdot \zeta_{[\delta_i \wedge \delta_j]}.$$

(SH_3) *For any pairwise distinct $1 \le i, j, k \le m$, there exist $v \in \Gamma(\mathscr{O}_{\mathcal{V}}^*)$ and $w \in \Gamma(\mathscr{O}_{\mathcal{V}})$ such that*

$$\kappa_{ik} = v \cdot \kappa_{ij} + w,$$

and the following holds.

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- *If* $\langle\delta_i\rangle$, $\langle\delta_j\rangle$ *and* $\langle\delta_k\rangle$ *all lie on a maximal non-separating bridge* $\mathrm{B}[\mathfrak{p},\mathfrak{q}]$ *of the core of* C, *let* x_i *and* x_j *be smooth points on the core* F *that are sufficiently close to* $\langle\delta_i\rangle$ *and* $\langle\delta_j\rangle$, *respectively. Then, there exist* $f, g \in \Gamma(\mathscr{O}_{\mathcal{V}}^*)$ *and* $f', g' \in \Gamma(\mathscr{O}_{\mathcal{V}})$ *such that*

$$\kappa_{ij} = f \cdot \prod_{q\in N_{[x_i,\mathfrak{p}]}\cap N_{[x_j,\mathfrak{p}]}} \zeta_q \;+\; g\cdot \prod_{q\in N_{[x_i,\mathfrak{q}]}\cap N_{[x_j,\mathfrak{q}]}} \zeta_q,$$
$$w = f' \cdot \prod_{q\in N_{[\delta_j,\mathfrak{p}]}\cap N_{[\delta_k,\mathfrak{p}]}} \zeta_q \;+\; g'\cdot \prod_{q\in N_{[\delta_j,\mathfrak{q}]}\cap N_{[\delta_k,\mathfrak{q}]}} \zeta_q;$$

  moreover, $\det\begin{bmatrix} f & f' \\ g & g' \end{bmatrix} \in \Gamma(\mathscr{O}_{\mathcal{V}}^*)$.
- *Otherwise, there exists* $u' \in \Gamma(\mathscr{O}_{\mathcal{V}}^*)$ *such that* $w = u' \cdot \zeta_{[\delta_j\wedge\delta_k]}$.

Here, (SH_1) is a restatement of Proposition 2.10. To establish (SH_2) and (SH_3), we shall first analyze the entries and 2×2 minors of φ along a maximal chain of smooth rational subcurves on a given tail in Sections 2.8 and 2.9, respectively; we will then complete the proof of Proposition 2.22 in Section 2.10.

Before delving into the technical proofs in Sections 2.8–2.10, we present a simple application of Proposition 2.22 in this subsection, which reveals where $\overline{M}_2(\mathbb{P}^n, d)$ is smooth. We follow the notation in Section 2.3.

PROPOSITION 2.23. *Assume* $(C, D) \in \mathfrak{M}_2^{\mathrm{div}}$ *satisfies one of the following:*

- *the core* F *of* C *is inseparable and* $D\cap F$ *contains a pair of non-conjugate points; or*
- F *is separable, and hence contains two genus 1 inseparable components* F_1 *and* F_2 *(see Definition 2.1), and* $D\cap F_i \neq \emptyset$ *for* $i = 1, 2$.

Then by shrinking $\mathcal{V}$ *if necessary, we see that* $\mathcal{U} = \mathcal{V}\times_{\mathfrak{M}_2^{\mathrm{div}}} U$ *is smooth. In particular, the main component of* $\overline{M}_2(\mathbb{P}^n, d)$ *is generically smooth. When* $d > 2$, *the main component is also of the expected dimension.*

Proof. We can trivialize $\rho_*\mathscr{M}(\mathcal{D})$ and $\rho_*\mathscr{O}_{\mathcal{A}}(\mathcal{A})$ so that

$$\mathscr{E}_{\mathcal{V}} \cong \mathcal{V}\times(\mathbb{A}^{d+1})^n.$$

Let $w_j^i \in \mathbb{A}^1$ for all $1\le i\le n$ and $0\le j\le d$. Recall that by (2.8), the homomorphism 'rest' of (2.6) can be identified with the homomorphism

$$(0\oplus\varphi)^{\oplus n} : (\mathscr{O}_{\mathcal{V}}\oplus\rho_*\mathscr{M}(\mathcal{D}))^{\oplus n} \longrightarrow (\rho_*\mathscr{O}_{\mathcal{A}}(\mathcal{A}))^{\oplus n}.$$

Suppose F is inseparable and $D\cap F$ contains a pair of non-conjugate points. By Proposition 2.22 (SH_1) and (SH_2), we can trivialize $\rho_*\mathscr{M}(\mathcal{D})$ and $\rho_*\mathscr{O}_{\mathcal{A}}(\mathcal{A})$ differently if necessary so that $0\oplus\varphi$ can be represented by

$$0\oplus\varphi = [0, I_2, 0\cdots, 0].$$

Suppose F contains two genus 1 inseparable components F_1 and F_2, and $D\cap F_i\neq\emptyset$ for $i = 1, 2$. Without loss of generality we assume

$$\delta_1,\ a_1 \in F_1, \quad \delta_2,\ a_2 \in F_2.$$

Therefore, $\zeta_{[\delta_1,a_1]} = \zeta_{[\delta_2,a_2]} = 1$ (see the sentence below (2.12)), whereas $\zeta_{[\delta_1,a_2]}$ and $\zeta_{[\delta_2,a_1]}$ vanish at (C, D). By Proposition 2.22 (SH_1), $0\oplus\varphi$ can thus be represented by

$$0\oplus\varphi = [0, I_2, 0\cdots, 0].$$

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Hence, in either case, the equation $\Phi = 0$ is equivalent to the following equations

$$w_1^i = w_2^i = 0 \quad \text{for all } 1 \le i \le n, \tag{2.28}$$

which obviously implies that $\mathcal{U}$ is isomorphic to an open subset of $\mathcal{V} \times (\mathbb{A}^{d-1})^n$. Note that $\dim(\mathcal{V} \times (\mathbb{A}^{d-1})^n) = 3 + d + (d-1)n = d(n+1) - n + 3$, which is the expected dimension of $\overline{M}_2(\mathbb{P}^n, d)$ when $d > 2$. The assertion then follows. □

All the sequential blowups in Section 3 will be performed over the loci of $\mathfrak{M}_2^{\mathrm{wt}}$ or $\mathfrak{M}_2^{\mathrm{div}}$ *outside* the images of the loci described in Proposition 2.23 under the morphism $\overline{M}_2(\mathbb{P}^n, d) \longrightarrow \mathfrak{M}_2^{\mathrm{wt}}$ as in (1.6) and the local morphisms $\overline{M}_2(\mathbb{P}^n, d) \supset U \longrightarrow \mathfrak{M}_2^{\mathrm{div}}$ as in (1.2).

2.8 Structural homomorhpisms: applying semi-stabilization

In this subsection, we aim to extract information about the matrix (2.15) of the homomorphism φ. The main result of this subsection, Proposition 2.27, provides a strengthened form (compared with Proposition 2.10) of each row of the restriction $\varphi_{\le r}$ of the homomorphism φ. This is the first step towards the proof of Proposition 2.22.

We fix $(C, D) \in \mathfrak{M}_2^{\mathrm{div}}$ and an affine smooth chart $\mathcal{V} \longrightarrow \mathfrak{M}_2^{\mathrm{div}}$ containing (C, D). We also fix a rational tail T of C. We focus on a maximal chain $R \subset T$ of smooth rational subcurves

$$R = R_1 \cup \cdots \cup R_h$$

of C attached to a smooth point of the core $F \subset C$. Recall that a bridge is always a part of the core F of C, whilst a rational tail meets the core F only at one point (indeed a pivotal node) as in the definition. In particular, the chain R is not contained in any bridge.

We fix some notation. First, we index the irreducible components of R so that R_1 is attached to F, and $R_{\mu+1}$ is attached to R_μ. Let

$$q_1 = F \cap R_1, \quad q_{\mu+1} = R_\mu \cap R_{\mu+1} \quad \forall\, 1 \le \mu \le h-1. \tag{2.29}$$

In particular, the node q_1 is a pivotal node.

Next, recall that D is simple and effective. We fix a sub-divisor of D (i.e. a subset of D):

$$\{\delta_1, \ldots, \delta_r\} \subset D.$$

Since $\deg D = m$, we have $1 \le r \le m$. Throughout this subsection, we assume that

$$\begin{aligned} &\{\delta_1, \ldots, \delta_r\} \subset R, \\ &\{\delta_1, \ldots, \delta_{r_1}\} \subset R_1, \quad \{\delta_{r_1+1}, \ldots, \delta_{r_1+r_2}\} \subset R_2, \quad \ldots, \\ &\{\delta_{r_1+\cdots+r_{h-1}+1}, \ldots, \delta_{r_1+\cdots+r_{h-1}+r_h} = \delta_r\} \subset R_h. \end{aligned} \tag{2.30}$$

This implies that $r = r_1 + \cdots + r_h$. We emphasize that every r_μ is possibly zero. Intuitively, the assumption (2.30) means that $\{\delta_1, \ldots, \delta_r\}$ is a subset of $D \cap R$, and the larger the index of an element of this subset is, the 'farther away' it is from the core F.

Let

$$\begin{aligned} &e : \{1, \ldots, r\} \longrightarrow \{1, \ldots, h\} \\ &e(i) = \mu \ \text{ if } \delta_i \in R_\mu \text{ for some } 1 \le \mu \le h, \ \forall\, 1 \le i \le r. \end{aligned} \tag{2.31}$$

Thus,

$$\delta_i \in R_{e(i)} \ \ \forall\, 1 \le i \le r \quad \text{and} \quad e(i) \le e(j) \ \ \forall\, 1 \le i < j \le r. \tag{2.32}$$

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We denote by

$$\varphi_{\le r} : \bigoplus_{1\le i\le r} \rho_*\mathscr{M}(\mathcal{D}_i) \longrightarrow \rho_*\mathscr{O}_{\mathcal{A}}(\mathcal{A}) \tag{2.33}$$

the restriction homomorphism.

Remark 2.24. To avoid possible confusion, we mention firmly here that we typically index a member of D or $\mathcal{D}$ by an integer $1 \le i$ (or j) $\le m$, e.g. δ_i or $\mathcal{D}_j$; we typically index a member of the nodes on R or its corresponding smooth rational component by an integer $1 \le \mu \le h$, e.g. q_μ or R_μ.

Recall that the modular parameter associated with the node q_μ is denoted by ζ_{q_μ} for all $1 \le \mu \le h$. Since q_μ is separating,

$$\mathcal{C} \times_{\mathcal{V}} (\zeta_{q_\mu} = 0) = \mathcal{G}_\mu \cup \mathcal{R}_{\ge\mu}$$

is a union of two families of nodal curves over $(\zeta_{q_\mu} = 0)$ intersecting at (i.e. glued along) a section of nodes

$$\Sigma_\mu = \mathcal{G}_\mu \cap \mathcal{R}_{\ge\mu} \quad \text{with} \quad \Sigma_\mu \cap C = q_\mu, \quad \mathcal{R}_{\ge\mu} \cap C \supset R_\mu \cup \cdots \cup R_h.$$

In other words, $\mathcal{G}_\mu \cap C$ and $\mathcal{R}_{\ge\mu} \cap C$ are the two components obtained by normalizing C at the node q_μ, and $\mathcal{R}_{\ge\mu}$ is the component that, before the normalization, is contained in the tail T.

Let $\mathfrak{M}_{2,\mathrm{ss}}$ be the Artin stack of semi-stable genus 2 nodal curves. (A nodal curve is semi-stable if its smooth rational subcurves contain at least two nodes of the nodal curve.) By (successively) contracting rational subcurves that contain only one node of the ambient curves, we obtain a semi-stabilization $\mathfrak{M}_2^{\mathrm{div}} \longrightarrow \mathfrak{M}_{2,\mathrm{ss}}$. Possibly after shrinking and base changing $\mathcal{V}$, we can assume that there is a smooth chart $\underline{\mathcal{V}} \to \mathfrak{M}_{2,\mathrm{ss}}$ with $\mathcal{Y} \to \underline{\mathcal{V}}$ its universal family so that the composite $\mathcal{V} \to \mathfrak{M}_2^{\mathrm{div}} \to \mathfrak{M}_{2,\mathrm{ss}}$ factors through a morphism $g : \mathcal{V} \to \underline{\mathcal{V}}$ together with a semi-stable contraction $\widetilde{g}$ as shown in (2.34).

$$\begin{array}{ccc} \mathcal{V} & \longrightarrow & \mathfrak{M}_2^{\mathrm{div}} \\ \downarrow{\scriptstyle g} & & \downarrow \\ \underline{\mathcal{V}} & \longrightarrow & \mathfrak{M}_{2,\mathrm{ss}} \end{array}, \quad \widetilde{g} : \mathcal{C} \longrightarrow \overline{\mathcal{C}} = \mathcal{Y} \times_{\underline{\mathcal{V}}} \mathcal{V}. \tag{2.34}$$

Note that $\widetilde{g}$ contracts all rational tails of the curves in the family $\mathcal{C} \to \mathcal{V}$. Let $\overline{\mathcal{D}}_i = \widetilde{g}(\mathcal{D}_i)$, which is a family of smooth divisors of $\overline{\mathcal{C}} \to \mathcal{V}$, and let

$$\mathcal{D}_{[j]} = \sum_{i=1}^{j} \mathcal{D}_i \subset \mathcal{C}, \quad \overline{\mathcal{D}}_{[j]} = \sum_{i=1}^{j} \overline{\mathcal{D}}_i \subset \overline{\mathcal{C}} \quad \forall\, 1 \le j \le r.$$

Let $\bar{\rho} : \overline{\mathcal{C}} = \mathcal{Y} \times_{\underline{\mathcal{V}}} \mathcal{V} \to \mathcal{V}$ be the projection to the second component. By construction, the divisors $\mathcal{R}_{\ge 1}, \ldots, \mathcal{R}_{\ge h}$ are contracted under the morphism $\widetilde{g}$, and

$$\widetilde{g}^{-1}(\overline{\mathcal{D}}_i) = \sum_{\mu=1}^{e(i)} \mathcal{R}_{\ge\mu} + \mathcal{D}_i,$$

where e is the index function (2.31).

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Recall that each $r_\mu \geq 0$ is the number of points in $R_\mu \cap D$, and satisfies $\sum_{\mu=1}^{h} r_\mu = r$. For convenience, we set

$$r_{[\mu]} := r_1 + \cdots + r_\mu \quad \forall\, 1 \leq \mu \leq h, \quad r_{[0]} := 0.$$

For every $1 \leq j \leq r$, let

$$\mathcal{R}_{[j]} = \sum_{i=1}^{j} \sum_{\mu=1}^{e(i)} \mathcal{R}_{\geq \mu} = \sum_{\mu=1}^{e(j)} (j - r_{[\mu-1]}) \mathcal{R}_{\geq \mu}. \tag{2.35}$$

As a particular case, we write

$$\mathcal{R} := \mathcal{R}_{[r]} \quad \left(= \sum_{\mu=1}^{h} (r_\mu + \cdots + r_h) \mathcal{R}_{\geq \mu} \right).$$

Then, one calculates that for each $1 \leq i \leq r$,

$$\widetilde{g}^* \mathscr{O}_{\overline{\mathcal{C}}}(\overline{\mathcal{D}}_{[i]} + \overline{\mathcal{A}} - \overline{\mathcal{B}}) = \mathscr{O}_{\mathcal{C}}(\mathcal{R}_{[i]} + \mathcal{D}_{[i]} + \mathcal{A} - \mathcal{B}). \tag{2.36}$$

Thus, we have

$$\rho_* \mathscr{O}_{\mathcal{C}}(\mathcal{D}_{[i]} + \mathcal{A} - \mathcal{B}) \xrightarrow{\subset} \rho_* \mathscr{O}_{\mathcal{C}}(\mathcal{R}_{[i]} + \mathcal{D}_{[i]} + \mathcal{A} - \mathcal{B}) = \bar{\rho}_* \mathscr{O}_{\overline{\mathcal{C}}}(\overline{\mathcal{D}}_{[i]} + \overline{\mathcal{A}} - \overline{\mathcal{B}}). \tag{2.37}$$

Here the identity follows from combining (2.36) and the identities

$$\rho_* \widetilde{g}^* \mathscr{O}_{\overline{\mathcal{C}}}(\overline{\mathcal{D}}_{[i]} + \overline{\mathcal{A}} - \overline{\mathcal{B}}) = \bar{\rho}_* \widetilde{g}_* \widetilde{g}^* \mathscr{O}_{\overline{\mathcal{C}}}(\overline{\mathcal{D}}_{[i]} + \overline{\mathcal{A}} - \overline{\mathcal{B}}) = \bar{\rho}_* \mathscr{O}_{\overline{\mathcal{C}}}(\overline{\mathcal{D}}_{[i]} + \overline{\mathcal{A}} - \overline{\mathcal{B}}),$$

where the last equality holds since both $\mathcal{C}$ and $\overline{\mathcal{C}}$ are smooth and $\widetilde{g} : \mathcal{C} \to \overline{\mathcal{C}}$ is a proper divisorial contraction.

For convenience, we denote

$$\mathbf{M}_j := \Gamma(\rho_* \mathscr{O}_{\mathcal{C}}(\mathcal{D}_{[j]} + \mathcal{A} - \mathcal{B})) \ \subset \ \overline{\mathbf{M}}_j := \Gamma(\bar{\rho}_* \mathscr{O}_{\overline{\mathcal{C}}}(\overline{\mathcal{D}}_{[j]} + \overline{\mathcal{A}} - \overline{\mathcal{B}})), \quad \forall\, 1 \leq j \leq r. \tag{2.38}$$

In particular, we write

$$\mathbf{M} = \mathbf{M}_r, \quad \overline{\mathbf{M}} = \overline{\mathbf{M}}_r.$$

As before, we write $a_s = F \cap \mathcal{A}_s$, $a = a_1 + a_2$, and $b = F \cap \mathcal{B}$. Since they are in general position (see Assumption 2.2), we have

$$H^1(\mathscr{O}_F(\delta_1 + \cdots + \delta_j + a - b)) = 0 \quad \text{and} \quad H^1(\mathscr{O}_F(jq_1 + a - b)) = 0 \quad \forall\, 1 \leq j \leq r,$$

where q_1 is as in (2.29). Thus, for every $1 \leq j \leq r$, $\mathbf{M}_j$ and $\overline{\mathbf{M}}_j$ are both free $\Gamma(\mathscr{O}_{\mathcal{V}})$-module of rank j.

The main goal of Section 2.8 is to find a basis of $\mathbf{M}$ as in Lemma 2.26 below. To this end, we begin with a basis of $\overline{\mathbf{M}}$. We start with the point $(C, D) \in \mathcal{V}$. Since a and b are in general positions, by a simple vanishing argument, we see that $H^0(\mathscr{O}_F(a - b)) = H^1(\mathscr{O}_F(a - b)) = 0$. Hence, the restriction homomorphism

$$H^0(\mathscr{O}_F(rq_1 + a - b)) \longrightarrow H^0(\mathscr{O}_F(rq_1)|_{rq_1}) \tag{2.39}$$

is an isomorphism. Moreover, for every $1 \leq j \leq r$, since $H^1(\mathscr{O}_F(jq_1 + a - b)) = 0$, we have the short exact sequence

$$0 \to H^0(\mathscr{O}_F(jq_1 + a - b)) \to H^0(\mathscr{O}_F(rq_1 + a - b)) \to H^0(\mathscr{O}_F(rq_1 + a - b)|_{(r-j)q_1}) \to 0. \tag{2.40}$$

This determines a complete flag

$$0 = H^0(\mathscr{O}_F(a - b)) \subset H^0(\mathscr{O}_F(q_1 + a - b)) \subset H^0(\mathscr{O}_F(2q_1 + a - b)) \subset \cdots \subset H^0(\mathscr{O}_F(rq_1 + a - b)).$$

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By choosing

$$s_i \in H^0(\mathscr{O}_F(iq_1+a-b))\backslash H^0(\mathscr{O}_F((i-1)q_1+a-b)) \quad \forall\, 1\le i\le r,$$

it is routine to show they form a basis

$$s_1,\ \ldots,\ s_r$$

of the vector space $H^0(\mathscr{O}_F(rq_1+a-b))$, satisfying, for every $1\le j\le r$:

- s_j, as an element of $H^0(\mathscr{O}_F(rq_1+a-b))$, has vanishing order $(r-j)$ at q_1; and
- $s_1,\ldots,s_j$ forms a basis of $H^0(\mathscr{O}_F(jq_1+a-b))$.

Remark 2.25. Notice that for every $1\le i\le j\le r$, s_i, as an element of $H^0(\mathscr{O}_F(jq_1+a-b))$, has vanishing order $(j-i)$ at q_1. Therefore, when studying the vanishing order of a given element (e.g. s_i), we need to specify the module (e.g. $H^0(\mathscr{O}_F(jq_1+a-b))$ versus $H^0(\mathscr{O}_F(rq_1+a-b))$) containing it. The same remark applies to the extensions of $s_1,\ldots,s_r$ to $\mathcal{V}$ as follows.

Next, for any $1\le\mu\le h$, let $\mathcal{H}_\mu=(\zeta_{q_\mu}=0)\subset\mathcal{V}$ and $\mathcal{H}=\bigcup_{\mu=1}^h\mathcal{H}_\mu$. Let

$$\bar\rho_{\mathcal{H}}:\overline{\mathcal{C}}_{\mathcal{H}}:=\overline{\mathcal{C}}\times_{\mathcal{V}}\mathcal{H}\longrightarrow\mathcal{H}$$

be the projection and $\bar\Sigma_\mu=\widetilde{g}(\Sigma_\mu)$, where $\Sigma_\mu=\mathcal{G}_\mu\cap\mathcal{R}_{\ge\mu}$. For the same reason as earlier, $\overline{\mathbf{M}}_{\mathcal{H}}:=\Gamma((\bar\rho_{\mathcal{H}})_*(\mathscr{O}_{\overline{\mathcal{C}}_{\mathcal{H}}}(\overline{\mathcal{D}}_{[r]}+\overline{\mathcal{A}}-\overline{\mathcal{B}})))$ is a free $\Gamma(\mathscr{O}_{\mathcal{H}})$-module and has a basis $\bar S_{\mathcal{H},1},\ldots,\bar S_{\mathcal{H},r}$ extending the basis $s_1,\ldots,s_r$, satisfying, for every $1\le j\le r$:

- $\bar S_{\mathcal{H},j}$, as an element of $\overline{\mathbf{M}}_{\mathcal{H}}$, has vanishing order $(r-j)$ along $\bar\Sigma_\mu$ for all $1\le\mu\le e(j)$; and
- $\bar S_{\mathcal{H},1},\ldots,\bar S_{\mathcal{H},j}$ forms a basis of $\Gamma((\bar\rho_{\mathcal{H}})_*(\mathscr{O}_{\overline{\mathcal{C}}_{\mathcal{H}}}(\overline{\mathcal{D}}_{[j]}+\overline{\mathcal{A}}-\overline{\mathcal{B}})))$.

By shrinking $\mathcal{V}$ if necessary, we can extend $\bar S_{\mathcal{H},i}$ to $\bar S_i\in\overline{\mathbf{M}}$ so that each $\overline{\mathbf{M}}_j$ of (2.38), as a free $\Gamma(\mathscr{O}_{\mathcal{V}})$-module, is generated freely by $\bar S_1,\ldots,\bar S_j$. Thus, by (2.37), this implies that $\widetilde{g}^*\bar S_1,\ldots,\widetilde{g}^*\bar S_j$ freely generate the free $\Gamma(\mathscr{O}_{\mathcal{V}})$-module $\Gamma(\rho_*\mathscr{O}_{\mathcal{C}}(\mathcal{R}_{[j]}+\mathcal{D}_{[j]}+\mathcal{A}-\mathcal{B}))$ for all $1\le j\le r$.

LEMMA 2.26. *Let*

$$\zeta_{[[\delta_i]]}=\prod_{\ell=1}^{i}\zeta_{[\delta_\ell]},\qquad S_i=\zeta_{[[\delta_i]]}\cdot\widetilde{g}^*\bar S_i,\quad 1\le i\le r.$$

Then, each $\mathbf{M}_j$ *of (2.38) is a free* $\Gamma(\mathscr{O}_{\mathcal{V}})$*-module of rank* j *generated by* $S_1,\ldots,S_j$*. In particular,* $\mathbf{M}$ $(=\mathbf{M}_r)$ *is a free* $\Gamma(\mathscr{O}_{\mathcal{V}})$*-module of rank* r *generated by* $S_1,\ldots,S_r$*.*

Proof. Throughout this proof, we fix an arbitrary $1\le j\le r$.

Since $\mathbf{M}_j$ is a free $\Gamma(\mathscr{O}_{\mathcal{V}})$-module of rank j, it suffices to prove:

(1) $S_1,\ldots,S_j$ lie in $\mathbf{M}_j$; and

(2) $\mathbf{M}_j$ is generated by $S_1,\ldots,S_j$.

(1) For every $1\le i\le j$, we show $S_i\in\mathbf{M}_j$. Since $\mathbf{M}_i$ is naturally included in $\mathbf{M}_j$, it suffices to show that $S_i\in\mathbf{M}_i$. By (2.37), this is equivalent to showing

$$S_i|_{\mathcal{R}_{[i]}}=0\in\Gamma(\rho_*\mathscr{O}_{\mathcal{R}_{[i]}}(\mathcal{R}_{[i]}+\mathcal{D}_{[i]}+\mathcal{A}-\mathcal{B})).\tag{2.41}$$

To establish (2.41), recall that, for every $1\le\mu\le e(i)$, $\zeta_{[[\delta_i]]}=\prod_{\ell=1}^{i}\zeta_{[\delta_\ell]}$ and $r_{[\mu-1]}=\sum_{\nu=1}^{\mu-1}r_\nu$ (which is 0 if $\mu=1$). Thus, $\rho^*\zeta_{[[\delta_i]]}$ has vanishing order

$$\#\{1\le\ell\le i:e(\ell)\ge\mu\}=i-r_{[\mu-1]}\quad\text{along }\mathcal{R}_{\ge\mu}.\tag{2.42}$$

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In addition, by Remark 2.25 and our choice of the vanishing order of $\bar{S}_i$ along $\bar{\Sigma}_\mu$, we see that $\widetilde{g}^*\bar{S}_i$, as an element of $\Gamma(\rho_*\mathscr{O}_{\mathcal{C}}(\mathcal{R}_{[i]}+\mathcal{D}_{[i]}+\mathcal{A}-\mathcal{B}))$, has vanishing order 0 along $\mathcal{R}_{\geq\mu}$ for all $1\leq\mu\leq e(i)$. Therefore, $S_i=\zeta_{[[\delta_i]]}\cdot\widetilde{g}^*\bar{S}_i$, as an element of $\Gamma(\rho_*\mathscr{O}_{\mathcal{C}}(\mathcal{R}_{[i]}+\mathcal{D}_{[i]}+\mathcal{A}-\mathcal{B}))$, has vanishing order $(i-r_{[\mu-1]})$ along $\mathcal{R}_{\geq\mu}$ for all $1\leq\mu\leq e(i)$. Taking (2.35) into consideration, we obtain (2.41).

(2) We now prove that $\mathbf{M}_j$ is generated by $S_1,\ldots,S_j$. Take an arbitrary $S'\in\mathbf{M}_j$, which is identified with its image in $\Gamma(\rho_*\mathscr{O}_{\mathcal{C}}(\mathcal{R}_{[j]}+\mathcal{D}_{[j]}+\mathcal{A}-\mathcal{B}))$ via the inclusion in (2.37). Using the identity in (2.37) and the fact that $\overline{\mathbf{M}}_j$ is a free $\Gamma(\mathscr{O}_{\mathcal{V}})$-module *freely generated* by $\bar{S}_1,\ldots,\bar{S}_j$, S' can be uniquely expressed as a linear combination

$$S'=f_1\cdot\widetilde{g}^*\bar{S}_1+\cdots+f_j\cdot\widetilde{g}^*\bar{S}_j,\quad f_i\in\Gamma(\mathscr{O}_{\mathcal{V}}). \tag{2.43}$$

(Here, some f_i are allowed to be zero.) Since $S_i=\zeta_{[[\delta_i]]}\cdot\widetilde{g}^*\bar{S}_i$ for every $1\leq i\leq j$, part (2) will follow from Claim A below.

Claim A. *For every $1\leq i\leq j$, $f_i/\zeta_{[[\delta_i]]}$ is regular over $\mathcal{V}$.*

To establish Claim A, notice that for every $1\leq i\leq j$, $\zeta_{[[\delta_i]]}$ only vanishes along $\bigcup_{\mu=1}^{e(i)}\mathcal{H}_\mu$, and hence we analyze the vanishing order of f_i along these $\mathcal{H}_\mu$. The goal is to show that the vanishing order of ρ^*f_i along each $\mathcal{R}_{\geq\mu}$ is at least $i-r_{[\mu-1]}$, which is the vanishing order of $\rho^*\zeta_{[[\delta_i]]}$ along $\mathcal{R}_{\geq\mu}$ (see (2.42)). As the analysis is local, we will localize $\mathcal{C}$ at $\mathcal{R}_{\geq\mu}$ (respectively, $\mathcal{V}$ at $\mathcal{H}_\mu$) for each $1\leq\mu\leq e(i)$. Below are some necessary setups.

For every $1\leq\mu\leq h$, let $\mathcal{U}_\mu\subset\mathcal{C}$ be an affine open subset that contains the generic point ξ_μ of $\mathcal{R}_{\geq\mu}$. Let $u_\mu\in\Gamma(\mathscr{O}_{\mathcal{U}_\mu})$ be such that $\mathcal{R}_{\geq\mu}\cap\mathcal{U}_\mu=(u_\mu=0)\subset\mathcal{U}_\mu$. Since $\mathcal{R}_{\geq\mu}\to\mathcal{H}_\mu$ is a family of rational curves and has general fibers isomorphic to $\mathbb{P}^1$, we can find a rational function v_μ on $\mathcal{U}_\mu\cap\mathcal{R}_{\geq\mu}$ so that v_μ restricted to general fibers of $\mathcal{R}_{\geq\mu}\cap\mathcal{U}_\mu\to\mathcal{H}_\mu$ are birational maps to $\mathbb{A}^1$. We next let ξ'_μ be the generic point of $\bar{\Sigma}_\mu$. Since $\widetilde{g}(\xi_\mu)=\xi'_\mu$, the field $K_\mu=\mathscr{O}_{\xi_\mu}$ contains $K'_\mu=\mathscr{O}_{\xi'_\mu}$ as its subfield. Because of our choice of v_μ, we have $K_\mu=K'_\mu(v_\mu)$.

Let $\hat{\mathscr{O}}_{\mathcal{U}_\mu}$ be the formal completion of $\mathscr{O}_{\mathcal{U}_\mu}$ along ξ_μ. Then $\hat{\mathscr{O}}_{\mathcal{U}_\mu}=K_\mu[[\hat{u}_\mu]]$, where $\hat{u}_\mu$ is the image of u_μ in $\hat{\mathscr{O}}_{\mathcal{U}_\mu}$. For any $f\in\Gamma(\mathscr{O}_{\mathcal{V}})$, we denote by $\hat{f}$ its image in $\hat{\mathscr{O}}_{\mathcal{U}_\mu}$ via the pullback $\mathscr{O}_{\mathcal{V}}\to\mathscr{O}_{\mathcal{U}_\mu}$ and the completion map. In this way, we obtain

$$\Gamma(\mathscr{O}_{\mathcal{V}})\longrightarrow\hat{\mathscr{O}}_{\mathcal{U}_\mu},\quad f\mapsto\hat{f}.$$

We next pick an open $\mathcal{U}'_\mu\subset\overline{\mathcal{C}}$ that contains the generic point ξ'_μ of $\bar{\Sigma}_\mu$. By shrinking $\mathcal{U}_\mu$ if necessary, we can assume $\widetilde{g}(\mathcal{U}_\mu)\subset\mathcal{U}'_\mu$.

For every $1\leq i\leq j$ and $1\leq\mu\leq e(i)$, fix a trivialization $\mathscr{O}_{\mathcal{U}'_\mu}(\overline{\mathcal{D}}_{[i]}+\overline{\mathcal{A}}-\overline{\mathcal{B}})\cong\mathscr{O}_{\mathcal{U}'_\mu}$, and form the induced trivialization

$$\mathscr{O}_{\mathcal{U}_\mu}(\mathcal{R}_{[i]}+\mathcal{D}_{[i]}+\mathcal{A}-\mathcal{B})\cong\widetilde{g}^*\mathscr{O}_{\mathcal{U}'_\mu}(\overline{\mathcal{D}}_{[i]}+\overline{\mathcal{A}}-\overline{\mathcal{B}})\cong\mathscr{O}_{\mathcal{U}_\mu}. \tag{2.44}$$

Using this trivialization, we can identify any section in $\Gamma(\mathscr{O}_{\mathcal{U}_\mu}(\mathcal{R}_{[i]}+\overline{\mathcal{D}}_{[i]}+\overline{\mathcal{A}}-\overline{\mathcal{B}}))$ with a regular function in $\Gamma(\mathscr{O}_{\mathcal{U}_\mu})$, thus obtaining its image in $\hat{\mathscr{O}}_{\mathcal{U}_\mu}$. Thus, this process provides

$$\Gamma(\mathscr{O}_{\mathcal{U}_\mu}(\mathcal{R}_{[i]}+\mathcal{D}_{[i]}+\mathcal{A}-\mathcal{B}))\longrightarrow\mathscr{O}_{\mathcal{U}_\mu}(\mathcal{R}_{[i]}+\mathcal{D}_{[i]}+\mathcal{A}-\mathcal{B})\otimes_{\mathscr{O}_{\mathcal{U}_\mu}}\hat{\mathscr{O}}_{\mathcal{U}_\mu}\cong\hat{\mathscr{O}}_{\mathcal{U}_\mu},\quad S\mapsto[S].$$

For $1\leq i\leq j$ and $1\leq\mu\leq e(i)$ as above, by Remark 2.25 and our choice of the vanishing order of $\bar{S}_i$ along $\bar{\Sigma}_\mu$, we see that $\widetilde{g}^*\bar{S}_i|\mathcal{U}_\mu$, as an element of $\Gamma(\rho_*\mathscr{O}_{\mathcal{U}_\mu}(\mathcal{R}_{[j]}+\mathcal{D}_{[j]}+\mathcal{A}-\mathcal{B}))$, has vanishing order $(j-i)$ along $\mathcal{R}_{\geq\mu}$. So, there exists (nonzero) $\lambda_i\in\hat{\mathscr{O}}_{\mathcal{U}_\mu}=K_\mu[[\hat{u}_\mu]]$ such that

$$[\widetilde{g}^*\bar{S}_i]=\lambda_i\,(\hat{u}_\mu)^{j-i}\quad\text{and}\quad\hat{u}_\mu\not|\,\lambda_i. \tag{2.45}$$

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Now back to (2.43). Since $S' \in \mathbf{M}_j$, $S'|_{\mathcal{R}_{[j]}} = 0$. In particular, $(u_\mu)^{j-r_{[\mu-1]}}$ divides $S'|_{\mathcal{U}_\mu}$ for all $1 \le \mu \le e(j)$. To establish Claim A, we need the following claim.

Claim B. *For every $1 \le i \le j$, every $1 \le \mu \le e(i)$, and every positive integer α, u_μ^α divides $S'|_{\mathcal{U}_\mu}$ if and only if $(\rho^* f_i)|_{\mathcal{U}_\mu}$ has vanishing order at least $\max(0, \alpha-(j-i))$ along $\mathcal{R}_{\ge\mu}$.*

The 'if' direction of Claim B holds because each $f_i \cdot \widetilde{g}^* \bar{S}_i$, as an element of $\Gamma(\rho_* \mathscr{O}_{\mathcal{C}}(\mathcal{R}_{[j]} + \mathcal{D}_{[j]} + \mathcal{A} - \mathcal{B}))$, has vanishing order

$$\max(0, \alpha-(j-i)) + (j-i) \ge \alpha - (j-i) + (j-i) = \alpha$$

along $\mathcal{R}_{\ge\mu}$ for any $1 \le \mu \le e(i)$.

Next, we prove the 'only if' part of Claim B. Taking their respective images in $\hat{\mathscr{O}}_{\mathcal{U}_\mu}$, u_μ^α dividing $S'|_{\mathcal{U}_\mu}$ implies that

$$(\hat{u}_\mu)^\alpha \quad \text{divides} \quad [S'] = \hat{f}_1 \cdot [\widetilde{g}^* \bar{S}_1] + \cdots + \hat{f}_j \cdot [\widetilde{g}^* \bar{S}_j] \in K_\mu[[\hat{u}_\mu]]. \tag{2.46}$$

By replacing v_μ by v_μ^{-1} if necessary, we can assume that $\hat{\zeta}_{q_\mu} = c_\mu \hat{u}_\mu v_\mu$, where $\zeta_{q_\mu} \in \Gamma(\mathscr{O}_{\mathcal{V}})$ is the modular parameter corresponding to the node q_μ, $\hat{\zeta}_{q_\mu}$ is its image in $\hat{\mathscr{O}}_{\mathcal{U}_\mu}$, and $c_\mu \ne 0 \in K'_\mu$. Let α_i be the order by which f_i is divisible by ζ_{q_μ}; in other words,

$$f_i/\zeta_{q_\mu}^{\alpha_i} \in \Gamma(\mathscr{O}_{\mathcal{V}}) \quad \text{and} \quad f_i/\zeta_{q_\mu}^{\alpha_i+1} \notin \Gamma(\mathscr{O}_{\mathcal{V}}).$$

Then, $\hat{f}_i = c'_i(\hat{\zeta}_{q_\mu})^{\alpha_i}$, where $c'_i \ne 0 \in K'_\mu$. (The case $f_i = 0$ is trivially true and is ignored.)

By (2.45), we see that (2.46) translates to the statement that $(\hat{u}_\mu)^\alpha$ divides

$$\sum_{i=1}^{j} \hat{f}_i \cdot [\widetilde{g}^* \bar{S}_i] = \sum_{i=1}^{j} c'_i (c_\mu \hat{u}_\mu v_\mu)^{\alpha_i} \cdot \lambda_i (\hat{u}_\mu)^{j-i} = \sum_{k \ge 0} v_\mu^k \sum_{1 \le i \le j \text{ s.t. } \alpha_i = k} (c'_i (c_\mu)^k \lambda_i)(\hat{u}_\mu)^{k+j-i}$$

in $K_\mu[[\hat{u}_\mu]] = K'_\mu(v_\mu)[[\hat{u}_\mu]]$. As $\hat{u}_\mu$ does not divide $c'_i(c_\mu)^k \lambda_i$, this divisibility holds if and only if $(\hat{u}_\mu)^\alpha$ divides each individual $v_\mu^k (\hat{u}_\mu)^{k+j-i}$, which holds if and only if $\alpha \le \alpha_i + j - i$ for each i where $f_i \ne 0$. This completes the 'only if' part, so Claim B is proved.

We continue to prove Claim A. For every $1 \le i \le j$ and $1 \le \mu \le e(i)$, recall that we have shown that $(u_\mu)^{j-r_{[\mu-1]}}$ divides $S'|_{\mathcal{U}_\mu}$. By applying Claim B to $\alpha = j - r_{[\mu-1]}$, we conclude that

$$\alpha_i \ge (j - r_{[\mu-1]}) - (j-i) \; = \; i - r_{[\mu-1]}.$$

This shows that $f_i/\zeta_{[[\delta_i]]}$, which is a meromorphic function on $\mathcal{V}$, is regular over an open subset of $\mathcal{V}$ containing the generic point of $\mathcal{H}_\mu$, for any $1 \le \mu \le e(i)$. Since $f_i/\zeta_{[[\delta_i]]}$ is regular over $\mathcal{V} - \bigcup_{\mu=1}^{e(i)} \mathcal{H}_\mu$ and regular at the generic point of $\mathcal{H}_\mu$ for all $1 \le \mu \le e(i)$, by Hartogs' Lemma it is regular over $\mathcal{V}$. This proves Claim A, hence completes part (2) and thus the entire proof of Lemma 2.26. □

We are ready to present a strengthened form of the restriction homomorphism

$$\varphi_{\le r} : \bigoplus_{1 \le i \le r} \rho_* \mathscr{M}(\mathcal{D}_i) \longrightarrow \rho_* \mathscr{O}_{\mathcal{A}}(\mathcal{A})$$

given in (2.33). By Proposition 2.10, possibly under basis changes of $\mathbf{M}$ and (2.14), we see that $\varphi_{\le r}$ takes the form

$$A = \begin{bmatrix} c_{11}\zeta_{[\delta_1,a_1]} & c_{12}\zeta_{[\delta_2,a_1]} & c_{13}\zeta_{[\delta_3,a_1]} & \cdots \\ c_{21}\zeta_{[\delta_1,a_2]} & c_{22}\zeta_{[\delta_2,a_2]} & c_{23}\zeta_{[\delta_3,a_2]} & \cdots \end{bmatrix}, \tag{2.47}$$

with $c_{ij} \in \Gamma(\mathscr{O}_{\mathcal{V}}^*)$.

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PROPOSITION 2.27. *Assume $\delta_1, \dots, \delta_r$ are on R and ordered according to (2.30). After choosing a suitable trivialization*

$$\bigoplus_{1\le i\le r} \rho_*\mathscr{M}(\mathcal{D}_i) \cong \mathscr{O}_{\mathcal{V}}^{\oplus r} \quad \text{and} \quad \rho_*\mathscr{O}_{\mathcal{A}_s}(\mathcal{A}_s) \cong \mathscr{O}_{\mathcal{V}}, \quad s=1,2,$$

the restriction homomorphism $\varphi_{\le r}$ also takes the form

$$\varphi_{\le r} = \begin{bmatrix} \widehat{c}_{11}\zeta_{[\delta_1,a_1]} & \widehat{c}_{12}\zeta_{[\delta_1,a_1]}\zeta_{[\delta_2]} & \widehat{c}_{13}\zeta_{[\delta_1,a_1]}\zeta_{[\delta_2]}\zeta_{[\delta_3]} & \cdots \\ \widehat{c}_{21}\zeta_{[\delta_1,a_2]} & \widehat{c}_{22}\zeta_{[\delta_1,a_2]}\zeta_{[\delta_2]} & \widehat{c}_{23}\zeta_{[\delta_1,a_2]}\zeta_{[\delta_2]}\zeta_{[\delta_3]} & \cdots \end{bmatrix} \tag{2.48}$$

where $\widehat{c}_{11} = c_{11}, \widehat{c}_{21} = c_{21} \in \Gamma(\mathscr{O}_{\mathcal{V}}^*)$, and $\widehat{c}_{ij} \in \Gamma(\mathscr{O}_{\mathcal{V}}), \forall\, j > 1$.

Proof. We follow the notation of Lemma 2.26. Let

$$b_{ij} = \bar{S}_j|_{\bar{\mathcal{A}}_i} \in \Gamma(\rho_*\mathscr{O}_{\mathcal{A}_i}(\mathcal{A}_i)) \cong \Gamma(\mathscr{O}_{\mathcal{V}}).$$

Then under $\{S_1, \cdots, S_r\}$ and (2.14), φ^R takes the form

$$B = \begin{bmatrix} b_{11}\zeta_{[\delta_1]} & b_{12}\zeta_{[\delta_1]}\zeta_{[\delta_2]} & b_{13}\zeta_{[\delta_1]}\zeta_{[\delta_2]}\zeta_{[\delta_3]} & \cdots \\ b_{21}\zeta_{[\delta_1]} & b_{22}\zeta_{[\delta_1]}\zeta_{[\delta_2]} & b_{23}\zeta_{[\delta_1]}\zeta_{[\delta_2]}\zeta_{[\delta_3]} & \cdots \end{bmatrix}. \tag{2.49}$$

Therefore, $B = A(\mu_{ij})$ for a matrix $(\mu_{ij}) \in GL(r, \mathscr{O}_{\mathcal{V}})$.

Moreover, the choices of the sections S_i, $1 \le i \le r$, in Lemma 2.26 imply that (μ_{ij}) is upper triangular, because with $\sigma_1, \dots, \sigma_r$ denoting a basis of $\mathbf{M}$ that gives rise to the matrix A (i.e. each σ_i is a nonzero element of $\Gamma(\rho_*\mathscr{O}_{\mathcal{C}}(\mathcal{D}_i + \mathcal{A} - \mathcal{B}))$), we conclude from Lemma 2.26 that for each $1 \le j \le r$, both $\sigma_1, \dots, \sigma_j$ and $S_1, \dots, S_j$ are bases of $\mathbf{M}_j = \Gamma(\rho_*\mathscr{O}_{\mathcal{C}}(\mathcal{D}_{[j]} + \mathcal{A} - \mathcal{B}))$. Multiplying each b_{ij} by a unit if necessary, we may further assume that the entries on the diagonal of (μ_{ij}) are all 1.

Let $x_{q_1} \in F$ be a smooth point of the curve C that is sufficiently close to the node q_1. Since $\zeta_{[x_{q_1},a_1]}$ divides the first row of (2.47), it divides the first row of (2.49) as well. Since $\zeta_{[x_{q_1},a_1]}$ is coprime with $\zeta_{[\delta_j]}$ for all $1 \le j \le r$, we conclude that $\zeta_{[x_{q_1},a_1]}$ divides b_{1j}, thus $b_{1j} = \widehat{c}_{1j}\zeta_{[x_{q_1},a_1]}$ for some $\widehat{c}_{1j} \in \Gamma(\mathscr{O}_{\mathcal{V}}^*)$.

Similar arguments can be applied to the second row, so we obtain

$$b_{sj} = \widehat{c}_{sj}\zeta_{[x_{q_1},a_s]} \quad \forall\ s=1,2, \quad 1 \le j \le r. \tag{2.50}$$

Note that $\zeta_{[\delta_j,a_s]} = \zeta_{[x_{q_1},a_s]}\zeta_{[\delta_j]}$. This brings both rows of (2.49) to the desired form.

It remains to check $\widehat{c}_{11}$ and $\widehat{c}_{21}$. Let $(\nu_{ij}) = (\mu_{ij})^{-1}$. Then (ν_{ij}) is also upper triangular with all diagonal entries equal to 1, and for $s = 1, 2$ and $1 \le i \le r$, we have

$$c_{si}\zeta_{[\delta_i,a_s]} = \nu_{1i}\widehat{c}_{s1}\zeta_{[\delta_1,a_s]} + \nu_{2i}\widehat{c}_{s2}\zeta_{[\delta_1,a_s]}\zeta_{[\delta_2]} + \cdots + \nu_{ii}\widehat{c}_{si}\zeta_{[\delta_1,a_s]}\zeta_{[\delta_2]}\cdots\zeta_{[\delta_i]}. \tag{2.51}$$

Hence for $s = 1, 2$, we have $c_{s1} = \nu_{11}\cdot\widehat{c}_{s1} = \widehat{c}_{s1}$ and $\widehat{c}_{s1} \in \Gamma(\mathscr{O}_{\mathcal{V}}^*)$. □

2.9 Structural homomorphisms: 2×2 minors

We continue with the notation of Section 2.8. For conciseness, we let

$$z_0 = (C, D) \in \mathcal{V} \subset \mathfrak{M}_2^{\mathrm{div}}$$

and always assume that $\mathcal{V}$ is sufficiently small. In this subsection, we still focus on δ_i, $1 \le i \le r$, that are on a maximal chain of tail rational subcurves R, attached to the core F of C at the node q_1.

Below we analyze the 2×2 minors of $\varphi_{\le r}$, which is the second step towards the proof of Proposition 2.22.

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For $s=1,2$ and $1\le i\le r$, let c_{si} and $\widehat{c}_{si}$ be as in (2.47) and (2.48), respectively. For $1\le i<j\le r$, we set

$$\theta_{ij}=\det\begin{bmatrix}c_{1i}&c_{1j}\\c_{2i}&c_{2j}\end{bmatrix},\quad \widehat{\theta}_{ij}=\det\begin{bmatrix}\widehat{c}_{1i}&\widehat{c}_{1j}\\\widehat{c}_{2i}&\widehat{c}_{2j}\end{bmatrix}\quad\in\Gamma(\mathscr{O}_{\mathcal{V}}).\tag{2.52}$$

For distinct $1\le i,j\le r$, let $\mathcal{K}_{ij}\subset\mathfrak{M}_2^{\mathrm{div}}$ be the locus of points z such that $\mathcal{D}_i(z)$ $(=\mathcal{D}_i\cap\mathcal{C}_z)$ and $\mathcal{D}_j(z)$ $(=\mathcal{D}_j\cap\mathcal{C}_z)$ are a pair of conjugate points. By Proposition 2.17, $\mathcal{K}_{ij}$ is a Cartier divisor locally given by $\kappa_{ij}=0$ for some $\kappa_{ij}\in\Gamma(\mathscr{O}_{\mathcal{V}})$. All possible expressions of κ_{ij} are listed in Proposition 2.17.

PROPOSITION 2.28. *With notation as above, we have:*

(R_1) *there exists $u\in\Gamma(\mathscr{O}_{\mathcal{V}}^*)$ such that*

$$\theta_{12}=\zeta_{[\delta_1]}\cdot\widehat{\theta}_{12}\quad\text{and}\quad\widehat{\theta}_{12}=u\cdot\kappa_{12};$$

(R_2) *there exists $v\in\Gamma(\mathscr{O}_{\mathcal{V}})$ such that*

$$\theta_{13}=\zeta_{[\delta_1]}\zeta_{[\delta_2]}\widehat{\theta}_{13}+v\cdot\theta_{12};$$

(R_3) *if q_1 is not on any non-separating bridge of C, then*

$$\operatorname{rank}\begin{bmatrix}\widehat{c}_{11}&\widehat{c}_{12}&\widehat{c}_{13}\\\widehat{c}_{21}&\widehat{c}_{22}&\widehat{c}_{23}\end{bmatrix}(z_0)=2,$$

or equivalently, $\widehat{\theta}_{13}$ is invertible along $\mathcal{K}_{12}$; and

(R_4) *if q_1 is on a maximal non-separating bridge $\mathrm{B}[\mathfrak{p},\mathfrak{q}]$ of C, with $x_{q_1}\in F$ denoting a smooth point of the core F that is sufficiently close to the node q_1, then there exist $f_2,g_2\in\Gamma(\mathscr{O}_{\mathcal{V}}^*)$ and $f_3,g_3\in\Gamma(\mathscr{O}_{\mathcal{V}})$ such that*

$$\widehat{\theta}_{12}=f_2\zeta_{[x_{q_1},\mathfrak{p}]}+g_2\zeta_{[x_{q_1},\mathfrak{q}]},\quad\widehat{\theta}_{13}=f_3\zeta_{[x_{q_1},\mathfrak{p}]}+g_3\zeta_{[x_{q_1},\mathfrak{q}]},\quad\det\begin{bmatrix}f_2&f_3\\g_2&g_3\end{bmatrix}\in\Gamma(\mathscr{O}_{\mathcal{V}}^*).$$

To establish Proposition 2.28, we need the following lemma.

LEMMA 2.29. *With notation as in Proposition 2.28, we have the following.*

(1) *If the core F of the curve C contains at most one node and that node, if it exists, is non-separating, then:*
   (1.a) $\widehat{\theta}_{12}(z_0)=0$ if and only if q_1 is Weierstrass, or equivalently, if and only if δ_1 and δ_2 are conjugate;
   (1.b) $\operatorname{rank}\begin{bmatrix}\widehat{c}_{11}&\widehat{c}_{12}&\widehat{c}_{13}\\\widehat{c}_{21}&\widehat{c}_{22}&\widehat{c}_{23}\end{bmatrix}(z_0)=2.$

(2) *For any $z\in\mathcal{V}$ such that $\mathcal{C}_z$ contains at most one node and that node, if it exists, is non-separating:*
   (2.a) $\theta_{12}(z)=0$ if and only if $z\in\mathcal{K}_{12}\cap\mathcal{V}$, that is, $\mathcal{D}_1(z)$ and $\mathcal{D}_2(z)$ are a pair of conjugate points;
   (2.b) $\operatorname{rank}\begin{bmatrix}c_{11}&c_{12}&c_{13}\\c_{21}&c_{22}&c_{23}\end{bmatrix}(z)=2.$

(3) *θ_{12} vanishes to the first order along $\mathcal{K}_{12}\cap\mathcal{V}$.*

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(4) *For any $z \in \mathcal{V}$ such that $\mathcal{C}_z$ consists of two smooth genus 1 subcurves attached at a (separating) node:*

(4.a) if $z \notin \mathcal{K}_{12}$, then $\theta_{12}(z) \neq 0$;
(4.b) if $z \in \mathcal{K}_{12}$, then $\theta_{13}(z) \neq 0$.

Proof. We continue to use the notation of Section 2.8.

(1.a) Since $s_1, s_2 \in H^0(\mathscr{O}_F(rq_1 + a - b))$ have vanishing order $r-1$ and $r-2$ respectively at q_1, one verifies directly that the image of $H^0(\mathscr{O}_F(2q_1 + a - b))$ in $H^0(\mathscr{O}_F(rq_1 + a - b))$ (via the inclusion) is the subspace spanned by s_1 and s_2. By identifying the image of $H^0(\mathscr{O}_F(2q_1 + a - b))$ with this subspace, we obtain a homomorphism

$$H^0(\mathscr{O}_F(2q_1 + a - b)) \xrightarrow{s_{12}} \mathbf{k}^2, \quad \text{where}$$
$$s_{12} = \begin{bmatrix} s_1(a_1) & s_2(a_1) \\ s_1(a_2) & s_2(a_2) \end{bmatrix} = \begin{bmatrix} \widehat{c}_{11} & \widehat{c}_{12} \\ \widehat{c}_{21} & \widehat{c}_{22} \end{bmatrix} (z_0).$$

The last equality above holds because F is assumed to be inseparable. Thus, the matrix $\begin{bmatrix} \widehat{c}_{11} & \widehat{c}_{12} \\ \widehat{c}_{21} & \widehat{c}_{22} \end{bmatrix} (z_0)$ is of rank 1 if and only if $h^0(\mathscr{O}_F(2q_1 - b)) = 1$. Since b is a general point, the evaluation homomorphism at the point b, $H^0(\mathscr{O}_F(2q_1)) \to \mathbf{k}$, is surjective. By the exact sequence

$$0 \longrightarrow H^0(\mathscr{O}_F(2q_1 - b)) \longrightarrow H^0(\mathscr{O}_F(2q_1)) \longrightarrow \mathbf{k} \longrightarrow 0,$$

$h^0(\mathscr{O}_F(2q_1 - b)) = 1$ is equivalent to $h^0(\mathscr{O}_F(2q_1)) = 2$, which is further equivalent to the fact that q_1 is a Weierstrass point because F is either smooth or has exactly one self-intersection and no other nodes; see the proof of [BC23, Proposition 3.12] for the details of the latter case. Thus, $\widehat{\theta}_{12}(z_0) = 0$ if and only if q_1 is Weierstrass, which by Lemma 2.20 is equivalent to δ_1 and δ_2 being conjugate. This proves part (1.a).

(1.b) Likewise, one sees that the natural image of $H^0(\mathscr{O}_F(3q_1 + a - b))$ in $H^0(\mathscr{O}_F(rq_1 + a - b))$ is the subspace spanned by s_1, s_2 and s_3. Assume $\operatorname{rank} \begin{bmatrix} \widehat{c}_{11} & \widehat{c}_{12} & \widehat{c}_{13} \\ \widehat{c}_{21} & \widehat{c}_{22} & \widehat{c}_{23} \end{bmatrix} (z_0) \leq 1$. By arguments similar to the ones used in (1.a), we obtain that $h^0(\mathscr{O}_F(3q_1 - b)) \geq 2$, and consequently $h^0(\mathscr{O}_F(3q_1)) \geq 3$, which is impossible by Riemann–Roch because F is either smooth or has exactly one self-intersection and no other nodes. This completes the proof of part (1.b).

(2) Let $z \in \mathcal{V}$ be such that the curve $\mathcal{C}_z$ is either smooth or has exactly one self-intersection and no other nodes. Then, the evaluation homomorphisms are given by

$$H^0(\mathscr{O}_{\mathcal{C}_z}(\mathcal{D}_1(z) + \mathcal{D}_2(z) + \mathcal{A}(z) - \mathcal{B}(z))) \xrightarrow{\Theta_{12}(z)} \mathbf{k}^2,$$
$$H^0(\mathscr{O}_{\mathcal{C}_z}(\mathcal{D}_1(z) + \mathcal{D}_2(z) + \mathcal{D}_3(z) + \mathcal{A}(z) - \mathcal{B}(z))) \xrightarrow{\Theta_{13}(z)} \mathbf{k}^2,$$
$$\text{where} \quad \Theta_{12}(z) = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} (z), \quad \Theta_{13}(z) = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix} (z),$$

respectively. Thus,

$$\begin{aligned} \theta_{12}(z) = 0 \quad &\Longleftrightarrow \quad \operatorname{rank} \Theta_{12}(z) = 1 \\ &\Longleftrightarrow \quad h^0(\mathscr{O}_{\mathcal{C}_z}(\mathcal{D}_1(z) + \mathcal{D}_2(z) - \mathcal{B}(z))) = 1; \\ \operatorname{rank} \Theta_{13}(z) \leq 1 \quad &\Longleftrightarrow \quad h^0(\mathscr{O}_{\mathcal{C}_z}(\mathcal{D}_1(z) + \mathcal{D}_2(z) + \mathcal{D}_3(z) - \mathcal{B}(z))) \geq 2. \end{aligned}$$

Since $\mathcal{B}(z)$ is a general point on the curve $\mathcal{C}_z$, the evaluation homomorphism at the point $\mathcal{B}(z)$, $H^0(\mathscr{O}_{\mathcal{C}_z}(\mathcal{D}_1(z) + \mathcal{D}_2(z))) \to \mathbf{k}$, is surjective. By the exact sequence

$$0 \to H^0(\mathscr{O}_{\mathcal{C}_z}(\mathcal{D}_1(z) + \mathcal{D}_2(z) - \mathcal{B}(z))) \to H^0(\mathscr{O}_{\mathcal{C}_z}(\mathcal{D}_1(z) + \mathcal{D}_2(z))) \to \mathbf{k} \to 0,$$

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we see that $h^0(\mathscr{O}_{\mathcal{C}_z}(\mathcal{D}_1(z)+\mathcal{D}_2(z)-\mathcal{B}(z)))=1$ if and only if $h^0(\mathscr{O}_{\mathcal{C}_z}(\mathcal{D}_1(z)+\mathcal{D}_2(z))=2$, which is equivalent to the statement that $\mathcal{D}_1(z)$ and $\mathcal{D}_2(z)$ are a pair of conjugate points, for the same reason as in the proof of part (1.a).

Similarly, $h^0(\mathscr{O}_{\mathcal{C}_z}(\mathcal{D}_1(z)+\mathcal{D}_2(z)+\mathcal{D}_3(z)-\mathcal{B}(z)))\geq 2$ if and only if $h^0(\mathscr{O}_{\mathcal{C}_z}(\mathcal{D}_1(z)+\mathcal{D}_2(z)+\mathcal{D}_3(z))\geq 3$, which is impossible for the same reason as in the proof of part (1.b). This proves part (2).

(3) By part (2), θ_{12}/κ_{12} is regular on the open subset of $\mathcal{V}$ consisting of $z\in\mathcal{V}$ such that $\mathcal{C}_z$ is smooth. Notice that Proposition 2.17 implies that κ_{12} is irreducible and coprime to any node-smoothing parameter, so θ_{12}/κ_{12} is regular on the entire $\mathcal{V}$. To show part (3), it thus suffices to prove that θ_{12} vanishes at the point z to the first order for any $z\in\mathcal{K}_{12}\cap\mathcal{V}$ such that $\mathcal{C}_z$ is smooth.

Let $\Delta\subset\mathcal{V}$ be a small curve through z, meeting $\mathcal{K}_{12}\cap\mathcal{V}$ transversely at z. Let $\mathcal{C}_\Delta=\mathcal{C}\times_{\mathcal{V}}\Delta$, $\mathcal{D}_{\Delta,i}=\mathcal{D}_i\times_{\mathcal{V}}\Delta$ for $i=1,2$, $\mathcal{A}_\Delta=\mathcal{A}\times_{\mathcal{V}}\Delta$, and $\mathcal{B}_\Delta=\mathcal{B}\times_{\mathcal{V}}\Delta$. Then, we obtain the evaluation homomorphism

$$\rho_*\mathscr{O}_{\mathcal{C}_\Delta}(\mathcal{D}_{\Delta,1}+\mathcal{D}_{\Delta,2}+\mathcal{A}_\Delta-\mathcal{B}_\Delta)\xrightarrow{\Theta_{12}|_\Delta}\rho_*\mathscr{O}_{\mathcal{A}_\Delta}(\mathcal{A}_\Delta),$$
$$\text{where}\quad \Theta_{12}|_\Delta=\begin{bmatrix}c_{11}&c_{12}\\c_{21}&c_{22}\end{bmatrix}\Big|_\Delta.$$

By long exact sequence of cohomology and Riemann–Roch, the co-kernel of $\Theta_{12}|_\Delta$ is a skyscraper sheaf with one-dimensional stalk at z. This implies that $\theta_{12}|_\Delta$ vanishes at the point z to the first order, hence θ_{12} vanishes along $\mathcal{K}_{12}$ to the first order.

(4) The proofs of (4.a) and (4.b) are both similar to that of part (3). Given $z\in\mathcal{V}$ such that $\mathcal{C}_z$ consists of two smooth genus 1 subcurves attached at a node $q(z)$, The node-smoothing parameter on $\mathcal{V}$ corresponding to $q(z)$ is denoted by ζ_q.

(4.a) Assume that $z\notin\mathcal{K}_{12}$. Let $\Delta\subset\mathcal{V}$ be a small curve through z and disjoint from $\mathcal{K}_{12}$, meeting $(\zeta_q=0)$ transversely at z. Mimicking the proof of part (3), we define $\mathcal{C}_\Delta$, $\mathcal{D}_{\Delta,1}$, $\mathcal{D}_{\Delta,2}$, $\mathcal{A}_\Delta$, and $\mathcal{B}_\Delta$ likewise. Then, we obtain the evaluation homomorphism

$$\rho_*\mathscr{O}_{\mathcal{C}_\Delta}(\mathcal{D}_{\Delta,1}+\mathcal{D}_{\Delta,2}+\mathcal{A}_\Delta-\mathcal{B}_\Delta)\xrightarrow{\widetilde{\Theta}_{12}|_\Delta}\rho_*\mathscr{O}_{\mathcal{A}_\Delta}(\mathcal{A}_\Delta),$$
$$\text{where}\quad \widetilde{\Theta}_{12}|_\Delta=\begin{bmatrix}c_{11}&c_{12}\\c_{21}\zeta_q&c_{22}\zeta_q\end{bmatrix}\Big|_\Delta;$$

see Proposition 2.10 for the factor ζ_q in the second row of $\widetilde{\Theta}_{12}|_\Delta$. By long exact sequence of cohomology and Riemann–Roch, the co-kernel of $\widetilde{\Theta}_{12}|_\Delta$ is still a skyscraper sheaf with one-dimensional stalk at z. This implies that $\det\widetilde{\Theta}_{12}|_\Delta$ vanishes at the point z to the first order, and hence $\det\widetilde{\Theta}_{12}$ vanishes along $(\zeta_q=0)$ to the first order. Notice that $\det\widetilde{\Theta}_{12}=\zeta_q\theta_{12}$, which implies that θ_{12} is invertible on a neighborhood of z.

(4.b) Assume that $z\in\mathcal{K}_{12}$. Since $q(z)$ is the only node of $\mathcal{C}_z$, we see from Proposition 2.17 that near z the locus $\mathcal{K}_{12}$ is a smooth Cartier divisor, meeting $(\zeta_q=0)$ transversely at z. We can thus take $\Delta\subset\mathcal{K}_{12}$ to be a small curve through z, meeting $(\zeta_q=0)$ transversely at z. Let $\mathcal{C}_\Delta$, $\mathcal{D}_{\Delta,1}$, $\mathcal{D}_{\Delta,2}$, $\mathcal{D}_{\Delta,3}$, $\mathcal{A}_\Delta$, and $\mathcal{B}_\Delta$ be analogous to part (4.a). Then, we obtain the evaluation homomorphism

$$\rho_*\mathscr{O}_{\mathcal{C}_\Delta}(\mathcal{D}_{\Delta,1}+\mathcal{D}_{\Delta,2}+\mathcal{D}_{\Delta,3}+\mathcal{A}_\Delta-\mathcal{B}_\Delta)\xrightarrow{\widetilde{\Theta}_{13}|_\Delta}\rho_*\mathscr{O}_{\mathcal{A}_\Delta}(\mathcal{A}_\Delta),$$
$$\text{where}\quad \widetilde{\Theta}_{13}|_\Delta=\begin{bmatrix}c_{11}&c_{12}&c_{13}\\c_{21}\zeta_q&c_{22}\zeta_q&c_{23}\zeta_q\end{bmatrix}\Big|_\Delta;$$

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once again see Proposition 2.10 for the factor ζ_q in the second row of $\widetilde{\Theta}_{13}|_\Delta$. By long exact sequence of cohomology and Riemann–Roch, the co-kernel of $\widetilde{\Theta}_{13}|_\Delta$ is still a skyscraper sheaf with one-dimensional stalk at z.

Meanwhile, note that after an elementary row operation, $\widetilde{\Theta}_{13}|_\Delta$ can be rewritten as

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ 0 & \theta_{12}\zeta_q & \theta_{13}\zeta_q \end{bmatrix}\Big|_\Delta.$$

By part (3) and the assumption that $\Delta \subset \mathcal{K}_{12}$, we have that $\theta_{12}|_\Delta$ is identically zero. Taking into account that c_{11} is invertible, we conclude that the entry $\theta_{13}\zeta_q|_\Delta$ vanishes at the point z to the first order, and hence $\theta_{13}\zeta_q$ vanishes along $(\zeta_q = 0)$ to the first order, which implies that θ_{13} is invertible on a neighborhood of z. □

Proof of Proposition 2.28. (R_1) Taking $i = 2$ in (2.51), we have

$$c_{s2}\zeta_{[\delta_2,a_s]} = \nu_{12}\widehat{c}_{s1}\zeta_{[\delta_1,a_s]} + \widehat{c}_{s2}\zeta_{[\delta_1,a_s]}\zeta_{[\delta_2]} = \nu_{12}\widehat{c}_{s1}\zeta_{[\delta_1,a_s]} + \widehat{c}_{s2}\zeta_{[\delta_2,a_s]}\zeta_{[\delta_1]}.$$

The last equality holds because $\zeta_{[\delta_2,a_s]} = \zeta_{[\delta_1,a_2]}\zeta_{[\delta_2,\delta_1]}$ and $\zeta_{[\delta_2]} = \zeta_{[\delta_1]}\zeta_{[\delta_2,\delta_1]}$. Consequently, we have

$$c_{s2} = \zeta_{[\delta_1]}\widehat{c}_{s2} + \frac{\nu_{12}}{\zeta_{[\delta_1,\delta_2]}}\widehat{c}_{s1} = \zeta_{[\delta_1]}\widehat{c}_{s2} + \frac{\nu_{12}}{\zeta_{[\delta_1,\delta_2]}}c_{s1}, \quad s = 1, 2.$$

The last equality holds because $\widehat{c}_{s1} = c_{s1}$, $s = 1, 2$, by Proposition 2.27. Then, by computing θ_{12} using the above, we obtain $\theta_{12} = \zeta_{[\delta_1]}\widehat{\theta}_{12}$.

To show $\widehat{\theta}_{12}/\kappa_{12} \in \Gamma(\mathscr{O}_{\mathcal{V}}^*)$, we observe that Lemma 2.29 (3) implies that θ_{12}/κ_{12} is regular on $\mathcal{V}$. In other words, $\zeta_{[\delta_1]}\widehat{\theta}_{12}/\kappa_{12}$ is regular on $\mathcal{V}$. However, $\zeta_{[\delta_1]}$ and κ_{12} are coprime, and hence we can write $\widehat{\theta}_{12} = u \cdot \kappa_{12}$ for some $u \in \Gamma(\mathscr{O}_{\mathcal{V}})$.

To show that $u \in \Gamma(\mathscr{O}_{\mathcal{V}}^*)$, we fix $z \in \mathcal{V}$. If $\mathcal{C}_z$ contains at most one node and that node, if it exists, is non-separating, then Lemma 2.29 (2) implies that $u(z) = 0$ can possibly be true only if $z \in \mathcal{K}_{12}$. If $\mathcal{C}_z$ contains one separating node $q(z)$ and no other nodes, there are two possibilities: either the core F_z of $\mathcal{C}_z$ is smooth, or $\mathcal{C}_z$ consists of two smooth genus 1 subcurves attached at $q(z)$. By Lemma 2.29 (1.a) and (4.a), we always conclude that $u(z)$ can possibly be 0 only if $z \in \mathcal{K}_{12}$.

In sum, let $\mathcal{Z} \subset \mathcal{V}$ consist of all $z \in \mathcal{V}$ satisfying the requirement that $\mathcal{C}_z$ contains at least two nodes. The above paragraph ensures that

$$u(z) \neq 0 \quad \forall\, z \in \mathcal{V}\backslash(\mathcal{K}_{12} \cup \mathcal{Z}).$$

Since $\mathcal{Z}$ is of codimension 2 and κ_{12} is irreducible, we have that $\widehat{\theta}_{12} = \widetilde{u} \cdot \kappa_{12}^\ell$ for some $\widetilde{u} \in \Gamma(\mathscr{O}_{\mathcal{V}}^*)$ and $\ell \in \mathbb{Z}_{>0}$. By Lemma 2.29 (3), we then see that $\ell = 1$. This completes the proof of part (R_1).

(R_2) Taking $i = 1, 2, 3$ in (2.51) and considering (2.32), we see that

$$\begin{aligned} c_{s3} &= \zeta_{[\delta_1]}\zeta_{[\delta_2]}\widehat{c}_{s3} + \frac{\nu_{23}\zeta_{[\delta_1]}}{\zeta_{[\delta_2,\delta_3]}}\widehat{c}_{s2} + \frac{\nu_{13}}{\zeta_{[\delta_1,\delta_3]}}\widehat{c}_{s1} \\ &= \zeta_{[\delta_1]}\zeta_{[\delta_2]}\widehat{c}_{s3} + \frac{\nu_{23}\zeta_{[\delta_1]}}{\zeta_{[\delta_2,\delta_3]}}\widehat{c}_{s2} + \frac{\nu_{13}}{\zeta_{[\delta_1,\delta_3]}}c_{s1}, \quad s = 1, 2. \end{aligned}$$

The last equality above holds because $\widehat{c}_{11} = c_{11}$ and $\widehat{c}_{11} = c_{11}$; cf. Proposition 2.27. Therefore,

$$\theta_{13} = \zeta_{[\delta_1]}\zeta_{[\delta_2]}\widehat{\theta}_{13} + \frac{\nu_{23}\zeta_{[\delta_1]}}{\zeta_{[\delta_2,\delta_3]}}\widehat{\theta}_{12} = \zeta_{[\delta_1]}\zeta_{[\delta_2]}\widehat{\theta}_{13} + \frac{\nu_{23}}{\zeta_{[\delta_2,\delta_3]}}\theta_{12}.$$

The last equality above follows from part (R_1).

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Note that $\zeta_{[\delta_1]}\kappa_{12}$, which equals θ_{12} up to a unit by part (R_1), and $\zeta_{[\delta_2,\delta_3]}$ are coprime, hence $\nu_{23}/\zeta_{[\delta_2,\delta_3]} \in \Gamma(\mathscr{O}_{\mathcal{V}})$. This proves part ($R_2$).

(R_3) To see that the two statements of (R_3) are equivalent, just observe that

$$\begin{aligned}\operatorname{rank}\begin{bmatrix}\widehat{c}_{11} & \widehat{c}_{12} & \widehat{c}_{13}\\ \widehat{c}_{21} & \widehat{c}_{22} & \widehat{c}_{23}\end{bmatrix}(z) &= \operatorname{rank}\begin{bmatrix}\widehat{c}_{11} & \widehat{c}_{12} & \widehat{c}_{13}\\ 0 & \widehat{\theta}_{12} & \widehat{\theta}_{13}\end{bmatrix}(z)\\ &= \operatorname{rank}\begin{bmatrix}\widehat{c}_{11} & \widehat{c}_{12} & \widehat{c}_{13}\\ 0 & \kappa_{12} & \widehat{\theta}_{13}\end{bmatrix}(z).\end{aligned}$$

The last equality above follows from part (R_1).

When q_1 is not on any non-separating bridge of C, we conclude from Proposition 2.17 that $\mathcal{K}_{12} \cap \mathcal{V}$ is a smooth Cartier divisor of $\mathcal{V}$ (recall that $\mathcal{V}$ is assumed to be sufficiently small). Our goal is to show that $\widehat{\theta}_{13}(z) \neq 0$ for all $z \in \mathcal{K}_{12} \cap \mathcal{V}$. Below we fix $z \in \mathcal{K}_{12} \cap \mathcal{V}$.

If $\mathcal{C}_z$ is either smooth or contains exactly one node and that node is non-separating, we consider the evaluation homomorphism of $H^0(\mathscr{O}_{\mathcal{C}_z}(\mathcal{D}_1(z) + \mathcal{D}_2(z) + \mathcal{D}_3(z) + \mathcal{A}(z) - \mathcal{B}(z)))$ to $\mathbf{k}^2$. Following an argument parallel to the proof of Lemma 2.29(1.b), we conclude that

$$\operatorname{rank}\begin{bmatrix}c_{11} & c_{12} & c_{13}\\ c_{21} & c_{22} & c_{23}\end{bmatrix}(z) = \operatorname{rank}\begin{bmatrix}c_{11} & c_{12} & c_{13}\\ 0 & \theta_{12} & \theta_{13}\end{bmatrix}(z) = 2.$$

Since $\kappa_{12}(z) = 0$, we have $\theta_{12}(z) = 0$ and thus $\theta_{13}(z) \neq 0$. Taking part (R_2) into account, we see that $\widehat{\theta}_{13}(z) \neq 0$.

If $\mathcal{C}_z$ contains exactly one node $q(z)$ and that node is separating, once again there are two possibilities: either the core F_z of $\mathcal{C}_z$ is smooth, or $\mathcal{C}_z$ consists of two smooth genus 1 subcurves attached at $q(z)$. In the former case, $\widehat{\theta}_{13}(z) \neq 0$ because of Lemma 2.29 (1.b). In the latter case, Lemma 2.29 (4.b) implies that $\theta_{13}(z) \neq 0$, which by part (R_2) further implies $\widehat{\theta}_{13}(z) \neq 0$.

In sum, let $\mathcal{Z} \subset \mathcal{V}$ consist of all $z \in \mathcal{V}$ such that $\mathcal{C}_z$ contains at least two nodes. The previous two paragraphs ensure that

$$\widehat{\theta}_{13}(z) \neq 0 \quad \forall\, z \in \mathcal{K}_{12} \backslash \mathcal{Z}.$$

Since κ_{12} is coprime to any node-smoothing parameter, we see that $\mathcal{Z} \cap \mathcal{K}_{12}$ is of codimension 2 in $\mathcal{K}_{12}$, hence $\widehat{\theta}_{13}$ is invertible on $\mathcal{K}_{12}$. This establishes part (R_3).

(R_4) Assume q_1 is on a maximal non-separating bridge $\mathrm{B}[\mathfrak{p}, \mathfrak{q}]$. By part ($R_1$), there exists $u \in \Gamma(\mathscr{O}_{\mathcal{V}}^*)$ such that $\widehat{\theta}_{12} = u \cdot \kappa_{12}$. Since $\langle \delta_1 \rangle = \langle \delta_2 \rangle = q_1$ in this case, we choose a smooth point $x_{q_1} \in \mathrm{B}[\mathfrak{p}, \mathfrak{q}]$ near q_1 and conclude from Proposition 2.17(iii) that there exist $f_2, g_2 \in \Gamma(\mathscr{O}_{\mathcal{V}}^*)$ such that

$$\widehat{\theta}_{12} = f_2 \zeta_{[x_{q_1}, \mathfrak{p}]} + g_2 \zeta_{[x_{q_1}, \mathfrak{q}]}. \tag{2.53}$$

Below, we show the remaining statements of part (R_4) in the following steps.

- (R_4.a) We first show that (R_4) can be reduced to the case when $\mathrm{B}[\mathfrak{p}, \mathfrak{q}]$ is smooth.
- (R_4.b) Under the assumption that $\mathrm{B}[\mathfrak{p}, \mathfrak{q}]$ is smooth, we then show that $\widehat{\theta}_{13} = u' \widehat{\theta}_{12} + v' \zeta_{\mathfrak{p}}$ for some $u' \in \Gamma(\mathscr{O}_{\mathcal{V}}^*)$ and $v' \in \Gamma(\mathscr{O}_{\mathcal{V}})$.
- (R_4.c) Finally, we show that v' in (R_4.b) is invertible.

The above three steps together imply (R_4).

(R_4.a) Let $C_1 \subset \mathrm{B}[\mathfrak{p}, \mathfrak{q}]$ be the irreducible component containing q_1. Reordering the points of $D = \delta_1 + \cdots + \delta_m$ if necessary, we assume that $\delta_{\ell+1}, \ldots, \delta_m$, for some $r \leq \ell \leq m$, are all the points of D such that either they themselves or the pivotal nodes of the rational tails containing

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them lie on $\mathrm{B}[\mathfrak{p},\mathfrak{q}]\backslash C_1$. If $m=\ell$, then the stability of (C,D) implies that $\mathrm{B}[\mathfrak{p},\mathfrak{q}]=C_1$, so we assume $m>\ell$ below.

Shrinking $\mathcal{V}$ if necessary, we can find a small chart $\overline{\mathcal{V}}$ of $\mathfrak{M}_2^{\mathrm{div}}(m-1)$, the connected component of $\mathfrak{M}_2^{\mathrm{div}}$ whose divisors are of degree $m-1$, such that the forgetful morphism $\varpi:\mathcal{V}\longrightarrow\overline{\mathcal{V}}$ that forgets the last marked point (corresponding to δ_m) is well defined. With $\overline{\rho}:\overline{\mathcal{C}}\to\overline{\mathcal{V}}$ denoting the universal family, and $\overline{\mathcal{D}}\subset\overline{\mathcal{C}}$ denoting the tautological divisor, the forgetful morphism ϖ leads to a contraction $\widetilde{\varpi}:\mathcal{C}\longrightarrow\overline{\mathcal{C}}$ so that the following diagram commutes.

$$\begin{array}{ccc}\mathcal{C} & \xrightarrow{\widetilde{\varpi}} & \overline{\mathcal{C}}\\ \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\overline{\rho}}\\ \mathcal{V} & \xrightarrow{\varpi} & \overline{\mathcal{V}}\end{array}\tag{2.54}$$

By writing $\overline{\mathcal{D}}_i=\widetilde{\varpi}(\mathcal{D}_i)$ for all $1\le i\le\ell$, the tautological divisor $\overline{\mathcal{D}}\subset\overline{\mathcal{C}}$ can then be written as $\overline{\mathcal{D}}=\overline{\mathcal{D}}_1+\cdots+\overline{\mathcal{D}}_\ell$.

Since none of $\mathcal{D}_1,\dots,\mathcal{D}_r$, $\mathcal{A}$, or $\mathcal{B}$ meets the contracted locus of $\widetilde{\varpi}$, we have

$$\widetilde{\varpi}^{-1}(\overline{\mathcal{D}}_i)=\mathcal{D}_i\ \ \forall\,1\le i\le r,\quad \widetilde{\varpi}^{-1}(\overline{\mathcal{A}})=\mathcal{A},\quad\text{and}\quad \widetilde{\varpi}^{-1}(\overline{\mathcal{B}})=\mathcal{B}.$$

Hence

$$\widetilde{\varpi}^*\mathscr{O}_{\overline{\mathcal{C}}}(\overline{\mathcal{D}}_i+\overline{\mathcal{A}}-\overline{\mathcal{B}})=\mathscr{O}_{\mathcal{C}}(\mathcal{D}_i+\mathcal{A}-\mathcal{B})\ \ \forall\,1\le i\le r,\quad \widetilde{\varpi}^*\mathscr{O}_{\overline{\mathcal{A}}}(\overline{\mathcal{A}})=\mathscr{O}_{\mathcal{A}}(\mathcal{A}).$$

Therefore,

$$\begin{aligned}&\rho_*\mathscr{O}_{\mathcal{C}}(\mathcal{D}_i+\mathcal{A}-\mathcal{B})=\rho_*\widetilde{\varpi}^*\mathscr{O}_{\overline{\mathcal{C}}}(\overline{\mathcal{D}}_i+\overline{\mathcal{A}}-\overline{\mathcal{B}})=\varpi^*\overline{\rho}_*\mathscr{O}_{\overline{\mathcal{C}}}(\overline{\mathcal{D}}_i+\overline{\mathcal{A}}-\overline{\mathcal{B}}),\\ &\rho_*\mathscr{O}_{\mathcal{A}}(\mathcal{A})=\rho_*\widetilde{\varpi}^*\mathscr{O}_{\overline{\mathcal{A}}}(\overline{\mathcal{A}})=\varpi^*\overline{\rho}_*\mathscr{O}_{\overline{\mathcal{A}}}(\overline{\mathcal{A}}),\end{aligned}$$

where the last equality of each line follows from direct computation (for instance, see [SS16, Theorem 4.4]) since ρ and $\overline{\rho}$ are both separated and locally proper. Consequently, the restriction homomorphism on the target

$$\overline{\varphi}_{\le r}:\bigoplus_{1\le i\le r}\overline{\rho}_*\mathscr{O}_{\overline{\mathcal{C}}}(\overline{\mathcal{D}}_i+\overline{\mathcal{A}}-\overline{\mathcal{B}})\longrightarrow\overline{\rho}_*\mathscr{O}_{\overline{\mathcal{A}}}(\overline{\mathcal{A}})$$

exactly pulls back to the restriction homomorphism on the source

$$\varphi_{\le r}:\bigoplus_{1\le i\le r}\rho_*\mathscr{O}_{\mathcal{C}}(\mathcal{D}_i+\mathcal{A}-\mathcal{B})\longrightarrow\rho_*\mathscr{O}_{\mathcal{A}}(\mathcal{A}).$$

Then, we apply the above forgetful morphism inductively so as to drop all the marked points labeled by $\ell+1,\dots,m$ (corresponding to $\delta_{\ell+1},\dots,\delta_m$), so, in particular, $\mathrm{B}[\mathfrak{p},\mathfrak{q}]\backslash C_1$ is contracted. Slightly abusing the notation, we still denote by $\overline{\varphi}_{\le r}$ the resulting restriction homomorphism. If the statements of part (R_4) hold for $\overline{\varphi}_{\le r}$, then by applying Lemma 2.18 inductively, we see that the statements of part (R_4) hold for the original $\varphi_{\le r}$.

(R_4.b) Assume $B[\mathfrak{p},\mathfrak{q}]$ is smooth. Then (2.53) can be rewritten as

$$\widehat{\theta}_{12}=f_2\zeta_{\mathfrak{p}}+g_2\zeta_{\mathfrak{q}}.\tag{2.55}$$

With φ_i as in (2.4), we consider $\varphi_1\oplus\varphi_3$. On the one hand, the determinant θ_{13} remains unchanged because the c_{si} are determined by Proposition 2.10 (hence independent of the choice of the S_i in Lemma 2.26); on the other hand, by applying part (R_1) of Proposition 2.28 to $\varphi_1\oplus\varphi_3$, we observe that θ_{13} should take the same form as θ_{12} does in $\varphi_{\le r}$, so, in particular,

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it contains κ_{13} as a factor. Thus by Proposition 2.17, one can write

$$\theta_{13} = f'\zeta_{\mathfrak{p}} + g'\zeta_{\mathfrak{q}} \tag{2.56}$$

for some $f', g' \in \Gamma(\mathscr{O}_{\mathcal{V}})$. Then, by ($R_2$) and the fact that the node-smoothing parameters are independent, we have

$$\widehat{\theta}_{13} = f_3\zeta_{\mathfrak{p}} + g_3\zeta_{\mathfrak{q}}$$

for some $f_3, g_3 \in \Gamma(\mathscr{O}_{\mathcal{V}})$. By setting

$$\eta := \det \begin{bmatrix} f_3 & g_3 \\ f_2 & g_2 \end{bmatrix},$$

we can further write $\widehat{\theta}_{13}$ as

$$\widehat{\theta}_{13} = \frac{\eta}{g_2}\zeta_{\mathfrak{p}} + \frac{g_3}{g_2}\widehat{\theta}_{12}. \tag{2.57}$$

(R_4.c) It suffices to show that $\eta(z_0) \neq 0$ because $\mathcal{V}$ is assumed small. Consider the locus

$$\mathcal{Z} := (\zeta_{q_1} = 0,\ \kappa_{12} = 0) \quad \subset \mathcal{V},$$

which is smooth and contains z_0. Indeed, we claim that $\eta|_{\mathcal{Z}}$ is nowhere zero.

To establish the above claim, consider the sub-locus $\mathcal{Z}' \subset \mathcal{Z}$ consisting of $z \in \mathcal{Z}$ such that the core F_z of $\mathcal{C}_z$ contains at most one node. Then, F_z does not have any non-separating bridge (so without loss of generality we assume $\zeta_{\mathfrak{p}}(z) \neq 0$), and the marked points corresponding to $\delta_1, \ldots, \delta_r$ are on a rational tail whose pivotal node (corresponding to q_1) is a Weierstrass point on F_z. By (R_3) and (2.57), we have $\eta(z) \neq 0$. Therefore, $\eta|_{\mathcal{Z}}$ is nowhere zero outside the codimension 2 subset $\mathcal{Z}\backslash\mathcal{Z}'$, hence is nowhere zero. □

We conclude this subsection with the following implications for the relations between κ_{12}, κ_{13} and κ_{23}, provided $\{\delta_1, \delta_2, \delta_3\}$ is on a chain R of tail rational subcurves of the center $z_0 = (C, D)$ of $\mathcal{V}$, ordered according to (2.30).

LEMMA 2.30. *With notation as in Proposition 2.28, there exist $v \in \Gamma(\mathscr{O}_{\mathcal{V}}^*)$ and $w \in \Gamma(\mathscr{O}_{\mathcal{V}})$ such that*

$$\kappa_{13} = v \cdot \kappa_{12} + w,$$

and the following hold.

(a) *If q_1 does not lie on any non-separating bridge of the core of C, then there exists $u \in \Gamma(\mathscr{O}_{\mathcal{V}}^*)$ such that $w = u \cdot \zeta_{[\delta_2]}$.*

(b) *If q_1 lies on a maximal non-separating bridge $\mathrm{B}[\mathfrak{p}, \mathfrak{q}]$ of the core F of C, let $x_{q_1} \in F$ be an arbitrary smooth point that is sufficiently close to the node q_1. Then, there exist $f, g \in \Gamma(\mathscr{O}_{\mathcal{V}}^*)$ and $f', g' \in \Gamma(\mathscr{O}_{\mathcal{V}})$ so that*

$$\kappa_{12} = f\zeta_{[x_{q_1},\mathfrak{p}]} + g\zeta_{[x_{q_1},\mathfrak{q}]}, \quad w = f' \cdot \prod_{q \in N_{[\delta_2,\mathfrak{p}]}} \zeta_q \ + \ g' \cdot \prod_{q \in N_{[\delta_2,\mathfrak{q}]}} \zeta_q;$$

moreover, $\det \begin{bmatrix} f & f' \\ g & g' \end{bmatrix} \in \Gamma(\mathscr{O}_{\mathcal{V}}^*)$.

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Proof. Without loss of generality we assume that $\zeta_{[q_1,a_1]}$ divides $\zeta_{[q_1,a_2]}$ evenly in this proof. Moreover, we assume that $z_0 \in \mathcal{K}_{12}$ and consequently $z_0 \in \mathcal{K}_{13}$ (i.e. $\kappa_{12}(z_0) = \kappa_{13}(z_0) = 0$), for otherwise we may shrink $\mathcal{V}$ if necessary so that κ_{12} and κ_{13} are both units on $\mathcal{V}$, thus all statements of the corollary trivially hold.

We first fix the trivialization $\rho_*\mathscr{O}_{\mathcal{A}_s}(\mathcal{A}_s) \cong \mathscr{O}_{\mathcal{V}}$, $s = 1, 2$. Taking $r = 2$ in Proposition 2.28 and choosing a suitable trivialization $\rho_*\mathcal{M}(\mathcal{D}_1) \oplus \rho_*\mathcal{M}(\mathcal{D}_2) \cong \mathscr{O}_{\mathcal{V}}^{\oplus 2}$ so that the first component of $\mathscr{O}_{\mathcal{V}}^{\oplus 2}$ corresponds to $\rho_*\mathcal{M}(\mathcal{D}_1)$, we have

$$\varphi_{\leq 2} = \begin{bmatrix} \widehat{c}_{11}\zeta_{[\delta_1,a_1]} & 0 \\ \widehat{c}_{21}\zeta_{[\delta_1,a_2]} & \kappa_{12}\zeta_{[\delta_1,a_2]}\zeta_{[\delta_2]} \end{bmatrix} : \rho_*\mathcal{M}(\mathcal{D}_1) \oplus \rho_*\mathcal{M}(\mathcal{D}_2) \longrightarrow \rho_*\mathscr{O}_{\mathcal{A}}(\mathcal{A}).$$

Similarly, by applying Proposition 2.28 to $\{\delta_1, \delta_3\}$ (i.e. dropping δ_2 and all δ_j with $j > 3$) and choosing a suitable trivialization $\rho_*\mathcal{M}(\mathcal{D}_1) \oplus \rho_*\mathcal{M}(\mathcal{D}_3) \cong \mathscr{O}_{\mathcal{V}}^{\oplus 2}$ so that the first component of $\mathscr{O}_{\mathcal{V}}^{\oplus 2}$ corresponds to $\rho_*\mathcal{M}(\mathcal{D}_1)$, we can write the restriction of the structural homomorphism as

$$\varphi_{1,3} = \begin{bmatrix} \widehat{c}_{11}\zeta_{[\delta_1,a_1]} & 0 \\ \widehat{c}_{21}\zeta_{[\delta_1,a_2]} & \kappa_{13}\zeta_{[\delta_1,a_2]}\zeta_{[\delta_3]} \end{bmatrix} : \rho_*\mathcal{M}(\mathcal{D}_1) \oplus \rho_*\mathcal{M}(\mathcal{D}_3) \longrightarrow \rho_*\mathscr{O}_{\mathcal{A}}(\mathcal{A}).$$

We emphasize that the coefficients $\widehat{c}_{11}$ and $\widehat{c}_{21}$ in $\varphi_{1,3}$ are the same as those in $\varphi_{\leq 2}$. Therefore, one can choose a suitable trivialization $\bigoplus_{i=1}^{3} \rho_*\mathcal{M}(\mathcal{D}_i) \cong \mathscr{O}_{\mathcal{V}}^{\oplus 3}$ and $\rho_*\mathscr{O}_{\mathcal{A}}(\mathcal{A}) \cong \mathscr{O}_{\mathcal{V}}^{\oplus 2}$ so that

$$\varphi_{\leq 3} : \bigoplus_{1 \leq i \leq 3} \rho_*\mathcal{M}(\mathcal{D}_i) \longrightarrow \rho_*\mathscr{O}_{\mathcal{A}}(\mathcal{A}),$$
$$\varphi_{\leq 3} = \begin{bmatrix} \widehat{c}_{11}\zeta_{[\delta_1,a_1]} & 0 & 0 \\ 0 & \kappa_{12}\zeta_{[\delta_1,a_2]}\zeta_{[\delta_2]} & \kappa_{13}\zeta_{[\delta_1,a_2]}\zeta_{[\delta_3]} \end{bmatrix}.$$

Alternatively, we can take $r = 2$ in Proposition 2.28 and choose a suitable trivialization $\bigoplus_{i=1}^{3} \rho_*\mathcal{M}(\mathcal{D}_i) \cong \mathscr{O}_{\mathcal{V}}^{\oplus 3}$ and $\rho_*\mathscr{O}_{\mathcal{A}}(\mathcal{A}) \cong \mathscr{O}_{\mathcal{V}}^{\oplus 2}$ so that

$$\varphi_{\leq 3} : \bigoplus_{1 \leq i \leq 3} \rho_*\mathcal{M}(\mathcal{D}_i) \longrightarrow \rho_*\mathscr{O}_{\mathcal{A}}(\mathcal{A}),$$
$$\varphi_{\leq 3} = \begin{bmatrix} \widehat{c}_{11}\zeta_{[\delta_1,a_1]} & 0 & 0 \\ 0 & \kappa_{12}\zeta_{[\delta_1,a_2]}\zeta_{[\delta_2]} & \widehat{\theta}_{13}\zeta_{[\delta_1,a_2]}\zeta_{[\delta_2]}\zeta_{[\delta_3]} \end{bmatrix}.$$

The above two expressions of $\varphi_{\leq 3}$ together imply that there exist $v' \in \Gamma(\mathscr{O}_{\mathcal{V}})$ and $u' \in \Gamma(\mathscr{O}_{\mathcal{V}}^*)$ such that

$$\kappa_{13}\zeta_{[\delta_1,a_2]}\zeta_{[\delta_3]} = v' \cdot \kappa_{12}\zeta_{[\delta_1,a_2]}\zeta_{[\delta_2]} + u' \cdot \widehat{\theta}_{13}\zeta_{[\delta_1,a_2]}\zeta_{[\delta_2]}\zeta_{[\delta_3]}.$$

By Proposition 2.17, κ_{12} is coprime with ζ_q for any $q \in N_{[\delta_2,\delta_3]}$ regardless of the position of the pivotal node q_1. Taking into account that the node-smoothing parameters are independent, we conclude that

$$v := \frac{v'}{\zeta_{[\delta_2,\delta_3]}} \in \Gamma(\mathscr{O}_{\mathcal{V}}) \quad \text{and} \quad \kappa_{13} = v \cdot \kappa_{12} + u' \cdot \widehat{\theta}_{13}\zeta_{[\delta_2]}. \tag{2.58}$$

If q_1 does not lie on any non-separating bridge, then part (R_3) of Proposition 2.28 implies that $\widehat{\theta}_{13}$ is invertible. Hence by writing $u = u' \cdot \widehat{\theta}_{13} \in \Gamma(\mathscr{O}_{\mathcal{V}}^*)$, we have $\kappa_{13} = v \cdot \kappa_{12} + u \cdot \zeta_{[\delta_2]}$. Moreover, Proposition 2.17 implies κ_{12} and κ_{13}, as well as their pullbacks under any forgetful morphism, are always smooth local parameters, hence v must be invertible. This completes the proof of part (a).

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If q_1 is on a maximal non-separating bridge $\mathrm{B}[\mathfrak{p},\mathfrak{q}]$, then part (R_3) of Proposition 2.28 implies that

$$\kappa_{12} = f\cdot\zeta_{[x_{q_1},\mathfrak{p}]} + g\cdot\zeta_{[x_{q_1},\mathfrak{q}]}, \quad \widehat{\theta}_{13} = f'\cdot\zeta_{[x_{q_1},\mathfrak{p}]} + g'\cdot\zeta_{[x_{q_1},\mathfrak{q}]},$$
$$\det\begin{bmatrix} f & f' \\ g & g' \end{bmatrix} \in \Gamma(\mathscr{O}_{\mathcal{V}}^*).$$

Hence, by (2.58), we have

$$\kappa_{13} = v\cdot f\cdot\zeta_{[x_{q_1},\mathfrak{p}]} + v\cdot g\cdot\zeta_{[x_{q_1},\mathfrak{q}]} + u'\cdot f'\cdot\zeta_{[\delta_2,\mathfrak{p}]} + u'\cdot g'\cdot\zeta_{[\delta_2,\mathfrak{q}]},$$
$$\det\begin{bmatrix} f & u'\cdot f' \\ g & u'\cdot g' \end{bmatrix} \in \Gamma(\mathscr{O}_{\mathcal{V}}^*).$$

Finally, by applying Proposition 2.17 to κ_{13}, we see that v must be invertible. This completes the proof of part (b). □

Lemma 2.30 reveals the relation between κ_{12} and κ_{13}, provided $\{\delta_1,\delta_2,\delta_3\}$ is on a chain R of tail rational subcurves, ordered according to (2.30). Indeed, analogous relations between κ_{12} and κ_{23} and between κ_{13} and κ_{23} also hold. These relations are summarized below.

Recall that for any smooth points δ_i and δ_j on R, $\delta_i\wedge\delta_j$ refers to the node in (2.12); intuitively it is a node between the core and both δ_i and δ_j, and is the 'farthest away' from the core among all such nodes.

COROLLARY 2.31. *With notation as in Proposition 2.28, let (i,j,k) be an arbitrary permutation of $\{1,2,3\}$. There then exist $v\in\Gamma(\mathscr{O}_{\mathcal{V}}^*)$ and $g\in\Gamma(\mathscr{O}_{\mathcal{V}})$ so that*

$$\kappa_{ik} = v\cdot\kappa_{ij} + w,$$

and the following holds.

- *If q_1 does not lie on any non-separating bridge of the core of C, then there exists $u\in\Gamma(\mathscr{O}_{\mathcal{V}}^*)$ such that $w = u\cdot\zeta_{[\delta_j\wedge\delta_k]}$.*
- *If q_1 lies on a maximal non-separating bridge $\mathrm{B}[\mathfrak{p},\mathfrak{q}]$ of the core F of C, let $x_{q_1}\in F$ be an arbitrary smooth point that is sufficiently close to the node q_1. Then, there exist $f,g\in\Gamma(\mathscr{O}_{\mathcal{V}}^*)$ and $f',g'\in\Gamma(\mathscr{O}_{\mathcal{V}})$ so that*

$$\kappa_{ij} = f\cdot\prod_{q\in N_{[x_{q_1},\mathfrak{p}]}}\zeta_q + g\cdot\prod_{q\in N_{[x_{q_1},\mathfrak{q}]}}\zeta_q,$$
$$w = f'\cdot\prod_{q\in N_{[\delta_j,\mathfrak{p}]}\cap N_{[\delta_k,\mathfrak{p}]}}\zeta_q + g'\cdot\prod_{q\in N_{[\delta_j,\mathfrak{q}]}\cap N_{[\delta_k,\mathfrak{q}]}}\zeta_q;$$

  moreover, $\det\begin{bmatrix} f & f' \\ g & g' \end{bmatrix}\in\Gamma(\mathscr{O}_{\mathcal{V}}^*)$.

Proof. Since $\kappa_{ij}=\kappa_{ji}$ for all $1\le i,j\le r$, it suffices to show Corollary 2.31 for the permutations $(1,2,3)$, $(2,3,1)$ and $(3,1,2)$. In addition, the $(i,j,k)=(1,2,3)$ case is covered in Lemma 2.30.

If $(i,j,k)=(2,3,1)$ (i.e. to find the relations between κ_{12} and κ_{23}), notice that by applying Proposition 2.28 to $\{\delta_2,\delta_3\}$ (i.e. dropping δ_1 and all δ_j with $j>3$) and choosing a suitable trivialization $\rho_*\mathcal{M}(\mathcal{D}_2)\oplus\rho_*\mathcal{M}(\mathcal{D}_3)\cong\mathscr{O}_{\mathcal{V}}^{\oplus 2}$ so that the first component of $\mathscr{O}_{\mathcal{V}}^{\oplus 2}$ corresponds to $\rho_*\mathcal{M}(\mathcal{D}_2)$, we can write the restriction of the structural homomorphism as

$$\varphi_{2,3} = \begin{bmatrix} c_{12}\zeta_{[\delta_2,a_1]} & 0 \\ c_{22}\zeta_{[\delta_2,a_2]} & \kappa_{23}\zeta_{[\delta_2,a_2]}\zeta_{[\delta_3]} \end{bmatrix} : \rho_*\mathcal{M}(\mathcal{D}_2)\oplus\rho_*\mathcal{M}(\mathcal{D}_3)\longrightarrow\rho_*\mathscr{O}_{\mathcal{A}}(\mathcal{A}).$$

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The coefficients c_{12} and c_{12} in $\varphi_{2,3}$ are the same as in Proposition 2.10. Therefore, one can choose a suitable trivialization $\bigoplus_{i=1}^3 \rho_*\mathcal{M}(\mathcal{D}_i) \cong \mathscr{O}_{\mathcal{V}}^{\oplus 3}$ and $\rho_*\mathscr{O}_{\mathcal{A}}(\mathcal{A}) \cong \mathscr{O}_{\mathcal{V}}^{\oplus 2}$ so that

$$\varphi_{\le 3} : \bigoplus_{1\le i\le 3} \rho_*\mathcal{M}(\mathcal{D}_i) \longrightarrow \rho_*\mathscr{O}_{\mathcal{A}}(\mathcal{A}),$$
$$\varphi_{\le 3} = \begin{bmatrix} \widehat{c}_{11}\zeta_{[\delta_1,a_1]} & 0 & 0 \\ 0 & \kappa_{12}\zeta_{[\delta_1,a_2]}\zeta_{[\delta_2]} & \kappa_{23}\zeta_{[\delta_2,a_2]}\zeta_{[\delta_3]} \end{bmatrix}.$$

Parallel to (2.58), we can find $v \in \Gamma(\mathscr{O}_{\mathcal{V}})$ and $u' \in \Gamma(\mathscr{O}_{\mathcal{V}}^*)$ so that

$$\kappa_{23} = v \cdot \kappa_{12} + u' \cdot \widehat{\theta}_{13}\zeta_{[\delta_1]}.$$

The remainder of the proof of the $(i,j,k) = (2,3,1)$ case is analogous to the proof of Lemma 2.30 after (2.58), hence is omitted.

If $(i,j,k) = (3,1,2)$ (i.e. to find the relations between κ_{13} and κ_{23}), we observe that it suffices to verify the statements after forgetting all the marked points except δ_1, δ_2 and δ_3. (Recall that locally we identify $\mathcal{V}$ with a smooth chart of $\overline{\mathcal{M}}_{2,m}$.) Then the fact that the κ_{ij} are defined via pullbacks and Lemma 2.18 lead to the statements for the $(i,j,k) = (3,1,2)$ case.

To show the $(i,j,k) = (3,1,2)$ case after forgetting all the marked points except δ_1, δ_2 and δ_3, just notice that under the forgetful map, either the images of δ_1, δ_2 and δ_3 are on a smooth rational subcurve C', or the images of δ_2 and δ_3 are on a smooth rational subcurve C', which attaches to another smooth rational subcurve C'' that contains the image of δ_1 and attaches to the core at a pivotal node. In either case, however, the symmetry between δ_2 and δ_3, as well as the statements for κ_{12} and κ_{23} that have already been established, give rise to the relations between κ_{13} and κ_{23}. □

2.10 Proof of Proposition 2.22

In this subsection, we complete the proof of Proposition 2.22 using Propositions 2.27 and 2.28 and Corollary 2.31.

(SH_1) is a restatement of Proposition 2.10.

To show (SH_2), we divide the arguments into three cases, depending on the location of δ_i and δ_j.

Case (2.a): *δ_i and δ_j belong to a chain of tail rational subcurves.* This case simply follows from replacing δ_1 and δ_2 by δ_i and δ_j in Proposition 2.28 (R_1).

Case (2.b): *δ_i and δ_j belong to the same tail, but lie on different maximal chains of tail rational subcurves.*

We write the maximal chains as

$$R^i := R^i_1 \cup \cdots \cup R^i_{h_i} \quad \text{and} \quad R^j := R^j_1 \cup \cdots \cup R^j_{h_j},$$

respectively, with their rational subcurves being ordered in the same way as in Section 2.8. Obviously, $R^i_1 = R^j_1$. Let

$$\widehat{\mu} := \max\{\mu \in \mathbb{Z}_{>0} : 1 \le \mu \le \min\{h_i, h_j\},\ R^i_\mu = R^j_\mu\}, \quad q_{\widehat{\mu}} := q^i_{\widehat{\mu}} = q^j_{\widehat{\mu}},$$

where the nodes $q^i_{\widehat{\mu}}$ and $q^j_{\widehat{\mu}}$ are as in (2.29). It is straightforward that $q_{\widehat{\mu}} = \delta_i \wedge \delta_j$.

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Next, we apply an argument parallel to the proof of Proposition 2.28 (R_4). Without loss of generality we assume that

$$D \cap (R^i_{\hat{\mu}} \cup R^i_{\hat{\mu}+1} \cup \cdots \cup R^i_{h_i}) = \delta_i + (\delta_{\ell+1} + \delta_{\ell+2} + \cdots + \delta_m)$$

for some ℓ satisfying $\max\{i,j\} \le \ell \le m$. We then forget the marked points $\delta_{\ell+1}, \ldots, \delta_m$, take stabilization, and define the points $(\overline{C}(z), \overline{D}(z)) \in \mathfrak{M}_2^{\mathrm{div}}$, the universal family $\overline{\rho} : \overline{\mathcal{C}} \to \overline{\mathcal{V}}$, the tautological divisor $\overline{\mathcal{D}} \subset \overline{\mathcal{C}}$, the forgetful map $\varpi : \mathcal{V} \to \overline{\mathcal{V}}$, and the contraction $\widetilde{\varpi} : \mathcal{C} \to \overline{\mathcal{C}}$ analogously so that (2.54) commutes.

In the current scenario, notice that we still have

$$\varpi^{-1}(\overline{\mathcal{D}}_k) = \mathcal{D}_k \quad \forall\, 1 \le k \le \ell, \qquad \varpi^{-1}(\overline{\mathcal{A}}) = \mathcal{A}, \quad \text{and} \quad \varpi^{-1}(\overline{\mathcal{B}}) = \mathcal{B}.$$

As in the proof of Proposition 2.28 (R_4), we conclude that $\overline{\varphi}_{\le \ell}$ pulls back to $\varphi_{\le \ell}$. In particular, $\overline{\theta}_{ij}$ pulls back to θ_{ij}.

Meanwhile, observe that on $\overline{C}(z_0)$, the rational subcurves of the maximal chain R^j are not contracted by $\widetilde{\varpi}$, and the images $\overline{\delta}_i$ and $\overline{\delta}_j$ of δ_i and δ_j both belong to this chain. So by replacing $\varphi_{\le r}$, δ_1, and δ_2 in Proposition 2.28 with $\overline{\varphi}_{\le \ell}$, $\overline{\delta}_i$, and $\overline{\delta}_j$, respectively, we have

$$\overline{\theta}_{ij} = \overline{u} \cdot \zeta_{[\overline{\delta}_i]} \cdot \overline{\kappa}_{ij} = \overline{u} \cdot \zeta_{[q_{\hat{\mu}}]} \cdot \overline{\kappa}_{ij} \quad \text{with} \quad \overline{u} \in \Gamma(\mathscr{O}^*_{\overline{\mathcal{V}}}).$$

Since $\zeta_{[q_{\hat{\mu}}]}$ and $\overline{\kappa}_{ij}$ pull back to $\zeta_{[\delta_i \wedge \delta_j]}$ and κ_{ij}, respectively, Case (2.b) of (SH_2) is thus established.

Case (2.c): *either δ_i and δ_j belong to different tails, or at least one of them is on the core of C.* In this case, notice that $\zeta_{[\delta_i \wedge \delta_j]}$ is set to be 1 (see the sentence below (2.11)). Also notice that the statements (2.a), (3), and (4.a) of Lemma 2.29 remain valid by replacing δ_1 and δ_j with δ_i and δ_j, whereas (1.a) can be replaced by the following: $\theta_{ij}(z_0) = 0$ if and only if δ_i and δ_j are conjugate. Mimicking the proof of Proposition 2.28(R_1), we then observe that θ_{ij} still takes the form $u \cdot \kappa_{ij}$ for some $u \in \Gamma(\mathscr{O}_{\mathcal{V}})$, and such u is still nowhere vanishing on $\mathcal{V}$ except possibly on a codimension 2 subset, hence u is invertible. This completes the proof of (SH_2).

To establish the statements in (SH_3), first observe that if one of κ_{ik} and κ_{ij} is invertible on $\mathcal{V}$, then $\langle \delta_i \rangle$, $\langle \delta_j \rangle$, and $\langle \delta_k \rangle$ cannot be on a common non-separating bridge, so one can always choose invertible functions v and u' on $\mathcal{V}$ such that

$$\kappa_{ik} = v \cdot \kappa_{ij} + u' \cdot \zeta_{[\delta_j \wedge \delta_k]}$$

holds. So hereafter we assume

$$\kappa_{ik}(z_0) = \kappa_{ij}(z_0) = 0. \tag{2.59}$$

Moreover, since $\overline{\mathcal{K}}_{ik}$ and $\overline{\mathcal{K}}_{ij}$ are defined using pullbacks via the forgetful morphisms, it suffices to justify (SH_3) by forgetting all the remaining elements of D (recall we consider D as a finite set of points). In other words, we assume

$$D = \{\delta_i, \delta_j, \delta_k\}$$

and treat (C, D) as a point in $\overline{\mathcal{M}}_{2,3}$.

Below we divide the argument into five cases.

Case (3.a): *δ_i, δ_j and δ_k belong to the same tail.* Notice that in this scenario, there exists a (unique) maximal chain of tail rational subcurves containing these three points. This case simply follows from replacing δ_1, δ_2 and δ_3 by δ_i, δ_j and δ_k in Corollary 2.31.

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Case (3.b): *δ_k is on the core F of C, whereas δ_i and δ_j belong to the same tail.* In this scenario, we have

$$\delta_k = \langle \delta_k \rangle \quad \text{and} \quad \langle \delta_i \rangle = \langle \delta_j \rangle \ =: q_i.$$

Let $x_i \in F$ be a smooth point that is sufficiently close to the node q_i. Notice that the assumption (2.59) only holds when q_i and δ_k are on a maximal non-separating bridge, denoted by $\mathrm{B}[\mathfrak{p}, \mathfrak{q}]$. Notice that $\mathrm{B}[\mathfrak{p}, \mathfrak{q}]$ consists of either one or two (rational) subcurves.

If $\mathrm{B}[\mathfrak{p}, \mathfrak{q}]$ consists of two rational subcurves $R_\mathfrak{p}$ and $R_\mathfrak{q}$, respectively containing $\mathfrak{p}$ and $\mathfrak{q}$, we denote by $\mathfrak{n}$ the node $R_\mathfrak{p} \cap R_\mathfrak{q}$. Without loss of generality we assume $\delta_k \in R_\mathfrak{p}$ and $q_i \in R_\mathfrak{q}$. Then by Proposition 2.17, there exist $f, g, \widetilde{f}, \widetilde{g} \in \Gamma(\mathscr{O}^*_\mathcal{V})$ such that

$$\kappa_{ij} = f\zeta_\mathfrak{n}\zeta_\mathfrak{p} + g\zeta_\mathfrak{q}, \quad \kappa_{ik} = \widetilde{f}\zeta_\mathfrak{p} + \widetilde{g}\zeta_\mathfrak{q}.$$

Therefore,

$$\kappa_{ik} = \tfrac{\widetilde{g}}{g}\kappa_{ij} + f' \cdot \zeta_\mathfrak{q}, \quad \text{where } f' := \widetilde{f} - \tfrac{f\cdot\widetilde{g}}{g}\zeta_\mathfrak{n}.$$

Since $\widetilde{f}$ is invertible, $\zeta_\mathfrak{n}(z_0) = 0$ and $\mathcal{V}$ is sufficiently small, we see that $w = f' \cdot \zeta_\mathfrak{p} + 0 \cdot \zeta_\mathfrak{q}$, f' is invertible, and so is $\det \begin{bmatrix} f & f' \\ g & 0 \end{bmatrix}$.

If $\mathrm{B}[\mathfrak{p}, \mathfrak{q}]$ consists of only one rational subcurve R_0 (thus $\mathfrak{p}, \mathfrak{q}, q_i \in R_0$), there exist $f, g, \widetilde{f}, \widetilde{g} \in \Gamma(\mathscr{O}^*_\mathcal{V})$ such that

$$\kappa_{ij} = f\zeta_\mathfrak{p} + g\zeta_\mathfrak{q}, \quad \kappa_{ik} = \widetilde{f}\zeta_\mathfrak{p} + \widetilde{g}\zeta_\mathfrak{q}.$$

Therefore,

$$\kappa_{ik} = \tfrac{\widetilde{g}}{g}\kappa_{ij} + f' \cdot \zeta_\mathfrak{q}, \quad \text{where } f' := \widetilde{f} - \tfrac{f\cdot\widetilde{g}}{g}.$$

It remains to show that f' is invertible.

To this end, we mimic the last two paragraphs of the proof of Proposition 2.28 (R_4). More precisely, by Proposition 2.17, $\mathcal{K}_{ij}$ is still a smooth Cartier divisor of $\mathcal{V}$, so we can take a small curve $\Delta \subset \mathcal{K}_{ij}$ through z_0, meeting $(\zeta_\mathfrak{q} = 0)$ (and hence $(\zeta_\mathfrak{p} = 0)$ because of the expression of κ_{ij} above) transversely at z_0 so that the node q_i is not smoothed out along Δ. Let $\mathcal{C}_\Delta$, $\mathcal{Q}_{\Delta,i}$, $\mathcal{D}_{\Delta,k}$, $\mathcal{A}_\Delta$, and $\mathcal{B}_\Delta$ be the analogues of those in the proof of Proposition 2.28 (R_4).

If the core F is inseparable, then we obtain the evaluation homomorphism

$$\rho_*\mathscr{O}_{\mathcal{C}_\Delta}(2\mathcal{Q}_{\Delta,i} + \mathcal{D}_{\Delta,k} + \mathcal{A}_\Delta - \mathcal{B}_\Delta) \xrightarrow{\Theta_{i,k}|_\Delta} \rho_*\mathscr{O}_{\mathcal{A}_\Delta}(\mathcal{A}_\Delta),$$
$$\text{where } \Theta_{i,k}|_\Delta = \begin{bmatrix} c_{1i} & \widehat{c}_{1j} & c_{1k} \\ c_{2i} & \widehat{c}_{2j} & c_{2k} \end{bmatrix}\Bigg|_\Delta.$$

which after an elementary row operation can be rewritten as

$$\begin{bmatrix} c_{11} & \widehat{c}_{12} & c_{13} \\ 0 & 0 & f'\zeta_\mathfrak{q} \end{bmatrix}\Bigg|_\Delta.$$

Following the argument in the penultimate paragraph of Proposition 2.28 (R_4), we see that $f'\zeta_\mathfrak{q}|_\Delta$ vanishes at z_0 to the first order, hence $f'(z_0) \neq 0$ and f' is invertible on $\mathcal{V}$ (which is assumed small).

If the core F contains a separating node, we consider the locus $\mathcal{Z} := (\zeta_{q_i} = \kappa_{ij} = 0)$, which is smooth and contains z_0. From the last paragraph we see that $f'(z) \neq 0$ for all $z \in \mathcal{Z}$ for which the core of $\mathcal{C}_z$ is inseparable. For any $z \in \mathcal{Z}$ for which the core of $\mathcal{C}_z$ contains only one node

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and that node is separating, $\mathcal{Q}_i(z)$ is a Weierstrass point by Lemma 2.20, so $\mathcal{D}_k(z)$ cannot be conjugate to $\mathcal{D}_i(z)$. Thus, $\kappa_{ik}(z)\neq 0$, so $f'(z)\neq 0$. This implies that f' is nowhere zero on $\mathcal{Z}$ except possibly on a codimension 2 subset of $\mathcal{Z}$, hence is invertible on $\mathcal{Z}$ and thus on $\mathcal{V}$. This completes the proof of Case (3.b).

Case (3.c): *δ_j is on the core F of C, whereas δ_i and δ_k belong to the same tail.* This case is parallel to Case (3.b) because of the symmetry between δ_j and δ_k in the statements of (SH_3).

Case (3.d): *δ_i is on the core F of C, whereas δ_j and δ_k belong to the same tail.* In this case, notice that

$$\delta_i=\langle\delta_i\rangle \quad\text{and}\quad \langle\delta_j\rangle=\langle\delta_k\rangle \;=:q_j.$$

Let $x_j\in F$ be a smooth point sufficiently close to the node q_j.

- If δ_i and q_j are on a maximal non-separating bridge $\mathrm{B}[\mathfrak{p},\mathfrak{q}]$, then by Proposition 2.17, there exist $f,g,\widetilde{f},\widetilde{g}\in\Gamma(\mathscr{O}_{\mathcal{V}}^*)$ such that
$$\kappa_{ij}=f\zeta_{\mathfrak{p}}+g\zeta_{\mathfrak{q}},\quad \kappa_{ik}=\widetilde{f}\zeta_{\mathfrak{p}}+\widetilde{g}\zeta_{\mathfrak{q}}, \tag{2.60}$$
regardless of the number of irreducible components of $\mathrm{B}[\mathfrak{p},\mathfrak{q}]$. Mimicking the $(i,j,k)=(2,3,1)$ case of the proof of Corollary 2.31, we apply (SH_2) and choose a suitable trivialization of $\rho_*\mathcal{M}(\mathcal{D}_i)\oplus\rho_*\mathcal{M}(\mathcal{D}_j)\oplus\rho_*\mathcal{M}(\mathcal{D}_k)$ and $\rho_*\mathscr{O}_{\mathcal{A}}(\mathcal{A})$ so that the structural homomorphism
$$\varphi\colon\ \rho_*\mathcal{M}(\mathcal{D}_i)\oplus\rho_*\mathcal{M}(\mathcal{D}_j)\oplus\rho_*\mathcal{M}(\mathcal{D}_k)\longrightarrow\rho_*\mathscr{O}_{\mathcal{A}}(\mathcal{A}),$$
can be written as
$$\varphi=\begin{bmatrix} c_{1i}\zeta_{[\delta_i,a_1]} & 0 & 0\\ 0 & \kappa_{ij}\zeta_{[\delta_j,a_2]} & \kappa_{jk}\zeta_{q_j}\zeta_{[\delta_k,a_2]}\end{bmatrix}.$$
Alternatively, still by (SH_2), we can choose a different trivialization so that φ takes the form
$$\varphi=\begin{bmatrix} c_{1i}\zeta_{[\delta_i,a_1]} & 0 & 0\\ 0 & \kappa_{ij}\zeta_{[\delta_j,a_2]} & \kappa_{ik}\zeta_{[\delta_k,a_2]}\end{bmatrix}.$$
In addition, we have $\zeta_{[\delta_j,a_2]}=\zeta_{[\delta_k,a_2]}$. Consequently, there exist $\nu\in\Gamma(\mathscr{O}_{\mathcal{V}})$ and $\lambda\in\Gamma(\mathscr{O}_{\mathcal{V}}^*)$ such that
$$\kappa_{ik}=\lambda\cdot\zeta_{q_j}\kappa_{jk}+\nu\cdot\kappa_{ij}. \tag{2.61}$$
By (2.60) and the fact that $\zeta_{q_j}(z_0)=0$, we see that $\nu\in\Gamma(\mathscr{O}_{\mathcal{V}}^*)$. Moreover, Case (3.b) above implies there exist $\lambda',\nu'\in\Gamma(\mathscr{O}_{\mathcal{V}}^*)$ such that
$$\kappa_{jk}=\lambda'\kappa_{ik}+\nu'\zeta_{\mathfrak{q}}.$$
Combining the above two equalities, we conclude that there exist $v,g'\in\Gamma(\mathscr{O}_{\mathcal{V}}^*)$ such that
$$\kappa_{ik}=v\cdot\kappa_{ij}+g'\zeta_{q_j}\zeta_{\mathfrak{q}}.$$
- If δ_i is not on any non-separating bridge, then the assumption (2.59) implies that q_j is not on any non-separating bridge either, which further implies $\kappa_{jk}(z_0)\neq 0$. Therefore, (2.61) takes a simpler form:
$$\kappa_{ik}=\lambda\cdot\zeta_{q_j}+\nu\cdot\kappa_{ij}$$

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for some $\lambda \in \Gamma(\mathscr{O}_{\mathcal{V}}^*)$ and $\nu \in \Gamma(\mathscr{O}_{\mathcal{V}})$. The symmetry between δ_j and δ_k in Case (3.d) also guarantees that $\nu \in \Gamma(\mathscr{O}_{\mathcal{V}}^*)$.

Case (3.e): *δ_i, δ_j and δ_k are all on the core F of C.* In this scenario, we have

$$\delta_i = \langle \delta_i \rangle, \quad \delta_j = \langle \delta_j \rangle, \quad \delta_k = \langle \delta_k \rangle.$$

Once again the assumption (2.59) only holds when the three points are on a common maximal non-separating bridge $\mathrm{B}[\mathfrak{p}, \mathfrak{q}]$. Depending on the number of the smooth rational subcurve(s) of $\mathrm{B}[\mathfrak{p}, \mathfrak{q}]$ and the distribution of δ_i, δ_j and δ_k, the proof of Case (3.e) is analogous to, but simpler than, the proof of either Case (3.b) or Case (3.d). We omit further details.

3. Sequential modular blowups of $\mathfrak{M}_2^{\mathrm{wt}}$ and $\mathfrak{M}_2^{\mathrm{div}}$

The key idea for resolving $\overline{M}_2(\mathbb{P}^n, d)$ is to globally blow up $\mathfrak{M}_2^{\mathrm{div}}$ to obtain $\widetilde{\mathfrak{M}}_2^{\mathrm{div}}$ so that the derived object $\mathbf{R}\pi_* \mathfrak{f}^* \mathscr{O}_{\mathbb{P}^n}(k)$ becomes locally diagonalizable in the sense of Definition 2.7 upon pulling back to any

$$\widetilde{\mathcal{U}}_{U,H} := U \times_{f_{U,H};\, \mathfrak{M}_2^{\mathrm{div}}} \widetilde{\mathfrak{M}}_2^{\mathrm{div}}.$$

The fact that these $\widetilde{\mathcal{U}}_{U,H}$ glue to form $\widetilde{M}_2(\mathbb{P}^n, d)/\overline{M}_2(\mathbb{P}^n, d)$ is relatively straightforward, and this will be handled in Section 5. As soon as the global sequential blowups of $\mathfrak{M}_2^{\mathrm{div}}$ are rigorously described, the verification of the local diagonalizability of the object $\mathbf{R}\pi_* \mathfrak{f}^* \mathscr{O}_{\mathbb{P}^n}(k)$, or, equivalently, the diagonalizability of the structural homomorphisms φ, is local.

As mentioned in the introduction, there are three rounds of sequential blowups, denoted by (r_1), (r_2), and (r_3). The first and third rounds each contain several phases, which are written as $(\mathsf{r}_i \mathsf{p}_j)$.

The first two rounds (r_1) and (r_2), as well as the first two phases $(\mathsf{r}_3\mathsf{p}_1)$ and $(\mathsf{r}_3\mathsf{p}_2)$ of the third round, can be introduced and performed on $\mathfrak{M}_2^{\mathrm{wt}}$. After $(\mathsf{r}_3\mathsf{p}_2)$, we pull them back to $\mathfrak{M}_2^{\mathrm{div}}$, and then describe the remaining phases of the sequential blowups.

The global blowup process is guided by the desire to make the local structural homomorphism

$$\varphi = \begin{bmatrix} c_{11}\zeta_{[\delta_1,a_1]} & c_{12}\zeta_{[\delta_2,a_1]} & \cdots & c_{1i}\zeta_{[\delta_i,a_1]} & \cdots \\ c_{21}\zeta_{[\delta_1,a_2]} & c_{22}\zeta_{[\delta_2,a_2]} & \cdots & c_{2i}\zeta_{[\delta_i,a_2]} & \cdots \end{bmatrix}. \tag{3.1}$$

as in Proposition 2.22 (SH_1) diagonalizable in the sense of Definition 2.6.

We focus first on one of the two rows of φ, although any blowup will inevitably affect the other row, but in any case, the purpose of the first round (r_1) of blowups is to make one row, say the first row, diagonalized. This is parallel to the blowups in [VZ08] and [HL10], although the blowup centers are somewhat more involved. Then, the task in the remaining rounds of the sequential blowups is to make the other (say, the second) row and hence the matrix (3.1) diagonalized.

More precisely, when (r_1) terminates, we denote the final stack by $\widetilde{\mathfrak{M}}^{\mathsf{r}_1}$. Every point of $\mathfrak{M}_2^{\mathrm{wt}}$ has a neighborhood $\mathcal{V}$ so that the pullback $\widetilde{\varphi}^{\mathsf{r}_1}$ of (3.1) to $\widetilde{\mathcal{V}}^{\mathsf{r}_1} = \mathcal{V} \times_{\mathfrak{M}_2^{\mathrm{wt}}} \widetilde{\mathfrak{M}}^{\mathsf{r}_1}$ has one row that contains an element that divides all the other entries of $\widetilde{\varphi}^{\mathsf{r}_1}$. However, $\widetilde{\varphi}^{\mathsf{r}_1}|_{\widetilde{\mathcal{V}}^{\mathsf{r}_1}}$ may *not* be locally diagonalizable (cf. Definition 2.6) because of the existence of (1) some distinctive directions in the exceptional divisors obtained in (r_1); and/or (2) the Weierstrass and conjugate points of genus 2 curves.

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The second round, (r_2), solves the former case; the third round, (r_3), solves the latter. In the following subsections, we precisely describe the three rounds of sequential blowups.

Remark 3.1. As mentioned in Section 1, throughout this paper, we fix an arbitrary positive integer m. For any point $(C, \mathbf{w}) \in \mathfrak{M}_2^{\mathrm{wt}}$, m corresponds to the total weight of $(C, \mathbf{w})$; for any point $(C, D) \in \mathfrak{M}_2^{\mathrm{div}}$, $m = \deg D$. Further, when applying to $\mathbf{R}\pi_*\mathfrak{f}^*\mathscr{O}_{\mathbb{P}^n}(k)$ over $\overline{M}_2(\mathbb{P}^n, d)$, $m = kd$.

In Section 3, we describe the global blowup centers of $\widetilde{\mathfrak{M}}_2^{\mathrm{div}}/\mathfrak{M}_2^{\mathrm{div}}$ phase by phase. Later in Section 4, we will show that each step of each phase is a smooth blowup, on a case-by-case basis.

3.1 The first round (r_1)

The blowup centers in (r_1) are the closures of the following strata

$$\mathfrak{M}_{(1,k)},\ \mathfrak{M}_{(2,k)},\ \mathfrak{M}_{(3,k)},\ \mathfrak{M}_{(4,k)},\ \mathfrak{M}_{(5,k)}, \quad k \in \mathbb{Z}_{>0}, \tag{3.2}$$

of $\mathfrak{M}_2^{\mathrm{wt}}$. Recall that we denote by F the core of a curve C. For each k:

- $(C, \mathbf{w}) \in \mathfrak{M}_{(1,k)}$ if F is a smooth genus 2 curve of weight 0 and the tails of C are k smooth rational curves of positive weights;
- $(C, \mathbf{w}) \in \mathfrak{M}_{(2,k)}$ if F consists of two smooth genus 1 curves of weight 0 and one smooth rational curve of positive weight, and the tails of C are $(k-1)$ smooth rational curves of positive weights;
- $(C, \mathbf{w}) \in \mathfrak{M}_{(3,k)}$ if F consists of one smooth genus 1 curve of weight 0 and one smooth rational curve of positive weight, and the tails of C are $(k-1)$ smooth rational curves of positive weights;
- $(C, \mathbf{w}) \in \mathfrak{M}_{(4,k)}$ if F consists of one smooth genus 1 curve of weight 0 and one smooth genus 1 curve of positive weight, and the tails of C are $(k-1)$ smooth rational curves of positive weights; and
- $(C, \mathbf{w}) \in \mathfrak{M}_{(5,k)}$ if F is a smooth genus 2 curve of weight 1 and the tails of C are k smooth rational curves of positive weights.

In Figure 1, we illustrate $\mathfrak{M}_{(i,k)}$ with $k \leq 3$. An unshaded irreducible component indicates that the weight is 0.

The closure $\overline{\mathfrak{M}}_{(i,k)}$ of $\mathfrak{M}_{(i,k)}$ in $\mathfrak{M}_2^{\mathrm{wt}}$ can be described as the images of natural node-identifying surjective immersions from smooth domains. Given a finite set S, let $\overline{\mathcal{M}}_{g;S}$ be the moduli space of genus g stable curves whose marked points are indexed by S. Let

$$\mathfrak{M}_{g,S}^{\mathrm{wt}}$$

be the Artin stack of stable pairs $(C, \mathbf{w})$ of genus g nodal curves C whose marked points are indexed by S and weights $\mathbf{w} \in H^2(C; \mathbb{Z})$ satisfying $\mathbf{w}(\Sigma) \geq 0$ for all irreducible $\Sigma \subset C$. Here $(C, \mathbf{w})$ is said to be **stable** if any smooth rational (respectively, genus 1) irreducible component of weight 0 contains at least three (respectively, one) nodal and/or marked points. In particular, $\mathfrak{M}_2^{\mathrm{wt}} = \mathfrak{M}_{2,\emptyset}^{\mathrm{wt}}$. Each connected component

$$\mathfrak{M}_{g,S}^{\mathrm{wt}}(\ell)$$

of $\mathfrak{M}_{g,S}^{\mathrm{wt}}$ is determined by the total weight ℓ of any element.

For conciseness, throughout this article, we write

$$[k] := [1, k] \cap \mathbb{Z} \quad \forall\, k \in \mathbb{Z}_{>0}. \tag{3.3}$$

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With the notation as above, we define

$$\widehat{\mathfrak{M}}_{(1,k)} := \overline{\mathcal{M}}_{2,k} \times \prod_{i=1}^{k} \mathfrak{M}^{\mathrm{wt}}_{0,\{i\}},$$

$$\widehat{\mathfrak{M}}_{(2,k)} := \bigsqcup_{\ell=1}^{k} \left(\overline{\mathcal{M}}_{1,[\ell-1]\sqcup\{k+1\}} \times \overline{\mathcal{M}}_{1,([k]\setminus[\ell])\sqcup\{k+2\}} \times \mathfrak{M}^{\mathrm{wt}}_{0,\{k+1,k+2\}} \times \prod_{i\in[k]\setminus\{\ell\}} \mathfrak{M}^{\mathrm{wt}}_{0,\{i\}} \right),$$

$$\widehat{\mathfrak{M}}_{(3,k)} := \overline{\mathcal{M}}_{1,k+1} \times \mathfrak{M}^{\mathrm{wt}}_{0,\{k,k+1\}} \times \prod_{i=1}^{k-1} \mathfrak{M}^{\mathrm{wt}}_{0,\{i\}},$$

$$\widehat{\mathfrak{M}}_{(4,k)} := \overline{\mathcal{M}}_{1,k} \times \mathfrak{M}^{\mathrm{wt}}_{1,\{k\}} \times \prod_{i=1}^{k-1} \mathfrak{M}^{\mathrm{wt}}_{0,\{i\}},$$

$$\widehat{\mathfrak{M}}_{(5,k)} := \mathfrak{M}^{\mathrm{wt}}_{2,[k]}(1) \times \prod_{i=1}^{k} \mathfrak{M}^{\mathrm{wt}}_{0,\{i\}}.$$

Here, each $\{i\}$ is the set with i as its unique element; it does not mean i unordered marked points. For each $(i,k) \in [5] \times \mathbb{Z}_{>0}$, there is a natural node-identifying immersion

$$\iota_{(i,k)} : \widehat{\mathfrak{M}}_{(i,k)} \longrightarrow \overline{\mathfrak{M}}_{(i,k)} \subset \mathfrak{M}^{\mathrm{wt}}_2 \tag{3.4}$$

obtained by identifying the marked points that share the same index. Such $\iota_{(i,k)}$ is surjective onto $\overline{\mathfrak{M}}_{(i,k)}$ and descends to the quotient

$$\bar{\iota}_{(i,k)} : \widehat{\mathfrak{M}}_{(i,k)}/S_{(i,k)} \longrightarrow \overline{\mathfrak{M}}_{(i,k)} \subset \mathfrak{M}^{\mathrm{wt}}_2, \tag{3.5}$$

where $S_{(i,k)}$ is the symmetric group of the dual graph of $\mathfrak{M}_{(i,k)}$. Notice that $\bar{\iota}_{(i,k)}$ is generally not an isomorphism to its image; cf. [VZ08, Figure 3] with the genus 1 component replaced by a genus 2 curve, as well as the example at the end of Example 4.11. Thus $\overline{\mathfrak{M}}_{(i,k)}$ is generally not smooth.

Let

$$\overline{\mathfrak{M}}_{(i)} := \bigcup_{k\geq 1} \overline{\mathfrak{M}}_{(i,k)}, \quad \mathfrak{M}^{\mathrm{mn}}_{(i)} := \overline{\mathfrak{M}}_{(i)} \setminus \left(\bigcup_{h=1}^{i-1} \overline{\mathfrak{M}}_{(h)} \right), \ 1 \leq i \leq 4; \quad \mathfrak{M}^{\mathrm{mn}} := \mathfrak{M}^{\mathrm{wt}}_2 \setminus \left(\bigcup_{i=1}^{4} \overline{\mathfrak{M}}_{(i)} \right). \tag{3.6}$$

In particular, $\mathfrak{M}^{\mathrm{mn}}_{(1)} = \overline{\mathfrak{M}}_{(1)}$. Notice that $\overline{\mathfrak{M}}_{(i,k)} \cap \mathfrak{M}_{(j,\ell)} = \emptyset$ whenever $(j,\ell) > (i,k)$ (with respect to the lexicographical order), so for each $1 \leq i \leq 4$, we have $\mathfrak{M}^{\mathrm{mn}}_{(i)} \supset \bigsqcup_{k\geq 1} \mathfrak{M}_{(i,k)}$. Then we obtain a partition of $\mathfrak{M}^{\mathrm{wt}}_2$:

$$\mathfrak{M}^{\mathrm{wt}}_2 = \mathfrak{M}^{\mathrm{mn}}_{(1)} \sqcup \mathfrak{M}^{\mathrm{mn}}_{(2)} \sqcup \mathfrak{M}^{\mathrm{mn}}_{(3)} \sqcup \mathfrak{M}^{\mathrm{mn}}_{(4)} \sqcup \mathfrak{M}^{\mathrm{mn}}.$$

In particular, $\mathfrak{M}_{(5,k)} \subset \mathfrak{M}^{\mathrm{mn}}$ for all $k \geq 1$.

In (r_1), we blow up $\mathfrak{M}^{\mathrm{wt}}_2$ along (the proper transforms of) $\overline{\mathfrak{M}}_{(i,k)}$ with respect to the lexicographical order and obtain

$$\widetilde{\mathfrak{M}}^{\mathsf{r}_1} \longrightarrow \mathfrak{M}^{\mathrm{wt}}_2.$$

More concretely, we start with $\widetilde{\mathfrak{M}}^{\mathsf{r}_1\mathsf{p}_1\mathsf{s}_0} = \mathfrak{M}^{\mathrm{wt}}_2$. Inductively, suppose that $\widetilde{\mathfrak{M}}^{\mathsf{r}_1\mathsf{p}_i\mathsf{s}_{k-1}}$ is constructed for some $i, k > 0$; we then take

$$\widetilde{\mathfrak{M}}^{\mathsf{r}_1\mathsf{p}_i\mathsf{s}_k} \longrightarrow \widetilde{\mathfrak{M}}^{\mathsf{r}_1\mathsf{p}_i\mathsf{s}_{k-1}}$$

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to be the blowup of $\widetilde{\mathfrak{M}}^{\mathsf{r}_1\mathsf{p}_i\mathsf{s}_{k-1}}$ along the proper transform of $\overline{\mathfrak{M}}_{(i,k)}$. We call $\widetilde{\mathfrak{M}}^{\mathsf{r}_1\mathsf{p}_i\mathsf{s}_k}$ the resulting blowup stack in **Round 1** (r_1), **Phase** i (p_i), and **Step** k (s_k). Here, to reconcile the notation, for $i>1$, we set $\widetilde{\mathfrak{M}}^{\mathsf{r}_1\mathsf{p}_i\mathsf{s}_0}$ to be the stack obtained when Phase $i-1$ is complete.

Recall that the connected components of $\mathfrak{M}_2^{\mathrm{wt}}$ are determined by the total weights of the curves, thus the stability of $\mathfrak{M}_2^{\mathrm{wt}}$ (cf. Section 2.1) implies that the process of sequential blowups (r_1) terminates after finitely many steps.

Remark 3.2 (The order of the blowups in (r_1)). The reason we choose the blowup centers of (r_1) to be the proper transforms of $\overline{\mathfrak{M}}_{(i,k)}$ is based on the way the node-smoothing parameters appear in the structural homomorphism (3.1). As mentioned earlier in this subsection, $\overline{\mathfrak{M}}_{(i,k)}$ are generally not smooth in $\mathfrak{M}_2^{\mathrm{wt}}$. The order of the blowups is thus chosen so that, in each step, we always have a smooth blowup on which the pullbacks of (3.1) can be systematically analyzed.

As mentioned at the beginning of Section 3, we will prove the smoothness of the blowups of (r_1) in Section 4. Here, we provide the general idea; see Example 3.4 below for a concrete example.

Within each phase of (r_1), the order of the blowups resembles that in [VZ08] and [HL10]. The order among the phases of (r_1) is based on the following observations.

- First, observe that
$$\overline{\mathfrak{M}}_{(2)}\cap\overline{\mathfrak{M}}_{(3)}\subset\overline{\mathfrak{M}}_{(1)}, \tag{3.7}$$
hence ($\mathsf{r}_1\mathsf{p}_1$) should be performed before ($\mathsf{r}_1\mathsf{p}_2$) and ($\mathsf{r}_1\mathsf{p}_3$). When ($\mathsf{r}_1\mathsf{p}_1$) terminates, the proper transforms of $\overline{\mathfrak{M}}_{(2)}$ and $\overline{\mathfrak{M}}_{(3)}$ are disjoint. It then makes no difference if we perform ($\mathsf{r}_1\mathsf{p}_3$) prior to ($\mathsf{r}_1\mathsf{p}_2$).
- Next, note that
$$\overline{\mathfrak{M}}_{(2)}\subset\overline{\mathfrak{M}}_{(4)}.$$
In particular, some singularities of $\overline{\mathfrak{M}}_{(4)}$ are contained in $\overline{\mathfrak{M}}_{(2)}$; for instance, $\overline{\mathfrak{M}}_{(4,1)}$ has a normal crossing singularity at a general point of $\overline{\mathfrak{M}}_{(2,1)}$, so ($\mathsf{r}_1\mathsf{p}_2$) should be carried out prior to ($\mathsf{r}_1\mathsf{p}_4$).
- Similarly, ($\mathsf{r}_1\mathsf{p}_5$) should be performed after ($\mathsf{r}_1\mathsf{p}_1$). For instance, $\overline{\mathfrak{M}}_{(5,1)}$ contains a general point of $\overline{\mathfrak{M}}_{(1,2)}$ whose tails are both of weight 1 as a normal crossing singularity.
- It is worth mentioning that it is feasible to put ($\mathsf{r}_1\mathsf{p}_4$) before ($\mathsf{r}_1\mathsf{p}_3$), which is indeed the order chosen in [HN21], although the resulting resolution of $\overline{M}_2(\mathbb{P}^n,d)$ will be different from the one constructed in this paper. It is also possible to perform ($\mathsf{r}_1\mathsf{p}_5$) before any ($\mathsf{r}_1\mathsf{p}_i$), $2\le i\le 4$, and obtain a different resolution of $\overline{M}_2(\mathbb{P}^n,d)$.

Remark 3.3 (The reason for ($\mathsf{r}_1\mathsf{p}_5$)). As mentioned in the introduction, the last phase of the first round, ($\mathsf{r}_1\mathsf{p}_5$), is solely to treat the boundary components of $\overline{M}_2(\mathbb{P}^n,d)$. As all the modular blowups are performed on $\mathfrak{M}_2^{\mathrm{wt}}$ and $\mathfrak{M}_2^{\mathrm{div}}$, more or less, we are forced to include the loci whose general points have smooth core subcurves of weight 1, even though the images of the primary component of $\overline{M}_2(\mathbb{P}^n,d)$, under either the morphism $\overline{M}_2(\mathbb{P}^n,d)\longrightarrow\mathfrak{M}_2^{\mathrm{wt}}$ as in (1.6) or the local morphisms $\overline{M}_2(\mathbb{P}^n,d)\supset U\longrightarrow\mathfrak{M}_2^{\mathrm{div}}$ as in (1.2), miss such loci.

The following example suggests how the various $\overline{\mathfrak{M}}_{(i,k)}$ intersect, and how the blowups of (r_1) affect them locally.

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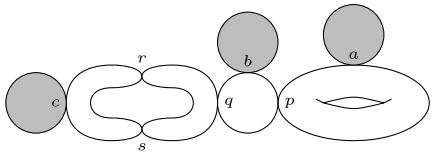


FIGURE 5. An example of the intersection of various $\overline{\mathfrak{M}}_{(i,k)}$.

Example 3.4. Let $x=(C,\mathbf{w})\in\mathfrak{M}_2^{\mathrm{wt}}$ be as in Figure 5, where the shaded components are of positive weight, and the unshaded ones are of weight 0. One can check directly that

$$x\in\overline{\mathfrak{M}}_{(1,3)}\cap\overline{\mathfrak{M}}_{(2,3)}\cap\overline{\mathfrak{M}}_{(3,3)}\cap\overline{\mathfrak{M}}_{(4,2)}\cap\overline{\mathfrak{M}}_{(4,3)};$$

see Figure 1 for an illustration of the $\overline{\mathfrak{M}}_{(i,k)}$ above. In fact, if any of the shaded components is of weight 1, then $x\in\overline{\mathfrak{M}}_{(5,2)}$ as well; for simplicity, we assume that no component is of weight 1 in this example.

Let $\mathcal{V}\longrightarrow\mathfrak{M}_2^{\mathrm{wt}}$ be a sufficiently small smooth chart containing x, and ζ_e, $e=a,b,c,p,q,r,s$, be the corresponding node-smoothing parameters. Then,

$$\begin{aligned}
&\overline{\mathfrak{M}}_{(1,3)}\cap\mathcal{V}=\{\zeta_a=\zeta_b=\zeta_c=0\},\quad \overline{\mathfrak{M}}_{(2,3)}\cap\mathcal{V}=\{\zeta_p=\zeta_q=\zeta_a=\zeta_c=0\},\\
&\overline{\mathfrak{M}}_{(3,3)}\cap\mathcal{V}=\{\zeta_r=\zeta_s=\zeta_a=\zeta_b=0\},\quad \overline{\mathfrak{M}}_{(4,2)}\cap\mathcal{V}=\{\zeta_p=\zeta_a=0\}\cup\{\zeta_q=\zeta_c=0\},\\
&\overline{\mathfrak{M}}_{(4,3)}\cap\mathcal{V}=\{\zeta_q=\zeta_a=\zeta_b=0\}\cup\{\zeta_p=\zeta_b=\zeta_c=0\};
\end{aligned}$$

any other $\overline{\mathfrak{M}}_{(i,k)}$ does not meet x, and hence is disjoint from $\mathcal{V}$ (because $\mathcal{V}$ is assumed small). By direct computation, we obtain

$$(\overline{\mathfrak{M}}_{(2,3)}\cap\overline{\mathfrak{M}}_{(3,3)}\cap\mathcal{V})\subset(\overline{\mathfrak{M}}_{(1,3)}\cap\mathcal{V}),\quad (\overline{\mathfrak{M}}_{(2,3)}\cap\mathcal{V})\subset(\overline{\mathfrak{M}}_{(4,2)}\cap\mathcal{V}).$$

Moreover:

- after blowing up $\mathcal{V}$ along $\overline{\mathfrak{M}}_{(1,3)}\cap\mathcal{V}$, the proper transforms of $\overline{\mathfrak{M}}_{(2,3)}\cap\mathcal{V}$ and $\overline{\mathfrak{M}}_{(2,3)}\cap\mathcal{V}$ become disjoint;
- after blowing up $\mathcal{V}$ along $\overline{\mathfrak{M}}_{(1,3)}\cap\mathcal{V}$, then along the proper transform of $\overline{\mathfrak{M}}_{(2,3)}\cap\mathcal{V}$, the proper transforms of the two irreducible components of $\overline{\mathfrak{M}}_{(4,2)}\cap\mathcal{V}$ become disjoint, and hence the proper transform of $\overline{\mathfrak{M}}_{(4,2)}\cap\mathcal{V}$ becomes smooth; and
- after blowing up $\mathcal{V}$ along $\overline{\mathfrak{M}}_{(1,3)}\cap\mathcal{V}$, the proper transforms of the two irreducible components of $\overline{\mathfrak{M}}_{(4,3)}\cap\mathcal{V}$ become disjoint, and hence the proper transform of $\overline{\mathfrak{M}}_{(4,3)}\cap\mathcal{V}$ becomes smooth.

3.2 The second round (r_2)

In this subsection, we describe the blowups of (r_2), whose blowup centers lie in the proper transforms of the exceptional divisors obtained in $(\mathsf{r}_1\mathsf{p}_1)$. Therefore, we start with analyzing these exceptional divisors. The key observation is that in any step $(\mathsf{r}_1\mathsf{p}_1\mathsf{s}_k)$ of $(\mathsf{r}_1\mathsf{p}_1)$, the blowup center $\overline{\mathfrak{M}}_{(1,k)}^{\mathsf{r}_1\mathsf{p}_1\mathsf{s}_{k-1}}$, which is the proper transform of $\overline{\mathfrak{M}}_{(1,k)}$ in $\widetilde{\mathfrak{M}}^{\mathsf{r}_1\mathsf{p}_1\mathsf{s}_{k-1}}$, has its normal bundle equal to a direct sum of k line bundles, with each corresponding to the smoothing of the corresponding node.

To proceed, we use the following terminology from [VZ08, Section 3.1]. For an arbitrary stack $\overline{\mathfrak{M}}$, denote by $TC\overline{\mathfrak{M}}$ its tangent cone, which may be related to the intrinsic normal cone of

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Behrend and Fantechi [BF97]. If X is a smooth stack, a morphism $\iota_X : X \longrightarrow \overline{\mathfrak{M}}$ is an **immersion** if its differential

$$\mathrm{d}\iota_X : \mathrm{TX} \longrightarrow \iota_X^* \mathrm{TC}\overline{\mathfrak{M}}$$

is injective at every point of X. Let

$$\mathcal{N}_{\iota_X} := \iota_X^* TC\overline{\mathfrak{M}}/\mathrm{Im}\mathrm{d}\iota_X$$

be the **normal cone** of ι_X in $\overline{\mathfrak{M}}$.

For $k \geq 1$, the normal cone of the immersion $\iota_{(1,k)} : \widehat{\mathfrak{M}}_{(1,k)} \longrightarrow \mathfrak{M}_2^{\mathrm{wt}}$ is the direct sum of k line bundles:

$$\mathcal{N}_{\iota_{(1,k)}} = \bigoplus_{i=1}^{k} (\mathrm{pr}_1^* \mathcal{L}_{k;i} \otimes \mathrm{pr}_{2;i}^* \mathfrak{L}) := \bigoplus_{i=1}^{k} \widehat{L}_{k;i} \longrightarrow \widehat{\mathfrak{M}}_{(1,k)}, \tag{3.8}$$

where $\mathcal{L}_{k;i}$ and $\mathfrak{L}$ are, respectively, the universal tangent line bundle at the marked point of $\overline{\mathcal{M}}_{2,k}$ labeled by i and the marked point of $\mathfrak{M}_{0,\{i\}}^{\mathrm{wt}}$, and $\mathrm{pr}_1 : \widehat{\mathfrak{M}}_{(1,k)} \to \overline{\mathcal{M}}_{2,k}$ and $\mathrm{pr}_{2;i} : \widehat{\mathfrak{M}}_{(1,k)} \to \mathfrak{M}_{0,\{i\}}^{\mathrm{wt}}$, $1 \leq i \leq k$, are the projections onto components.

The immersion $\bar{\iota}_{(i,k)}$ in (3.5) induced by $\iota_{(i,k)}$ is generally not an isomorphism to its image. For $j < k$, the immersion $\iota_{(1,k)}$ induces an immersion and the corresponding quotient:

$$\begin{aligned} \iota_{(1,k)}^{\mathsf{r}_1\mathsf{p}_1\mathsf{s}_j} &: \widehat{\mathfrak{M}}_{(1,k)}^{\mathsf{r}_1\mathsf{p}_1\mathsf{s}_j} \longrightarrow \overline{\mathfrak{M}}_{(1,k)}^{\mathsf{r}_1\mathsf{p}_1\mathsf{s}_j} \subset \widetilde{\mathfrak{M}}^{\mathsf{r}_1\mathsf{p}_1\mathsf{s}_j}, \\ \bar{\iota}_{(1,k)}^{\mathsf{r}_1\mathsf{p}_1\mathsf{s}_j} &: \widehat{\mathfrak{M}}_{(1,k)}^{\mathsf{r}_1\mathsf{p}_1\mathsf{s}_j}/S_{(1,k)} \longrightarrow \overline{\mathfrak{M}}_{(1,k)}^{\mathsf{r}_1\mathsf{p}_1\mathsf{s}_j} \subset \widetilde{\mathfrak{M}}^{\mathsf{r}_1\mathsf{p}_1\mathsf{s}_j}. \end{aligned} \tag{3.9}$$

The domain $\widehat{\mathfrak{M}}_{(1,k)}^{\mathsf{r}_1\mathsf{p}_1\mathsf{s}_j}$ is constructed by applying [VZ08, Lemma 3.3.(1)] repeatedly. It is analogous to the smooth variety introduced in [VZ08, Section 2.3]. To be precise, for an arbitrary nonempty finite set S, let $\mathcal{M}_{2,S;j} \subset \overline{\mathcal{M}}_{2,S}$ be comprised of marked curves that have exactly j special points (pivotal nodes and/or elements of S) on its core, and $\overline{\mathcal{M}}_{2,S;j}$ be its closure in $\overline{\mathcal{M}}_{2,S}$. We blow up $\overline{\mathcal{M}}_{2,S}$ successively along the proper transforms of

$$\overline{\mathcal{M}}_{2,S;1}, \ldots, \overline{\mathcal{M}}_{2,S;|S|-1}.$$

Let $\overline{\mathcal{M}}_{2,S}^{\mathsf{s}_j}$ be the blowup of $\overline{\mathcal{M}}_{2,S}$ after the jth step. For $S = [k]$ and $1 \leq j \leq k-1$, let

$$\widehat{\mathfrak{M}}_{(1,k)}^{\mathsf{r}_1\mathsf{p}_1\mathsf{s}_j} := \overline{\mathcal{M}}_{2,k}^{\mathsf{s}_j} \times \prod_{i=1}^{k} \mathfrak{M}_{0,\{i\}}^{\mathrm{wt}}, \tag{3.10}$$

which will turn out to be smooth by Lemma 4.10. For $j = 0$, we set $\iota_{(i,k)}^{\mathsf{r}_1\mathsf{p}_1\mathsf{s}_0} := \iota_{(i,k)}$.

The immersion $\bar{\iota}_{(1,k)}^{\mathsf{r}_1\mathsf{p}_1\mathsf{s}_{k-1}}$ in (3.9) is an embedding. This follows from [VZ08, Lemma 3.3.(2)] repeatedly; cf. [VZ08, Section 4.3 $(I13)$]. The normal cone of $\iota_{(1,k)}^{\mathsf{r}_1\mathsf{p}_1\mathsf{s}_{k-1}}$ is still a direct sum of k line bundles:

$$\mathcal{N}_{\iota_{(1,k)}^{\mathsf{r}_1\mathsf{p}_1\mathsf{s}_{k-1}}} = \bigoplus_{j=1}^{k} L_{k;j} \longrightarrow \widehat{\mathfrak{M}}_{(1,k)}^{\mathsf{r}_1\mathsf{p}_1\mathsf{s}_{k-1}}; \tag{3.11}$$

cf. [VZ08, Lemma 3.5]. Each line bundle $L_{k;\ell}$ above is the pullback of $\widehat{L}_{k;\ell}$ twisted by some exceptional divisors of the previous steps. The immersion $\iota_{(1,k)}^{\mathsf{r}_1\mathsf{p}_1\mathsf{s}_{k-1}}$ induces an immersion

$$\iota_{(1,k)}^{\varepsilon} : \mathbb{P}\mathcal{N}_{\iota_{(1,k)}^{\mathsf{r}_1\mathsf{p}_1\mathsf{s}_{k-1}}} \longrightarrow \mathcal{E}_{(1,k)} := \mathbb{P}\mathcal{N}_{\widetilde{\mathfrak{M}}^{\mathsf{r}_1\mathsf{p}_1\mathsf{s}_{k-1}}} \overline{\mathfrak{M}}_{(1,k)}^{\mathsf{r}_1\mathsf{p}_1\mathsf{s}_{k-1}} \subset \widetilde{\mathfrak{M}}^{\mathsf{r}_1\mathsf{p}_1\mathsf{s}_k}, \tag{3.12}$$

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which determines an isomorphism

$$\bar{\iota}^{\varepsilon}_{(1,k)} : \mathbb{P}\mathcal{N}_{\bar{\iota}_{(1,k)}^{\mathsf{r}_1\mathsf{p}_1\mathsf{s}_{k-1}}} \longrightarrow \mathcal{E}_{(1,k)}. \tag{3.13}$$

Notice that $\mathcal{E}_{(1,k)}$ is exactly the exceptional divisor obtained in $(\mathsf{r}_1\mathsf{p}_1\mathsf{s}_k)$.

We are ready to describe the blowup centers of (r_2), which lie in the proper transforms of $\mathcal{E}_{(1,k)}$. Let m be as in Remark 3.1. Consider the index set

$$\mathfrak{I}_1 = \{(k,j,\sigma) : (k,j) \in \mathbb{Z}\times\mathbb{Z},\ 1 \le j \le k,\ \sigma : \{2,\dots,j\} \longrightarrow \mathbb{Z}\cap[2,+\infty)\}.$$

Here, if $j=1$, then the domain of σ is empty. The subscript of $\mathfrak{I}_1$ indicates there is one special direction, given by $1\in[k]$, in $\mathcal{E}_{(1,k)}$.

Fix $\rho=(k,j,\sigma)\in\mathfrak{I}_1$. For every $\ell\in\mathbb{Z}_{>0}$, let

$$\mathfrak{M}^{\mathrm{wt}}_{0,S;\underline{\ell}} \subset \mathfrak{M}^{\mathrm{wt}}_{0,S} \tag{3.14}$$

be the closed subset whose general point consists of a smooth rational subcurve C_0 of weight 0 that contains all the marked points, as well as ℓ positively weighted unmarked smooth rational subcurves all attached to C_0. Then, we set

$$\widehat{\mathfrak{M}}_\rho = \overline{\mathcal{M}}_{2,k}\times\mathfrak{M}^{\mathrm{wt}}_{0,\{1\}}\times\prod_{i=2}^{j}\mathfrak{M}^{\mathrm{wt}}_{0,\{i\};\underline{\sigma(i)}}\times\prod_{i=j+1}^{k}\mathfrak{M}^{\mathrm{wt}}_{0,\{i\}} \quad (\subset\widehat{\mathfrak{M}}_{(1,k)}). \tag{3.15}$$

For $\rho=(k,j,\sigma)$ as above, with $\widehat{\mathfrak{M}}_{(1,k)}^{\mathsf{r}_1\mathsf{p}_1\mathsf{s}_{k-1}}$ as in (3.9), we denote by

$$\widehat{\mathfrak{M}}_\rho^{\mathsf{r}_1\mathsf{p}_1\mathsf{s}_{k-1}} \subset \widehat{\mathfrak{M}}_{(1,k)}^{\mathsf{r}_1\mathsf{p}_1\mathsf{s}_{k-1}}$$

the proper transform of $\widehat{\mathfrak{M}}_\rho$ in $\widehat{\mathfrak{M}}_{(1,k)}^{\mathsf{r}_1\mathsf{p}_1\mathsf{s}_{k-1}}$. Set

$$\widehat{X}_\rho := \mathbb{P}\Big(\bigoplus_{i=1}^{j}L_{k;i}\Big)\Big|_{\widehat{\mathfrak{M}}_\rho^{\mathsf{r}_1\mathsf{p}_1\mathsf{s}_{k-1}}} \subset \mathbb{P}\mathcal{N}_{\iota_{(1,k)}^{\mathsf{r}_1\mathsf{p}_1\mathsf{s}_{k-1}}},\quad X_\rho := \iota^{\varepsilon}_{(1,k)}(\widehat{X}_\rho)\subset\mathcal{E}_{(1,k)}. \tag{3.16}$$

If $j=1$, then $\widehat{X}_\rho$ is isomorphic to $\widehat{\mathfrak{M}}_\rho^{\mathsf{r}_1\mathsf{p}_1\mathsf{s}_{k-1}}$.

Intuitively, a general point of $\widehat{X}_\rho$ satisfies the following criteria:

- for $j+1\le i\le k$, the ith component of the fibers of $\mathbb{P}\mathcal{N}_{\iota_{(1,k)}^{\mathsf{r}_1\mathsf{p}_1\mathsf{s}_{k-1}}}$ is identically 0; and
- for $2\le i\le j$, the ith tail of the underlying curve C consists of a smooth rational subcurve C_i of weight 0, attaching to the core of C, as well as $\sigma(i)$ positively weighted smooth rational subcurves attaching to C_i.

Notice that $\sum_{i=2}^{j}\sigma(i)\ge(j-1)$ for every $(k,j,\sigma)\in\mathfrak{I}_1$. So, for every $1\le k\le k'$, we set

$$\mathfrak{I}_1(k,k') = \{(k,j,\sigma)\in\mathfrak{I}_1 : k-(j-1)+\sum_{i=2}^{j}\sigma(i)=k'\},\quad X_{k,k'} = \bigcup_{\rho\in\mathfrak{I}_1(k,k')}X_\rho. \tag{3.17}$$

(In particular, each $\mathfrak{I}_1(k,k)$ consists of a unique element $(k,1,\sigma:\emptyset\to\mathbb{Z}\cap[2,+\infty))$.) The proper transforms of the above $X_{k,k'}$ will be the blowup centers of (r_2). Some of these loci are illustrated in Figure 2, where, for conciseness, $\mathcal{E}_{(1,k)}$ is written as $\mathcal{E}_k$, and the proper transform $\widehat{\mathfrak{M}}_\rho^{\mathsf{r}_1\mathsf{p}_1\mathsf{s}_{k-1}}$ is abbreviated to $\widehat{\mathfrak{M}}_\rho^{\bullet}$.

Remark 3.5. The reason for introducing the index sets $\mathfrak{I}_1$ and $\mathfrak{I}_1(k,k')$, and particularly the map σ, to describe blowup centers of (r_2), comes solely from the expressions of the structural

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homomorphisms φ. In Example 3.11 of Section 3.7, for instance, the second locus in (3.28) corresponds to X_ρ with $\rho = (2, 2, \sigma(2) = 2)$.

Next, we describe the order of the sequential blowups within (r_2). Let the set

$$\{(k, k') \in \mathbb{Z} \times \mathbb{Z} : 1 \le k \le k'\}$$

be endowed with the following order:

$$(k, k') < (\ell, \ell') \iff \text{either} \quad k > \ell, \quad \text{or} \quad k = \ell \quad \text{and} \quad k' < \ell'. \tag{3.18}$$

In other words, (3.18) is inherited from the lexicographic order on $\mathbb{Z}^{\mathrm{op}} \times \mathbb{Z}$, where $\mathbb{Z}^{\mathrm{op}}$ is $\mathbb{Z}$ with the opposite order.

We blow up $\widetilde{\mathfrak{M}}^{\mathsf{r}_1}$ along the proper transforms of $X_{k,k'}$, $1 \le k \le k'$, with respect to the order (3.18). More precisely, for every $\rho \in \mathfrak{I}_1(k, k')$, notice that the total weight of any point of $\widetilde{\mathrm{M}}_\rho$ is at least k', and hence on the connected component of $\widetilde{\mathfrak{M}}^{\mathsf{r}_1}$ whose underlying weighted curves are of weight m, the relevant blowup centers of (r_2) are indexed by $1 \le k \le k' \le m$. For every (k, k') with $1 \le k \le k' \le m$, we denote by $(k, k') - 1 = (h, h')$ its immediate predecessor with respect to (3.18) that still satisfies $1 \le h \le h' \le m$. The stack obtained after Step $(k, k') - 1$ is written as $\widetilde{\mathfrak{M}}^{\mathsf{r}_2\mathsf{s}_{(k,k')-1}}$. Then, inductively, we blow up $\widetilde{\mathfrak{M}}^{\mathsf{r}_2\mathsf{s}_{(k,k')-1}}$ along the proper transform of $X_{k,k'}$ and obtain $\widetilde{\mathfrak{M}}^{\mathsf{r}_2\mathsf{s}_{(k,k')}}$. In this way, we obtain

$$\widetilde{\mathfrak{M}}^{\mathsf{r}_2} \longrightarrow \widetilde{\mathfrak{M}}^{\mathsf{r}_1}.$$

These are the sequential blowups in (r_2). This procedure is illustrated in Figure 2. The proper transforms of $X_{1,k'}$ are not included in Figure 2 because $X_{1,1}$ is isomorphic to the proper transform of $\overline{\mathfrak{M}}_{(1,1)}$, which is a Cartier divisor, while the remaining $X_{1,k'}$ are all empty.

Since on each connected component of $\widetilde{\mathfrak{M}}^{\mathsf{r}_1}$, the blowups in (r_2) are indexed by a finite set $\{(k, k') : 1 \le k \le k' \le m\}$, we see that (r_2) terminates after finitely many steps.

Remark 3.6. The reason for choosing the order (3.18) is twofold: to ensure the blowup centers of (r_2) are all smooth, and to make φ closer to the desired diagonalized form. For the former, notice that for each fixed k, the blowup centers, i.e. the proper transforms of $X_{k,k'}$, $k' \ge k$, intersect in a way similar to those of $(\mathsf{r}_1\mathsf{p}_1)$, so one naturally wonders whether the usual lexicographic order (i.e. that inherited from $\mathbb{Z} \times \mathbb{Z}$) is a good candidate for both goals. Example 3.13 of Section 3.7, however, will show that the usual lexicographic order is unsuitable for diagonalizing φ. This forces us to consider the seemingly unnatural order (3.18).

The following example shows how the $X_{k,k'}$, with various k', intersect, and how (r_2) affects them locally; see also Examples 3.11 and 3.12 in Section 3.7 for an example of $X_{2,2} \cap X_{2,3}$. In addition, Example 3.12 includes an example of $X_{2,3} \cap X_{3,3}$.

Example 3.7. Let $x = (C, \mathbf{w})$ be as in Figure 6, where the weight of each shaded component is no less than 2, while the unshaded components are of weight 0.

Let $\mathcal{V} \longrightarrow \mathfrak{M}_2^{\mathrm{wt}}$ be a small affine chart containing x, and $\zeta_\bullet$, $\bullet = a, b, c, d, e, f, g, p, q$, be, respectively, the node-smoothing parameters. In $(\mathsf{r}_1\mathsf{p}_1)$, since $\overline{\mathfrak{M}}_{(1,k)} \cap \mathcal{V} = \emptyset$ for $k = 1, 2$, we see that $(\mathsf{r}_1\mathsf{p}_1\mathsf{s}_1)$ and $(\mathsf{r}_1\mathsf{p}_1\mathsf{s}_2)$ do not affect $\mathcal{V}$, and the locus

$$\overline{\mathfrak{M}}_{(1,3)} \cap \mathcal{V} = \{\zeta_a = \zeta_b = \zeta_c = 0\}$$

is blown up in $(\mathsf{r}_1\mathsf{p}_1\mathsf{s}_3)$.

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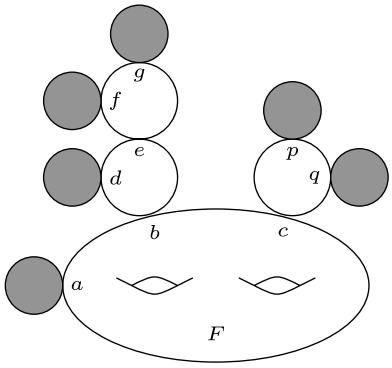


FIGURE 6. The underlying curve C in Example 3.7.

After $(\mathsf{r}_1\mathsf{p}_1\mathsf{s}_3)$, consider a lift $\widetilde{x}$ of x lying in a small chart $\widetilde{\mathcal{V}}_{\widetilde{x}}$ on which

$$\zeta_a = \varepsilon_a, \quad \zeta_b = u_b\varepsilon_a, \quad \zeta_c = \varepsilon_a\check{\zeta}_c,$$

where u_b is invertible, while $\check{\zeta}_c$, the proper transform of ζ_c, is not invertible. Then, one sees that the proper transforms of $\overline{\mathfrak{M}}_{(1,k)} \cap \mathcal{V}$, $k \geq 4$, as well as $\overline{\mathfrak{M}}_{(i,k)} \cap \mathcal{V}$, $i \geq 2$, $k \geq 1$, are disjoint from $\widetilde{\mathcal{V}}_{\widetilde{x}}$, and hence that from $(\mathsf{r}_1\mathsf{p}_1\mathsf{s}_4)$ until the end of (r_1), the chart $\widetilde{\mathcal{V}}_{\widetilde{x}}$ is not blown up.

To see how $\widetilde{\mathcal{V}}_{\widetilde{x}}$ changes in (r_2), observe that

$$\begin{aligned}
&\widetilde{x} \in X_{\rho_1} \cap X_{\rho_2} \cap X_{\rho_3} \cap X_{\rho_4}, \quad \text{where} \\
&\rho_1 = (3, 2, \sigma(2) = 2) \in \mathfrak{I}_1(3, 4), && \rho_2 = (3, 2, \sigma(2) = 3) \in \mathfrak{I}_1(3, 5), \\
&\rho_3 = (3, 3, \sigma(2) = 2, \sigma(3) = 2) \in \mathfrak{I}_1(3, 5), && \rho_4 = (3, 3, \sigma(2) = 3, \sigma(3) = 2) \in \mathfrak{I}_1(3, 6).
\end{aligned}$$

So, $\widetilde{x}$ lies in the intersection of $X_{3,k'}$, $k' = 4, 5, 6$. The remaining $X_{k,k'}$ do not contain $\widetilde{x}$, hence are disjoint from $\widetilde{\mathcal{V}}_{\widetilde{x}}$ (because $\widetilde{\mathcal{V}}_{\widetilde{x}}$ is assumed to be small).

Locally on $\widetilde{\mathcal{V}}_{\widetilde{x}}$, we have

$$\begin{aligned}
&\widetilde{x} \in \{\varepsilon_a = \check{\zeta}_c = \zeta_d = \zeta_e = \zeta_f = \zeta_g = \zeta_p = \zeta_q = 0\}, \\
&X_{\rho_1} \cap \widetilde{\mathcal{V}}_{\widetilde{x}} = \{\varepsilon_a = \check{\zeta}_c = \zeta_d = \zeta_e = 0\}, && X_{\rho_2} \cap \widetilde{\mathcal{V}}_{\widetilde{x}} = \{\varepsilon_a = \check{\zeta}_c = \zeta_d = \zeta_f = \zeta_g = 0\}, \\
&X_{\rho_3} \cap \widetilde{\mathcal{V}}_{\widetilde{x}} = \{\varepsilon_a = \zeta_d = \zeta_e = \zeta_p = \zeta_q = 0\}, && X_{\rho_4} \cap \widetilde{\mathcal{V}}_{\widetilde{x}} = \{\varepsilon_a = \zeta_d = \zeta_f = \zeta_g = \zeta_p = \zeta_q = 0\}.
\end{aligned}$$

The blowups of (r_2) blow up $\widetilde{\mathcal{V}}_{\widetilde{x}}$ first along $X_{\rho_1} \cap \widetilde{\mathcal{V}}_{\widetilde{x}}$, then along the proper transform of $(X_{\rho_2} \cup X_{\rho_3}) \cap \widetilde{\mathcal{V}}_{\widetilde{x}}$, and finally along the proper transform of $X_{\rho_4} \cap \widetilde{\mathcal{V}}_{\widetilde{x}}$. By direct computation, we have

$$(X_{\rho_2} \cap X_{\rho_3} \cap \widetilde{\mathcal{V}}_{\widetilde{x}}) \subset (X_{\rho_1} \cap \widetilde{\mathcal{V}}_{\widetilde{x}}),$$

and after blowing up $\widetilde{\mathcal{V}}_{\widetilde{x}}$ along $X_{\rho_1} \cap \widetilde{\mathcal{V}}_{\widetilde{x}}$, the proper transforms of $X_{\rho_1} \cap \widetilde{\mathcal{V}}_{\widetilde{x}}$ and $X_{\rho_1} \cap \widetilde{\mathcal{V}}_{\widetilde{x}}$ become disjoint from each other and are both smooth. One can also check directly that the proper transform of $X_{\rho_4} \cap \widetilde{\mathcal{V}}_{\widetilde{x}}$ is smooth after any of the above blowups.

After (r_2) is complete, the only issue with achieving the diagonalizability of the structural homomorphism lies on the second row, which involves the proper transforms of node-smoothing and/or exceptional parameters and the local parameters for the proper transforms of Weierstrass or conjugate loci: the latter are the proper transforms of κ_{ij} as characterized in Proposition 2.22 (see also Propositions 2.17, 2.19 and 2.28).

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There are four phases of sequential blowups in (r_3), which we will describe separately in the next four subsections.

3.3 The first phase of the third round $(\mathsf{r}_3\mathsf{p}_1)$

In $(\mathsf{r}_3\mathsf{p}_1)$, we consider the index set

$$\mathfrak{I}_2 = \{(k,j,\sigma):\, (k,j)\in\mathbb{Z}\times\mathbb{Z},\ 2\le j\le k,\ \sigma:\{3,\ldots,j\}\longrightarrow\mathbb{Z}\cap[2,+\infty)\}.$$

Here, if $j=2$, then the domain of σ is empty. The subscript of $\mathfrak{I}_2$ indicates there are two special directions, given by $1,2\in[k]$, in $\mathcal{E}_{(1,k)}$.

For every $\rho=(k,j,\sigma)\in\mathfrak{I}_2$, recall that $\overline{\mathcal{K}}_{k;1,2}$ denotes the locus in $\overline{\mathcal{M}}_{2,k}$ whose first two marked points are conjugate. By Proposition 2.17, it is a Cartier divisor of $\overline{\mathcal{M}}_{2,k}$. Also, for every $\ell\in\mathbb{Z}_{>0}$ and every finite set S, recall that $\mathfrak{M}^{\mathrm{wt}}_{0,S;\underline{\ell}}$ is given in (3.14). We set

$$\widehat{\mathfrak{K}}_\rho := \overline{\mathcal{K}}_{k;1,2}\times\mathfrak{M}^{\mathrm{wt}}_{0,\{1\}}\times\mathfrak{M}^{\mathrm{wt}}_{0,\{2\}}\times\prod_{i=3}^{j}\mathfrak{M}^{\mathrm{wt}}_{0,\{i\};\underline{\sigma(i)}}\times\prod_{i=j+1}^{k}\mathfrak{M}^{\mathrm{wt}}_{0,\{i\}}\quad(\subset\widehat{\mathfrak{M}}_{(1,k)}).$$

With $\widehat{\mathfrak{M}}_{(1,k)}^{\mathsf{r}_1\mathsf{p}_1\mathsf{s}_{k-1}}$ as in (3.9), we denote by

$$\widehat{\mathfrak{K}}_\rho^{\mathsf{r}_1\mathsf{p}_1\mathsf{s}_{k-1}}\subset\widehat{\mathfrak{M}}_{(1,k)}^{\mathsf{r}_1\mathsf{p}_1\mathsf{s}_{k-1}}$$

the proper transform of $\widehat{\mathfrak{K}}_\rho$ in $\widehat{\mathfrak{M}}_{(1,k)}^{\mathsf{r}_1\mathsf{p}_1\mathsf{s}_{k-1}}$. Let

$$\widehat{K}_\rho := \mathbb{P}\Big(\bigoplus_{i=1}^{j}L_{k;i}\Big)\Big|_{\widehat{\mathfrak{K}}_\rho^{\mathsf{r}_1\mathsf{p}_1\mathsf{s}_{k-1}}}\subset\mathbb{P}\mathcal{N}_{\iota_{(1,k)}^{\mathsf{r}_1\mathsf{p}_1\mathsf{s}_{k-1}}},\quad K_\rho := \iota_{(1,k)}^{\varepsilon}(\widehat{K}_\rho)\subset\mathcal{E}_{(1,k)}.$$

Intuitively, a general point of $\widehat{K}_\rho$ satisfies the following criteria:

- for $j+1\le i\le k$, the ith component of the fibers of $\mathbb{P}\mathcal{N}_{\iota_{(1,k)}^{\mathsf{r}_1\mathsf{p}_1\mathsf{s}_{k-1}}}$ is identically 0;
- for $3\le i\le j$, the ith tail of the underlying curve C consists of a smooth rational subcurve C_i of weight 0, attaching to the core of C, as well as $\sigma(i)$ positively weighted smooth rational subcurves attaching to C_i; and
- the first two tails of the underlying curve C are attached to the core of C at conjugate points.

Notice that $\sum_{i=3}^{j}\sigma(i)>(j-2)$ for every $(k,j,\sigma)\in\mathfrak{I}_2$, so for every $2\le k\le k'$, we set

$$\mathfrak{I}_2(k,k') = \{(k,j,\sigma)\in\mathfrak{I}_2:\, k-(j-2)+\sum_{i=3}^{j}\sigma(i)=k'\},\quad K_{k,k'} = \bigcup_{\rho\in\mathfrak{I}_2(k,k')}K_\rho.$$

In $(\mathsf{r}_3\mathsf{p}_1)$, we blow up $\widetilde{\mathfrak{M}}^{\mathsf{r}_2}$ successively along the proper transforms of $K_{k,k'}$, $2\le k\le k'$, with respect to the order inherited from the usual lexicographic order on $\mathbb{Z}\times\mathbb{Z}$, and obtain

$$\widetilde{\mathfrak{M}}^{\mathsf{r}_3\mathsf{p}_1}\longrightarrow\widetilde{\mathfrak{M}}^{\mathsf{r}_2}.$$

For every $\rho\in\mathfrak{I}_2(k,k')$, the total weight of every point of $\widetilde{\mathrm{K}}_\rho$ is at least k', hence on the connected component of $\widetilde{\mathfrak{M}}^{\mathsf{r}_2}$ whose underlying weighted curves are of weight m, the relevant blowup centers of $(\mathsf{r}_3\mathsf{p}_1)$ are indexed by $2\le k\le k'\le m$, and hence $(\mathsf{r}_3\mathsf{p}_1)$ terminates after finitely many steps on each connected component of $\widetilde{\mathfrak{M}}^{\mathsf{r}_2}$.

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Remark 3.8. In $(\mathsf{r}_3\mathsf{p}_1)$, it is also possible to use the order (3.18) instead of the usual lexicographic order and obtain a different resolution of $\overline{M}_2(\mathbb{P}^n, d)$. The usual lexicographic orders in $(\mathsf{r}_3\mathsf{p}_1)$, and later in $(\mathsf{r}_3\mathsf{p}_2)$ and $(\mathsf{r}_3\mathsf{p}_4)$, are just practical choices.

3.4 The second phase of the third round $(\mathsf{r}_3\mathsf{p}_2)$

The blowup centers of $(\mathsf{r}_3\mathsf{p}_2)$ are within the proper transforms of the exceptional divisors $\mathcal{E}_{k,k'}$, $1 \le k \le k'$, obtained in (r_2), so we begin by analyzing these $\mathcal{E}_{k,k'}$.

First, given $\rho = (k, j, \sigma) \in \mathfrak{I}_1$, note that for every $2 \le i \le j$, there is a natural immersion

$$\overline{\mathcal{M}}_{0,\sigma(i)+1} \times \prod_{h=1}^{\sigma(i)} \mathfrak{M}^{\mathrm{wt}}_{0,\{(i,h)\}} \longrightarrow \mathfrak{M}^{\mathrm{wt}}_{0,\{i\};\underline{\sigma(i)}}$$

(where each $\{(i,h)\}$ is the set containing the pair (i,h) as its unique element), given by identifying the hth marked point of $\overline{\mathcal{M}}_{0,\sigma(i)+1}$ with the unique marked point of $\mathfrak{M}^{\mathrm{wt}}_{0,\{(i,h)\}}$ for all $1 \le h \le \sigma(i)$, and then labeling the last marked point of $\overline{\mathcal{M}}_{0,\sigma(i)+1}$ by the number i. Such immersions are surjective. They together give rise to an immersion

$$\widetilde{\mathrm{M}}_\rho := \overline{\mathcal{M}}_{2,k} \times \mathfrak{M}^{\mathrm{wt}}_{0,\{1\}} \times \prod_{i=2}^{j} \left(\overline{\mathcal{M}}_{0,\sigma(i)+1} \times \prod_{h=1}^{\sigma(i)} \mathfrak{M}^{\mathrm{wt}}_{0,\{(i,h)\}} \right) \times \prod_{i=j+1}^{k} \mathfrak{M}^{\mathrm{wt}}_{0,\{i\}} \longrightarrow \widehat{\mathfrak{M}}_\rho, \tag{3.19}$$

where $\widehat{\mathfrak{M}}_\rho$ is as in (3.15). Mimicking (3.10), we replace $\overline{\mathcal{M}}_{(2,k)}$ with $\overline{\mathcal{M}}^{\mathsf{s}_{k-1}}_{(2,k)}$ in the above construction of $\widetilde{\mathrm{M}}_\rho$ to obtain $\widetilde{\mathrm{M}}^{\mathsf{r}_1\mathsf{p}_1\mathsf{s}_{k-1}}_\rho$, along with the immersion

$$\widetilde{\mathrm{M}}^{\mathsf{r}_1\mathsf{p}_1\mathsf{s}_{k-1}}_\rho \longrightarrow \widehat{\mathfrak{M}}^{\mathsf{r}_1\mathsf{p}_1\mathsf{s}_{k-1}}_\rho. \tag{3.20}$$

With $\widehat{X}_\rho$ and X_ρ as in (3.16), this gives rise to

$$\widetilde{\mathrm{X}}_\rho \xrightarrow{\nu_\rho} \widehat{X}_\rho \quad \left(\xrightarrow{\iota^{\varepsilon}_{(1,k)}} X_\rho \right), \tag{3.21}$$

where $\widetilde{\mathrm{X}}_\rho$ denotes the pullback of $\widehat{X}_\rho$ via (3.20), and ν_ρ is the natural morphism.

With notation as above, let

$$\begin{aligned} \iota_\rho &:= \iota^{\varepsilon}_{(1,k)} \circ \nu_\rho : \widetilde{\mathrm{X}}_\rho \longrightarrow X_\rho \subset \widetilde{\mathfrak{M}}^{\mathsf{r}_1\mathsf{p}_1\mathsf{s}_k}, \\ \bar{\iota}_\rho &:= \bar{\iota}^{\varepsilon}_{(1,k)} \circ \nu_\rho : \widetilde{\mathrm{X}}_\rho / G_\rho \longrightarrow X_\rho \subset \widetilde{\mathfrak{M}}^{\mathsf{r}_1\mathsf{p}_1\mathsf{s}_k}, \end{aligned} \tag{3.22}$$

where $\iota^{\varepsilon}_{(1,k)}$ and $\bar{\iota}^{\varepsilon}_{(1,k)}$ are as in (3.12) and (3.13), respectively, and G_ρ is the symmetry group of the dual graph of a general point of $\widetilde{\mathrm{M}}_\rho$. With the line bundles $L_{k;i}$ as in (3.11), for every $j \le k$, let

$$\gamma_{k,j} \longrightarrow \mathbb{P}\left(\bigoplus_{i=1}^{j} L_{k;i} \right)$$

be the tautological line bundle. By (3.11), (3.16), and a local computation (cf. [VZ08, Lemma 3.5]), the normal cone of ι_ρ can be written explicitly as a sum of line bundles:

$$\mathcal{N}_{\iota_\rho} \simeq \nu_\rho^* \gamma_{k,j} \oplus \bigoplus_{i=2}^{j} \bigoplus_{h=1}^{\sigma(i)} \widehat{L}_{\rho;(i,h)} \oplus \bigoplus_{i=j+1}^{k} \nu_\rho^*(\gamma_{k,j}^\vee \otimes L_{k;i}), \tag{3.23}$$

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where each $\widehat{L}_{\rho;(i,h)}$ is the line bundle corresponding to the smoothing of the node labeled by (i,h); these line bundles are analogous to those in (3.8).

For every $k \le k' \le m$, recall that $(k,k')-1$ denotes the immediate predecessor of (k,k') with respect to the order (3.18). For every $\rho \in \mathfrak{I}_1(k,k')$ (see (3.17) for notation), we observe that, after $(\mathsf{r}_2\mathsf{s}_{(k,k')-1})$, the immersion ι_ρ as in (3.22) induces the immersion

$$\iota_\rho^{\mathsf{r}_2\mathsf{s}_{(k,k')-1}} : \widetilde{\mathrm{X}}_\rho^{\mathsf{r}_2\mathsf{s}_{(k,k')-1}} \longrightarrow X_\rho^{\mathsf{r}_2\mathsf{s}_{(k,k')-1}} \subset \mathcal{E}_{(1,k)}^{\mathsf{r}_2\mathsf{s}_{(k,k')-1}} \subset \widetilde{\mathfrak{M}}^{\mathsf{r}_2\mathsf{s}_{(k,k')-1}},$$

where $\widetilde{\mathrm{X}}_\rho^{\mathsf{r}_2\mathsf{s}_{(k,k')-1}}$ is obtained by blowing up $\widetilde{\mathrm{X}}_\rho$ successively along some smooth substacks, analogously to the construction of $\widehat{\mathfrak{M}}_{(1,k)}^{\mathsf{r}_1\mathsf{p}_1\mathsf{s}_j}$ in (3.10). The key fact is that the normal cone of $\iota_\rho^{\mathsf{r}_2\mathsf{s}_{(k,k')-1}}$ is still a sum of line bundles:

$$\mathcal{N}_{\iota_\rho^{\mathsf{r}_2\mathsf{s}_{(k,k')-1}}} = L_{\rho;1} \oplus \bigoplus_{i=2}^{j} \bigoplus_{h=1}^{\sigma(i)} L_{\rho;(i,h)} \oplus \bigoplus_{i=j+1}^{k} L_{\rho;i}, \tag{3.24}$$

where each line bundle is the pullback of the corresponding line bundle in (3.23) twisted by some exceptional divisors of the previous steps. Furthermore, the immersion $\iota_\rho^{\mathsf{r}_2\mathsf{s}_{(k,k')-1}}$ determines

$$\bar{\iota}_\rho^{\mathsf{r}_2\mathsf{s}_{(k,k')-1}} : \widetilde{\mathrm{X}}_\rho^{\mathsf{r}_2\mathsf{s}_{(k,k')-1}}/G_\rho \longrightarrow X_\rho^{\mathsf{r}_2\mathsf{s}_{(k,k')-1}} \subset \widetilde{\mathfrak{M}}^{\mathsf{r}_2\mathsf{s}_{(k,k')-1}},$$

which is isomorphic to its image. Thus, the immersion

$$\iota_\rho^{\mathcal{E}} : \mathbb{P}\mathcal{N}_{\iota_\rho^{\mathsf{r}_2\mathsf{s}_{(k,k')-1}}} \longrightarrow \mathcal{E}_{k,k'}$$

determines an isomorphism

$$\bar{\iota}_\rho^{\mathcal{E}} : \mathbb{P}\mathcal{N}_{\bar{\iota}_\rho^{\mathsf{r}_2\mathsf{s}_{(k,k')-1}}} \longrightarrow \mathcal{E}_{k,k'}.$$

In sum, we obtain the desired form of $\mathcal{E}_{k,k'}$.

Next, we describe the blowup center of $(\mathsf{r}_3\mathsf{p}_2)$ within the $\mathcal{E}_{k,k'}$. Recall that $\overline{\mathcal{W}}_{k;1}$ denotes the locus in $\overline{\mathcal{M}}_{2,k}$ whose first marked point is Weierstrass, which by Proposition 2.19 is a Cartier divisor of $\overline{\mathcal{M}}_{2,k}$. Notice that every $\rho = (k,j,\sigma) \in \mathfrak{I}_1(k,k')$ determines a set

$$\begin{aligned}\aleph(\rho) := \{(j',\alpha,\beta) : j' \in \{j,\dots,k\},\ \alpha : \{2,\dots,j\} \longrightarrow \mathbb{Z}_{\ge 0} \text{ s.t. } \alpha \le \sigma,\\ \beta : \{(i,h) : 2 \le i \le j,\ 1 \le h \le \alpha(i)\} \sqcup \{j+1,\dots,j'\} \longrightarrow [2,+\infty) \cap \mathbb{Z}\}.\end{aligned}$$

For every $\vartheta = (j',\alpha,\beta) \in \aleph(\rho)$ and $2 \le i \le j$, we set

$$\begin{aligned}\widetilde{\mathrm{W}}_{\rho,\vartheta} = \overline{\mathcal{W}}_{k;1} \times \mathfrak{M}_{0,\{1\}}^{\mathrm{wt}} \times \prod_{i=2}^{j} \left(\overline{\mathcal{M}}_{0,\sigma(i)+1} \times \prod_{h=1}^{\alpha(i)} \mathfrak{M}_{0,\{(i,h)\};\underline{\beta(i,h)}}^{\mathrm{wt}} \times \prod_{h=\alpha(i)+1}^{\sigma(i)} \mathfrak{M}_{0,\{(i,h)\}}^{\mathrm{wt}} \right)\\ \times \prod_{i=j+1}^{j'} \mathfrak{M}_{0,\{i\};\underline{\beta(i)}}^{\mathrm{wt}} \times \prod_{i=j'+1}^{k} \mathfrak{M}_{0,\{i\}}^{\mathrm{wt}} \subset \widetilde{\mathrm{M}}_\rho,\end{aligned}$$

where $\widetilde{\mathrm{M}}_\rho$ is as in (3.19). The components of a general point of $\widetilde{\mathrm{W}}_{\rho,\vartheta}$ when $\rho = (3,2,\sigma(2)=3)$ and $\vartheta = (3,\alpha(2)=1,\beta(2,1)=3,\beta(3)=2)$ is illustrated in Figure 7, where the marked points that are identified under (3.4) (respectively, (3.19)) are indicated by the dashed (respectively, dotted) arrows. The proper transform of $\widetilde{\mathrm{W}}_{\rho,\vartheta}$ after $(\mathsf{r}_1\mathsf{p}_1\mathsf{s}_{k-1})$ is written as $\widetilde{\mathrm{W}}_{\rho,\vartheta}^{\mathsf{r}_1\mathsf{p}_1\mathsf{s}_{k-1}}$.

For the above ρ and ϑ, with $\widetilde{\mathrm{X}}_\rho$ as in (3.21), let

$$\widetilde{\mathrm{X}}_{\mathrm{w};\rho,\vartheta} := \widetilde{\mathrm{X}}_\rho|_{\widetilde{\mathrm{W}}_{\rho,\vartheta}^{\mathsf{r}_1\mathsf{p}_1\mathsf{s}_{k-1}}};$$

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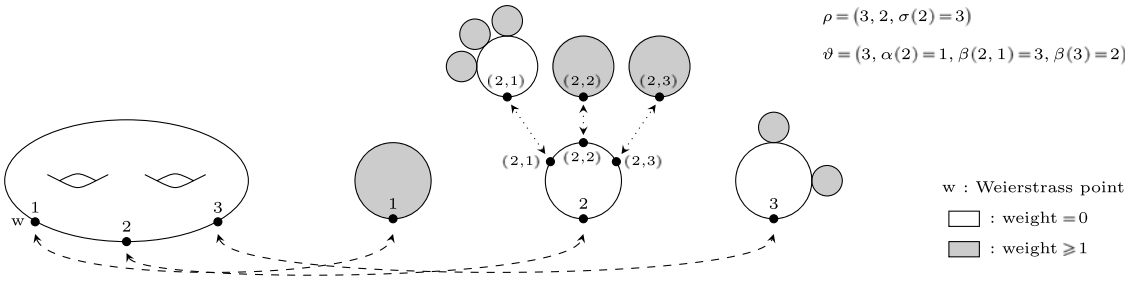


FIGURE 7. The components of a general point of $\widehat{\mathrm{W}}_{\rho,\vartheta}$.

its proper transform in $\widetilde{\mathrm{X}}_{\rho}^{\mathsf{r}_2\mathsf{s}_{(k,k')-1}}$ is denoted by $\widetilde{\mathrm{X}}_{\mathrm{w};\rho,\vartheta}^{\mathsf{r}_2\mathsf{s}_{(k,k')-1}}$. With the line bundles $L_{\rho;i}$ and $L_{\rho;(i,h)}$ as in (3.24), we set

$$\begin{aligned}\widetilde{\mathrm{W}}_{\rho,\vartheta}&:=\mathbb{P}\left(L_{\rho;1}\oplus\bigoplus_{i=2}^{j}\bigoplus_{h=1}^{\alpha(i)}L_{\rho;(i,h)}\oplus\bigoplus_{i=j+1}^{j'}L_{\rho;i}\right)\Big|_{\widetilde{\mathrm{X}}_{\mathrm{w};\rho,\vartheta}^{\mathsf{r}_2\mathsf{s}_{(k,k')-1}}}\subset\mathbb{P}\mathcal{N}_{\iota_\rho^{\mathsf{r}_2\mathsf{s}_{(k,k')-1}}},\\ W_{\rho,\vartheta}&:=\iota_\rho^{\varepsilon}(\widetilde{\mathrm{W}}_{\rho,\vartheta})=\overline{\iota}_\rho^{\varepsilon}(\widetilde{\mathrm{W}}_{\rho,\vartheta}/G_\rho)\subset\mathcal{E}_{k,k'}.\end{aligned}\tag{3.25}$$

Notice that for every $\rho=(k,j,\sigma)\in\mathfrak{I}_1(k,k')$ and $\vartheta=(j',\alpha,\beta)\in\aleph(\rho)$, we have

$$\left(\sum_{i=2}^{j}\sum_{h=1}^{\alpha(i)}\beta(i,h)\right)+\left(\sum_{i=j+1}^{j'}\beta(i)\right)\geq\left(\sum_{i=2}^{j}\alpha(i)\right)+(j'-j).$$

Therefore, for every $k''\geq k'$ $(\geq k\geq 1)$, we set

$$\begin{aligned}\mathfrak{I}_{\mathrm{w}}(k,k',k'')=\Bigg\{&(\rho,\vartheta):\ \rho\in\mathfrak{I}_1(k,k'),\ \vartheta\in\aleph(\rho),\\ &k-(j'-1)+\sum_{i=2}^{j}\left(\sigma(i)-\alpha(i)+\sum_{h=1}^{\alpha(i)}\beta(i,h)\right)+\sum_{i=j+1}^{j'}\beta(i)=k''\Bigg\},\\ W_{k,k',k''}=&\bigcup_{(\rho,\vartheta)\in\mathfrak{I}_{\mathrm{w}}(k,k',k'')}W_{\rho,\vartheta}.\end{aligned}$$

The proper transforms of these $W_{k,k',k''}$ will be the blowup centers of $(\mathsf{r}_3\mathsf{p}_2)$.

Remark 3.9. The reason for introducing the index set $\mathfrak{I}_{\mathrm{w}}(k,k',k'')$ once again comes from the expressions of the structural homomorphisms φ. In Example 3.12 of Section 3.7, for instance, the second locus in (3.29) corresponds to $W_{\rho,\vartheta}$ with $(\rho=(2,1,\emptyset),\vartheta=(2,\emptyset,\beta(2)=2))\in\mathfrak{I}_{\mathrm{w}}(2,2,3)$ (where $\emptyset$ refers to the empty map).

In $(\mathsf{r}_3\mathsf{p}_2)$, we blow up $\widetilde{\mathfrak{M}}^{\mathsf{r}_3\mathsf{p}_1}$ successively along the proper transforms of $W_{k,k',k''}$, $1\leq k\leq k'\leq k''$, with respect to the order inherited from the usual lexicographic order on $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$, and obtain

$$\widetilde{\mathfrak{M}}^{\mathsf{r}_3\mathsf{p}_2}\longrightarrow\widetilde{\mathfrak{M}}^{\mathsf{r}_3\mathsf{p}_1}.$$

On the connected component of $\widetilde{\mathfrak{M}}^{\mathsf{r}_3\mathsf{p}_1}$ whose underlying weighted curves are of weight m, the relevant blowup centers of $(\mathsf{r}_3\mathsf{p}_2)$ are indexed by the finite set $1\leq k\leq k'\leq k''\leq m$, and hence the process of the sequential blowups $(\mathsf{r}_3\mathsf{p}_2)$ terminates after finitely many steps.

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3.5 The third phase of the third round ($\mathsf{r}_3\mathsf{p}_3$)

The blowup loci of $(\mathsf{r}_3\mathsf{p}_3)$ lie in the stack $\mathfrak{M}_2^{\mathrm{div}}$ over $\mathfrak{M}_2^{\mathrm{wt}}$ instead of the stack $\mathfrak{M}_2^{\mathrm{wt}}$ itself; see Section 2.1 for the definition. Let

$$\mathfrak{H} \subset \mathfrak{M}_2^{\mathrm{div}}$$

be the closed substack consisting of all $(C, D) \in \mathfrak{M}_2^{\mathrm{div}}$ such that $\deg D = 2$ and the two points of D are conjugate. In particular, $\mathfrak{H}$ is a Cartier divisor of $\mathfrak{M}_2^{\mathrm{div}}$ by Proposition 2.17.

For any $k \geq 0$, let $\mathcal{H}_k$ be the closed substack of $\mathfrak{M}_2^{\mathrm{div}}$ whose *general* points are pairs (C, D) such that, with F denoting the core of C, $(F, D \cap F) \in \mathfrak{H}$ and C has k tails attached to F. After $(\mathsf{r}_3\mathsf{p}_2)$, let

$$\widetilde{\mathfrak{M}}^{\mathrm{div},\mathsf{r}_3\mathsf{p}_2} := \mathfrak{M}_2^{\mathrm{div}} \times_{\mathfrak{M}_2^{\mathrm{wt}}} \widetilde{\mathfrak{M}}^{\mathsf{r}_3\mathsf{p}_2}, \quad \mathcal{H}_k^{\mathsf{r}_3\mathsf{p}_2} := \mathrm{PT}_{\widetilde{\mathfrak{M}}^{\mathrm{div},\mathsf{r}_3\mathsf{p}_2}}(\mathcal{H}_k), \quad k \geq 0, \tag{3.26}$$

i.e. $\mathcal{H}_k^{\mathsf{r}_3\mathsf{p}_2}$ is the proper transform of $\mathcal{H}_k$ in $\widetilde{\mathfrak{M}}^{\mathrm{div},\mathsf{r}_3\mathsf{p}_2}$ for every $k \geq 0$. In $(\mathsf{r}_3\mathsf{p}_3)$, we blow up the stack $\widetilde{\mathfrak{M}}^{\mathrm{div},\mathsf{r}_3\mathsf{p}_2}$ successively along the proper transforms of $\mathcal{H}_k^{\mathsf{r}_3\mathsf{p}_2}$ with $k \geq 0$, starting from $k = 0$, and obtain

$$\widetilde{\mathfrak{M}}^{\mathrm{div},\mathsf{r}_3\mathsf{p}_3} \longrightarrow \widetilde{\mathfrak{M}}^{\mathrm{div},\mathsf{r}_3\mathsf{p}_2}.$$

The stability requirement of $\mathfrak{M}_2^{\mathrm{div}}$ implies that on each connected component of $\widetilde{\mathfrak{M}}^{\mathrm{div},\mathsf{r}_3\mathsf{p}_2}$, the process of the sequential blowups $(\mathsf{r}_3\mathsf{p}_3)$ terminates after finitely many steps.

3.6 The fourth phase of the third round ($\mathsf{r}_3\mathsf{p}_4$)

The process of $(\mathsf{r}_3\mathsf{p}_4)$ is analogous to the round (r_2), but the blowup loci lie in the proper transforms of $\mathfrak{M}_2^{\mathrm{div}}$ instead of $\mathfrak{M}_2^{\mathrm{wt}}$.

Notice that the main difference between $\overline{\mathfrak{M}}_{(1)}$ and $\overline{\mathfrak{M}}_{(5)}^{\mathsf{r}_1\mathsf{p}_4}$ in (3.6) is the weight of the core curve. Thus, the exceptional divisors obtained in the phases $(\mathsf{r}_1\mathsf{p}_1)$ and $(\mathsf{r}_1\mathsf{p}_5)$ share similar properties. In particular, we define

$$X'_\rho \subset \mathcal{E}_{(5,k)}, \quad \rho = (k, j, \sigma) \in \mathfrak{I}_1,$$

to be parallel to X_ρ in (3.16). For each $\rho = (k, j, \sigma) \in \mathfrak{I}_1$ and every $x \in X'_\rho$, we denote by F_x the core of the underlying curve C_x, by q_x the node labeled by 1, and by $p_x = \langle q_x \rangle$ the pivotal node corresponding to q_x. Let

$$\mathcal{Q}_\rho \subset \widetilde{\mathfrak{M}}^{\mathrm{div},\mathsf{r}_1\mathsf{p}_5\mathsf{s}_k} := \mathfrak{M}_2^{\mathrm{div}} \times_{\mathfrak{M}_2^{\mathrm{wt}}} \widetilde{\mathfrak{M}}^{\mathsf{r}_1\mathsf{p}_5\mathsf{s}_k}, \quad \rho = (k, j, \sigma) \in \mathfrak{I}_1,$$

be the closed substack of $\widetilde{\mathfrak{M}}^{\mathrm{div},\mathsf{r}_1\mathsf{p}_5\mathsf{s}_k}$ whose *general* points are pairs (x, D) satisfying

$$x \in X'_\rho, \quad (F_x,\ (D \cap F_x) + p_x) \in \mathfrak{H},$$

where $\mathfrak{H}$ is as in Section 3.5.

In $(\mathsf{r}_3\mathsf{p}_4)$, we blow up $\widetilde{\mathfrak{M}}^{\mathrm{div},\mathsf{r}_3\mathsf{p}_3}$ successively along the proper transforms of

$$\mathcal{Q}_{k,k'} = \bigcup_{\rho \in \mathfrak{I}_1(k,k')} \mathcal{Q}_\rho \ \ (\subset \widetilde{\mathfrak{M}}^{\mathrm{div},\mathsf{r}_1\mathsf{p}_5\mathsf{s}_k}), \quad 1 \leq k \leq k', \tag{3.27}$$

with respect to the order inherited from the usual lexicographic order on $\mathbb{Z} \times \mathbb{Z}$, and obtain

$$\widetilde{\mathfrak{M}}_2^{\mathrm{div}} := \widetilde{\mathfrak{M}}^{\mathrm{div},\mathsf{r}_3\mathsf{p}_4} \longrightarrow \widetilde{\mathfrak{M}}^{\mathrm{div},\mathsf{r}_3\mathsf{p}_3}.$$

The stability of $\mathfrak{M}_2^{\mathrm{div}}$ once again implies that on each connected component of $\widetilde{\mathfrak{M}}^{\mathrm{div},\mathsf{r}_3\mathsf{p}_3}$, $(\mathsf{r}_3\mathsf{p}_4)$ terminates after finitely many steps.

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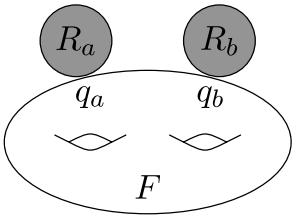


FIGURE 8. The curve C in Example 3.10.

3.7 Examples of modular blowups

We conclude Section 3 with a set of examples illustrating the sequential modular blowups.

Example 3.10. We begin with an example that appears simple enough yet still possibly involves all three rounds of blowups.

Let $z_0 = (C, D) \in \mathfrak{M}_2^{\mathrm{div}}$ be such that C consists of a smooth core F and two smooth rational subcurves R_a and R_b attached to F at q_a and q_b, respectively, and D satisfies

$$D \cap R_a = \{\delta_1, \dots, \delta_{\ell-1}\}, \quad D \cap R_b = \{\delta_\ell, \dots, \delta_m\}, \quad D \cap F = \emptyset,$$

where $1 < \ell \le m$. In particular, $D \cap R_a$ and $D \cap R_b$ are both nonempty. An illustration of C is provided in Figure 8, where we follow the convention of Figure 2 by shading the irreducible components of C that are positively weighted (i.e. whose intersections with D are nonempty). The same convention applies to the remaining figures in Section 3.7.

By Proposition 2.22 (SH_1), locally on a small affine chart $\mathcal{V} \longrightarrow \mathfrak{M}_2^{\mathrm{div}}$ containing the point z_0, we have

$$\varphi = \begin{bmatrix} c_{11}\zeta_a & \cdots & c_{1,\ell-1}\zeta_a & c_{1\ell}\zeta_b & \cdots & c_{1m}\zeta_b \\ c_{21}\zeta_a & \cdots & c_{2,\ell-1}\zeta_a & c_{2\ell}\zeta_b & \cdots & c_{2m}\zeta_b \end{bmatrix},$$

where ζ_a and ζ_b are the node-smoothing parameters (i.e. the modular parameters) corresponding to q_a and q_b, respectively, and c_{ij} are invertible functions on $\mathcal{V}$.

In the above matrix of φ, the commas in the subscriptions of $c_{1,\ell-1}$ and $c_{2,\ell-1}$ are only for separation. Hereafter, if necessary, we will use commas to separate the indices in subscriptions should no ambiguity occur.

In $(\mathsf{r}_1\mathsf{p}_1)$, $\mathcal{V}$ is blown up along the locus $(\zeta_a = \zeta_b = 0)$. After $(\mathsf{r}_1\mathsf{p}_1)$, up to switching the symbols a and b, the pullback $\widetilde{\varphi}^{\mathsf{r}_1\mathsf{p}_1}$ of φ locally takes the form

$$\varepsilon \begin{bmatrix} c_{11} & \cdots & c_{1,\ell-1} & c_{1\ell}\check{\zeta}_b & \cdots & c_{1m}\check{\zeta}_b \\ c_{21} & \cdots & c_{2,\ell-1} & c_{2\ell}\check{\zeta}_b & \cdots & c_{2m}\check{\zeta}_b \end{bmatrix},$$

where ε is the local parameter corresponding to the exceptional divisor, and $\check{\zeta}_b$ is the proper transform of ζ_b. In other words, up to a multiplicative unit, ζ_a (respectively, ζ_b) pulls back to ε (respectively, $\varepsilon\check{\zeta}_b$).

Recall that κ_{ij}, $1 \le i < j \le m$, are the local parameters on $\mathcal{V}$ whose zero loci consist of $z \in \mathcal{V}$ such that $\mathcal{D}_i(z)$ and $\mathcal{D}_j(z)$ are conjugate points on $\mathcal{C}_z$. By Proposition 2.22 (SH_2), one can apply elementary row and column operations to the above matrix of $\widetilde{\varphi}^{\mathsf{r}_1\mathsf{p}_1}$ and obtain

$$\varepsilon \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \check{\kappa}_{12}\cdot\varepsilon & \check{\kappa}_{13}\cdot\varepsilon & \cdots & \check{\kappa}_{1,\ell-1}\cdot\varepsilon & \check{\kappa}_{1\ell}\cdot\check{\zeta}_b & \check{\kappa}_{\ell,\ell+1}\cdot\varepsilon\cdot\check{\zeta}_b^2 & \cdots & \check{\kappa}_{\ell m}\cdot\varepsilon\cdot\check{\zeta}_b^2 \end{bmatrix},$$

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where $\check{\kappa}_{ij}$ refers to the proper transform of κ_{ij}, which is equivalent to its pullback in this case. Notice that, depending on the value of ℓ and m, some columns above may not exist.

Below, we analyze $\widetilde{\varphi}^{\mathsf{r}_1\mathsf{p}_1}$ depending on the location of q_a and q_b on the core F of C.

Case (A): q_a is in general position. This means q_a is neither Weierstrass nor conjugate to q_b. Hence κ_{1i} with $2 \le i \le \ell$ are all invertible and so are their pullbacks. Therefore, the matrix of $\widetilde{\varphi}^{\mathsf{r}_1\mathsf{p}_1}$, after further elementary column operations, can be rewritten as

$$\varepsilon \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \varepsilon & 0 & \cdots & 0 & \check{\zeta}_b & 0 & \cdots & 0 \end{bmatrix}.$$

If $\ell = 2$ (i.e. $|D \cap R_a| = 1$), then the second column $[0, \varepsilon]^T$ does not exist and the matrix is already diagonalized.

If $\ell \ge 3$ (i.e. $|D \cap R_a| \ge 2$), we observe that the remaining phases of (r_1) do not affect this chart, and the blowups in (r_2) that may affect this chart have the center $X_{2,2}$ (cf. Figure 2), which locally is given by $(\varepsilon = \check{\zeta}_b = 0)$. Therefore, after (r_2), the pullback $\widetilde{\varphi}^{\mathsf{r}_2}$ of φ becomes diagonalized.

Case (B): q_a and q_b are a pair of conjugate points on the core F. Since F is smooth, neither q_a nor q_b is a Weierstrass point of F, because of Lemma 2.14. Lemma 2.20 then implies that κ_{1i} with $2 \le i \le \ell - 1$ and $\kappa_{\ell k}$ with $\ell + 1 \le k \le m$ are all invertible and so are their proper transforms. Therefore, the matrix of $\widetilde{\varphi}^{\mathsf{r}_1\mathsf{p}_1}$, after further elementary column operations, can be rewritten as

$$\varepsilon \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \varepsilon & 0 & \cdots & 0 & \check{\kappa}_{1\ell}\check{\zeta}_b & \varepsilon\check{\zeta}_b^2 & 0 & \cdots & 0 \end{bmatrix}.$$

If $\ell = m = 2$ (i.e. $|D \cap R_a| = |D \cap R_b| = 1$), then only the first and the ℓth columns exist, and the matrix takes the form

$$\begin{bmatrix} 1 & 0 \\ 0 & \check{\kappa}_{1\ell}\check{\zeta}_b \end{bmatrix},$$

which is already diagonalized.

If $2 = \ell < m$ (i.e. $1 = |D \cap R_a| < |D \cap R_b|$), then the second column $[0, \varepsilon]^T$ does not exist and the matrix takes the form

$$\begin{bmatrix} \varepsilon & 0 \\ 0 & \varepsilon\check{\zeta}_b \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \check{\kappa}_{1\ell} & \varepsilon\check{\zeta}_b & 0 & \cdots & 0 \end{bmatrix}.$$

The blowups in $(\mathsf{r}_1\mathsf{p}_2)$ to $(\mathsf{r}_1\mathsf{p}_4)$ do not affect the chart. In $(\mathsf{r}_1\mathsf{p}_5)$, the locus $(\check{\zeta}_b = 0)$ is blown up, which does not yield anything new because the blowup center is a smooth Cartier divisor. The blowups in (r_2), $(\mathsf{r}_3\mathsf{p}_2)$ and $(\mathsf{r}_3\mathsf{p}_3)$ do not affect the chart either, whereas in $(\mathsf{r}_3\mathsf{p}_1)$ and $(\mathsf{r}_3\mathsf{p}_4)$, the blowup centers are locally given by

$$(\check{\kappa}_{1\ell} = \varepsilon = 0) \quad \text{and} \quad (\check{\kappa}_{1\ell} = \check{\zeta}_b = 0),$$

respectively. It is easy to check that after $(\mathsf{r}_3\mathsf{p}_4)$, the pullback $\widetilde{\varphi}^{\mathsf{r}_3\mathsf{p}_4}$ of φ becomes diagonalized.

If $\ell \ge 3$ (i.e. $|D \cap R_a| \ge 2$), then the second column $[0, \varepsilon]^T$ exists and the matrix of $\varphi^{\mathsf{r}_1\mathsf{p}_1}$, after another column addition, takes the form

$$\varepsilon \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \varepsilon & 0 & \cdots & 0 & \check{\kappa}_{1\ell}\check{\zeta}_b & 0 & \cdots & 0 \end{bmatrix}.$$

The blowups in $(\mathsf{r}_1\mathsf{p}_2)$ to $(\mathsf{r}_1\mathsf{p}_5)$ and $(\mathsf{r}_3\mathsf{p}_2)$ to $(\mathsf{r}_3\mathsf{p}_4)$ do not affect (the pullback of) the chart. In (r_2) and $(\mathsf{r}_3\mathsf{p}_1)$, the blowup centers are locally given by

$$(\varepsilon = \check{\zeta}_b = 0) \quad \text{and} \quad \mathrm{PT}(\varepsilon = \check{\kappa}_{1\ell} = 0),$$

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respectively, where PT refers to the proper transform. So after $(\mathsf{r}_3\mathsf{p}_1)$, the pullback $\widetilde{\varphi}^{\mathsf{r}_3\mathsf{p}_1}$ of φ becomes diagonalized.

Case (C): q_a is a Weierstrass point on the core F. Since F is smooth, q_a and q_b cannot be conjugate, by Lemma 2.14, and hence κ_{1k}, $\ell \le k \le m$, are all invertible and so are their proper transforms. Taking Proposition 2.22 (SH_3) into consideration, we see that the matrix of $\widetilde{\varphi}^{\mathsf{r}_1\mathsf{p}_1}$, after further elementary column operations, can be rewritten as

$$\varepsilon \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \check{\kappa}_{12}\varepsilon & \varepsilon^2 & 0 & \cdots & 0 & \check{\zeta}_b & 0 & \cdots & 0 \end{bmatrix}.$$

If $\ell = 2$ (i.e. $|D \cap R_a| = 1$), then the second and third columns $[0, \check{\kappa}_{12}\varepsilon]^T$ and $[0, \varepsilon^2]^T$ do not exist, so the matrix is already diagonalized.

If $\ell \ge 3$ (i.e. $|D \cap R_a| \ge 2$), notice that the blowups in $(\mathsf{r}_1\mathsf{p}_2)$ to $(\mathsf{r}_1\mathsf{p}_5)$ and $(\mathsf{r}_3\mathsf{p}_3)$ to $(\mathsf{r}_3\mathsf{p}_4)$ do not affect (the pullback of) the chart. The blowups in (r_2) may affect the chart only when $\check{\zeta}_b$ is not invertible (otherwise the matrix is already diagonalized), in which case the relevant blowup center is $(\varepsilon = \check{\zeta}_b = 0)$. After (r_2), on the new chart given by $\check{\zeta}_b = \varepsilon'$ and $\varepsilon = \varepsilon'\check{\varepsilon}$ (where $\varepsilon' = 0$ defines the exceptional divisor obtained in (r_2)), the matrix becomes diagonalized. On the new chart given by $\varepsilon = \varepsilon'$ and $\check{\zeta}_b = \varepsilon'\check{\zeta}'_b$, the matrix can be written as

$$\begin{bmatrix} \varepsilon' & 0 \\ 0 & (\varepsilon')^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \check{\kappa}_{12} & \varepsilon' & 0 & \cdots & 0 & \check{\zeta}'_b & 0 & \cdots & 0 \end{bmatrix}.$$

We remark that the third column $[0, \varepsilon']^T$ of the right-hand matrix above does not exist if $\ell = 3$ (i.e. $|D \cap R_a| = 2$).

The blowups in $(\mathsf{r}_3\mathsf{p}_1)$ do not affect this new chart locally. In $(\mathsf{r}_3\mathsf{p}_2)$ (and $(\mathsf{r}_3\mathsf{p}_3)$ if $\ell = 3$), the blowup center is locally given by

$$(\check{\kappa}_{12} = \varepsilon' = \check{\zeta}'_b = 0) \quad (\text{and} \quad \mathrm{PT}(\check{\kappa}_{12} = \check{\zeta}'_b = 0) \quad \text{if} \quad \ell = 3),$$

where PT refers to the proper transform. So after $(\mathsf{r}_3\mathsf{p}_2)$ ($(\mathsf{r}_3\mathsf{p}_3)$ if $\ell = 3$), the pullback of φ becomes diagonalized.

Example 3.11. Let $z_0 = (C, D)$, where C is made of a smooth core curve F with weight zero and two smooth rational tails T_1 and T_2 with positive weights, attached to F at q_a and q_b, respectively. Further, let T_1 be made of a single smooth rational curve R_a, and T_2 be made of three rational curves R_b, R_c and R_d such that R_b is attached to F and R_c and R_d are attached to R_b at two distinct points q_c and q_d, respectively; see Figure 9.

We assume that q_a and q_b are in general positions on F. In other words, they are not conjugate points, and neither is Weierstrass.

By Proposition 2.22 (SH_1), locally on some affine chart $\mathcal{V}$ containing the point z_0, we have

$$\varphi = \begin{bmatrix} a_{11}\zeta_a & c_{11}\zeta_b\zeta_c & d_{11}\zeta_b\zeta_d & \cdots \\ a_{21}\zeta_a & c_{21}\zeta_b\zeta_c & d_{21}\zeta_b\zeta_d & \cdots \end{bmatrix},$$

where ζ_a (respectively, $\zeta_b, \zeta_c, \zeta_d$) is the node-smoothing parameter for q_a (respectively, q_b, q_c, q_d), and a_{ij}, c_{ij} and d_{ij} are invertible functions on $\mathcal{V}$. Each of the omitted entries can also be divided by ζ_a, $\zeta_b\zeta_c$, or $\zeta_b\zeta_d$. Moreover, by Proposition 2.22 (SH_2), the vanishing orders of 2×2 minors of φ can be higher: under the assumption of this example, we have

$$\det \begin{bmatrix} a_{11} & c_{11} \\ a_{21} & c_{21} \end{bmatrix}, \ \det \begin{bmatrix} a_{11} & d_{11} \\ a_{21} & d_{21} \end{bmatrix} \in \Gamma(\mathscr{O}^*_{\mathcal{V}}), \quad \det \begin{bmatrix} c_{11} & d_{11} \\ c_{21} & d_{21} \end{bmatrix} \Big/ \zeta_b \in \Gamma(\mathscr{O}^*_{\mathcal{V}}).$$

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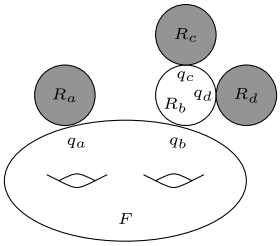


FIGURE 9. The curve C in Example 3.11.

In addition, if R_a has weight greater than one, the 2×2 minors corresponding to the points of $D\cap R_a$ may need to be taken into consideration, and these contain ζ_a^2 as a factor. Similarly, the term $\zeta_b^2\zeta_c^2$ (respectively, $\zeta_b^2\zeta_d^2$) may need to be taken into consideration if R_c (respectively, R_d) has weight greater than one.

Here, we point out the following. In $(\mathsf{r}_1\mathsf{p}_1)$, for any $z_0=(C,\mathbf{w})\in\mathfrak{M}_2^{\mathrm{wt}}$ whose core is of weight 0, the farther the separating nodes are away from the core, the later the corresponding parameters appear in the local equations of (the proper transforms of) the blowup centers. For instance, in the above example, for the nodes q_a and q_b that are closest to the core, locally at z_0,

$$(\zeta_a=\zeta_b=0)$$

defines the first blowup center $\overline{\mathfrak{M}}_{(1,2)}\cap\mathcal{V}$. After this blowup, on the chart given by $\zeta_b=\varepsilon_b$ and $\zeta_a=\varepsilon_b\check{\zeta}_a$, the locus

$$(\check{\zeta}_a=\zeta_c=\zeta_d=0)$$

locally defines the next blowup center, the proper transform of $\overline{\mathfrak{M}}_{(1,3)}$. This generalizes to all separating nodes that are farther from the core, if any.

Going back to the current example, the rooted tree of this example is identical to that of [HL10, Example 5.17]. The calculation within each phase of (r_1) is also similar to [HL10].

More precisely, assume that R_a, R_c and R_d all have weight one. Then, using similar calculations to those of [HL10, Example 5.17] and then applying Proposition 2.22, one finds that after (r_1), up to switching the symbols c and d, the pullback $\widetilde{\varphi}^{\mathsf{r}_1}$ of φ may take one of the following forms:

$$\begin{bmatrix}\varepsilon_a^{\mathsf{r}_1\mathsf{p}_1} & 0\\ 0 & \varepsilon_a^{\mathsf{r}_1\mathsf{p}_1}\varepsilon_b^{\mathsf{r}_1\mathsf{p}_5}\varepsilon_c^{\mathsf{r}_1\mathsf{p}_5}\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&\check{\zeta}_d\end{bmatrix},\qquad \begin{bmatrix}\varepsilon_a^{\mathsf{r}_1\mathsf{p}_1} & 0\\ 0 & \varepsilon_a^{\mathsf{r}_1\mathsf{p}_1}\varepsilon_c^{\mathsf{r}_1\mathsf{p}_5}\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&\check{\zeta}_d\end{bmatrix},$$

$$\begin{bmatrix}\varepsilon_b^{\mathsf{r}_1\mathsf{p}_1}\varepsilon_a^{\mathsf{r}_1\mathsf{p}_1} & 0\\ 0 & \varepsilon_b^{\mathsf{r}_1\mathsf{p}_1}\varepsilon_a^{\mathsf{r}_1\mathsf{p}_1}\varepsilon_c^{\mathsf{r}_1\mathsf{p}_5}\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&\check{\zeta}_d\end{bmatrix},\qquad \varepsilon_b^{\mathsf{r}_1\mathsf{p}_1}\varepsilon_c^{\mathsf{r}_1\mathsf{p}_1}\begin{bmatrix}0&1&0\\1&0&\varepsilon_b^{\mathsf{r}_1\mathsf{p}_1}\check{\zeta}_d\end{bmatrix},$$

$$\varepsilon_b^{\mathsf{r}_1\mathsf{p}_1}\varepsilon_c^{\mathsf{r}_1\mathsf{p}_1}\begin{bmatrix}0&1&0\\\check{\zeta}_a&0&\varepsilon_b^{\mathsf{r}_1\mathsf{p}_1}\end{bmatrix},\qquad \begin{bmatrix}\varepsilon_b^{\mathsf{r}_1\mathsf{p}_1}\varepsilon_c^{\mathsf{r}_1\mathsf{p}_1} & 0\\ 0 & \varepsilon_b^{\mathsf{r}_1\mathsf{p}_1}\varepsilon_c^{\mathsf{r}_1\mathsf{p}_1}\varepsilon_a^{\mathsf{r}_1\mathsf{p}_5}\end{bmatrix}\begin{bmatrix}0&1&0\\1&0&\varepsilon_b^{\mathsf{r}_1\mathsf{p}_1}\check{\zeta}_d\end{bmatrix},$$

$$\text{or}\quad \begin{bmatrix}\varepsilon_b^{\mathsf{r}_1\mathsf{p}_1}\varepsilon_c^{\mathsf{r}_1\mathsf{p}_1} & 0\\ 0 & \varepsilon_b^{\mathsf{r}_1\mathsf{p}_1}\varepsilon_c^{\mathsf{r}_1\mathsf{p}_1}\varepsilon_d^{\mathsf{r}_1\mathsf{p}_5}\end{bmatrix}\begin{bmatrix}0&1&0\\\check{\zeta}_a&0&\varepsilon_b^{\mathsf{r}_1\mathsf{p}_1}\end{bmatrix},$$

where the ε are local parameters corresponding to the proper transforms of the exceptional divisors, the superscripts of ε indicate the corresponding phases in (r_1), and $\check{\zeta}_a$ and $\check{\zeta}_d$ are proper transforms of ζ_a and ζ_d, respectively. Among the above seven possibilities of $\widetilde{\varphi}^{\mathsf{r}_1}$, five are already diagonalized; in either of the remaining two cases, the only blowup center of (r_2) that may affect

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$\widetilde{\varphi}^{\mathsf{r}_1}$ is locally defined by

$$(\varepsilon_b^{\mathsf{r}_1\mathsf{p}_1} = \check{\zeta}_a = 0),$$

and hence the pullback of φ after (r_2) becomes diagonalized.

Next, consider the case when at least one of R_a, R_b and R_c, say R_a, has weight greater than one. In such a case, by Proposition 2.22 (SH_2), entries containing ζ_a^2 may occur in φ. For instance, one finds that after $(\mathsf{r}_1\mathsf{p}_1)$ (hence after (r_1)), on the chart given by $\zeta_a = \varepsilon_a$ and $\zeta_b = \varepsilon_a\check{\zeta}_b$, the pullback of φ takes the following form:

$$\varepsilon_a \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \varepsilon_a & \check{\zeta}_b\zeta_c & \check{\zeta}_b\zeta_d & 0 & \cdots \end{bmatrix}.$$

The blowup in (r_2) that may affect this chart has the centers locally defined by

$$(\varepsilon_a = \check{\zeta}_b = 0) \quad \text{and} \quad \mathrm{PT}(\varepsilon_a = \zeta_c = \zeta_d = 0), \tag{3.28}$$

where PT refers to the proper transform. Globally the two loci correspond to $X_{2,2}$ and $X_{2,3}$, respectively; see Figure 2. (The derived tree in this case is labeled (3) in Figure 18.) Then, following similar calculations as in (r_1), one finds that after (r_2), the pullback of φ becomes diagonalized.

Similarly, one finds that after (r_2), the pullback of φ becomes diagonalized on other charts.

Example 3.12. Let $z_0 = (C, D)$ be the same as in Example 3.11 but with one difference: we assume that q_a is a Weierstrass point of the core F of C. Since F is assumed to be smooth, q_a and q_b cannot be conjugate. We further assume that $D \cap R_a = \{\delta_1, \ldots, \delta_k\}$ with $k \geq 3$.

After (r_1), on the chart given by $\zeta_a = \varepsilon_a$ and $\zeta_b = \varepsilon_a\check{\zeta}_b$, the pullback of φ takes the following form:

$$\varepsilon_a \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \check{\kappa}_{12}\varepsilon_a & \varepsilon_a^2 & \check{\zeta}_b\zeta_c & \check{\zeta}_b\zeta_d & 0 & \cdots \end{bmatrix},$$

where $\check{\kappa}_{12}$ is the proper transform of κ_{12} as in Example 3.10. The blowup in (r_2) that may affect this chart still has the centers locally defined by

$$(\varepsilon_a = \check{\zeta}_b = 0) \quad \text{and} \quad \mathrm{PT}(\varepsilon_a = \zeta_c = \zeta_d = 0).$$

Following similar calculations as in Examples 3.10 and 3.11, one finds that after (r_2), on the new chart given by $\varepsilon_a = \varepsilon'$ and $\check{\zeta}_b = \varepsilon'\check{\zeta}_b'$, the pullback of φ takes the form

$$\begin{bmatrix} \varepsilon' & 0 \\ 0 & (\varepsilon')^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \check{\kappa}_{12} & \varepsilon' & \check{\zeta}_b'\zeta_c & \check{\zeta}_b'\zeta_d & 0 & \cdots \end{bmatrix}.$$

The blowup in $(\mathsf{r}_3\mathsf{p}_2)$ that may affect this chart has the centers locally defined by

$$(\check{\kappa}_{12} = \varepsilon' = \check{\zeta}_b' = 0) \quad \text{and} \quad \mathrm{PT}(\check{\kappa}_{12} = \varepsilon' = \zeta_c = \zeta_d = 0). \tag{3.29}$$

Therefore, after $(\mathsf{r}_3\mathsf{p}_2)$, the pullback of φ becomes diagonalized.

Example 3.13. Let $z_0 = (C, D)$ be the same as in Example 3.11 but with one difference: we assume that q_a and q_b are conjugate points on the core F of C. Since F is assumed to be smooth, neither q_a nor q_b is Weierstrass. We further assume that $|D \cap R_\bullet| \geq 3$ for $\bullet = a, c, d$. Moreover, we assume that $\delta_1 \in R_a$, $\delta_2 \in R_b$ and $\delta_3 \in R_c$.

After (r_1), on the chart given by

$$\zeta_a = \varepsilon_1\varepsilon_2, \quad \zeta_b = \varepsilon_1, \quad \zeta_c = \varepsilon_1\varepsilon_2\check{\zeta}_c, \quad \text{and} \quad \zeta_d = \varepsilon_1\varepsilon_2\check{\zeta}_d,$$

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(i.e. locally one first blows up along $(\zeta_a=\zeta_b=0)$, then on the chart given by $\zeta_b=\varepsilon_1$ and $\zeta_a=\varepsilon_1\check{\zeta}_a$, one blows up along $(\check{\zeta}_a=\zeta_c=\zeta_d=0)$ and considers the new chart given by $\check{\zeta}_a=\varepsilon_2$, $\zeta_c=\varepsilon_2\check{\zeta}_c$ and $\zeta_d=\varepsilon_2\check{\zeta}_d$,) the pullback of φ takes the following form:

$$\varepsilon_1\varepsilon_2\begin{bmatrix}1 & 0 & 0 & 0 & 0 & \cdots\\ 0 & \varepsilon_1\varepsilon_2 & \check{\kappa}_{12}\check{\zeta}_c & \check{\kappa}_{13}\check{\zeta}_d & 0 & \cdots\end{bmatrix}, \tag{3.30}$$

where $\check{\kappa}_{ij}$ is the pullback of κ_{ij} as in Example 3.10. Proposition 2.22 (SH_3) implies that

$$\kappa_{13}=f'\kappa_{12}+g'\zeta_b, \quad\Longrightarrow\quad \check{\kappa}_{13}=f\check{\kappa}_{12}+g\varepsilon_1,$$

where f', g', f and g are some units.

The blowup in (r_2) that may affect this chart still has the centers locally defined by

$$(\varepsilon_2=\check{\zeta}_c=\check{\zeta}_d=0) \quad\text{and}\quad \mathrm{PT}(\varepsilon_1=\check{\zeta}_c=\check{\zeta}_d=0).$$

Globally, these two loci correspond to the substacks $X_{3,3}$ and $X_{2,3}$ of $\widetilde{\mathfrak{M}}^{\mathsf{r}_2}$; see (3.17) for the notation and Figure 2 for an illustration. (The derived tree in this case is labeled (1) in Figure 18.)

Following the order (3.18), we first blow up along $(\varepsilon_2=\check{\zeta}_c=\check{\zeta}_d=0)$.

On the chart given by $\varepsilon_2=\varepsilon'$, $\check{\zeta}_c=\varepsilon'\check{\zeta}'_c$ and $\check{\zeta}_d=\varepsilon'\check{\zeta}'_d$, the second row of the pullback of φ can be rewritten as

$$\varepsilon_1(\varepsilon')^2[\,0\quad \varepsilon_1\quad \check{\kappa}_{12}\check{\zeta}'_c\quad \check{\kappa}_{13}\check{\zeta}'_d\quad \cdots\,] \;=\; \varepsilon_1(\varepsilon')^2[\,0\quad \varepsilon_1\quad \check{\kappa}_{12}\check{\zeta}'_c\quad (f\check{\kappa}_{12}+g\varepsilon_1)\check{\zeta}'_d\quad \cdots\,].$$

The remaining blowups in (r_2), as well as those in $(\mathsf{r}_3\mathsf{p}_1)$, that may affect this chart have centers locally given by

$$\mathrm{PT}(\varepsilon_1=\check{\zeta}_c=\check{\zeta}_d=0)=(\varepsilon_1=\check{\zeta}'_c=\check{\zeta}'_d=0) \quad\text{and}\quad \mathrm{PT}(\varepsilon_1=\check{\kappa}_{12}=0),$$

respectively. Therefore, the pullback of φ is diagonalized after $(\mathsf{r}_3\mathsf{p}_1)$.

On the chart given by $\check{\zeta}_c=\varepsilon'$, $\varepsilon_2=\varepsilon'\check{\varepsilon}_2$ and $\check{\zeta}_d=\varepsilon'\check{\zeta}'_d$, the second row of the pullback of φ can be rewritten as

$$\varepsilon_1(\varepsilon')^2[\,0\quad \varepsilon_1\check{\varepsilon}_2\quad \check{\kappa}_{12}\quad \check{\kappa}_{13}\check{\zeta}'_d\quad \cdots\,] \;=\; \varepsilon_1(\varepsilon')^2[\,0\quad \varepsilon_1\check{\varepsilon}_2\quad \check{\kappa}_{12}\quad \varepsilon_1\check{\zeta}'_d\quad \cdots\,].$$

The proper transform of $(\varepsilon_1=\check{\zeta}_c=\check{\zeta}_d=0)$ does not affect this chart, whereas the relevant blowup centers in $(\mathsf{r}_3\mathsf{p}_1)$ are

$$(\check{\varepsilon}_2=\check{\zeta}'_d=\check{\kappa}_{12}=0) \quad\text{and}\quad \mathrm{PT}(\varepsilon_1=\check{\kappa}_{12}=0),$$

respectively. Therefore, the pullback of φ is diagonalized after $(\mathsf{r}_3\mathsf{p}_1)$.

The argument on the chart given by $\check{\zeta}_d=\varepsilon'$, $\varepsilon_2=\varepsilon'\check{\varepsilon}_2$ and $\check{\zeta}_c=\varepsilon'\check{\zeta}'_c$ is parallel.

At the end of this example, we remark that in (r_2), if the order (3.18) were replaced by the usual lexicographic order on $\mathbb{Z}\times\mathbb{Z}$, then after the entire three rounds of blowups terminate, the pullback of φ would fail to be diagonalizable on some chart. To see this, we blow up $X_{2,3}$ prior to $X_{3,3}$ in this example. Locally, this means that the locus $(\varepsilon_1=\check{\zeta}_c=\check{\zeta}_d=0)$ is blown up before $(\varepsilon_2=\check{\zeta}_c=\check{\zeta}_d=0)$. Then, on the chart given by $\varepsilon_1=\varepsilon'$, $\check{\zeta}_c=\varepsilon'\check{\zeta}'_c$ and $\check{\zeta}_d=\varepsilon'\check{\zeta}'_d$, the second row of the pullback of (3.30) can be rewritten as

$$\varepsilon_2(\varepsilon')^2[\,0\quad \varepsilon_2\quad \check{\kappa}_{12}\check{\zeta}'_c\quad (f\check{\kappa}_{12}+g\varepsilon')\check{\zeta}'_d\quad \cdots\,].$$

After blowing up $\mathrm{PT}(\varepsilon_2=\check{\zeta}_c=\check{\zeta}_d=0)=(\varepsilon_2=\check{\zeta}'_c=\check{\zeta}'_d=0)$, on the chart given by $\check{\zeta}'_c=\varepsilon''$, $\varepsilon_2=\varepsilon''\check{\varepsilon}_2$ and $\check{\zeta}'_d=\varepsilon''\check{\zeta}''_d$, the second row of the pullback of φ can be further rewritten as

$$\varepsilon_2(\varepsilon'\varepsilon'')^2[\,0\quad \check{\varepsilon}_2\quad \check{\kappa}_{12}\quad \varepsilon'\check{\zeta}''_d\quad \cdots\,].$$

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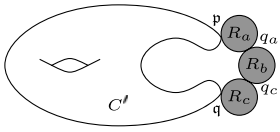


FIGURE 10. The curve C in Example 3.15.

The problem is that the locus $(\check{\varepsilon}_2 = \varepsilon' = \check{\kappa}_{12} = 0)$, which must be blown up so as to simplify the matrix above, does not correspond to any blowup center in (r_3). Indeed, such loci lie deeply in the intersections of the exceptional divisors from the previous steps, so it seems formidable to organize them in a systematic way should such blowups be included. However, by introducing the order (3.18) in (r_2), such issues can be avoided nicely.

Example 3.14. Let $z_0 = (C, D) \in \mathfrak{M}_2^{\mathrm{div}}$ be such that the core F of C is comprised of a smooth rational subcurve B and a smooth genus 1 subcurve C_2, meeting at two points $\mathfrak{p}$ and $\mathfrak{q}$, the tail of C is a smooth rational subcurve C_a attached to C_2 at q_a, and D satisfies

$$D \cap B = \{\delta_1, \delta_2, \delta_3\}, \quad D \cap C_2 = \emptyset, \quad D \cap C_a \neq \emptyset.$$

In particular, B is a (maximal) non-separating bridge $\mathrm{B}[\mathfrak{p}, \mathfrak{q}]$. Such (C, D) lies over a general point in $\overline{\mathfrak{M}}_{(3,2)}$; see Figure 1 for an illustration.

By Proposition 2.22, under a suitable trivialization, φ on a chart $\mathcal{V}$ containing (C, D) can be written as

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & f\zeta_{\mathfrak{p}} + g\zeta_{\mathfrak{q}} & f'\zeta_{\mathfrak{p}} + g'\zeta_{\mathfrak{q}} & \zeta_a & 0 & \cdots \end{bmatrix},$$

where $\zeta_{\mathfrak{p}}$ and $\zeta_{\mathfrak{q}}$ are node-smoothing parameters corresponding to $\mathfrak{p}$ and $\mathfrak{q}$, and f, f', g, and g' are invertible functions on $\mathcal{V}$ satisfying

$$\det \begin{bmatrix} f & f' \\ g & g' \end{bmatrix} \in \Gamma(\mathscr{O}_{\mathcal{V}}^*). \tag{3.31}$$

In $(\mathsf{r}_1\mathsf{p}_3)$, locally the blowup center is given by $(\zeta_{\mathfrak{p}} = \zeta_{\mathfrak{q}} = \zeta_a = 0)$, corresponding to (the proper transform of) $\overline{\mathfrak{M}}_{(3,2)}$. When $(\mathsf{r}_1\mathsf{p}_3)$ terminates, let y be a fixed lift of x.

On the chart given by $\zeta_a = \varepsilon$, $\zeta_{\mathfrak{p}} = \varepsilon\check{\zeta}_{\mathfrak{p}}$ and $\zeta_{\mathfrak{q}} = \varepsilon\check{\zeta}_{\mathfrak{q}}$, the pullback of φ is obviously diagonalizable near y.

On the chart given by $\zeta_{\mathfrak{p}} = \varepsilon$, $\zeta_a = \varepsilon\check{\zeta}_a$ and $\zeta_{\mathfrak{q}} = \varepsilon\check{\zeta}_{\mathfrak{q}}$, the pullback of φ can be rewritten as

$$\begin{bmatrix} 1 & 0 \\ 0 & \varepsilon \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & f + g\check{\zeta}_{\mathfrak{q}} & f' + g'\check{\zeta}_{\mathfrak{q}} & \check{\zeta}_a & 0 & \cdots \end{bmatrix}.$$

By (3.31), $f + g\check{\zeta}_{\mathfrak{q}}$ and $f' + g'\check{\zeta}_{\mathfrak{q}}$ cannot vanish simultaneously, and hence the pullback of φ is diagonalizable near y.

Example 3.15. Let $z_0 = (C, D)$ be such that $C = F$ consists of a smooth genus 1 curve C' having weight zero and an inseparable bridge $\mathrm{B}[\mathfrak{p}, \mathfrak{q}]$ made of three smooth rational curves R_a, R_b and R_c that are all positively weighted. We assume that R_a is attached to the genus 1 curve at $\mathfrak{p}$ and R_c is attached to the genus 1 curve at $\mathfrak{q}$. Further, we write $q_a = R_a \cap R_b$ and $q_c = R_b \cap R_c$; see Figure 10.

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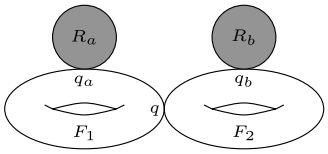


FIGURE 11. The curve C in Example 3.16.

By Proposition 2.17, using a point from each of $D \cap R_a$, $D \cap R_b$ and $D \cap R_c$, we have

$$\begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & f_2\zeta_{\mathfrak{p}} + g_2\zeta_c\zeta_{\mathfrak{q}} & f_3\zeta_{\mathfrak{p}} + g_3\zeta_{\mathfrak{q}} & \cdots \end{bmatrix},$$

where $\zeta_{\mathfrak{p}}$, $\zeta_{\mathfrak{q}}$ and ζ_c are, respectively, the node-smoothing parameters corresponding to $\mathfrak{p}$, $\mathfrak{q}$ and q_c, while f_2, f_3, g_2 and g_3 are all units. In $(\mathsf{r}_1\mathsf{p}_3)$, this point lies in the blowup center locally defined by $(\zeta_{\mathfrak{p}} = \zeta_{\mathfrak{q}} = 0)$. Then, one calculates and finds that after $(\mathsf{r}_1\mathsf{p}_3)$, the pullback of φ on any chart takes one of the following two forms:

$$\begin{bmatrix} 1 & 0 \\ 0 & \varepsilon_{\mathfrak{p}} \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots \\ 0 & 1 & \cdots \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon_{\mathfrak{q}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & \check{\zeta}_{\mathfrak{p}} & 1 & \cdots \end{bmatrix},$$

where $\check{\zeta}_{\mathfrak{p}}$ is the proper transform of $\zeta_{\mathfrak{p}}$. The above calculations generalize to all cases of $\overline{\mathfrak{M}}_{(3,1)}$ (cf. Figure 1). Only the two nodes $\mathfrak{p}$ and $\mathfrak{q}$ in the maximal bridge $\mathrm{B}[\mathfrak{p}, \mathfrak{q}]$ play deciding roles in the process of the blowup; the remaining node-smoothing parameters, such as ζ_a and ζ_c in the above example, do not play a decisive role, even though they may appear in φ.

Example 3.16. Let $z_0 = (C, D)$ be such that the core F of C is made of two smooth genus 1 curves F_1 and F_2, each of which is of weight zero, and the tails of C consists of two smooth rational curves R_a and R_b, positively weighted and attached to F_1 and F_2 at points q_a and q_b, respectively. We write $q = F_1 \cap F_2$; see Figure 11.

In such a case, q_a and q_b cannot be conjugate. We assume that neither of them is Weierstrass. Further, we assume that the weights of R_a and R_b are both greater than one.

Then, by Proposition 2.22, locally on some affine chart $\mathcal{V}$ containing the point z_0, we have

$$\varphi = \begin{bmatrix} c_{11}\zeta_a & 0 & \cdots & d_{11}\zeta_q\zeta_b & \zeta_q\zeta_b^2 & \cdots \\ c_{21}\zeta_q\zeta_a & \zeta_q\zeta_a^2 & \cdots & d_{21}\zeta_b & 0 & \cdots \end{bmatrix}$$

where ζ_q is the node-smoothing parameter for the separating node q, ζ_a (respectively, ζ_b) is the node-smoothing parameter for q_a (respectively, q_b), and c_{ij} and d_{ij} are some units.

After (r_1), on any small affine chart over z_0, either the pullback of φ is already diagonalized, or else, up to switching the symbols a and b, the pullback of φ may take the form

$$\begin{bmatrix} \varepsilon_a & 0 \\ 0 & \varepsilon_a\varepsilon_q \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & \varepsilon_a & \check{\zeta}_b & \cdots \end{bmatrix}$$

where ε_a and ε_q are exceptional parameters from $(\mathsf{r}_1\mathsf{p}_1)$ and $(\mathsf{r}_1\mathsf{p}_4)$ (corresponding to q_a and q), respectively, and $\check{\zeta}_b$ is the proper transform of ζ_b.

As in Case (A) of Example 3.10, one sees that after (r_2), the pullback of the above matrix becomes diagonalizable on any small chart over z_0.

Example 3.17. Let $z_0 = (C, D)$ be such that C is made of a smooth core curve F satisfying $F \cap D = \{\delta_1, \delta_2\}$ and one smooth rational curve R attached to F at a point $q \neq \delta_1, \delta_2$; see Figure 12.

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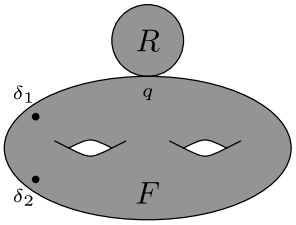


FIGURE 12. The point (C, D) in Example 3.17.

When δ_1, δ_2 are not conjugate, z_0 is already a smooth point of $\mathfrak{M}_2^{\mathrm{div}}$, so we study the case when δ_1, δ_2 are conjugate.

By Proposition 2.22, locally on some affine chart $\mathcal{V}$ containing the point z_0, we have

$$\varphi = \begin{bmatrix} c_{11} & c_{12} & c_{13}\zeta & \cdots \\ c_{21} & c_{22} & c_{23}\zeta & \cdots \end{bmatrix}$$

where ζ is the node-smoothing parameter for the node q. By Proposition 2.22 (SH_2), under a suitable new basis, this matrix can be represented as

$$\begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & \kappa_{12} & \zeta & \cdots \end{bmatrix}$$

where $\kappa_{12} = 0$ defines the locus where $\mathcal{D}_1(z)$ and $\mathcal{D}_2(z)$ are conjugate points on the curve $\mathcal{C}_z$.

In $(\mathsf{r}_3\mathsf{p}_3)$, z_0 lies in the blowup center locally given by $(\kappa_{12} = \zeta = 0)$, so after $(\mathsf{r}_3\mathsf{p}_3)$, one sees that the pullback of the above matrix becomes diagonalizable on any small chart over z_0.

4. Local diagonalizability of the structural homomorphisms on $\widetilde{\mathfrak{M}}_2^{\mathrm{div}}$

4.1 Main statement and the structure of its proof

Throughout Section 4, $\widetilde{\mathfrak{M}}_2^{\mathrm{div}}$ denotes the final stack after the three rounds of blowups and $\pi : \widetilde{\mathfrak{M}}_2^{\mathrm{div}} \to \mathfrak{M}_2^{\mathrm{div}}$ the projection. The following commutative diagram shows the relation between the relevant stacks, where the horizontal arrow is given by (2.1).

$$\begin{array}{ccc} \widetilde{\mathfrak{M}}_2^{\mathrm{div}} & & \\ \downarrow{\scriptstyle \pi} & \searrow{\scriptstyle \varpi} & \\ \mathfrak{M}_2^{\mathrm{div}} & \longrightarrow & \mathfrak{M}_2^{\mathrm{wt}} \end{array}$$

In this section, we investigate the pullback $\widetilde{\varphi}$ of the structural homomorphism φ of (2.4) locally. The main statement is as follows.

PROPOSITION 4.1. *For every $\widetilde{x} \in \widetilde{\mathfrak{M}}_2^{\mathrm{div}}$, there exists an affine smooth chart $\widetilde{\mathcal{V}}_{\widetilde{x}} \to \widetilde{\mathfrak{M}}_2^{\mathrm{div}}$ containing $\widetilde{x}$ so that the pullback $\widetilde{\varphi}$ of φ to $\widetilde{\mathcal{V}}_{\widetilde{x}}$ takes the form*

$$\begin{bmatrix} z_1 & 0 & 0 & \cdots & 0 \\ 0 & z_2 & 0 & \cdots & 0 \end{bmatrix} \quad \text{with} \quad z_1 | z_2 \in \Gamma(\mathscr{O}_{\widetilde{\mathcal{V}}_{\widetilde{x}}}),$$

where z_1 and z_2 are both products of some smooth parameters corresponding to (the proper transforms of) the exceptional divisors of the sequential blowups $\widetilde{\mathfrak{M}}_2^{\mathrm{div}} \to \mathfrak{M}_2^{\mathrm{div}}$. Consequently, $\widetilde{\varphi}$ is diagonalizable.

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Recall that in (3.6), there is a partition of $\mathfrak{M}_2^{\mathrm{wt}}$ into five disjoint substacks:

$$\mathfrak{M}_2^{\mathrm{wt}} = \mathfrak{M}_{(1)}^{\mathrm{mn}} \sqcup \mathfrak{M}_{(2)}^{\mathrm{mn}} \sqcup \mathfrak{M}_{(3)}^{\mathrm{mn}} \sqcup \mathfrak{M}_{(4)}^{\mathrm{mn}} \sqcup \mathfrak{M}^{\mathrm{mn}}.$$

Proposition 4.1 will be proved depending on which of the above substacks contains $\varpi(\widetilde{x})$. More concretely, throughout Section 4, we fix an arbitrary $\widetilde{x} \in \widetilde{\mathfrak{M}}_2^{\mathrm{div}}$ and write

$$\widehat{x} = (C,\ D = \delta_1 + \cdots + \delta_m) = \pi(\widetilde{x}) \in \mathfrak{M}_2^{\mathrm{div}}, \quad x = (C, c_1(D)) = \varpi(\widetilde{x}) \in \mathfrak{M}_2^{\mathrm{wt}}.$$

Based on the topology of F and the locations of the points in D, we divide the proof of Proposition 4.1 into the following cases.

Case A. $x \in \mathfrak{M}^{\mathrm{mn}}$, satisfying one of the following:

1. the core F of C is separable;
2. F is inseparable and $\deg D \cap F \geq 3$;
3. F is inseparable, $\deg D \cap F = 2$, and the elements of $D \cap F$ are not conjugate to each other;
4. F is inseparable, $\deg D \cap F = 2$, and the elements of $D \cap F$ are conjugate to each other;
5. F is inseparable and $\deg D \cap F = 1$.

Case B. $x \in \mathfrak{M}_{(4)}^{\mathrm{mn}}$, satisfying one of the following:

1. $\deg D \cap F \geq 3$;
2. $\deg D \cap F = 2$ and the elements of $D \cap F$ are not conjugate to each other;
3. $\deg D \cap F = 2$ and the elements of $D \cap F$ are conjugate to each other;
4. $\deg D \cap F = 1$.

Case C. $x \in \mathfrak{M}_{(3)}^{\mathrm{mn}}$, satisfying one of the following:

1. F is inseparable and $\deg D \cap F \geq 3$;
2. F is inseparable and $\deg D \cap F = 2$;
3. F is inseparable and $\deg D \cap F = 1$;
4. F is separable.

Case D. $x \in \mathfrak{M}_{(2)}^{\mathrm{mn}}$.

Case E. $x \in \mathfrak{M}_{(1)}^{\mathrm{mn}}$.

We remark that Case E is the most complicated case. We need to introduce certain notation and terminology before dividing it into sub-cases, so we postpone this until Section 4.12.

Remark 4.2. Let $\mathfrak{N}$ be the image of $\overline{M}_2(\mathbb{P}^n, d)$ in $\mathfrak{M}_2^{\mathrm{wt}}$ under the morphism (1.6). Then $\mathfrak{N}$ is neither open nor closed in $\mathfrak{M}_2^{\mathrm{wt}}$. For instance, consider $\mathfrak{M}_{(i,j)} \subset \mathfrak{M}_2^{\mathrm{wt}}$ as in (3.2); see Figure 1 for an illustration.

- Every $x \in \mathfrak{M}_{(2,1)}$ with weight-1 core, which occurs in Case D, is contained in $\mathfrak{N}$. However, every open neighborhood $\mathcal{V}_x$ of x contains points of $\mathfrak{M}_{(4,1)}$ with weight-1 core, which occurs in Case B.4 and is not contained in $\mathfrak{N}$. Therefore, $\mathfrak{N}$ is not open.
- Every $x \in \mathfrak{M}_{(4,3)}$ with weight-1 core, which occurs in Case B.4, is not contained in $\mathfrak{N}$, but every open neighborhood $\mathcal{V}_x$ of x contains points of $\mathfrak{N}$ whose domain curves are smooth, and hence $\mathfrak{M}_2^{\mathrm{wt}} \backslash \mathfrak{N}$ is not open.

Likewise, the preimage $\mathfrak{N}^{\mathrm{div}}$ of $\mathfrak{N}$ is neither open nor closed in $\mathfrak{M}_2^{\mathrm{div}}$. Therefore, although in Proposition 4.1, $\mathcal{E}_{\mathcal{V}}$ could be replaced by its restriction to $\mathcal{V} \cap \mathfrak{N}^{\mathrm{div}}$, it would be difficult to construct a blowup $\widetilde{\mathfrak{N}}^{\mathrm{div}} \to \mathfrak{N}^{\mathrm{div}}$ and to analyze the pullbacks of the structural homomorphisms.

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This is why in Proposition 4.1 we choose the base stack to be the entire $\mathfrak{M}_2^{\mathrm{div}}$, which is smooth and has local parameters that are suitable for describing the structural homomorphisms, even though the points of $\mathfrak{M}_2^{\mathrm{wt}}$ satisfying Case A.5, Case B.4, or Case C.3 are not contained in $\mathfrak{N}$.

In Sections 4.2 to 4.16, we will prove Proposition 4.1 case by case.

4.2 Proof of Cases A.1 to 3

We begin with the three simplest sub-cases of Case A above.

Proposition 4.3. *Proposition 4.1 holds in Case B.1, Case A.2, and Case A.3.*

Proof. Let $\mathcal{V}\to\mathfrak{M}_2^{\mathrm{div}}$ be a small affine smooth chart containing $\widehat{x}$.

In Case A.1, let F_1 and F_2 be the two genus 1 inseparable components of F. Without loss of generality we assume that

$$\delta_1,\ a_1\in F_1,\quad \delta_2,\ a_2\in F_2.$$

Then, by (the proof of) Proposition 2.23, the structural homomorphism φ is diagonalizable on $\mathcal{V}$.

In Case A.2, observe that an inseparable F can still be reducible, for example if it is a smooth rational curve attached to two distinct points of a smooth genus 1 curve. Nonetheless, there always exist $\delta,\delta'\in D\cap F$ that are not conjugate to each other, for otherwise all the elements of $D\cap F$ would lie on a non-separating bridge of F, which would imply that $x\in\mathfrak{M}_{(3)}^{\mathrm{mn}}$. In addition, F being inseparable implies that $\zeta_{[\delta,a_s]},\zeta_{[\delta',a_s]}\in\Gamma(\mathscr{O}_{\mathcal{V}}^*)$ for $s=1,2$. Hence by (the proof of) Proposition 2.23, φ is diagonalizable on $\mathcal{V}$.

In Case A.3, the argument from the preceding paragraph still holds verbatim.

Finally, observe that x is away from the blowup centers of $\widetilde{\mathfrak{M}}_2^{\mathrm{div}}/\mathfrak{M}_2^{\mathrm{div}}$, hence we can take $\widetilde{x}=x$ and $\widetilde{\mathcal{V}}_{\widetilde{x}}=\mathcal{V}$, which is an affine smooth chart. □

In the remaining cases, φ is always not diagonalizable on any neighborhood of $\widehat{x}$, so we will study how each phase of the three rounds of blowups affect the pullback of φ on a case-by-case basis.

4.3 Local parameters for blowups

We continue with the setup in Section 4.1 and fix a small affine smooth chart $\mathcal{V}\to\mathfrak{M}_2^{\mathrm{div}}$ containing $\widehat{x}$.

In order to tackle Case A.4, when F is inseparable and $D\cap F$ consists of two conjugate points, which we without loss of generality assume to be δ_1 and δ_2, we first observe that δ_1 cannot be conjugate to any $\delta\in D\backslash F$, for otherwise δ_1 and hence δ_2 would be on a non-separating bridge, which would imply that $x\in\mathfrak{M}_{(3)}^{\mathrm{mn}}$. Thus, by Proposition 2.22, after suitable elementary row and column operations, φ can be written as

$$\varphi=\begin{bmatrix}1 & 0 & 0 & \cdots & 0\\ 0 & \kappa_{12} & \zeta_{[\delta_3]} & \cdots & \zeta_{[\delta_m]}\end{bmatrix}. \tag{4.1}$$

(Recall that $(\kappa_{12}=0)\subset\mathcal{V}$ defines the locus where $\mathcal{D}_1(z)$ and $\mathcal{D}_2(z)$ are conjugate, and that $\zeta_{[\delta_i]}$ is the product of the node-smoothing parameters ζ_q for all the nodes between δ_i and F.) Since $\deg D\cap F=2$, the modular blowups, up to the end of $(\mathsf{r}_3\mathsf{p}_2)$, do not affect $\mathcal{V}$. We thus do not distinguish $\mathcal{V}$ and its pullback $\widetilde{\mathcal{V}}^{\mathsf{r}_3\mathsf{p}_2}$ when $(\mathsf{r}_3\mathsf{p}_2)$ completes.

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Recall that $N(C)$ denotes the set of the nodes of C. By Proposition 2.17, κ_{12} and ζ_q, $q \in N(C)$, form a subset of a system of local parameters on $\mathcal{V}$. In order to analyze the pullback of the second row of (4.1), we need to study how κ_{12} and the ζ_q change in $(\mathsf{r}_3\mathsf{p}_3)$.

To this end, we introduce certain notation and terminology, which will be assumed throughout Section 4.

Given a stack $\mathfrak{M}$ and a closed substack $Z \subset \mathfrak{M}$, let $\widetilde{\mathfrak{M}}$ be the blowup of $\mathfrak{M}$ along Z, $\pi : \widetilde{\mathfrak{M}} \to \mathfrak{M}$ be the projection, and $\mathcal{E}$ be the exceptional divisor. For every $\mathfrak{N} \subset \mathfrak{M}$, we denote its **proper transform** and **pullback** in $\widetilde{\mathfrak{M}}$ by

$$\check{\mathfrak{N}} = \mathrm{PT}(\mathfrak{N}) := \overline{\pi^{-1}(\mathfrak{N} \backslash Z)} \quad \text{and} \quad \widetilde{\mathfrak{N}} := \pi^{-1}(\mathfrak{N}),$$

respectively. Here, $^-$ denotes the closure.

Assume that $\mathfrak{M}$ has an affine smooth chart $\mathcal{V} \to \mathfrak{M}$. In addition, assume that $\zeta_1, \ldots, \zeta_r$ is a system of local parameters on $\mathcal{V}$, and

$$Z \cap \mathcal{V} = \{\zeta_1 = \cdots = \zeta_\ell = 0\}.$$

Then, by writing

$$\widetilde{\zeta}_i := \zeta_i \circ \pi \in \Gamma(\mathscr{O}_{\pi^{-1}(\mathcal{V})}) \quad \forall\, i \in [r], \tag{4.2}$$

we have that every closed point $\widetilde{x} \in \mathcal{E} \cap \pi^{-1}(\mathcal{V})$ is in the form

$$\widetilde{x} = (0, \ldots, 0, \widetilde{\zeta}_{\ell+1}, \ldots, \widetilde{\zeta}_r;\ [u_1, \ldots, u_\ell]),$$

where $[u_1, \ldots, u_\ell]$ denotes the homogeneous coordinate of $\mathbb{P}^{\ell-1}$. Let

$$\Delta_{\widetilde{x}} = \{k \in [\ell] : u_k \neq 0\} \subset [\ell]. \tag{4.3}$$

Obviously, $\Delta_{\widetilde{x}} \neq \emptyset$.

LEMMA 4.4. *With $\widetilde{x}$ as above, there exists an affine smooth chart $\widetilde{\mathcal{V}}_{\widetilde{x}} \subset \pi^{-1}(\mathcal{V})$ containing $\widetilde{x}$ such that*

$$\widetilde{\zeta}_i \,|\, \widetilde{\zeta}_j \quad \text{and} \quad \mathrm{PT}\{\zeta_j = 0\} \cap \widetilde{\mathcal{V}}_{\widetilde{x}} = \{\widetilde{\zeta}_j/\widetilde{\zeta}_i = 0\} \quad \forall\, i \in \Delta_{\widetilde{x}},\ j \in [\ell].$$

Moreover, if $i \in \Delta_{\widetilde{x}}$ and $j \in [\ell] \backslash \Delta_{\widetilde{x}}$, then $\widetilde{\zeta}_j/\widetilde{\zeta}_i$ is a local parameter on $\widetilde{\mathcal{V}}_{\widetilde{x}}$, vanishing at $\widetilde{x}$; if $i, j \in \Delta_{\widetilde{x}}$, then $\widetilde{\zeta}_j/\widetilde{\zeta}_i$ does not vanish at $\widetilde{x}$, and $\widetilde{\zeta}_j/\widetilde{\zeta}_i - ((\widetilde{\zeta}_j/\widetilde{\zeta}_i)(\widetilde{x}))$ is a local parameter on $\widetilde{\mathcal{V}}_{\widetilde{x}}$.

Proof. Every point in $\widetilde{\mathcal{V}}_{\widetilde{x}}$ is in the form $(\widetilde{\zeta}_1, \ldots, \widetilde{\zeta}_r; [u_1, \ldots, u_\ell])$, so that

$$\widetilde{\zeta}_i\, u_j = \widetilde{\zeta}_j\, u_i \quad \forall\, i, j \in [\ell].$$

Shrinking $\widetilde{\mathcal{V}}_{\widetilde{x}}$ if necessary, we may assume u_i does not vanish on $\widetilde{\mathcal{V}}_{\widetilde{x}}$ for all $i \in \Delta_{\widetilde{x}}$. Thus, $\widetilde{\zeta}_j = (u_j/u_i)\widetilde{\zeta}_i$ for all $j \in [\ell]$. This leads to the two statements in the display. The last statement follows from a direct check. □

Let $\widetilde{\mathcal{V}}_{\widetilde{x}}$ be as in Lemma 4.4. For each $i \in \Delta_{\widetilde{x}}$, the zero locus of $\widetilde{\zeta}_i$ on $\widetilde{\mathcal{V}}_{\widetilde{x}}$ is the same as $\mathcal{E} \cap \widetilde{\mathcal{V}}_{\widetilde{x}}$. We select an arbitrary $i \in \Delta_{\widetilde{x}}$ and set

$$\varepsilon = \widetilde{\zeta}_i|_{\widetilde{\mathcal{V}}_{\widetilde{x}}} \in \Gamma(\mathscr{O}_{\widetilde{\mathcal{V}}_{\widetilde{x}}}).$$

COROLLARY 4.5. *Let $\widetilde{\mathcal{V}}_{\widetilde{x}}$ be the affine smooth chart in Lemma 4.4. Then,*

$$\varepsilon; \quad \frac{\widetilde{\zeta}_j}{\varepsilon} - \left(\frac{\widetilde{\zeta}_j}{\varepsilon}(\widetilde{x})\right),\ j \in \Delta_{\widetilde{x}} \backslash \{i\}; \quad \check{\zeta}_j := \frac{\widetilde{\zeta}_j}{\varepsilon},\ j \in [\ell] \backslash \Delta_{\widetilde{x}}; \quad \text{and} \quad \check{\zeta}_k := \widetilde{\zeta}_k,\ k \in [r] \backslash [\ell] \tag{4.4}$$

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form a system of local parameters on $\widetilde{\mathcal{V}}_{\widetilde{x}}$. *Moreover,* ε *and all* $\check{\zeta}_j$, $j \in [\ell]\backslash\Delta_{\widetilde{x}}$, *vanish at* $\widetilde{x}$, *whereas* $\widetilde{\zeta}_j/\varepsilon$ *does not vanish at* $\widetilde{x}$ *for any* $j \in \Delta_{\widetilde{x}}$.

Proof. By Lemma 4.4, ε and $\check{\zeta}_j$, $j \in [\ell]\backslash\Delta_{\widetilde{x}}$ can be considered as local parameters. In addition, it follows from a direct check that on $\pi^{-1}(\mathcal{V})$, $\widetilde{\zeta}_k$ is a local parameter satisfying

$$\{\widetilde{\zeta}_k = 0\} = \mathrm{PT}\{\zeta_k = 0\} \quad \forall\, k \in [r]\backslash[\ell].$$

Therefore, every regular function in (4.4) is a local parameter. It is straightforward that these local parameters are independent, i.e. they form a subset of a system of local parameters.

To see the last statement, notice the display above (4.3) shows that $\widetilde{\zeta}_i(\widetilde{x}) = 0$ for the chosen $i \in \Delta_{\widetilde{x}}$, i.e. $\varepsilon(\widetilde{x}) = 0$. In addition, for every $j \in [\ell]$, the last statement follows from Lemma 4.4. □

We point out that $\widetilde{\mathcal{V}}_{\widetilde{x}}$ contains other local parameters describing the positions of points in

$$\{\widetilde{x}' \in \mathcal{E} \cap \widetilde{\mathcal{V}}_{\widetilde{x}} : \Delta_{\widetilde{x}'} = \Delta_{\widetilde{x}}\} \subset \mathcal{E} \cap \widetilde{\mathcal{V}}_{\widetilde{x}}.$$

We will always denote by $\widetilde{\zeta}_j$, $j \in [r]$, the **pullback** of ζ_j as in (4.2), and by $\check{\zeta}_j$, $j \in [r]\backslash\Delta_{\widetilde{x}}$, the **proper transform** of ζ_j as in (4.4).

4.4 Locally tree-compatible blowups

Once again we go back to Case A.4. In the second row of (4.1), observe that $\zeta_{[\delta_i]}$ divides $\zeta_{[\delta_j]}$ whenever $N_{[\delta_i]} \subset N_{[\delta_j]}$, so we only need to study those $N_{[\delta_i]}$ that are *minimal* with respect to inclusion among all $N_{[\delta_j]}$, $3 \le j \le m$. Also observe that in the dual graph (in the usual sense; cf. [HN25, Section 23.4]) of $\widehat{x}$, after contracting (in the usual sense of graph theory) the edges corresponding to the core nodes, we obtain a rooted tree τ' whose root corresponds to the core of C. After further contracting every edge e' of τ' that has a vertex carrying at least one marked point and lying between e' and the root, the resulting rooted tree τ has the following property: each path of τ connecting the root and a leaf corresponds to some minimal $N_{[\delta_i]}$ as mentioned above, and vice versa. Therefore, in this subsection, we introduce certain terminology about rooted trees that is needed for the pullbacks of φ.

Recall that in the usual language of graph theory, a **tree** refers to a connected graph whose first Betti number is 0, and a **rooted tree** is a tree along with a selected vertex known as the **root**. On the set E of the edges of a rooted tree τ, consider the partial order $\preceq$ given by $e \preceq e'$ if any path containing the root and e must contain e' as well. We call $\preceq$ the **tree order** of τ. Notice that any elements e' and e'' of E are comparable if and only if there exists $e \in E$ such that $e \preceq e'$ and $e \preceq e''$ hold simultaneously.

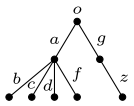
Conversely, any partially ordered set (poset) $(E, \preceq)$ satisfying the above property, up to isomorphisms of posets, uniquely determines a rooted tree τ whose set of edges is E, and the corresponding tree order is $\preceq$. The maximal and minimal elements of E are connected to the root and the **leaves** of τ, respectively.

The poset description turns out to be more convenient for the sequential blowups, and hence we utilize the following definition of rooted trees throughout the paper, which is identical to that used in [HN21].

Definition 4.6. A **rooted tree** is a finite (possibly empty) poset

$$\tau = (E, \preceq),$$

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$\tau = \{a, b, c, d, f, g, z\}$ satisfying $b, c, d, f \prec a$; $z \prec g$.

$\Xi(\tau) = \{\mathfrak{E}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}\}$, where $\mathfrak{E} := \{a, g\}$, $\mathfrak{F} := \{a, z\}$, $\mathfrak{G} := \{b, c, d, f, g\}$, $\mathfrak{H} := \{b, c, d, f, z\}$, satisfying $\mathfrak{H} \prec \mathfrak{F} \prec \mathfrak{E}$; $\mathfrak{H} \prec \mathfrak{G} \prec \mathfrak{E}$.

FIGURE 13. A rooted tree τ.

satisfying, for any $e', e'' \in E$,

$$(e' \text{ and } e'' \text{ are comparable}) \iff (\exists\, e \in E \text{ s.t. } e \preceq e' \text{ and } e \preceq e''). \tag{4.5}$$

We call the elements of E the **edges** of τ, and the order $\preceq$ the **tree order** of τ. When the context is clear, we identify τ with E and simply write $e \in \tau$ to refer to an edge of τ.

The set $\max(\tau)$ of the maximal elements of τ is called the **root** of τ. Each minimal element of τ is called a **leaf** of τ. The set of the leaves of τ is denoted by $\min(\tau)$.

A **root-to-leaf** path is a linearly ordered subset of τ that is maximal with respect to inclusion.

The edge-less rooted tree $(\emptyset, -)$ is said to be **trivial** and is denoted by $\tau_\bullet$.

The set of all rooted trees is denoted by $\mathbf{T}$.

Two rooted trees are said to be **isomorphic** if they are isomorphic as posets.

Remark 4.7. Although we consider the rooted trees as posets instead of graphs, it is convenient to illustrate them in the usual sense of graph theory; cf. Figure 13. The root, as per Definition 4.6, is exactly the set of the incident edges of the root vertex, which is labeled o (if needed). When the context is clear, the root vertex can also be called the root, for conciseness.

We emphasize that the trivial rooted tree $\tau_\bullet$, when visualized as a graph, is the edge-less graph that has exactly one vertex, instead of the empty graph.

For every $E' \subset \tau$, let

$$(E')^{\mathrm{R}} = \bigcup_{e' \in E'} \{e \in \tau : e \,\mathrm{R}\, e'\} \quad \text{with} \quad \mathrm{R} = \prec, \preceq, \succ, \text{ and } \succeq. \tag{4.6}$$

For instance, $(E')^{\prec}$ consists of $e \in \tau$ satisfying $e \prec e'$ for *some* $e' \in E'$ (not necessarily all $e' \in E'$). In Figure 13, for example, $\{a, g\}^{\prec} = \{b, c, d, f, z\}$ and $\{b, z\}^{\succeq} = \{a, b, g, z\}$.

The following notion plays a key role in the local equations of the blowup centers of any phase.

DEFINITION 4.8. Let $\tau \in \mathbf{T}$. A nonempty subset $\mathfrak{E} \subset \tau$ is called a **transverse section** of τ if it meets either of the following two equivalent conditions:

- $\mathfrak{E}$ meets every root-to-leaf path of τ at exactly one edge;
- $\mathfrak{E}$ is a maximal subset of incomparable edges of τ, i.e. every pair of distinct edges of $\mathfrak{E}$ are incomparable, and every edge of τ is comparable with an edge of $\mathfrak{E}$.

The set of all the transverse sections of τ is denoted by $\Xi(\tau)$.

The set $\Xi(\tau)$ is always nonempty as long as $\tau \neq \tau_\bullet$, because $\max(\tau) \in \Xi(\tau)$ and $\min(\tau) \in \Xi(\tau)$ always hold.

The tree order on τ induces a partial order, still denoted by $\preceq$, on $\Xi(\tau)$ such that for every $\mathfrak{E}, \mathfrak{E}' \in \Xi(\tau)$,

$$(\mathfrak{E} \prec \mathfrak{E}') \iff (\mathfrak{E} \neq \mathfrak{E}') \text{ and } (\mathfrak{E} \subset (\mathfrak{E}')^{\preceq}). \tag{4.7}$$

If $\tau \neq \tau_\bullet$, then the greatest and least elements of $\Xi(\tau)$ are, respectively, $\max(\tau)$ and $\min(\tau)$.

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For example, in Figure 13, we have

$$\Xi(\tau)=\{\mathfrak{E},\mathfrak{F},\mathfrak{G},\mathfrak{H}\},\quad \text{where}$$
$$\mathfrak{E}:=\{a,g\},\quad \mathfrak{F}:=\{a,z\},\quad \mathfrak{G}:=\{b,c,d,f,g\},\quad \mathfrak{H}:=\{b,c,d,f,z\}.$$

The partial order (4.7) is given by $\mathfrak{H}\prec\mathfrak{F}\prec\mathfrak{E}$ and $\mathfrak{H}\prec\mathfrak{G}\prec\mathfrak{E}$ in this case.

Let τ be a rooted tree and $\mathcal{V}$ be an affine smooth chart of a stack $\mathfrak{M}$. If there exists a system of local parameters on $\mathcal{V}$ as follows:

$$\{\zeta_e\in\Gamma(\mathscr{O}_{\mathcal{V}}):\ e\in\tau\}\sqcup\{\zeta_s\in\Gamma(\mathscr{O}_{\mathcal{V}}):\ s\in S'\},\tag{4.8}$$

then $\{\zeta_e\in\Gamma(\mathscr{O}_{\mathcal{V}}):e\in\tau\}$ is called a τ-**labeled subset of local parameters** on $\mathcal{V}$. In this case, we set

$$\zeta_{[e]}:=\prod_{e'\succeq e}\zeta_{e'}\quad\forall\, e\in\tau.\tag{4.9}$$

Every $E\subset\tau$ determines a closed locus $X_E^{\mathcal{V}}$ in $\mathcal{V}$ by

$$X_E^{\mathcal{V}}=\{\zeta_e=0:\ e\in E\}\quad\subset\mathcal{V}.$$

When the context is clear, we may simply write this as X_E.

With the notation as above, let

$$\pi:\widetilde{\mathfrak{M}}\longrightarrow\mathfrak{M}\quad\text{and}\quad\pi_{(k)}:\widetilde{\mathfrak{M}}_{(k)}\longrightarrow\mathfrak{M}\tag{4.10}$$

be the blowup of $\mathfrak{M}$ successively along the proper transforms of a sequence of closed substacks

$$Z_1,Z_2,\cdots\subset\mathfrak{M}$$

and the same blowup of $\mathfrak{M}$ after the kth step, respectively. We may call π a sequential blowup to emphasize that it is comprised of a sequence of blowups.

Definition 4.9. Let $\tau\in\mathbf{T}$ and $\pi:\widetilde{\mathfrak{M}}\to\mathfrak{M}$ be as in (4.10). Then, π is said to be τ-**compatible on** $\mathcal{V}$ if there exists a partition of $\Xi(\tau)$:

$$\Xi(\tau)=\bigsqcup_{k\geq1}\Xi_k(\tau),$$

and a τ-labeled subset of local parameters on $\mathcal{V}$ such that:

(C1) for every $k\geq1$,

$$Z_k\cap\mathcal{V}=\bigcup_{\mathfrak{E}\in\Xi_k(\tau)}X_{\mathfrak{E}}^{\mathcal{V}};$$

(C2) if $\mathfrak{E}'\in\Xi_{k'}(\tau)$, $\mathfrak{E}''\in\Xi_{k''}(\tau)$, and $\mathfrak{E}'\succ\mathfrak{E}''$, then $k'<k''$.

As $\Xi(\tau)$ is a finite set, there exist only *finitely* many $k\in\mathbb{Z}_{\geq0}$ with $Z_k\cap\mathcal{V}\neq\emptyset$.

Locally, the blowup centers are naturally given by the unions of some $X_{\mathfrak{E}}^{\mathcal{V}}$ as in (C1). Example 4.11 below will demonstrate this point.

The condition (C2) has three immediate conclusions. First, a τ-compatible blowup of $\mathcal{V}$ must begin with $X_{\max(\tau)}$, unless τ is trivial (i.e. edge-less). Second, in each $\Xi_k(\tau)$, distinct elements are always not comparable. The final conclusion is the following lemma, which shows that every τ-compatible blowup on $\mathcal{V}$ is always a *smooth* blowup of $\mathcal{V}$, although certain blowup centers $Z_k\cap\mathcal{V}$ may be singular a priori (in fact, $Z_k\cap\mathcal{V}$ is singular whenever $|\Xi_k(\tau)|\geq2$). This is a

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crucial observation that guarantees that the final stack $\widetilde{\mathfrak{M}}_2^{\mathrm{div}}$ proposed in Theorem 1.1 is a smooth stack.

LEMMA 4.10. *Let $\pi:\widetilde{\mathfrak{M}}\to\mathfrak{M}$ be τ-compatible on $\mathcal{V}$ as in Definition 4.9. Then for each $k\geq 1$, the proper transforms of the irreducible components $X_{\mathfrak{E}}^{\mathcal{V}}$ of $Z_k\cap\mathcal{V}$ after the $(k-1)$th step are all smooth and pairwise disjoint. Therefore, the proper transform of $Z_k\cap\mathcal{V}$ after the $(k-1)$th step is smooth, and hence the pullback of $\mathcal{V}$ after each step is smooth.*

Proof. For distinct $\mathfrak{E},\mathfrak{E}'\in\Xi_k(\tau)$, $\mathfrak{E}$ and $\mathfrak{E}'$ are not comparable by (C2). Let

$$\mathfrak{F}=\max(\mathfrak{E}\cup\mathfrak{E}'),$$

which is nonempty because $\mathfrak{E}$ and $\mathfrak{E}'$ are nonempty as transverse sections. Distinct elements of $\mathfrak{F}$ are incomparable because they are maximal elements of $\mathfrak{E}\cup\mathfrak{E}'$. Finally, for every $e'\in\tau$, there exist $e\in\mathfrak{E}$ comparable with e', as well as $e''\in\max(\mathfrak{E}\cup\mathfrak{E}')$ satisfying $e\preceq e''$ (because $\mathfrak{E}\cup\mathfrak{E}'$ is finite). Therefore, if $e'\preceq e$, then $e'\preceq e''$; if $e'\succ e$, then e' and e'' are still comparable because of (4.5). In sum, $\mathfrak{F}$ satisfies the latter criterion (hence both criteria) of Definition 4.8, and hence $\mathfrak{F}\in\Xi(\tau)$. Indeed, $\mathfrak{F}$ is the least common successor of $\mathfrak{E}$ and $\mathfrak{E}'$ with respect to the order (4.7).

Since $\mathfrak{F}\subset\mathfrak{E}\cup\mathfrak{E}'$, we have

$$X_{\mathfrak{E}}\cap X_{\mathfrak{E}'}=X_{\mathfrak{E}\cup\mathfrak{E}'}\subset X_{\mathfrak{F}}.$$

(Here, we drop the superscript $\mathcal{V}$ of the $X_{\mathfrak{E}}$ for conciseness.) The restriction of the normal bundle of $X_{\mathfrak{F}}$ in $\mathcal{V}$ to $X_{\mathfrak{E}}\cap X_{\mathfrak{E}'}$ can be written as

$$\left(\bigoplus_{e\in\mathfrak{F}\cap\mathfrak{E}\cap\mathfrak{E}'}(\mathscr{O}_{\mathcal{V}}(X_e)|_{X_{\mathfrak{E}}\cap X_{\mathfrak{E}'}})\right)\oplus\left(\bigoplus_{e\in\mathfrak{F}\cap(\mathfrak{E}\setminus\mathfrak{E}')}(\mathscr{O}_{\mathcal{V}}(X_e)|_{X_{\mathfrak{E}}\cap X_{\mathfrak{E}'}})\right)\oplus\left(\bigoplus_{e\in\mathfrak{F}\cap(\mathfrak{E}'\setminus\mathfrak{E})}(\mathscr{O}_{\mathcal{V}}(X_e)|_{X_{\mathfrak{E}}\cap X_{\mathfrak{E}'}})\right),$$

where the leftmost direct sum is in the normal direction of both $X_{\mathfrak{E}}$ and $X_{\mathfrak{E}'}$, while the middle (respectively, rightmost) sum is in the normal direction of $X_{\mathfrak{E}}$ (respectively, $X_{\mathfrak{E}'}$) and the tangent direction of $X_{\mathfrak{E}'}$ (respectively, $X_{\mathfrak{E}}$).

Let $h\geq 1$ be such that $X_{\mathfrak{F}}\subset Z_h$. By (C2), $h<k$. The last paragraph implies that the proper transforms of $X_{\mathfrak{E}}$ and $X_{\mathfrak{E}'}$ become disjoint after $X_{\mathfrak{F}}$ is blown up in the hth step. The smoothness of the proper transforms of $X_{\mathfrak{E}}$ and $X_{\mathfrak{E}'}$ after step h follows from direct computation; see [VZ08, Lemma 2.3] for more detail.

Since we have shown that the blowup center in each step is smooth, the last statement of Lemma 4.10 holds immediately. □

For each $k\geq$ and $\mathfrak{E}\in\Xi_k(\tau)$, the exceptional divisor obtained by blowing up $\check{X}_{\mathfrak{E}}^{\mathcal{V}}$ is written as

$$\mathcal{E}_{\mathfrak{E}}\subset\pi_{(k)}^{-1}(\mathcal{V}). \tag{4.11}$$

By Lemma 4.10, these $\mathcal{E}_{\mathfrak{E}}$, $\mathfrak{E}\in\Xi_k(\tau)$, are disjoint. Moreover, by a direct check, we see that each $\mathcal{E}_{\mathfrak{E}}$ is transverse to the proper transforms of $X_{\mathfrak{E}'}^{\mathcal{V}}$ for all $\mathfrak{E}'\in\Xi_{k'}(\tau)$, $k'>k$. Thus, the proper transform of $\mathcal{E}_E$ in $\pi^{-1}(\mathcal{V})$ is the same as its pullback, i.e.

$$\check{\mathcal{E}}_{\mathfrak{E}}=\widetilde{\mathcal{E}}_{\mathfrak{E}}\quad(\subset\pi^{-1}(\mathcal{V}))\quad\forall\,\mathfrak{E}\in\Xi(\tau). \tag{4.12}$$

Finally, we comment that the notions of $\mathcal{E}_{\mathfrak{E}}$ and $\check{\mathcal{E}}_{\mathfrak{E}}$ are intrinsic to $\mathcal{V}$ and τ. They are independent of the choice of the τ-compatible blowup on $\mathcal{V}$.

Example 4.11. An example of locally tree-compatible blowups is the the blowup $\widetilde{\mathfrak{M}}_1^{\mathrm{wt}}$ of the Artin stack $\mathfrak{M}_1^{\mathrm{wt}}$ in [HL10], which is performed successively along the proper transforms of the

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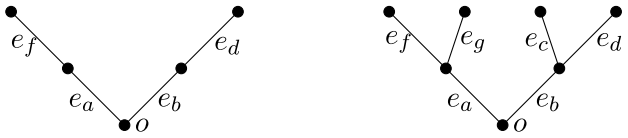


FIGURE 14. Rooted trees.

closed loci Θ_k, $k=1,2,\ldots$. Here, a general point of each Θ_k consists of a weight-0 genus 1 core and k positively weighted tails. In other words, the Θ_k are the genus 1 counterparts of the $\overline{\mathfrak{M}}_{(1,k)}$ in Figure 1.

For every $x=(C,\ \mathbf{w})\in\mathfrak{M}_1^{\mathrm{wt}}$, let τ be its terminally weighted tree in the sense of [HL10, Paragraph 3.16]. Then, there exists an affine smooth chart $\mathcal{V}\to\mathfrak{M}_1^{\mathrm{wt}}$ containing x such that the set of node-smoothing parameters $\{\zeta_e\}_{e\in\tau}$ is a τ-labeled subset of local parameters on $\mathcal{V}$. Each Θ_k satisfies

$$\Theta_k\cap\mathcal{V}=\bigcup_{\mathfrak{E}\in\Xi_k(\tau)}X^{\mathcal{V}}_{\mathfrak{E}},\quad\text{where }\Xi_k(\tau)=\{\mathfrak{E}\in\Xi(\tau):|\mathfrak{E}|=k\}.$$

The stability of $\mathfrak{M}_g^{\mathrm{wt}}$ implies that for every $\mathfrak{E},\mathfrak{E}'\in\Xi(\tau)$ with $\mathfrak{E}\succ\mathfrak{E}'$, we have $|\mathfrak{E}|<|\mathfrak{E}'|$. One can thus check directly that the blowup successively along the proper transforms of $\Theta_1,\Theta_2,\cdots$ is τ-compatible on $\mathcal{V}$, hence that the sequential blowup in [HL10] is locally tree-compatible.

We point out it is because of the stability of $\mathfrak{M}_g^{\mathrm{wt}}$ that the above partition of $\Xi(\tau)$ by the cardinals of the transverse sections actually leads to a locally τ-compatible blowup on $\mathcal{V}$. Generally, given an arbitrary rooted tree τ', there may exist $\mathfrak{E},\mathfrak{E}'\in\Xi(\tau')$ satisfying $\mathfrak{E}\succ\mathfrak{E}'$ yet $|\mathfrak{E}|=|\mathfrak{E}'|$; see the leftmost diagram of Figure 14.

For a concrete example, consider the tree τ illustrated in the rightmost diagram of Figure 14. The sequential blowup of $\mathfrak{M}_1^{\mathrm{wt}}$ determines a partition

$$\Xi(\tau)=\{\mathfrak{E}_2=\{e_a,e_b\}\}\sqcup\{\mathfrak{E}_3=\{e_a,e_c,e_d\},\ \mathfrak{E}_3'=\{e_b,e_f,e_g\}\}\sqcup\{\mathfrak{E}_4=\{e_f,e_g,e_c,e_d\}\}\tag{4.13}$$

such that

$$\Theta_2\cap\mathcal{V}=X^{\mathcal{V}}_{\mathfrak{E}_2},\quad\Theta_3\cap\mathcal{V}=X^{\mathcal{V}}_{\mathfrak{E}_3}\cup X^{\mathcal{V}}_{\mathfrak{E}_3'},\quad\Theta_4\cap\mathcal{V}=X^{\mathcal{V}}_{\mathfrak{E}_4},\quad\mathfrak{E}_2\succ\mathfrak{E}_3,\mathfrak{E}_3'\succ\mathfrak{E}_4,$$

thus (C1) and (C2) in Definition 4.9 are satisfied. Notice that $\Theta_3\cap\mathcal{V}=X^{\mathcal{V}}_{\mathfrak{E}_3}\cup X^{\mathcal{V}}_{\mathfrak{E}_3'}$ has normal crossing singularities at $X^{\mathcal{V}}_{\mathfrak{E}_3}\cap X^{\mathcal{V}}_{\mathfrak{E}_3'}$ $(=X^{\mathcal{V}}_{\tau})$. Nonetheless, after blowing up along $X^{\mathcal{V}}_{\mathfrak{E}_2}$, the proper transforms of $X^{\mathcal{V}}_{\mathfrak{E}_3}$ and $X^{\mathcal{V}}_{\mathfrak{E}_3'}$ are smooth and meet the exceptional divisor $\mathcal{E}_{\mathfrak{E}_2}$ at the loci given by $\{[u_a,u_b]=[0,1]\}$ and $\{[u_a,u_b]=[1,0]\}$, respectively, hence they become disjoint. Here, $[u_a,u_b]$ denotes the homogeneous coordinate on $\mathbb{P}^1$ satisfying $\widetilde{\zeta}_{e_a}u_b=\widetilde{\zeta}_{e_b}u_a$.

As this example reveals, given a blowup center Z_k and a chart $\mathcal{V}$, $Z_k\cap\mathcal{V}$ is possibly a union of $X^{\mathcal{V}}_{\mathfrak{E}}$ for some $\mathfrak{E}\in\Xi(\tau)$. This explains (C1) of Definition 4.9. Nonetheless, this example also demonstrates that the proper transform of each blowup center Z_k becomes smooth after the $(k-1)$th step, which is proved in Lemma 4.10.

Example 4.12. In Definition 4.9, (C1) implies that the partition of $\Xi(\tau)$ is solely determined by the sequence of the blowup centers Z_k, $k\geq 1$. As suggested by the following example, however, the same blowup $\widetilde{\mathfrak{M}}/\mathfrak{M}$ may be achieved by choosing a different sequence of blowup centers, which may yield a different partition of $\Xi(\tau)$.

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FIGURE 15. An example of grafting.

Let $\mathcal{V}\to\mathfrak{M}_1^{\mathrm{wt}}$ and τ be as in Example 4.11. In this example, we take the base stack $\mathfrak{M}$ of Definition 4.9 to be $\mathcal{V}$ instead of $\mathfrak{M}_1^{\mathrm{wt}}$, and consider $\widetilde{\mathcal{V}}/\mathcal{V}$, the restriction of the blowup $\widetilde{\mathfrak{M}}_1^{\mathrm{wt}}/\mathfrak{M}_1^{\mathrm{wt}}$ to $\mathcal{V}$. From Example 4.11, we already know that $\widetilde{\mathcal{V}}/\mathcal{V}$ can be achieved by blowing up $\mathcal{V}$ along the proper transforms of $\Theta_2\cap\mathcal{V}$, $\Theta_3\cap\mathcal{V}$, and $\Theta_4\cap\mathcal{V}$ successively, which gives the partition (4.13) of $\Xi(\tau)$. On the other hand, notice that $\widetilde{\mathcal{V}}/\mathcal{V}$ can also be achieved by blowing up $\mathcal{V}$ successively along the proper transforms of $X^{\mathcal{V}}_{\mathfrak{E}_2}$, $X^{\mathcal{V}}_{\mathfrak{E}_3}$, $X^{\mathcal{V}}_{\mathfrak{E}'_3}$, and $X^{\mathcal{V}}_{\mathfrak{E}_4}$, where the four transverse sections are the same as in (4.13). This also determines a partition of $\Xi(\tau)$ given by

$$\Xi(\tau)=\{\mathfrak{E}_2\}\sqcup\{\mathfrak{E}_3\}\sqcup\{\mathfrak{E}'_3\}\sqcup\{\mathfrak{E}_4\},$$

which is different from (4.13).

It is straightforward to show that the above two sequential blowups are both τ-compatible on $\mathcal{V}$, and indeed yield the same $\widetilde{\mathcal{V}}/\mathcal{V}$. As $\mathfrak{E}_3$ and $\mathfrak{E}'_3$ are not comparable, this example, in particular, shows that the converse of (C2) need not be true.

At the end of this subsection, we describe a sequential blowup induced from a τ-compatible sequential blowup on $\mathcal{V}$. This is related to the modular blowups in (r_3) and is needed for proving many cases of Proposition 4.1, including Case A.4.

For every rooted tree $\tau=(E,\preceq)$, let $\mathrm{gr}(\tau)$ be the rooted tree

$$\mathrm{gr}(\tau)=(E\sqcup\{e_{\mathrm{gr}}\},\ \preceq),$$

where $\preceq$ is the relation on $\mathrm{gr}(\tau)$ given by the tree order on τ as well as the following: every $e\in E$ is *not* comparable with e_{gr}. One can check directly that $\preceq$ is a tree order as per Definition 4.6, and that e_{gr} is both maximal and minimal. The rooted tree $\mathrm{gr}(\tau)$ is uniquely determined by τ (up to an isomorphism of rooted trees).

Intuitively, $\mathrm{gr}(\tau)$ is obtained from τ by **grafting** a new vertex to the root via a new edge e_{gr}. An example is illustrated in Figure 15.

Assume $\pi:\widetilde{\mathfrak{M}}\to\mathfrak{M}$ as in (4.10) is τ-compatible on $\mathcal{V}$. A blowup

$$\pi_{\mathrm{gr}}:\widetilde{\mathfrak{M}}_{\mathrm{gr}}\longrightarrow\mathfrak{M}$$

successively along the proper transforms of closed substacks $Z_{\mathrm{gr},k}$, $k=1,2,\ldots$, of $\mathfrak{M}$ is called a **grafted blowup** with respect to π on $\mathcal{V}$ if there exists a local parameter ζ_{gr} such that $\{\zeta_e:e\in\tau\}\sqcup\{\zeta_{\mathrm{gr}}\}$ is a subset of a system of local parameters on $\mathcal{V}$, and

$$Z_{\mathrm{gr},k}\cap\mathcal{V}=\bigcup_{\mathfrak{E}\in\Xi_k(\tau)}X^{\mathcal{V}}_{\mathfrak{E}\sqcup\{e_{\mathrm{gr}}\}}\quad\forall\,k\ge1.$$

Observe that for each $\mathfrak{E}\in\Xi(\tau)$, we have $\mathfrak{E}\sqcup\{e_{\mathrm{gr}}\}\in\Xi(\mathrm{gr}(\tau))$; conversely, every $\mathfrak{E}'\in\Xi(\mathrm{gr}(\tau))$ must contain e_{gr}, and $\mathfrak{E}'\backslash\{e_{\mathrm{gr}}\}\in\Xi(\tau)$. This leads immediately to the following statement.

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Lemma 4.13. *Let $\pi:\widetilde{\mathfrak{M}}\to\mathfrak{M}$ and $\pi_{\mathrm{gr}}:\widetilde{\mathfrak{M}}_{\mathrm{gr}}\to\mathfrak{M}$ be two sequential blowups, and $\mathcal{V}\to\mathfrak{M}$ be a small affine smooth chart. If π is τ-compatible on $\mathcal{V}$, and π_{gr} is a grafted blowup with respect to π on $\mathcal{V}$, then π_{gr} is $\mathrm{gr}(\tau)$-compatible on $\mathcal{V}$.*

4.5 Root-to-leaf sequences, dominant edges, and the proof of Case A.4

We continue with the notation from Section 4.4. Let $\pi:\widetilde{\mathfrak{M}}\to\mathfrak{M}$ be the blowup in (4.10) and $\tau\in\mathbf{T}$. Assume that π is τ-compatible on $\mathcal{V}$ as per Definition 4.9. For every $\mathfrak{E}\in\Xi(\tau)$, we still use $\check{\mathcal{E}}_{\mathfrak{E}}$ to denote the proper transform of the exceptional divisor $\mathcal{E}_{\mathfrak{E}}$ in (4.11), which, by (4.12), is the same as the pullback of $\mathcal{E}_{\mathfrak{E}}$.

In this subsection, we fix $\widehat{x}\in X_\tau^{\mathcal{V}}\subset\mathcal{V}$ (i.e. $\zeta_e(\widehat{x})=0$ for all $e\in\tau$) as well as a lift $\widetilde{x}$ of $\widehat{x}$ in $\widetilde{\mathcal{V}}=\pi^{-1}(\mathcal{V})$. If $\tau=\tau_\bullet$, which implies that $\Xi(\tau)=\emptyset$, then the blowup centers do not meet $\mathcal{V}$, so the blowup of $\mathcal{V}$ is equal to $\mathcal{V}$.

If $\tau\neq\tau_\bullet$, then

$$\overline{\mathbf{E}}:=\{\mathfrak{E}\in\Xi(\tau):\ \widetilde{x}\in\check{\mathcal{E}}_{\mathfrak{E}}\}\tag{4.14}$$

is nonempty.

Lemma 4.14. *Assume $\tau\neq\tau_\bullet$. Then we can order the elements of $\overline{\mathbf{E}}$ so that $\overline{\mathbf{E}}=\{\mathfrak{E}_1,\dots,\mathfrak{E}_{\mathbf{m}}\}$, satisfying*

$$\mathfrak{E}_1\succ\cdots\succ\mathfrak{E}_{\mathbf{m}}\quad\text{and}\quad\bigcup_{i\in[k]}\mathfrak{E}_i=\mathfrak{E}_k^{\succeq}\quad\forall\,k\in[\mathbf{m}].\tag{4.15}$$

Hereafter, we call $\overline{\mathbf{E}}$ the root-to-leaf sequence (RLS) of $\widetilde{x}$, and treat $\overline{\mathbf{E}}$ both as a subset of $\Xi(\tau)$ and as a finite sequence of the elements of $\Xi(\tau)$, should no ambiguity occur. The notion of RLS can be generalized to the $\tau=\tau_\bullet$ case, in which we set $\mathbf{m}=0$ and call $\emptyset$ the RLS of $\widetilde{x}$.

Proof of Lemma 4.14. We will begin with an inductive construction of a sequence $\mathfrak{E}_1,\dots,\mathfrak{E}_{\mathbf{m}}$ satisfying $\widetilde{x}\in\bigcap_{i\in[\mathbf{m}]}\check{\mathcal{E}}_{\mathfrak{E}_i}$ as well as (4.15). We will then prove $\widetilde{x}\notin\check{\mathcal{E}}_{\mathfrak{E}}$ for any other $\mathfrak{E}\in\Xi(\tau)$.

First, notice that any τ-compatible blowup on $\mathcal{V}$ must begin with the blowup $\mathcal{V}$ along $X^{\mathcal{V}}_{\max(\tau)}$. More precisely, in Definition 4.9, by setting $k_1:=\min\{k\geq1:\Xi_k(\tau)\neq\emptyset\}$, we see that $\Xi_{k_1}(\tau)$ contains the root $\max(\tau)$ as its unique element, because $\max(\tau)$ is the greatest element of $\Xi(\tau)$. By writing

$$\mathfrak{E}_1=\max(\tau),$$

we have $\mathfrak{E}_1=\mathfrak{E}_1^{\succeq}$. In addition, since $x\in X_\tau^{\mathcal{V}}\subset X^{\mathcal{V}}_{\mathfrak{E}_1}$, we have $\widetilde{x}\in\widetilde{\mathcal{E}}_{\mathfrak{E}_1}$, which is equivalent to $\widetilde{x}\in\check{\mathcal{E}}_{\mathfrak{E}_1}$ by (4.12).

Next, assume that we have constructed $\mathfrak{E}_1\succ\cdots\succ\mathfrak{E}_k$ satisfying $\widetilde{x}\in\bigcap_{i\in[k]}\check{\mathcal{E}}_{\mathfrak{E}_i}$ and $\bigcup_{i\in[k]}\mathfrak{E}_i=\mathfrak{E}_k^{\succeq}$ for some $k\geq1$. We denote by h_k the unique step of the sequential blowup such that $\mathfrak{E}_k\in\Xi_{h_k}(\tau)$. Recall that $\pi_{(h_k)}:\widetilde{\mathfrak{M}}_{(h_k)}\to\mathfrak{M}$ denotes the blowup after the h_kth step. Let $\widetilde{x}_k$ be the image of $\widetilde{x}$ in $\widetilde{\mathfrak{M}}_{(h_k)}$. By the inductive hypothesis, we have $\widetilde{x}_k\in\mathcal{E}_{\mathfrak{E}_k}$. We thus define

$$\Delta_{\widetilde{x},k}\subset\mathfrak{E}_k$$

analogously to (4.3), with $\widetilde{x}$ and $[\ell]$ in (4.3) replaced by $\widetilde{x}_k$ and $\mathfrak{E}_k$, respectively.

- If $\Delta_{\widetilde{x},k}\cap\min(\tau)\neq\emptyset$, we simply set $\mathbf{m}=k$ and $\overline{\mathbf{E}}=\{\mathfrak{E}_1,\dots,\mathfrak{E}_k\}$.

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- If $\Delta_{\widetilde{x},k} \cap \min(\tau) = \emptyset$, then for every $e \in \Delta_{\widetilde{x},k}$, the set $\{e\}^{\prec}$ ($= \{e' \in \tau : e' \prec e\}$) is nonempty. Let

$$\mathfrak{E}_{k+1} := (\mathfrak{E}_k \backslash \Delta_{\widetilde{x},k}) \sqcup \bigsqcup_{e \in \Delta_{\widetilde{x},k}} \max(\{e\}^{\prec}). \tag{4.16}$$

  We claim that this is the $(k+1)$th term of the proposed sequence $\mathfrak{E}_1, \ldots, \mathfrak{E}_{\mathbf{m}}$. Indeed, by a direct check, we have

$$\mathfrak{E}_{k+1} \in \Xi(\tau), \quad \mathfrak{E}_{k+1} \prec \mathfrak{E}_k, \quad \text{and} \quad \bigcup_{i \in [k+1]} \mathfrak{E}_i = \mathfrak{E}_k^{\succeq} \cup \mathfrak{E}_{k+1} = \mathfrak{E}_{k+1}^{\succeq}.$$

  To see that $\widetilde{x} \in \widetilde{\mathcal{E}}_{\mathfrak{E}_{k+1}}$, notice that on a small neighborhood $\widetilde{\mathcal{V}}_{\widetilde{x}_k}$ of $\widetilde{x}_k$, the proper transform $\check{X}_{\mathfrak{E}_{k+1};(h_k)}$ of $X_{\mathfrak{E}_{k+1}}$ is given by

$$\check{X}_{\mathfrak{E}_{k+1};(h_k)} \cap \widetilde{\mathcal{V}}_{\widetilde{x}_k} = \{\check{\zeta}_{k;e} = 0 : e \in \mathfrak{E}_{k+1}\},$$

  where the local parameters $\check{\zeta}_{k;e}$ are analogous to (4.4). By the last statement of Corollary 4.5, we have $\check{\zeta}_{k;e}$ for all $e \in \mathfrak{E}_k \backslash \Delta_{\widetilde{x},k}$; in addition, for every $e \in \Delta_{\widetilde{x},k}$ and $e' \in \max(\{e\}^{\prec})$, we have $\check{\zeta}_{k;e'} = \widetilde{\zeta}_{k;e'} = 0$, where $\widetilde{\zeta}_{k;e'}$ is the pullback of $\zeta_{e'}$ to $\widetilde{\mathcal{V}}_{\widetilde{x}_k}$. In sum, we obtain $\widetilde{x}_k \in \check{X}_{\mathfrak{E}_{k+1};(h_k)}$, which implies $\widetilde{x} \in \widetilde{\mathcal{E}}_{\mathfrak{E}_{k+1}}$.

It remains to show that, for every $\mathfrak{E} \in \Xi(\tau) \backslash \overline{\mathbf{E}}$, we have $\widetilde{x} \notin \check{\mathcal{E}}_{\mathfrak{E}}$. To see this, we claim that there exists $j \in [\mathbf{m}]$ satisfying

$$\mathfrak{E}_j \succeq \mathfrak{E} \quad \text{and} \quad \Delta_{\widetilde{x},j} \cap \mathfrak{E} \neq \emptyset. \tag{4.17}$$

Indeed, the set $\{\mathfrak{E}' \in \overline{\mathbf{E}} : \mathfrak{E}' \succeq \mathfrak{E}\}$ is nonempty because $\mathfrak{E}_1 = \max(\tau) \in \overline{\mathbf{E}}$; we take $\mathfrak{E}_j$ to be the least element (w.r.t. the order in (4.15)) of this set. Then, $\mathfrak{E}_j \succeq \mathfrak{E}$ but $\mathfrak{E}_{j+1} \not\succeq \mathfrak{E}$.

- If $j < \mathbf{m}$, then $\Delta_{\widetilde{x},j} = \mathfrak{E}_j \backslash \mathfrak{E}_{j+1}$. So if $\Delta_{\widetilde{x},j} \cap \mathfrak{E} = \emptyset$, then by (4.16), we would have $\mathfrak{E}_{j+1} \succeq \mathfrak{E}$, contradicting the fact $\mathfrak{E}_{j+1} \not\succeq \mathfrak{E}$. Therefore, $\Delta_{\widetilde{x},j} \cap \mathfrak{E} \neq \emptyset$.
- If $j = \mathbf{m}$, notice the inductive construction of $\mathfrak{E}_1, \ldots, \mathfrak{E}_{\mathbf{m}}$ implies that $\Delta_{\widetilde{x},\mathbf{m}} \cap \min(\tau) \neq \emptyset$. For any $e' \in \Delta_{\widetilde{x};\mathbf{m}} \cap \min(\tau)$, we have $e' \in \mathfrak{E}$ because e' is minimal and $\mathfrak{E} \preceq \mathfrak{E}_j$. Therefore, $\Delta_{\widetilde{x},j} \cap \mathfrak{E} \neq \emptyset$.

Now that (4.17) is established, consider the aforementioned lift $\widetilde{x}_j$ of x and its neighborhood $\widetilde{\mathcal{V}}_{\widetilde{x}_j}$. On $\widetilde{\mathcal{V}}_{\widetilde{x}_j}$, the pullback of $X_{\mathfrak{E}}^{\mathcal{V}}$ takes the form

$$\{\varepsilon_j = 0;\ \check{\zeta}_{j;e} = 0 \,\forall\, e \in \mathfrak{E} \backslash \Delta_{\widetilde{x},j}\} \quad \subset \{\varepsilon_j = 0\} = \mathcal{E}_{\mathfrak{E}_j} \cap \widetilde{\mathcal{V}}_{\widetilde{x}_j}, \tag{4.18}$$

where the local parameters ε_j and $\check{\zeta}_{j;e}$ are analogous to (4.4). Therefore, the proper transform of $X_{\mathfrak{E}}^{\mathcal{V}}$ is empty on $\widetilde{\mathcal{V}}_{\widetilde{x}_j}$, hence does not contain $\widetilde{x}_j$, the image of $\widetilde{x}$ on $\widetilde{\mathfrak{M}}_{(h_j)}$. Therefore, $\widetilde{x} \notin \check{\mathcal{E}}_{\mathfrak{E}}$. □

The inductive construction of $\mathfrak{E}_1, \ldots, \mathfrak{E}_{\mathbf{m}}$ in the proof of Lemma 4.14 implies, in particular, that

$$\Delta_{\widetilde{x},\mathbf{m}} \cap \min(\tau) \neq \emptyset \quad (\Longrightarrow \mathfrak{E}_{\mathbf{m}} \cap \min(\tau) \neq \emptyset).$$

DEFINITION 4.15. The edges in

$$\Delta_{\widetilde{x}} := \bigsqcup_{i \in [\mathbf{m}]} \Delta_{\widetilde{x},i} \quad \text{and} \quad \Delta_{\widetilde{x}}^{\min} := \Delta_{\widetilde{x}} \cap \min(\tau)\ (= \Delta_{\widetilde{x},\mathbf{m}} \cap \min(\tau))$$

are called, respectively, the dominant edges and dominant leaves of $\widetilde{x}$.

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Here, recall that the leaves of a rooted tree refer to the minimal edges, as per Definition 4.6.

For every $k \in [\mathbf{m}]$, the exceptional divisor $\mathcal{E}_{\mathfrak{E}_k}$ is the zero locus of a local parameter ε_k on an affine smooth chart $\widetilde{\mathcal{V}}_{\widetilde{x}_k}$ containing $\widetilde{x}_k$. The proper transform $\check{\mathcal{E}}_{\mathfrak{E}_k}$ is thus the zero locus of a local parameter $\check{\varepsilon}_k$ on an affine smooth chart $\widetilde{\mathcal{V}}_{\widetilde{x}}$ containing $\widetilde{x}$.

LEMMA 4.16. *Let τ, $\mathcal{V}$, and $\pi : \widetilde{\mathfrak{M}} \to \mathfrak{M}$ be as in Lemma 4.10, $\widetilde{x} \in \pi^{-1}(X_\tau^{\mathcal{V}})$, and $\overline{\mathbf{E}} = \{\mathfrak{E}_1, \ldots, \mathfrak{E}_{\mathbf{m}}\}$ be the RLS of $\widetilde{x}$. Then:*

(1) *for each $i \in [\mathbf{m}]$, ε_i can be taken as the pullback $\widetilde{\zeta}_{\mathsf{e}_i} \in \Gamma(\mathscr{O}_{\widetilde{\mathcal{V}}_{\widetilde{x}_i}})$ of $\zeta_{\mathsf{e}_i} \in \Gamma(\mathscr{O}_{\mathcal{V}})$ with an arbitrary edge $\mathsf{e}_i \in \Delta_{\widetilde{x},i}$;*

(2) *for each $i \in [\mathbf{m}]$, $\check{\varepsilon}_i$ can be taken as the pullback $\widetilde{\varepsilon}_i$ of ε_i;*

(3) *there is an affine smooth chart $\widetilde{\mathcal{V}}_{\widetilde{x}} \subset \pi^{-1}(\mathcal{V})$ containing $\widetilde{x}$ such that*

$$\check{\varepsilon}_1, \ldots, \check{\varepsilon}_{\mathbf{m}}; \quad \frac{\widetilde{\zeta}_e}{\prod_{i\in[\mathbf{m}] \text{ s.t. } e\in\mathfrak{E}_i} \check{\varepsilon}_i} - \Big(\frac{\widetilde{\zeta}_e}{\prod_{i\in[\mathbf{m}] \text{ s.t. } e\in\mathfrak{E}_i} \check{\varepsilon}_i}(\widetilde{x})\Big), \quad e \in \Delta_{\widetilde{x}}\backslash\{\mathsf{e}_1, \ldots, \mathsf{e}_{\mathbf{m}}\};$$
$$\check{\zeta}_e, \quad e \in \mathfrak{E}_{\mathbf{m}}\backslash\Delta_{\widetilde{x},\mathbf{m}}; \quad \check{\zeta}_e = \widetilde{\zeta}_e, \quad e \in \mathfrak{E}_{\mathbf{m}}^{\prec}; \quad \check{\zeta}_s = \widetilde{\zeta}_s, \quad s \in S';$$

form a system of local parameters on $\widetilde{\mathcal{V}}_{\widetilde{x}}$, where ζ_s, $s \in S'$, are as in (4.8); and

(4) *$\check{\varepsilon}_i$, $i \in [\mathbf{m}]$, and $\check{\zeta}_e$, $e \in ((\mathfrak{E}_{\mathbf{m}}\backslash\Delta_{\widetilde{x},\mathbf{m}}) \sqcup \mathfrak{E}_{\mathbf{m}}^{\prec})$ all vanish at $\widetilde{x}$, whereas $\widetilde{\zeta}_e/(\prod_{i\in[\mathbf{m}] \text{ s.t. } e\in\mathfrak{E}_i} \check{\varepsilon}_i)$ does not vanish at $\widetilde{x}$ for any $e \in \Delta_{\widetilde{x}}\backslash\{\mathsf{e}_1, \ldots, \mathsf{e}_{\mathbf{m}}\}$.*

Proof. The statements (1), (3), and (4) can be deduced from Corollary 4.5. The statement (2) follows from (4.12). □

The next statement plays a crucial rule in analyzing the change of the structural homomorphism φ throughout Section 4. Let τ, $\mathcal{V}$, π, $\widetilde{x}$, $\overline{\mathbf{E}}$, and $\widetilde{\mathcal{V}}_{\widetilde{x}}$ be as in Lemma 4.16. With the functions $\zeta_{[e]}$ as in (4.9), we write

$$\widetilde{\zeta}_{[e]} := \pi^*(\zeta_{[e]}) = \prod_{e'\succeq e} \widetilde{\zeta}_{e'} \in \Gamma(\mathscr{O}_{\widetilde{\mathcal{V}}_{\widetilde{x}}}) \quad \forall\, e \in \tau.$$

PROPOSITION 4.17. *With the notation as above, there exist invertible functions $u_e \in \Gamma(\mathscr{O}^*_{\widetilde{\mathcal{V}}_{\widetilde{x}}})$, $e \in \mathfrak{E}_{\mathbf{m}}$, such that*

$$\widetilde{\zeta}_{[e]} = u_e \cdot \check{\varepsilon}_1 \cdots \check{\varepsilon}_{\mathbf{m}}, \quad (\text{respectively } \widetilde{\zeta}_{[e]} = u_e \cdot \check{\varepsilon}_1 \cdots \check{\varepsilon}_{\mathbf{m}} \cdot \check{\zeta}_e)$$

for every $e \in \Delta_{\widetilde{x},\mathbf{m}}$ (respectively, $e \in \mathfrak{E}_{\mathbf{m}}\backslash\Delta_{\widetilde{x},\mathbf{m}}$). In particular, $\widetilde{\zeta}_{[e]} | \widetilde{\zeta}_{[e']}$ for all $e \in \Delta_{\widetilde{x}}^{\min}$ and $e' \in \min(\tau)$.

Proof. By (4.16), we have $\Delta_{\widetilde{x},k} = \mathfrak{E}_k\backslash\mathfrak{E}_{k+1}$ for any $k \in [\mathbf{m}-1]$. Along with the equality in (4.15), this implies

$$\mathfrak{E}_{\mathbf{m}}^{\succeq} = (\mathfrak{E}_{\mathbf{m}}\backslash\Delta_{\widetilde{x},\mathbf{m}}) \sqcup \bigsqcup_{k\in[\mathbf{m}]} \Delta_{\widetilde{x},k}.$$

Hence for every $e \in \mathfrak{E}_{\mathbf{m}}^{\succeq}$, by applying Lemma 4.16 and the middle equation in (4.4) repeatedly, we obtain

$$\widetilde{\zeta}_e = \begin{cases} u'_e \cdot \prod_{i\in[\mathbf{m}] \text{ s.t. } e\in\mathfrak{E}_i} \check{\varepsilon}_i & \text{if } e \in \bigsqcup_{k\in[\mathbf{m}]} \Delta_{\widetilde{x},k}, \\ u'_e \cdot \check{\zeta}_e \cdot \prod_{i\in[\mathbf{m}] \text{ s.t. } e\in\mathfrak{E}_i} \check{\varepsilon}_i & \text{if } e \in \mathfrak{E}_{\mathbf{m}}\backslash\Delta_{\widetilde{x},\mathbf{m}}, \end{cases}$$

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where $u'_e \in \Gamma(\mathscr{O}^*_{\widetilde{\mathcal{V}}_{\widetilde{x}}})$. This gives rise to the formulae for all $\widetilde{\zeta}_{[e]}$, $e \in \mathfrak{E}_{\mathbf{m}}$. The last statement then follows from the fact that $\Delta^{\min}_{\widetilde{x}} \neq \emptyset$. □

Remark 4.18. The statements of Proposition 4.17 are consistent with the displayed formulae in the paragraph after [HL10, (5.23)], but are more precise. In [HL10], it suffices to know the last statement of Proposition 4.17.

Now, we are ready to prove Case A.4.

PROPOSITION 4.19. *Proposition 4.1 holds in Case A.4.*

Proof. At the beginning of Section 4.3, we already observed that the blowups in $(\mathsf{r}_1\mathsf{p}_1)$ to $(\mathsf{r}_3\mathsf{p}_2)$ do not affect $\mathcal{V}$, and hence we can identify $\mathcal{V}$ with its pullback $\widetilde{\mathcal{V}}^{\mathsf{r}_3\mathsf{p}_2}$. The structural homomorphism takes the form of (4.1), where κ_{12} and ζ_q, $q \in N(C)$, form a subset of a system of local parameters on $\mathcal{V}$. Moreover, we only need to study the case when

$$m \geq 3,$$

for otherwise (4.1) is already diagonalized, and $(\mathsf{r}_3\mathsf{p}_3)$ to $(\mathsf{r}_3\mathsf{p}_4)$ do not affect $\mathcal{V}$ either.

Next, we interpret the rooted tree of the first paragraph of Section 4.4 using the language of Definition 4.6. Intuitively, the underlying set of the rooted tree is the union of those $N_{[\delta]}$ with $\delta \in D$ satisfying the condition that δ is on a tail and there is no other $\delta' \in D$ on the same tail that is closer to the core F than δ. More precisely, we set

$$\begin{aligned} &D_{\text{i-m}} := \{\delta \in D\backslash F :\ N_{[\delta]} \text{ is minimal w.r.t. inclusion among all } N_{[\delta']},\ \delta' \in D\backslash F\},\\ &E := \bigcup_{\delta \in D_{\text{i-m}}} N_{[\delta]} \quad (\subset N(C)). \end{aligned} \tag{4.19}$$

Here, the subscript of $D_{\text{i-m}}$ is short for 'inclusion-minimal'; see Definition 4.20 below. The set E is the underlying set of the proposed rooted tree.

Recall that we have assumed that $D \cap F = \{\delta_1, \delta_2\}$ and $m \geq 3$. Since $\delta_3, \ldots, \delta_m$ are all on tails, we have $N_{[\delta_i]} \neq \emptyset$ for any $3 \leq i \leq m$, thus $E \neq \emptyset$. On the set E, consider the relation $\preceq$ given by $e \preceq e'$ if and only if any connected subcurve of C containing F and e must contain e'. The fact that F is of arithmetic genus 2 guarantees that $\preceq$ satisfies (4.5), i.e.

$$\tau_2 := (E, \preceq) \tag{4.20}$$

is a rooted tree, whose root-to-leaf paths are exactly $N_{[\delta]}$, $\delta \in D_{\text{i-m}}$. In Figure 16, an example of such a constructed τ_2 of a point $\widehat{x}$ satisfying the hypothesis of Case A.4 is provided.

As described in Section 3.5, the blowup centers of $(\mathsf{r}_3\mathsf{p}_3)$ locally are given by

$$\begin{aligned} &\mathcal{H}_k^{\mathsf{r}_3\mathsf{p}_2} \cap \widetilde{\mathcal{V}}^{\mathsf{r}_3\mathsf{p}_2} = \mathcal{H}_k \cap \mathcal{V} = \bigcup_{\mathfrak{E} \in \Xi_k(\tau_2)} \{\kappa_{12} = 0;\ \zeta_e = 0\ \forall\, e \in \mathfrak{E}\},\\ &\text{where} \quad \Xi_k(\tau_2) = \{\mathfrak{E} \in \Xi(\tau_2) : |\mathfrak{E}| = k\}. \end{aligned} \tag{4.21}$$

One can thus check directly that, on $\mathcal{V}$, $(\mathsf{r}_3\mathsf{p}_3)$ is a grafted blowup with respect to a τ_2-compatible blowup of $\mathcal{V}$, hence, by Lemma 4.13, $(\mathsf{r}_3\mathsf{p}_3)$ is $\mathrm{gr}(\tau_2)$-compatible on $\mathcal{V}$, where the grafted edge corresponds to κ_{12}. Then, by Proposition 4.17, after $(\mathsf{r}_3\mathsf{p}_3)$, we have either:

- $\widetilde{\kappa}_{12}$ divides all $\widetilde{\zeta}_{[e]}$, $e \in \min(\tau_2)$, hence divides all $\widetilde{\zeta}_{[\delta_i]}$, $3 \leq i \leq m$; or
- there exists $e \in \Delta^{\min}_{\widetilde{x}}$ $(\subset \min(\tau_2))$ such that $\widetilde{\zeta}_{[e]}$ divides $\widetilde{\kappa}_{12}$ as well as all $\widetilde{\zeta}_{[e']}$, $e' \in \min(\tau_2)$, hence there exists $3 \leq i \leq m$ such that $\widetilde{\zeta}_{[\delta_i]}$ divides $\widetilde{\kappa}_{12}$ as well as all $\widetilde{\zeta}_{[\delta_j]}$, $3 \leq j \leq m$.

In sum, the pullback of the matrix (4.1) becomes diagonalized after $(\mathsf{r}_3\mathsf{p}_3)$ finishes.

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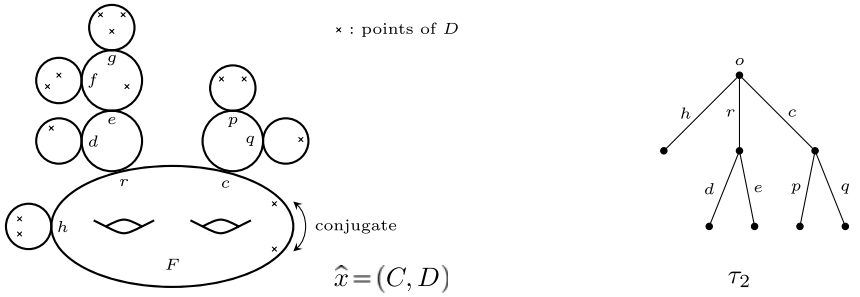


FIGURE 16. An example of the rooted tree τ_2 in Case A.4.

Finally, by Lemma 4.10, we see that the pullback $\widetilde{\mathcal{V}}^{\mathsf{r}_3\mathsf{p}_3}$ of $\mathcal{V}$ is smooth. In addition, since the blowup centers of $(\mathsf{r}_3\mathsf{p}_4)$ lie in the exceptional divisors of $(\mathsf{r}_1\mathsf{p}_5)$, and $(\mathsf{r}_1\mathsf{p}_5)$ does not affect $\mathcal{V}$, we conclude that the pullback of $\mathcal{V}$ in $\widetilde{\mathfrak{M}}_2^{\mathrm{div}}$ is equal to $\widetilde{\mathcal{V}}^{\mathsf{r}_3\mathsf{p}_3}$, hence is smooth. □

We remark that in (4.19), the reason for choosing *minimal* sets $N_{[\delta]}$ is the same as that for introducing the *terminally weighted trees* in [HL10, Section 3] (except that here the core is of weight 2 instead of 0): if $N_{[\delta]} \subset N_{[\delta']}$, then $\zeta_{[\delta]}$ divides $\zeta_{[\delta']}$ in (4.1), hence $\widetilde{\zeta}_{[\delta]}$ divides $\widetilde{\zeta}_{[\delta']}$. In order to distinguish the minimality with respect to inclusion from other types of minimality, e.g. with respect to the tree order on a rooted tree, we introduce the following notion for later use.

DEFINITION 4.20. Given a set I of sets, we say that $N \in I$ is inclusion-minimal if it is minimal with respect to the inclusions of subsets among all $N' \in I$.

4.6 Proof of Case A.5, Part I

We continue with the setup in Section 4.1 and fix a small affine smooth chart $\mathcal{V} \to \mathfrak{M}_2^{\mathrm{div}}$ containing $\widehat{x} = (C, D)$. Recall that the image of $\widehat{x}$ in $\mathfrak{M}_2^{\mathrm{wt}}$ is denoted by x.

Now we aim to prove Case A.5 of Proposition 4.1, when F is inseparable and $D \cap F$ consists of one point, which without loss of generality is assumed to be δ_1. Such a point $\widehat{x} \in \mathfrak{M}_2^{\mathrm{div}}$ is possibly the image of a point of the boundary of $\overline{M}_2(\mathbb{P}^n, d)$, e.g. when the core F of C consists of three smooth rational subcurves C_1, C_2 and C_3, satisfying $C_1 \cap C_2 = \{e_1, e_2\}$, $C_3 \cap C_2 = \{e_3, e_4\}$, $e_1, \ldots, e_4$ are all distinct, $D \cap C_1 = D \cap C_3 = \emptyset$ (there are tails attached to C_1 and C_3 by stability), and $D \cap C_2 = \{\delta_1\}$. As mentioned in Remark 3.3, such $\widehat{x}$ must be considered in order to resolve the entire $\overline{M}_2(\mathbb{P}^n, d)$.

By Proposition 2.22, the structural homomorphism can be written as

$$\varphi = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \kappa_{12}\zeta_{[\delta_2]} & \cdots & \kappa_{1m}\zeta_{[\delta_m]} \end{bmatrix}. \tag{4.22}$$

Since $x \in \mathfrak{M}^{\mathrm{mn}}$ and $\deg D \cap F = 1$, the phases $(\mathsf{r}_1\mathsf{p}_1)$ to $(\mathsf{r}_1\mathsf{p}_4)$ do not affect $\mathcal{V}$. Moreover, we only need to study the case when

$$m \geq 3,$$

for otherwise (4.22) is already diagonalized, and $(\mathsf{r}_1\mathsf{p}_5)$ to $(\mathsf{r}_3\mathsf{p}_4)$ do not affect $\mathcal{V}$ either.

By defining the subsets $D_{\text{i-m}} \subset D$ and $E \subset N(C)$ in exactly the same way as (4.19), and defining the relation $\preceq$ on E given by $e \preceq e'$ if and only if any connected subcurve of C containing

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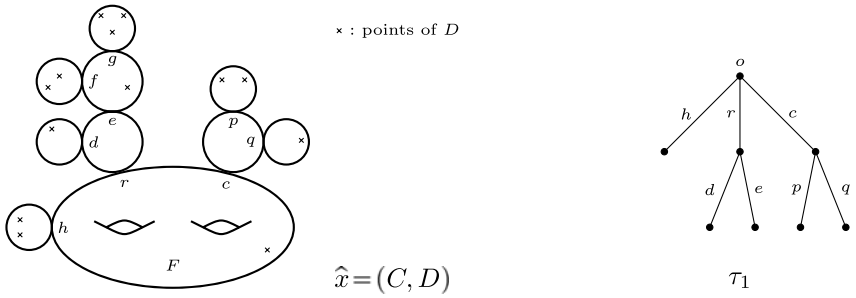


FIGURE 17. An example of the rooted tree τ_1 in Case A.5.

F and e must contain e', we see that

$$\tau_1 := (E, \preceq) \tag{4.23}$$

is a rooted tree, whose root-to-leaf paths are exactly $N_{[\delta]}$, $\delta \in D_{\text{i-m}}$. Although the constructions of τ_1 here and τ_2 in (4.20) are analogous, we emphasize that $D_{\text{i-m}} \subset \{\delta_2, \ldots, \delta_\ell\}$ for τ_1, whereas $D_{\text{i-m}} \subset \{\delta_3, \ldots, \delta_\ell\}$ for τ_2. An example of τ_1 is illustrated in Figure 17.

Mimicking the argument of the proof of Proposition 4.19, we see that $(\mathsf{r}_1\mathsf{p}_5)$ is τ-compatible on $\mathcal{V}$, with the partition $\Xi(\tau_1) = \bigsqcup_k \Xi_k(\tau_1)$ still given by $\Xi_k(\tau_1) = \{\mathfrak{E} \in \Xi(\tau_1) : |\mathfrak{E}| = k\}$. Lemma 4.10 then implies that the pullback $\widetilde{\mathcal{V}}^{\mathsf{r}_1\mathsf{p}_5}$ of $\mathcal{V}$ is smooth.

After $(\mathsf{r}_1\mathsf{p}_5)$, we fix an arbitrary lift $y \in \widetilde{\mathcal{V}}^{\mathsf{r}_1\mathsf{p}_5}$ of $\widehat{x}$, as well as a small neighborhood $\mathcal{V}_y$ of y in $\widetilde{\mathcal{V}}^{\mathsf{r}_1\mathsf{p}_5}$. By Definition 4.15, there exists $i \geq 2$ such that $N_{[\delta_i]} \subset \Delta_y$, i.e. every node lying between δ_i and the core F is dominant; such δ_i is naturally contained in $D_{\text{i-m}}$. For convenience, hereafter we write

$$D(\Delta_y) := \{\delta \in D \backslash F : N_{[\delta]} \subset \Delta_y\} \quad (= \{\delta_i \in D : i \geq 2,\ N_{[\delta_i]} \subset \Delta_y\}). \tag{4.24}$$

If in addition, there exists $\delta_i \in D(\Delta_y)$ that is not conjugate to δ_1 (i.e. κ_{1i} is invertible as $\mathcal{V}$ is assumed small), then by Proposition 4.17, on $\mathcal{V}_y$, the ith entry of the pullback of the second row of (4.22) divides the other entries of the second row, and hence the pullback of φ is diagonalized after $(\mathsf{r}_1\mathsf{p}_5)$. Moreover, (r_2) to $(\mathsf{r}_3\mathsf{p}_2)$ do not affect $\mathcal{V}_y$ because $(\mathsf{r}_1\mathsf{p}_1)$ does not affect $\mathcal{V}$; and $(\mathsf{r}_3\mathsf{p}_3)$ to $(\mathsf{r}_3\mathsf{p}_4)$ do not affect $\mathcal{V}_y$ because δ_i is not conjugate to δ_1. Therefore, the pullback of $\mathcal{V}_y$ in $\widetilde{\mathfrak{M}}_2^{\text{div}}$ is equal to $\mathcal{V}_y$, hence is smooth. In sum, we have established the following.

PROPOSITION 4.21. *Proposition 4.1 holds in Case A.5 if there exists $\delta \in D(\Delta_y)$ that is not conjugate to the point $D \cap F$.*

If δ_i and δ_1 are conjugate for all $\delta_i \in D(\Delta_y)$, then $(\mathsf{r}_3\mathsf{p}_3)$ is involved (and so is $(\mathsf{r}_3\mathsf{p}_4)$ if such i is unique). We shall deal with this situation in the following two subsections, which will complete the proof of Case A.5.

4.7 First-order derived trees and the proof of Case A.5, Part II

We continue with the proof of Case A.5 under the setting of Section 4.6. Recall that we have already assumed that $D \cap F = \{\delta_1\}$, and fixed a lift y of $\widehat{x} = (C, D)$ as well as a small neighborhood $\mathcal{V}_y$ of y in $\widetilde{\mathcal{V}}^{\mathsf{r}_1\mathsf{p}_5}$. The RL-sequence of y is written as $\overline{\mathbf{E}} = \{\mathfrak{E}_1, \ldots, \mathfrak{E}_{\mathbf{m}}\}$, and the set of dominant edges of y is denoted by Δ_y.

To complete the proof of Case A.5, it remains to handle the case when δ_i and δ_1 are conjugate for all $\delta_i \in D(\Delta_y)$. We will tackle the complicated $|D(\Delta_y)| = 1$ sub-case in the succeeding

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subsection; in this subsection, we assume

$$|D(\Delta_y)| \geq 2.$$

Without loss of generality we further assume $D(\Delta_y) = \{\delta_2, \dots, \delta_\ell\}$ for some $\ell \geq 3$.

The above assumptions imply that there exists a unique tail T containing $\delta_2, \dots, \delta_\ell$, for otherwise there would be multiple tails whose pivotal nodes are conjugate with δ_1, which could only occur when these tails and δ_1 are on a non-separating bridge of F, and hence the image of $\widehat{x}$ in $\mathfrak{M}_2^{\mathrm{wt}}$ would belong to $\mathfrak{M}_{(3)}^{\mathrm{mn}}$ instead of $\mathfrak{M}^{\mathrm{mn}}$, contradicting the hypothesis of Case A. Therefore, neither δ_1 nor the pivotal node p ($= \langle \delta_i \rangle$ for all $2 \leq i \leq \ell$) of T is on any non-separating bridge of F.

By Proposition 4.17, under a suitable trivialization, the pullback of (4.22) can be written as

$$\widetilde{\varphi}^{\mathsf{r}_1\mathsf{p}_5} = \begin{bmatrix} 1 & 0 \\ 0 & \prod_{k\in[\mathbf{m}]} \check{\varepsilon}_k \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \check{\kappa}_{12} & \cdots & \check{\kappa}_{1\ell} & \check{\kappa}_{1,\ell+1} \prod_{e\in N_{[\delta_{\ell+1}]}\setminus\Delta_y} \check{\zeta}_e & \cdots & \check{\kappa}_{1m} \prod_{e\in N_{[\delta_m]}\setminus\Delta_y} \check{\zeta}_e \end{bmatrix}.$$

Here, the $\check{\varepsilon}_k$ are the local parameters corresponding to the proper transforms $\check{\mathcal{E}}_{\mathfrak{E}_k}$ of the exceptional divisors obtained in $(\mathsf{r}_1\mathsf{p}_5)$; see Lemma 4.16.

Recall that for every $i \in [m]$, $N_{[\delta_2\wedge\delta_i]} = N_{[\delta_2]} \cap N_{[\delta_i]}$ as in (2.11). Hence by Proposition 2.22 (SH_3), there exist invertible u_i' and v_i' such that

$$\check{\kappa}_{1i} = u_i' \cdot \check{\kappa}_{12} + v_i' \cdot \prod_{k\in[\mathbf{m}]_{[\delta_2\wedge\delta_i]}} \check{\varepsilon}_k, \quad \text{where } [\mathbf{m}]_{[\delta_i\wedge\delta_j]} := \{k \in [\mathbf{m}] : N_{[\delta_i\wedge\delta_j]} \cap \mathfrak{E}_k \neq \emptyset\}. \tag{4.25}$$

In addition, by (4.24), we have

$$N_{[\delta_i]}\setminus\Delta_y = \emptyset \quad \forall\, 2 \leq i \leq \ell. \tag{4.26}$$

Therefore, after taking elementary column operations, we can rewrite the second row of $\widetilde{\varphi}^{\mathsf{r}_1\mathsf{p}_5}$ as

$$\Big(\prod_{k\in[\mathbf{m}]} \check{\varepsilon}_k\Big) \begin{bmatrix} 0 & \check{\kappa}_{12} & \eta_3' & \cdots & \eta_m' \end{bmatrix}, \quad \text{where } \eta_i' := \Big(\prod_{k\in[\mathbf{m}]_{[\delta_2\wedge\delta_i]}} \check{\varepsilon}_k\Big)\Big(\prod_{e\in N_{[\delta_i]}\setminus\Delta_y} \check{\zeta}_e\Big).$$

Since δ_2 is an arbitrary point of $D(\Delta_y)$, there should be a way to rewrite the second row of $\widetilde{\varphi}^{\mathsf{r}_1\mathsf{p}_5}$ so that the terms are independent of the choice of δ_2 (after suitable elementary column operations). This follows from the lemma below.

Lemma 4.22. *There exists $3 \leq j \leq \ell$ such that*

$$(N_{[\wedge D(\Delta_y)]} :=) \bigcap_{\delta\in D(\Delta_y)} N_{[\delta]} = N_{[\delta_2\wedge\delta_j]}.$$

Proof. Notice that the left-hand side of the above equality is equal to $\bigcap_{i=2}^{\ell} (N_{[\delta_2]} \cap N_{[\delta_i]})$. In addition, notice that, given three smooth points $\delta, \delta', \delta''$, one of $N_{[\delta]} \cap N_{[\delta']}$ and $N_{[\delta]} \cap N_{[\delta'']}$ must contain the other as a subset. Therefore, $\mathcal{N}_2' := \{N_{[\delta_2]} \cap N_{[\delta_i]} : 2 \leq \ell\}$ has a natural linear order given by the inclusion of subsets. As $\mathcal{N}_2'$ is a finite set, there exists $3 \leq j \leq \ell$ such that $N_{[\delta_2]} \cap N_{[\delta_j]}$ is contained in any element of $\mathcal{N}_2'$. Then,

$$\bigcap_{i=2}^{\ell} N_{[\delta_i]} = \bigcap_{i=2}^{\ell} (N_{[\delta_2]} \cap N_{[\delta_i]}) = N_{[\delta_2]} \cap N_{[\delta_j]} = N_{[\delta_2\wedge\delta_j]},$$

where the last equality follows from (2.11). □

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Let

$$[\mathbf{m}]_{[\wedge D(\Delta_y)]} := \{k \in [\mathbf{m}] : N_{[\wedge D(\Delta_y)]} \cap \mathfrak{E}_k \neq \emptyset\},$$
$$[\mathbf{m}]_{[\delta_i \wedge D(\Delta_y)]} := \{k \in [\mathbf{m}] : N_{[\delta_i]} \cap N_{[\wedge D(\Delta_y)]} \cap \mathfrak{E}_k \neq \emptyset\} \subset [\mathbf{m}]_{[\wedge D(\Delta_y)]}, \quad \forall\, 2 \le i \le m. \tag{4.27}$$

By Lemma 4.22 and (4.26), there exists $3 \le j \le \ell$ such that

$$\eta'_j = \prod_{k \in [\mathbf{m}]_{[\delta_j \wedge D(\Delta_y)]}} \check{\varepsilon}_k = \prod_{k \in [\mathbf{m}]_{[\wedge D(\Delta_y)]}} \check{\varepsilon}_k.$$

Moreover, for every $3 \le i \le m$ other than j, either:

- $N_{[\delta_2 \wedge \delta_i]} \supset N_{[\delta_2 \wedge \delta_j]}$, which, by Lemma 4.22, implies that $\eta'_j | \eta'_i$; or
- $N_{[\delta_2 \wedge \delta_i]} \subsetneq N_{[\delta_2 \wedge \delta_j]}$, which, also by Lemma 4.22, implies that

$$\eta'_i := \left(\prod_{k \in [\mathbf{m}]_{[\delta_i \wedge D(\Delta_y)]}} \check{\varepsilon}_k \right) \left(\prod_{e \in N_{[\delta_i]} \setminus \Delta_y} \check{\zeta}_e \right).$$

So, under suitable elementary column operations, we further rewrite the second row of $\widetilde{\varphi}^{\mathsf{r}_1\mathsf{p}_5}$ as

$$\left(\prod_{k \in [\mathbf{m}]} \check{\varepsilon}_k \right) \begin{bmatrix} 0 & \check{\kappa}_{12} & \eta_3 & \cdots & \eta_m \end{bmatrix}, \quad \text{where } \eta_i := \left(\prod_{k \in [\mathbf{m}]_{[\delta_i \wedge D(\Delta_y)]}} \check{\varepsilon}_k \right) \left(\prod_{e \in N_{[\delta_i]} \setminus \Delta_y} \check{\zeta}_e \right). \tag{4.28}$$

By Proposition 2.17 (ii) and Lemma 4.16(3), we see that $\check{\kappa}_{12}$, $\check{\varepsilon}_k$, $k \in [\mathbf{m}]$, and $\check{\zeta}_e$, $e \in N_{[\delta_i]} \setminus \Delta_y$, $3 \le i \le m$, together form a subset of a system of local parameters on $\mathcal{V}_y$. Since $|D \cap F| > 0$ and $|D(\Delta_y)| \ge 2$, we conclude that (r_2) to $(\mathsf{r}_3\mathsf{p}_3)$ do not affect $\mathcal{V}_y$. Like we did for Case A.4, we aim to construct a rooted tree ϱ_y whose root-to-leaf paths are exactly those inclusion-minimal $[\mathbf{m}]_{[\delta_i \wedge D(\Delta_y)]} \sqcup (N_{[\delta_i]} \setminus \Delta_y)$ among all $3 \le i \le m$, and to show that $(\mathsf{r}_3\mathsf{p}_4)$ is $\mathrm{gr}(\varrho_y)$-compatible on $\mathcal{V}_y$.

We begin with ϱ_y. In a similar way to (4.19), we set

$$D_{y,\text{i-m}} := \{\delta \in D \setminus F : [\mathbf{m}]_{[\delta \wedge D(\Delta_y)]} \sqcup (N_{[\delta]} \setminus \Delta_y) \text{ is inclusion-minimal among all } [\mathbf{m}]_{[\delta_i \wedge D(\Delta_y)]} \sqcup (N_{[\delta_i]} \setminus \Delta_y),\ 2 \le i \le m\}.$$

It is straightforward that $D_{y,\text{i-m}} \subset D_{\text{i-m}}$. So by writing

$$E_y := \bigcup_{\delta \in D_{y,\text{i-m}}} ([\mathbf{m}]_{[\delta \wedge D(\Delta_y)]} \sqcup (N_{[\delta]} \setminus \Delta_y)),$$

we see that the subset $E_y \setminus [\mathbf{m}]_{[\wedge D(\Delta_y)]}$ is also a subset of the underlying set of the rooted tree τ_1 of (4.23). On E_y, consider the partial order $\preceq_y$ determined by:

- for every $k, k' \in [\mathbf{m}]_{[\wedge D(\Delta_y)]}$, $k \prec_y k'$ if and only if $k > k'$;
- for every $e, e' \in \bigcup_{\delta \in D_{y,\text{i-m}}} (N_{[\delta]} \setminus \Delta_y)$, $e \preceq_y e'$ if and only if any connected subcurve of C containing F and e must contain e'; and
- for every $k \in [\mathbf{m}]_{[\wedge D(\Delta_y)]}$ and $e \in \bigcup_{\delta \in D_{y,\text{i-m}}} (N_{[\delta]} \setminus \Delta_y)$, $e \prec_y k$ if and only if $e \in (N_{[\wedge D(\Delta_y)]} \cap \mathfrak{E}_k)^{\prec}$ in τ_1.

One can check directly that $\preceq_y$ is a tree order on E_y as per (4.5). We call the rooted tree

$$\varrho_y := (E_y, \preceq_y) \tag{4.29}$$

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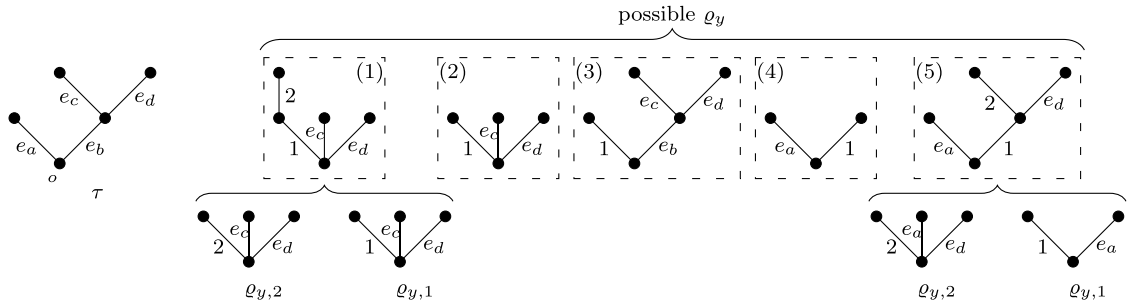


FIGURE 18. First-order derived trees in Example 4.23.

the first-order derived tree at y. Notice that restriction of the tree order $\prec_y$ to $E_y\backslash[\mathbf{m}]_{[\wedge D(\Delta_y)]}$ is consistent with the restriction of the tree order of (4.23). Also notice that the root-to-leaf paths of ϱ_y are exactly those inclusion-minimal $[\mathbf{m}]_{[\delta_i\wedge D(\Delta_y)]}\sqcup(N_{[\delta_i]}\backslash\Delta_y)$ among all $2\le i\le m$; in particular, $[\mathbf{m}]_{[\wedge D(\Delta_y)]}$ is a root-to-leaf path of ϱ_y.

In the following example, we provide an illustration of the notion of first-order derived trees. The rooted trees labeled as $\varrho_{y,1}$ and $\varrho_{y,2}$ in Figure 18 are irrelevant here; they will be introduced in Section 4.14.1.

Example 4.23. Let τ be the rooted tree given by the leftmost graph in Figure 18, whose root is $\{e_a,e_b\}$ (i.e. the root vertex o is the lowest one). As in Example 4.11, τ has two transverse sections:

$$\mathfrak{E}_1=\{e_a,e_b\}\succ\mathfrak{E}_2=\{e_a,e_c,e_d\}.$$

Given $\widehat{x}\in X_\tau^{\mathcal{V}}$ as well as y lifting $\widehat{x}$, we analyze ϱ_y for all possible RLS $\overline{\mathbf{E}}$ and Δ_y of y.

(1) If $\Delta_y=\{e_a,e_b\}$ and $\overline{\mathbf{E}}=\{\mathfrak{E}_1,\mathfrak{E}_2\}$, then ϱ_y is given by Graph (1) of Figure 18.
(2) If $\Delta_y=\{e_a,e_b\}$ and $\overline{\mathbf{E}}=\{\mathfrak{E}_1\}$, then ϱ_y is given by Graph (2) of Figure 18.
(3) If $\Delta_y=\{e_a\}$ and $\overline{\mathbf{E}}=\{\mathfrak{E}_1\}$, then ϱ_y is given by Graph (3) of Figure 18.
(4) If $\Delta_y=\{e_b,e_c,e_d\}$ and $\overline{\mathbf{E}}=\{\mathfrak{E}_1,\mathfrak{E}_2\}$, then ϱ_y is given by Graph (4) of Figure 18.
(5) If $\Delta_y=\{e_b,e_c\}$ and $\overline{\mathbf{E}}=\{\mathfrak{E}_1,\mathfrak{E}_2\}$ then ϱ_y is given by Graph (5) of Figure 18. The case when $\Delta_y=\{e_b,e_d\}$ and $\overline{\mathbf{E}}=\{\mathfrak{E}_1,\mathfrak{E}_2\}$ is parallel.
(6) In all other cases (e.g. $\Delta_y=\{e_a,e_b,e_c\}$ and $\overline{\mathbf{E}}=\{\mathfrak{E}_1,\mathfrak{E}_2\}$), $\varrho=\tau_\bullet$ because $N_{[\wedge D(\Delta_y)]}=\emptyset$.

In order to show that $(\mathsf{r}_3\mathsf{p}_4)$ is $\mathrm{gr}(\varrho_y)$-compatible on $\mathcal{V}_y$, first notice that each transverse section $\mathfrak{E}$ of ϱ_y contains a unique element $k_{\mathfrak{E}}\in[\mathbf{m}]_{[\wedge D(\Delta_y)]}$, and determines a transverse section

$$\mathfrak{E}^\vee:=(\mathfrak{E}\backslash\{k_{\mathfrak{E}}\})\sqcup(N_{[\wedge D(\Delta_y)]}\cap\mathfrak{E}_{k_{\mathfrak{E}}})$$

of the rooted tree τ_1. Since $(\mathfrak{E}\backslash\{k_{\mathfrak{E}}\})\subset(\tau_1\backslash\Delta_y)$, we have $\mathfrak{E}^\vee\preceq\mathfrak{E}_{k_{\mathfrak{E}}}$, hence $|\mathfrak{E}|=|\mathfrak{E}^\vee|\ge|\mathfrak{E}_{k_{\mathfrak{E}}}|$.

For every $1\le h\le h'$, with $\mathcal{Q}_{h,h'}$ as in (3.27), we denote by $\check{\mathcal{Q}}_{h,h'}$ its proper transform after $(\mathsf{r}_1\mathsf{p}_5)$. Then, one can check directly that

$$\check{\mathcal{Q}}_{h,h'}\cap\mathcal{V}_y=\bigcup_{\mathfrak{E}\in\Xi(\varrho_y)\text{ s.t. }|\mathfrak{E}_{k_{\mathfrak{E}}}|=h,\,|\mathfrak{E}|=h'}\{\check{\kappa}_{12}=0;\ \check{\varepsilon}_{k_{\mathfrak{E}}}=0;\ \check{\zeta}_e=0\,\forall\,e\in\mathfrak{E}\backslash\{k_{\mathfrak{E}}\}\}.\tag{4.30}$$

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This gives rise to a partition of $\Xi(\varrho_y)$, hence of $\Xi(\mathrm{gr}(\varrho_y))$. Observe that for every $\mathfrak{E},\mathfrak{E}'\in\Xi(\varrho_y)$ with $\mathfrak{E}\succ\mathfrak{E}'$, either we have $k_{\mathfrak{E}}\succ_y k_{\mathfrak{E}'}$, i.e. $k_{\mathfrak{E}}<k_{\mathfrak{E}'}$, hence $|\mathfrak{E}_{k_{\mathfrak{E}}}|<|\mathfrak{E}_{k_{\mathfrak{E}'}}|$; or we have $k_{\mathfrak{E}}=k_{\mathfrak{E}'}$ and $\mathfrak{E}^\vee\succ(\mathfrak{E}')^\vee$ in $\Xi(\tau_1)$, hence $|\mathfrak{E}|=|\mathfrak{E}^\vee|\le|(\mathfrak{E}')^\vee|=|\mathfrak{E}'|$. Therefore, the above partition of $\Xi(\mathrm{gr}(\varrho_y))$ satisfies Definition 4.9, i.e. $(\mathsf{r}_3\mathsf{p}_4)$ is $\mathrm{gr}(\varrho_y)$-compatible on $\mathcal{V}_y$.

So after $(\mathsf{r}_3\mathsf{p}_4)$, Proposition 4.17 implies that either the pullback of $\check\kappa_{12}$ or the pullback of some η_i, $3\le i\le m$, divides the other terms of the second row of the pullback $\widetilde\varphi^{\mathsf{r}_3\mathsf{p}_4}$ of $\widetilde\varphi^{\mathsf{r}_1\mathsf{p}_5}$, i.e. $\widetilde\varphi^{\mathsf{r}_3\mathsf{p}_4}$ is diagonalized. Moreover, Lemma 4.10 ensures that the pullback of $\mathcal{V}_y$ is smooth after $(\mathsf{r}_3\mathsf{p}_4)$. To summarize, in this subsection, we have established the following.

PROPOSITION 4.24. *Proposition 4.1 holds in Case A.5 if $|D(\Delta_y)|\ge 2$ and every point of $D(\Delta_y)$ is conjugate to the point $D\cap F$.*

4.8 Proper transforms of blowups and the proof of Case A.5, Part III

4.8.1 *Proof of Case A.5, Part III, up to $(\mathsf{r}_3\mathsf{p}_3)$.* We continue with the proof of Case A.5 under the setting of Sections 4.6 and 4.7. In particular, recall that we have assumed that $D\cap F=\{\delta_1\}$, and fixed a lift y of $\widehat x=(C,D)$ as well as a small neighborhood $\mathcal{V}_y$ of y in $\widetilde{\mathcal{V}}^{\mathsf{r}_1\mathsf{p}_5}$. The RL-sequence of y is written as $\overline{\mathbf{E}}=\{\mathfrak{E}_1,\dots,\mathfrak{E}_{\mathbf{m}}\}$, and the set of dominant edges of y is denoted by Δ_y.

In this subsection, we aim to show the only sub-case of Case A.5 that has not been established, namely when $|D(\Delta_y)|=1$ and the point of $D(\Delta_y)$ is conjugate to δ_1. Without loss of generality we further assume that $D(\Delta_y)=\{\delta_2\}$.

Under the above assumptions, the pullback $\widetilde\varphi^{\mathsf{r}_1\mathsf{p}_5}$ of the structural homomorphism φ is analogous to that in Section4.7, but simpler:

$$\widetilde\varphi^{\mathsf{r}_1\mathsf{p}_5}=\begin{bmatrix}1&0\\0&\prod_{k\in[\mathbf{m}]}\check\varepsilon_k\end{bmatrix}\begin{bmatrix}1&0&0&\cdots&0\\0&\check\kappa_{12}&\eta_3&\cdots&\eta_m\end{bmatrix},\qquad \text{where}\quad \eta_i:=\Bigg(\prod_{k\in[\mathbf{m}]_{[\delta_2\wedge\delta_i]}}\check\varepsilon_k\Bigg)\Bigg(\prod_{e\in N_{[\delta_i]}\setminus\Delta_y}\check\zeta_e\Bigg). \tag{4.31}$$

The blowups of (r_2) to $(\mathsf{r}_3\mathsf{p}_2)$ do not affect $\mathcal{V}_y$ because $|D\cap F|>0$, so we identify $\mathcal{V}_y$ with its pullback in $\widetilde{\mathfrak{M}}^{\mathsf{r}_3\mathsf{p}_2}$. The hypotheses $|D\cap F|=1$, $|D(\Delta_y)|=1$, and δ_1 and δ_2 are conjugate together imply that y lies in the intersection of certain $\mathcal{H}_h^{\mathsf{r}_3\mathsf{p}_2}$ as in (3.26), as well as the proper transforms $\check{\mathcal{Q}}_{h,h'}$ of certain $\mathcal{Q}_{h,h'}$ as in (3.27).

More precisely, consider the following two rooted trees: one is τ_2 as in (4.20), with $D_{\text{i-m}}$ of (4.19) replaced by

$$\{\delta_i\in D:\ N_{[\delta_i]}\setminus\Delta_y \text{ is inclusion-minimal among all } 3\le i\le m\};$$

the other is the first order derived tree ϱ_y as in (4.29). Notice that ϱ_y is still well defined when $|D(\Delta_y)|=1$; in this case

$$N_{[\wedge D(\Delta_y)]}=N_{[\delta_2]},\quad [\mathbf{m}]_{[\wedge D(\Delta_y)]}=[\mathbf{m}],\quad [\mathbf{m}]_{[\delta_i\wedge D(\Delta_y)]}=[\mathbf{m}]_{[\delta_2\wedge\delta_i]}.$$

One can then check directly that (4.21) and (4.30) remain valid on $\mathcal{V}_y$. Therefore, $(\mathsf{r}_3\mathsf{p}_3)$ is $\mathrm{gr}(\tau_2)$-compatible on $\mathcal{V}_y$, and, in particular, the pullback $\widetilde{\mathcal{V}}_y^{\mathsf{r}_3\mathsf{p}_3}\subset\widetilde{\mathfrak{M}}^{\mathrm{div},\mathsf{r}_3\mathsf{p}_3}$ of $\mathcal{V}_y$ is smooth. Moreover, the sequential blowup

$$(\widetilde{\mathfrak{M}}^{\mathrm{div},\mathsf{r}_3\mathsf{p}_4})'\longrightarrow\widetilde{\mathfrak{M}}^{\mathrm{div},\mathsf{r}_3\mathsf{p}_2} \tag{4.32}$$

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successively (w.r.t. the usual lexicographic order) along the proper transforms of $\mathcal{Q}_{h,h'}$ in $\widetilde{\mathfrak{M}}^{\mathrm{div},\mathsf{r}_3\mathsf{p}_2}$, which serves as the 'prototype' of $(\mathsf{r}_3\mathsf{p}_4)$, is $\mathrm{gr}(\varrho_y)$-compatible on $\mathcal{V}_y$.

After $(\mathsf{r}_3\mathsf{p}_3)$, fix an arbitrary lift $z\in\widetilde{\mathfrak{M}}^{\mathrm{div},\mathsf{r}_3\mathsf{p}_3}$ of y, which determines $\Delta_z\subset\mathrm{gr}(\tau_2)$ as in Definition 4.15. Let $\mathcal{V}_z\subset\widetilde{\mathcal{V}}_y^{\mathsf{r}_3\mathsf{p}_3}$ be a small neighborhood of z.

LEMMA 4.25. *For every $\mathfrak{E}\in\Xi(\mathrm{gr}(\varrho_y))$ satisfying $\mathfrak{E}\cap\Delta_z\neq\emptyset$, the proper transform (on $\widetilde{\mathcal{V}}_y^{\mathsf{r}_3\mathsf{p}_3}$) of the vanishing locus $X_{\mathfrak{E}}^{\mathcal{V}_y}$ is disjoint from $\mathcal{V}_z$.*

Proof. The statement holds because the pullback to $\mathcal{V}_z$ of $X_{\mathfrak{E}}^{\mathcal{V}_y}$ is contained in the proper transform of some exceptional divisor of $\mathcal{V}_z/\mathcal{V}_y$; this is similar to (4.18). □

By Lemma 4.25, on $\mathcal{V}_z$, the proper transforms of $\mathcal{Q}_{h,h'}$ of (3.27) on $\mathcal{V}_z$, which are exactly the blowup centers $(\mathsf{r}_3\mathsf{p}_4)$, are all unions of the proper transforms of $X_{\mathfrak{E}}^{\mathcal{V}_y}$ with $\mathfrak{E}\in\Xi(\mathrm{gr}(\varrho_y))$ satisfying $\mathfrak{E}\cap\Delta_z=\emptyset$. We thus expect such transverse sections of $\mathrm{gr}(\varrho_y)$ to form a new rooted tree, with which $(\mathsf{r}_3\mathsf{p}_4)$ is compatible on $\mathcal{V}_z$.

4.8.2 *Proper transforms of blowups and rooted trees.* In order to study the rooted tree with which $(\mathsf{r}_3\mathsf{p}_4)$ is compatible on $\mathcal{V}_z$, we take a detour and introduce the following notions under a more general setting.

DEFINITION 4.26. Let $\pi:\widetilde{\mathfrak{M}}\to\mathfrak{M}$ and $\pi':\widetilde{\mathfrak{M}}'\to\mathfrak{M}$, respectively, be the sequential blowups of a stack $\mathfrak{M}$ successively along the proper transforms of the closed substacks $Z_1,Z_2,\ldots\subset\mathfrak{M}$ and $Z_1',Z_2',\ldots\subset\mathfrak{M}$. Let $\check{Z}_1',\check{Z}_2',\ldots\subset\widetilde{\mathfrak{M}}$ be the proper transforms of $Z_1',Z_2',\ldots$, respectively. We call the sequential blowup of $\widetilde{\mathfrak{M}}$ successively along the proper transforms of $\check{Z}_1',\check{Z}_2',\ldots$ the **proper transform of π' (with respect to π)**, and denote it by

$$\mathrm{PT}_\pi(\pi'):\widetilde{\mathfrak{M}}''\longrightarrow\widetilde{\mathfrak{M}}.$$

For example, $(\mathsf{r}_3\mathsf{p}_4)$ is the proper transform of the sequential blowup (4.32) with respect to $(\mathsf{r}_3\mathsf{p}_3)$.

With notation as above, assume there exists an affine smooth chart $\mathcal{V}$ of $\mathfrak{M}$, two rooted trees τ and τ', and a $\tau\cup\tau'$-labeled subset of a system of local parameters on $\mathcal{V}$ as per (4.8), such that π and π' are, respectively, τ- and τ'-compatible. Here in $\tau\cup\tau'$, τ and τ' are considered as sets instead of posets. The intersection $\tau\cap\tau'$ may or may not be empty.

Fix a point $y\in X_\tau^{\mathcal{V}}\cap X_{\tau'}^{\mathcal{V}}$, as well as a lift $z\in\widetilde{\mathfrak{M}}$ of y (i.e. $\pi(z)=y$). By Lemma 4.16, $\{\check{\zeta}_e:e\in\tau'\backslash\Delta_z\}$ forms a subset of a system of local parameters on a neighborhood $\mathcal{V}_z\subset\widetilde{\mathfrak{M}}$ of z. Below, we construct the rooted tree with which $\mathrm{PT}_\pi(\pi')$ is compatible on $\mathcal{V}_z$.

Let

$$\Xi_z^{\complement}(\tau'):=\{\mathfrak{E}\in\Xi(\tau'):\ \mathfrak{E}\cap\Delta_z=\emptyset\},\quad E':=\bigcup_{\mathfrak{E}\in\Xi_z^{\complement}(\tau')}\mathfrak{E}\quad(\subset(\tau'\backslash\Delta_z)\subset\tau').\tag{4.33}$$

We denote by $\preceq'$ the tree order of τ', whose restriction to the subset E' is also written as $\preceq'$. It is straightforward that

$$\mathrm{PT}_z(\tau')=\mathrm{PT}_{\pi,z}(\tau'):=(E',\preceq')$$

is a rooted tree, called the **proper transform of τ' (with respect to π) at z**. (Indeed, every subposet of a rooted tree is naturally a rooted tree; cf. [HN21, Definition 2.10].)

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An example of the proper transform of a rooted tree is provided in Example 4.32 at the end of Section 4.8.

LEMMA 4.27. *With notation as above, we have* $\Xi(\mathrm{PT}_z(\tau')) = \Xi^{\mathbf{C}}_z(\tau')$.

Proof. Without loss of generality we assume that $\Xi^{\mathbf{C}}_z(\tau') \neq \emptyset$, which is equivalent to $\mathrm{PT}_z(\tau') \neq \tau_\bullet$.

Given $\mathfrak{E} \in \Xi^{\mathbf{C}}_z(\tau')$, any two edges of $\mathfrak{E}$ are incomparable in τ', hence in $\mathrm{PT}_z(\tau')$. Moreover, since $\mathrm{PT}_z(\tau') \subset \tau'$ as sets, we see that every edge of $\mathrm{PT}_z(\tau')$ is an edge of τ', hence is comparable with some edge of $\mathfrak{E}$. The second criterion of Definition 4.8 then implies that $\mathfrak{E} \in \Xi(\mathrm{PT}_z(\tau'))$, so $\Xi^{\mathbf{C}}_z(\tau') \subset \Xi(\mathrm{PT}_z(\tau'))$.

Conversely, given $\mathfrak{E} \in \Xi(\mathrm{PT}_z(\tau'))$, any two edges of $\mathfrak{E}$ are incomparable in $\mathrm{PT}_z(\tau')$, hence in τ'. It remains to show that for every $e' \in \tau'$, there exists $e \in \mathfrak{E}$ comparable with e'. Let $\wp'$ be an arbitrary root-to-leaf path of τ' that contains e'. By the first criterion of Definition 4.8, each $\mathfrak{E}' \in \Xi^{\mathbf{C}}_z(\tau')$ meets $\wp'$ at a unique edge $e_{\mathfrak{E}'}$. Since $\wp'$, along with the restriction of $\preceq'$, is a linear set, we denote by e_0 the least element of the set $\{e_{\mathfrak{E}'}\}$, which is comparable with e'. In addition, observe that $\wp' \cap \mathrm{PT}_z(\tau')$ is a root-to-leaf path of $\mathrm{PT}_z(\tau')$, so it meets $\mathfrak{E}$ at a unique edge e. Then, $e_0 \preceq' e$.

- If $e' \preceq' e_0$, then $e' \preceq' e$.
- If $e' \succ' e_0$, then (4.5) implies that e' and e are comparable.

In sum, we find $e \in \mathfrak{E}$ comparable with e'. By the second criterion of Definition 4.8, we thus have $\mathfrak{E} \in \Xi(\tau')$. Taking $\mathfrak{E} \in E' \subset \tau' \backslash \Delta_z$ into consideration, we conclude that $\Xi(\mathrm{PT}_z(\tau')) \subset \Xi^{\mathbf{C}}_z(\tau')$. □

COROLLARY 4.28. $\mathrm{PT}_\pi(\pi')$ *is* $\mathrm{PT}_z(\tau')$*-compatible on* $\mathcal{V}_z$.

Proof. Let $\Xi(\tau') = \bigsqcup_{k\geq 1} \Xi_k(\tau')$ be the partition determined by π' as per Definition 4.9. It induces a partition of $\Xi(\mathrm{PT}_z(\tau'))$:

$$\Xi(\mathrm{PT}_z(\tau')) = \Xi^{\mathbf{C}}_z(\tau') = \bigsqcup_{k\geq 1} \Xi^{\mathbf{C}}_{z,k}(\tau'), \quad \text{where } \Xi^{\mathbf{C}}_{z,k}(\tau') := \{\mathfrak{E} \in \Xi_k(\tau') : \mathfrak{E} \cap \Delta_z = \emptyset\}.$$

The first identity above follows from Lemma 4.27. By the analogue of Lemma 4.25 in the current setting, we see that each $\check{Z}'_k$ is the union of $X^{\mathcal{V}_z}_{\mathfrak{E}}$, $\mathfrak{E} \in \Xi^{\mathbf{C}}_{z,k}(\tau')$, which justifies (C1) of Definition 4.9 for $\mathrm{PT}_\pi(\pi')$. Since (C2) naturally holds for $\mathrm{PT}_\pi(\pi')$, the statement of the corollary is established. □

PROPOSITION 4.29. *Every root-to-leaf path of* $\mathrm{PT}_z(\tau')$ *is an inclusion-minimal element of*

$$S' := \{\wp' \backslash \Delta_z : \wp' \text{ is a root-to-leaf path of } \tau'\},$$

and vice versa.

We remark that each root-to-leaf path of $\mathrm{PT}_z(\tau')$, as a subposet of $\mathrm{PT}_z(\tau')$, is naturally a subposet of τ', because so is $\mathrm{PT}_z(\tau')$.

Proof of Proposition 4.29. If Δ_z contains a root-to-leaf path of τ', then, on the one hand, we have $\emptyset \in S'$, hence the only inclusion-minimal element of S' is $\emptyset$; on the other hand, we have $\Xi^{\mathbf{C}}_z(\tau') = \emptyset$ by Definition 4.8, so $\mathrm{PT}_z(\tau')$ is trivial, thus the only root-to-leaf path is the trivial poset. So Proposition 4.29 is true when Δ_z contains a root-to-leaf path of τ'.

Below, we assume that Δ_z does not contain any root-to-leaf path of τ' (which implies that τ' is not trivial).

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(a) We begin by showing that every inclusion-minimal element of S' is a root-to-leaf path of $\mathrm{PT}_z(\tau')$.

- To this end, we first show that every inclusion-minimal element $\widetilde{\wp}\backslash\Delta_z$ of S' is contained in $\mathrm{PT}_z(\tau')$. Notice that each element of S' is nonempty and thus has a unique least element (w.r.t. the tree order), hence

$$\mathfrak{E}' := \{\, \min(\varsigma') : \varsigma' \text{ is an inclusion-minimal element of } S' \}$$

is a well-defined nonempty subset of $\tau'\backslash\Delta_z$. We claim that

$$\mathfrak{E}' \in \Xi(\tau').$$

In fact, for every root-to-leaf path $\wp'$ of τ', there exists a root-to-leaf path $\wp''$ of τ' such that

$$\wp''\backslash\Delta_z \text{ is an inclusion-minimal element of } S' \quad \text{and} \quad (\wp''\backslash\Delta_z) \subset (\wp'\backslash\Delta_z).$$

With e'' denoting the least element of $\wp''\backslash\Delta_z$, we have

$$e'' \in (\mathfrak{E}' \cap (\wp''\backslash\Delta_z)) \subset (\mathfrak{E}' \cap (\wp'\backslash\Delta_z)) \subset (\mathfrak{E}' \cap \wp').$$

Suppose $\mathfrak{E}' \cap \wp'$ contains another edge $\widehat{e}$ distinct from e''. Then, either $\widehat{e} \succ e''$ or $\widehat{e} \prec e''$, because e'' and $\widehat{e}$ belong to the linearly ordered set $\wp'$. However, $\widehat{e} \in \mathfrak{E}'$ implies that there exists a root-to-leaf path $\widehat{\wp}$ of τ' such that $\widehat{\wp}\backslash\Delta_z$ is inclusion-minimal in S' and $\widehat{e}$ is the least element of $\widehat{\wp}\backslash\Delta_z$. Consequently, one of $\wp''\backslash\Delta_z$ and $\widehat{\wp}\backslash\Delta_z$ must contain the other as a strict subset, hence is not inclusion-minimal. This contradiction shows that e'' is the only element of $\mathfrak{E}' \cap \wp'$, which by Definition 4.8 implies that $\mathfrak{E}' \in \Xi(\tau')$. Now, consider an arbitrary inclusion-minimal element $\widetilde{\wp}\backslash\Delta_z$ of S', where $\widetilde{\wp}$ is a root-to-leaf path of τ'. For every $\widetilde{e} \in \widetilde{\wp}\backslash\Delta_z$, we claim that

$$\widetilde{e} \in \widetilde{\mathfrak{E}}' \in \Xi(\tau'), \quad \text{where } \widetilde{\mathfrak{E}}' := \max(\mathfrak{E}' \cup \{\widetilde{e}\}). \tag{4.34}$$

– The reason that $\widetilde{\mathfrak{E}}' \in \Xi(\tau')$ is as follows. Notice that distinct maximal elements of $\mathfrak{E}' \cup \{\widetilde{e}\}$ are incomparable. In addition, for every $e \in \tau'$, there exist $e' \in \mathfrak{E}'$ comparable with e, because $\mathfrak{E}' \in \Xi(\tau')$, as well as $e'' \in \widetilde{\mathfrak{E}}'$ satisfying $e'' \succeq e'$. If $e' \succeq e$, then $e'' \succeq e$; if $e' \prec e$, then, by (4.5), e and e'' are comparable. In sum, we have $\widetilde{\mathfrak{E}}' \in \Xi(\tau')$.

– Suppose $\widetilde{e} \notin \widetilde{\mathfrak{E}}'$, then there would exist $\widetilde{e}' \in \widetilde{\mathfrak{E}}'$ comparable with $\widetilde{e}$. From the definition of $\widetilde{\mathfrak{E}}'$ in (4.34), we conclude that $\widetilde{e} \prec \widetilde{e}'$ and $\widetilde{e}' \in \mathfrak{E}'$. However, the least element $\widetilde{e}^\flat$ of $\widetilde{\wp}\backslash\Delta_z$ is contained in $\mathfrak{E}'$, satisfying $\widetilde{e}^\flat \preceq \widetilde{e} \prec \widetilde{e}'$, which contradicts the fact that distinct elements of the transverse section $\mathfrak{E}'$ are incomparable. Therefore, $\widetilde{e} \in \widetilde{\mathfrak{E}}'$.

The above two paragraphs then give rise to (4.34). Taking $\mathfrak{E}' \cap \Delta_z = \emptyset$ and $\widetilde{e} \notin \Delta_z$ into consideration, we obtain

$$\widetilde{e} \in \widetilde{\mathfrak{E}}' \in \Xi_z^{\mathfrak{C}}(\tau'),$$

hence $\widetilde{e} \in \mathrm{PT}_z(\tau')$. In other words, the inclusion-minimal element $\widetilde{\wp}\backslash\Delta_z$ of S' is contained in $\mathrm{PT}_z(\tau')$.

- We then show that the above $\widetilde{\wp}\backslash\Delta_z$ is a root-to-leaf path of $\mathrm{PT}_z(\tau')$, i.e. a linearly ordered subset of $\mathrm{PT}_z(\tau')$ that is maximal with respect to inclusion. Indeed, it is linearly ordered because so is $\widetilde{\wp}$. Suppose it is not maximal with respect to inclusion, then there exists a root-to-leaf path $\wp$ of $\mathrm{PT}_z(\tau')$ containing $\widetilde{\wp}\backslash\Delta_z$ as a strict subset.

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Take $e \in \wp$ that is not in $\widetilde{\wp}\backslash\Delta_z$. By (4.33), there exists $\mathfrak{E} \in \Xi(\tau')$ that is disjoint from Δ_z and contains e. Since $\widetilde{\wp}$ is a root-to-leaf path of τ', it meets $\mathfrak{E}$ at a unique edge e_0.

- If $e_0 \in \Delta_z$, then $e_0 \in (\mathfrak{E} \cap \Delta_z)$, contradicting the fact that $\mathfrak{E}$ is disjoint from Δ_z.
- If $e_0 \in (\widetilde{\wp}\backslash\Delta_z)$, then $e_0 \neq e$ because $e \notin (\widetilde{\wp}\backslash\Delta_z)$. However, by the choice of $\wp$, we have $e_0 \in \wp$. Since $\wp$ is linearly ordered, e_0 and e are comparable, contradicting the fact that they are distinct elements of the transverse section $\mathfrak{E}$.

The above contradictions indicate that $\widetilde{\wp}\backslash\Delta_z$ is a root-to-leaf path of $\mathrm{PT}_z(\tau')$. This completes the proof of part (a).

(b) We next show that every root-to-leaf path $\wp$ of $\mathrm{PT}_z(\tau')$ is an inclusion-minimal element of S'. To see this, notice that $\wp$ is a priori a linearly ordered subset of τ'. Therefore, there exists a maximal linearly ordered subset $\wp'$ of τ', i.e. a root-to-leaf path of τ' such that $\wp \subset \wp'$. Since the underlying set of $\mathrm{PT}_z(\tau')$ is contained in $\tau'\backslash\Delta_z$, we have $\wp \subset (\wp'\backslash\Delta_z)$. Then, there exists a root-to-leaf path $\wp''$ of τ' such that

$$\wp''\backslash\Delta_z \text{ is an inclusion-minimal element of } S' \quad \text{and} \quad (\wp''\backslash\Delta_z) \subset (\wp'\backslash\Delta_z).$$

We will show $\wp = (\wp''\backslash\Delta_z)$.

Recall that, by part (a), $\wp''\backslash\Delta_z$ is a root-to-leaf path of $\mathrm{PT}_z(\tau')$. Therefore, in order to show that $\wp = (\wp''\backslash\Delta_z)$, it suffices to show that $\wp \subset (\wp''\backslash\Delta_z)$. Indeed, for every $e \in \wp$ $(\subset \mathrm{PT}_z(\tau'))$, by (4.33), there exists $\mathfrak{E} \in \Xi^{\mathfrak{C}}_z(\tau')$ containing e. As a transverse section of τ', $\mathfrak{E}$ meets $\wp''$ at a unique edge e''. Since $\mathfrak{E} \cap \Delta_z = \emptyset$, we have

$$e \in \wp \subset (\wp'\backslash\Delta_z), \quad e'' \in (\wp''\backslash\Delta_z) \subset (\wp'\backslash\Delta_z).$$

This implies that e and e'' both belong to $\wp' \cap \mathfrak{E}$, which is a singleton by Definition 4.8. Therefore, $e = e''$, and thus $\wp \subset (\wp''\backslash\Delta_z)$. □

4.8.3 *Proof of Case A.5, Part III, ($\mathsf{r}_3\mathsf{p}_4$).* We apply the results of Section 4.8.2 to the fixed lift $z \in \widetilde{\mathfrak{M}}^{\mathrm{div},\mathsf{r}_3\mathsf{p}_3}$ of $y \in \widetilde{\mathfrak{M}}^{\mathrm{div},\mathsf{r}_1\mathsf{p}_5}$. By Corollary 4.28, $(\mathsf{r}_3\mathsf{p}_4)$ is $\mathrm{PT}_z(\mathrm{gr}(\varrho_y))$-compatible on $\mathcal{V}_z$, so after $(\mathsf{r}_3\mathsf{p}_4)$ the pullback of $\mathcal{V}_z$ is smooth.

It remains to verify that the pullback of (4.31) is diagonalized after $(\mathsf{r}_3\mathsf{p}_4)$.

- If Δ_z contains the grafted edge (corresponding to the parameter $\check{\kappa}_{12}$) of $\mathrm{gr}(\tau_2)$, then the pullback of (4.31) to $\mathcal{V}_z$ is already diagonalized.
- If Δ_z does not contain the grafted edge of $\mathrm{gr}(\tau_2)$, then the second row of the pullback of (4.31) takes the form

$$\left(\prod_{k\in[\mathbf{m}]} \check{\varepsilon}_k\right)\left(\prod_{k\in[\mathbf{m}^{\mathsf{r}_3\mathsf{p}_3}]} \check{\varepsilon}_k^{\mathsf{r}_3\mathsf{p}_3}\right)\begin{bmatrix} 0 & \check{\kappa}_{12} & \check{\eta}_3 & \cdots & \check{\eta}_m \end{bmatrix}, \quad \text{where}$$

$$\check{\eta}_i := \left(\prod_{k\in[\mathbf{m}]_{[\delta_2\wedge\delta_i]}} \check{\varepsilon}_k\right)\left(\prod_{e\in N_{[\delta_i]}\backslash(\Delta_y\sqcup\Delta_z)} \check{\zeta}_e\right).$$

Here, the superscript $\mathsf{r}_3\mathsf{p}_3$ indicates the parameters corresponding to the exceptional divisors obtained in $(\mathsf{r}_3\mathsf{p}_3)$. Notice that for each $3 \le i \le m$ with $N_{[\delta_i]}\backslash\Delta_y \subset \Delta_z$, the corresponding $\check{\eta}_i = \prod_{k\in[\mathbf{m}]_{[\delta_2\wedge\delta_i]}} \check{\varepsilon}_k$. So by Proposition 4.29, the inclusion-minimal elements

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of the set

$$\{e_{\mathrm{gr}}\} \sqcup \{[\mathbf{m}]_{[\delta_2 \wedge \delta_i]} \sqcup (N_{[\delta_i]} \backslash (\Delta_y \sqcup \Delta_z)) : 3 \le i \le m\}$$

are exactly the root-to-leaf paths of $\mathrm{PT}_z(\mathrm{gr}(\varrho_y))$. Hence by Proposition 4.17, on the pullback of $\mathcal{V}_z$, the pullback of (4.31) is diagonalized.

To summarize, in this subsection, we have established the following.

Proposition 4.30. *Proposition 4.1 holds in Case A.5 if $|D(\Delta_y)| \ge 1$ and the point of $D(\Delta_y)$ is conjugate to the point $D \cap F$.*

Propositions 4.21, 4.24 and 4.30 together give rise to the following.

Proposition 4.31. *Proposition 4.1 holds in Case A.5.*

We conclude Section 4.8 with an example that demonstrates the idea of proper transforms of blowups as well as the structure of the proof of Part III of Case A.5.

Example 4.32. Let $\widehat{x} = (C, D) \in \mathfrak{M}_2^{\mathrm{div}}$ be such that C is the same rooted tree as in Figure 9, but the distribution of the elements of D is different:

$$D \cap F = \{\delta_1\}, \quad D \cap C_b = \{\delta_2\}, \quad D \cap C_a \ni \{\delta_3\}, \quad D \cap C_c \ni \{\delta_4\}, \quad D \cap C_d \ni \{\delta_5\}.$$

Moreover, we assume that δ_1 and δ_2 are conjugate. This implies that δ_1 and δ_3 cannot be conjugate, and nor can δ_3 be conjugate to δ_4 or δ_5.

By Proposition 2.22, under a suitable trivialization, φ on a chart $\mathcal{V}$ containing (C, D) can be written as

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \kappa_{12}\zeta_b & \zeta_a & \zeta_b^2\zeta_c & \zeta_b^2\zeta_d & 0 & \cdots \end{bmatrix}.$$

As shown in Section 4.6, $(\mathsf{r}_1\mathsf{p}_5)$ is compatible with τ_1 (4.23) near $\widehat{x}$. Here, τ_1 is given by the set $\{q_a, q_b\}$ along with the trivial order.

After $(\mathsf{r}_1\mathsf{p}_5)$, we take a lift y of $\widehat{x}$ whose RLS and dominant edges are, respectively, given by

$$\overline{\mathbf{E}} = \{\{q_a, q_b\}\}, \quad \Delta_y = \{q_b\}.$$

Locally near y, the pullback (4.31) of φ can be further simplified as

$$\begin{bmatrix} 1 & 0 \\ 0 & \varepsilon_1^{\mathsf{r}_1\mathsf{p}_5} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \check{\kappa}_{12} & \check{\zeta}_a & \varepsilon_1^{\mathsf{r}_1\mathsf{p}_5}\zeta_c & \varepsilon_1^{\mathsf{r}_1\mathsf{p}_5}\zeta_d & 0 & \cdots \end{bmatrix}.$$

Here, $\varepsilon_1^{\mathsf{r}_1\mathsf{p}_5}$ corresponds to the exceptional divisor after blowing up along $(\zeta_a = \zeta_b = 0)$.

Let τ_2 be the rooted tree given by the set $\{q_a, q_c, q_d\}$ along with the trivial order, which is the analogue of (4.20) in this situation. In addition, let ϱ_y be the first-order derived tree at y as in (4.23), which is given by the set $\{1, q_a\}$ along with the trivial order.

As shown in the beginning of Section 4.8.1, $(\mathsf{r}_3\mathsf{p}_3)$ is compatible with $\mathrm{gr}(\tau_2)$ near y, where the grafted edge corresponds to $\check{\kappa}_{12}$. After $(\mathsf{r}_3\mathsf{p}_3)$, we fix a lift z of y. As shown in Section 4.8.3, $(\mathsf{r}_3\mathsf{p}_4)$ is compatible with $\mathrm{PT}_z(\mathrm{gr}(\varrho_y))$ near z, where the grafted edge corresponds to $\check{\kappa}_{12}$.

If either $\check{\kappa}_{12}$ or q_a is contained in Δ_z, then, on the one hand, we immediately see that the pullback of φ becomes diagonalized near any lift of z. On the other hand, we have $\mathrm{PT}_z(\mathrm{gr}(\varrho_y)) = \tau_\bullet$ because the only transverse section of $\mathrm{gr}(\varrho_y)$ is $\{e_{\mathrm{gr}}, 1, q_a\}$; in other words, $(\mathsf{r}_3\mathsf{p}_4)$ does not affect a neighborhood of z.

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If neither $\check{\kappa}_{12}$ nor q_a is contained in Δ_z, then at least one of q_c and q_d, say q_c, is in Δ_z. On the one hand, near z, the pullback of φ can be rewritten as

$$\begin{bmatrix} 1 & 0 \\ 0 & \varepsilon_1^{\mathsf{r}_1\mathsf{p}_5}\varepsilon_1^{\mathsf{r}_3\mathsf{p}_3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \check{\kappa}_{12} & \check{\zeta}_a & \varepsilon_1^{\mathsf{r}_1\mathsf{p}_5} & 0 & \cdots \end{bmatrix}.$$

Here, $\varepsilon_1^{\mathsf{r}_1\mathsf{p}_5}$ corresponds to the exceptional divisor after blowing up along $(\check{\zeta}_a = \zeta_c = \zeta_d = 0)$. (By abuse of notation, we still write the proper transform of $\varepsilon_1^{\mathsf{r}_1\mathsf{p}_5}$ as $\varepsilon_1^{\mathsf{r}_1\mathsf{p}_5}$.) On the other hand, we have $\mathrm{PT}(\mathrm{gr}(\varrho_y)) = \mathrm{gr}(\varrho_y)$ because the only transverse section $\{e_{\mathrm{gr}}, 1, q_a\}$ of $\mathrm{gr}(\varrho_y)$ does not contain q_c or q_d. In $(\mathsf{r}_3\mathsf{p}_4)$, the locus $(\check{\kappa}_{12} = \varepsilon_1^{\mathsf{r}_1\mathsf{p}_5} = \check{\zeta}_a = 0)$ is blown up, hence the pullback of φ becomes diagonalized near any lift of z.

4.9 Proof of Case B

In this subsection, we prove Proposition 4.1 in Case B.

Let $\mathcal{V} \to \mathfrak{M}_2^{\mathrm{div}}$ be a small affine smooth chart containing $\widehat{x} = (C, D)$. In Case B, we assume that x, the image of $\widehat{x}$ in $\mathfrak{M}_2^{\mathrm{wt}}$, lies in $\mathfrak{M}_{(4)}^{\mathrm{mn}}$. This implies that the core F of $\widehat{x}$ can be written as

$$F = F_1 \cup \mathrm{B} \cup F_2,$$

where F_1 and F_2 are both genus 1 inseparable components, and B is either a separating bridge or a node, such that any two among F_1, F_2 and B do not share any common irreducible component. Without loss of generality we assume that

$$a_1 \in F_1, \quad a_2 \in F_2, \quad \text{and} \quad D \cap F = \{\delta_1, \ldots, \delta_\ell\} \subset F_1 \cup \mathrm{B}$$

for some $\ell \in [m]$.

Notice that $D \cap F$ cannot be entirely contained in B, for otherwise x would be in $\mathfrak{M}_{(2)}^{\mathrm{mn}}$. Hence D has an element, say δ_1, such that

$$\delta_1 \in F_1, \quad \text{i.e.} \quad N_{[\delta_1, a_1]} = \emptyset. \tag{4.35}$$

So by Proposition 2.22, after suitable elementary row and column operations, φ can be written as

$$\varphi = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \kappa_{12}\zeta_{[\delta_2, a_2]} & \cdots & \kappa_{1m}\zeta_{[\delta_m, a_2]} \end{bmatrix}. \tag{4.36}$$

Also notice that $D \cap F$ cannot belong to any non-separating bridge of F_1 (should there be one), for otherwise x would be in $\mathfrak{M}_{(3)}^{\mathrm{mn}}$. Therefore, $(\mathsf{r}_1\mathsf{p}_1)$ to $(\mathsf{r}_1\mathsf{p}_3)$ do not affect $\mathcal{V}$.

Mimicking (4.19), we set

$$D_{\text{i-m}} := \{\delta \in D : N_{[\delta, a_2]} \text{ is inclusion-minimal among all } N_{[\delta', a_2]},\ \delta' \in D\},$$
$$E := \bigcup_{\delta \in D_{\text{i-m}}} N_{[\delta, a_2]}\ (\subset N(C)).$$

On the set E, we define the relation $\preceq$ such that $e \preceq e'$ if and only if any connected subcurve of C containing F_2 and e must contain e', which once again satisfies (4.5), i.e. the poset

$$\tau^\dagger := (E, \preceq) \tag{4.37}$$

is a rooted tree. The root-to-leaf paths of $\tau^\dagger$ are exactly $N_{[\delta, a_2]}$, $\delta \in D_{\text{i-m}}$.

An example of $\tau^\dagger$ is provided in Figure 19.

LEMMA 4.33. *Given $\delta \in D \backslash F$, let $\langle\delta\rangle \in F$ be as in (2.17). If $\langle\delta\rangle \in F_1$, then $\delta \notin D_{\text{i-m}}$.*

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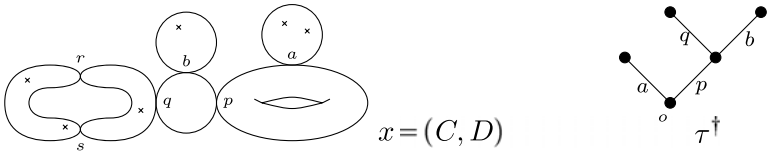


FIGURE 19. An example of $\tau^\dagger$.

Proof. If $\delta \notin F$ and $\langle\delta\rangle \in F_1$, then $N_{[\delta]} \neq \emptyset$ and $N_{[\delta,a_2]} = N_{[\delta]} \sqcup N_{[a_2,F_1]}$. Since $\delta_1 \in D \cap F$, we have $N_{[\delta_1,a_2]} \subset N_{[a_2,F_1]} \subsetneq N_{[\delta,a_2]}$, thus $\delta \notin D_{\text{i-m}}$. □

As described in Section 3.1, the blowup centers of $(\mathsf{r}_1\mathsf{p}_4)$ are locally given by

$$\overline{\mathfrak{M}}_{(4,k)} \cap \mathcal{V} = \bigcup_{\mathfrak{E}\in\Xi_k(\tau^\dagger)} \{\zeta_e = 0 : e \in \mathfrak{E}\}, \quad \text{where } \Xi_k(\tau^\dagger) = \{\mathfrak{E} \in \Xi(\tau^\dagger) : |\mathfrak{E}| = k\}.$$

From the same argument as in the proof of Proposition 4.19, we conclude that $(\mathsf{r}_1\mathsf{p}_4)$ is $\tau^\dagger$-compatible on $\mathcal{V}$. Lemma 4.10 then implies that the pullback $\widetilde{\mathcal{V}}^{\mathsf{r}_1\mathsf{p}_4}$ of $\mathcal{V}$ is smooth.

After $(\mathsf{r}_1\mathsf{p}_4)$, we fix an arbitrary lift $y \in \widetilde{\mathcal{V}}^{\mathsf{r}_1\mathsf{p}_4}$ of $\widehat{x}$, as well as a small neighborhood $\mathcal{V}_y$ of y in $\widetilde{\mathcal{V}}^{\mathsf{r}_1\mathsf{p}_4}$. The RL sequence of y is denoted by $\overline{\mathbf{E}} = \{\mathfrak{E}_1, \ldots, \mathfrak{E}_{\mathbf{m}}\}$ as before. By Definition 4.15, there exists $i \in [m]$ such that $N_{[\delta_i,a_2]} \subset \Delta_y$; such δ_i is naturally contained in $D_{\text{i-m}}$. Mimicking the notation of (4.24), we write

$$D(\Delta_y) := \{\delta \in D :\ N_{[\delta,a_2]} \subset \Delta_y\} \quad (= \{\delta \in D_{\text{i-m}} :\ N_{[\delta,a_2]} \subset \Delta_y\}).$$

- If $D(\Delta_y) \not\subset F$, there exists $i > \ell$ such that $\delta_i \in D(\Delta_y) \subset D_{\text{i-m}}$. Lemma 4.33 then implies that $\langle\delta_i\rangle \notin F_1$. From (4.35) and Lemma 2.14, we conclude that δ_1 and δ_i are not conjugate, i.e. the proper transform of κ_{1i} (which is equal to the pullback in this case) does not vanish at y, hence is invertible on the small neighborhood $\mathcal{V}_y$. So by Proposition 4.17, the pullback of $\kappa_{1i}\zeta_{[\delta_i,a_2]}$ divides all the other terms of the second row of (4.36). That is, after suitable elementary column operations, (4.36) becomes diagonalized. Moreover, it is straightforward that $(\mathsf{r}_1\mathsf{p}_5)$ to $(\mathsf{r}_3\mathsf{p}_4)$ do not affect $\mathcal{V}_y$, so the pullback of $\mathcal{V}_y$ to $\widetilde{\mathfrak{M}}_2^{\text{div}}$ is smooth.
- If $D(\Delta_y) \subset F$, then all the separating nodes of F are in Δ_y, hence

$$N_{[\delta_i,a_2]} \backslash \Delta_y = N_{[\delta_i]} \backslash \Delta_y \quad \forall\, i \in [m].$$

Therefore, the pullback of (4.36) to $\mathcal{V}_y$, after suitable elementary column operations, can be written as

$$\begin{bmatrix} 1 & 0 \\ 0 & \prod_{k\in[\mathbf{m}]} \check{\varepsilon}_k \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \check{\kappa}_{12} \prod_{e\in N_{[\delta_2]}\backslash\Delta_y} \check{\zeta}_e & \cdots & \check{\kappa}_{1m} \prod_{e\in N_{[\delta_m]}\backslash\Delta_y} \check{\zeta}_e \end{bmatrix}, \tag{4.38}$$

where $\check{\varepsilon}_k$ are the local parameters on $\mathcal{V}_y$ corresponding to the exceptional divisors obtained in $(\mathsf{r}_1\mathsf{p}_4)$; cf. Lemma 4.16. Without loss of generality we assume that there exists $h \in [\ell]$ such that

$$D(\Delta_y) = \{\delta_h, \ldots, \delta_\ell\} \quad (\subset F).$$

Then, we have

$$\prod_{e\in N_{[\delta_i]}\backslash\Delta_y} \check{\zeta}_e \in \Gamma(\mathscr{O}^*_{\mathcal{V}_y}) \quad \forall\, h \le i \le \ell. \tag{4.39}$$

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– In Case B.1, recall that $D \cap F$ cannot belong to any non-separating bridge of F_1 (should there be one), for otherwise x would be in $\mathfrak{M}^{\mathrm{mn}}_{(3)}$. Therefore, there exists $h \le j \le \ell$ such that δ_j is not conjugate to δ_1, i.e. $\check{\kappa}_{1j}$ is invertible on $\mathcal{V}_y$. Along with (4.39), this implies that in the second row of (4.38), the jth entry divides the other entries, hence (4.38) is diagonalized on $\mathcal{V}_y$. Moreover, it is straightforward that $(\mathsf{r}_1\mathsf{p}_5)$ to $(\mathsf{r}_3\mathsf{p}_4)$ do not affect $\mathcal{V}_y$, so the pullback of $\mathcal{V}_y$ to $\widetilde{\mathfrak{M}}^{\mathrm{div}}_2$ is smooth.
– In Case B.2, the same argument as above can be applied verbatim.
– In Case B.3, we have $h=1$, $\ell=2$, and $\check{\kappa}_{12}$ vanishes at y. Once again, every δ_i, $i \ge 3$, is not conjugate to δ_1 because δ_1 and δ_2 are not on any non-separating bridge. Taking (4.39) into consideration, we can rewrite (4.38) as

$$\begin{bmatrix} 1 & 0 \\ 0 & \prod_{k\in[\mathbf{m}]} \check{\varepsilon}_k \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \check{\kappa}_{12} & \prod_{e\in N_{[\delta_3]}\setminus\Delta_y} \check{\zeta}_e & \cdots & \prod_{e\in N_{[\delta_m]}\setminus\Delta_y} \check{\zeta}_e \end{bmatrix}, \tag{4.40}$$

In this situation, it is straightforward to show that $(\mathsf{r}_1\mathsf{p}_5)$ to $(\mathsf{r}_3\mathsf{p}_2)$ do not affect $\mathcal{V}_y$. Let τ_2 be the rooted tree as in (4.20) whose root-to-leaf paths are those inclusion-minimal $N_{[\delta_i]}$ among all $3 \le i \le m$. Then, by Corollary 4.28 and the proof of Proposition 4.19, we see that $(\mathsf{r}_3\mathsf{p}_3)$ is $\mathrm{PT}_y(\mathrm{gr}(\tau_2))$-compatible on $\mathcal{V}_y$, so the pullback of $\mathcal{V}_y$ to $\widetilde{\mathfrak{M}}^{\mathrm{div},\mathsf{r}_3\mathsf{p}_3}$, denoted by $\widetilde{\mathcal{V}}_y^{\mathsf{r}_3\mathsf{p}_3}$, is smooth. In addition, by Propositions 4.29 and 4.17, the pullback of (4.40) is diagonalized on $\widetilde{\mathcal{V}}_y^{\mathsf{r}_3\mathsf{p}_3}$. Moreover, it is straightforward to show that $(\mathsf{r}_3\mathsf{p}_4)$ does not affect $\widetilde{\mathcal{V}}_y^{\mathsf{r}_3\mathsf{p}_3}$ because $(\mathsf{r}_1\mathsf{p}_5)$ does not affect $\mathcal{V}_y$; therefore, the pullback of $\mathcal{V}_y$ to $\widetilde{\mathfrak{M}}^{\mathrm{div}}_2$ is smooth.
– In Case B.4, we have $h=\ell=1$. In this situation, let τ_1 be the rooted tree as in (4.23). Then, by the proof of Proposition 4.31, we see that $(\mathsf{r}_1\mathsf{p}_5)$ is $\mathrm{PT}_y(\tau_1)$-compatible on $\mathcal{V}_y$. Consequently, the three cases of the proof of Proposition 4.31, namely those of Propositions 4.21, 4.24, and 4.30, can be applied to show that the pullback of $\mathcal{V}_y$ to $\widetilde{\mathfrak{M}}^{\mathrm{div}}_2$ is smooth, on which the pullback of (4.38) is diagonalized.

To summarize, we have shown the following.

PROPOSITION 4.34. *Proposition 4.1 holds in Case B.*

4.10 Proof of Case C

In this subsection, we prove Proposition 4.1 in Case C.

Let $\mathcal{V} \to \mathfrak{M}^{\mathrm{div}}_2$ be a small affine smooth chart containing $\widehat{x} = (C, D)$. In Case C, we assume that x, the image of $\widehat{x}$ in $\mathfrak{M}^{\mathrm{wt}}_2$, lies in $\mathfrak{M}^{\mathrm{mn}}_{(3)}$. This implies that the core F of x contains a *maximal* non-separating bridge $\mathrm{B} = \mathrm{B}[\mathfrak{p}, \mathfrak{q}]$ such that

$$D \cap \mathrm{B} = D \cap F.$$

Without loss of generality we assume that there exist $1 \le h \le \ell \le m$ such that

$$D \cap F = \{\delta_1, \dots, \delta_h\}, \quad \{\delta \in D : \langle\delta\rangle \in \mathrm{B}\} = \{\delta_1, \dots, \delta_\ell\},$$

where $\langle\delta\rangle$ are as in (2.17). Furthermore, we assume that

$$N_{[\delta_1,\mathfrak{p}]} \subset N_{[\delta_i,\mathfrak{p}]} \quad \forall\, i \in [h].$$

In addition, if $|D \cap F| \ge 2$ (i.e. $h \ge 2$), then we assume

$$N_{[\delta_2,\mathfrak{q}]} \subset N_{[\delta_i,\mathfrak{q}]} \quad \forall\, i \in [h].$$

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Let F_1 be the subcurve of F, as in Definition 2.9, determined by $\mathrm{B}\cup F_1 = F$ and $\mathrm{B}\cap F_1 = \{\mathfrak{p},\mathfrak{q}\}$. The subcurve F_1 contains a unique genus 1 inseparable component, denoted by T_1. Notice that T_1 is strictly smaller than F_1 if and only if F is separable.

By Part (A) of Assumption 2.2, if F is inseparable, then a_1 and a_2 are both on T_1 $(=F_1)$; if F is separable, without loss of generality we assume that a_1 and B belong to the same genus 1 inseparable component, whereas $a_2\in T_1$. In either case, we have $N_{[\delta_1,a_1]}=\emptyset$, hence after suitable row and column operations, the structural homomorphism φ still takes the form of (4.36); that is,

$$\varphi=\begin{bmatrix}1 & 0 & \cdots & 0\\ 0 & \kappa_{12}\zeta_{[\delta_2,a_2]} & \cdots & \kappa_{1m}\zeta_{[\delta_m,a_2]}\end{bmatrix}. \tag{4.41}$$

Moreover, in either case,

$$N_{[\delta_i,a_2]}=N_{[a_1,a_2]}\quad\forall\, i\in[h]. \tag{4.42}$$

For the proof of Case C, we delve more deeply into (4.41). Notice that

$$\kappa_{1i}\in\Gamma(\mathscr{O}_{\mathcal{V}}^*)\quad\forall\,\ell<i\le m,\quad \kappa_{1i}(\widehat{x})=0\quad\forall\, i\in[\ell].$$

More precisely, for $i\in[\ell]$, we have the following. Recall that for every $\delta\in D$, $z_{\langle\delta\rangle}$ is a smooth point on B that is sufficiently close to $\langle\delta\rangle$; see Propositions 2.17 and 2.19.

- When $h\ge 2$, by Proposition 2.22 (SH_3), we have

$$\kappa_{12}=f_2\zeta_{[\delta_1,\mathfrak{p}]}+g_2\zeta_{[\delta_2,\mathfrak{q}]},\quad \kappa_{1i}=v_i\kappa_{12}+g_i'\zeta_{[\delta_2,\mathfrak{q}]},\quad 2<i\le h,$$
$$\kappa_{1i}=v_i\kappa_{12}+g_i'\beta_i,\quad h<i\le\ell,\quad \text{where } \beta_i:=\begin{cases}\zeta_{[z_{\langle\delta_i\rangle},\mathfrak{p}]} & \text{if } [z_{\langle\delta_i\rangle},\mathfrak{p}]\subsetneq[\delta_1,\mathfrak{p}],\\ \zeta_{[z_{\langle\delta_i\rangle},\mathfrak{q}]} & \text{if } [z_{\langle\delta_i\rangle},\mathfrak{q}]\subsetneq[\delta_2,\mathfrak{q}],\\ \zeta_{[\delta_2,\mathfrak{q}]} & \text{otherwise,}\end{cases} \tag{4.43}$$

  and f_2, g_2, v_i and g_i' are invertible functions on $\mathcal{V}$. Taking (4.42) into consideration, we can apply suitable column operations to (4.41) and obtain the following.

  – If $h\ge 3$, then the structural homomorphism φ can be rewritten as

$$\begin{bmatrix}1 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \vec{\mathbf{0}}\\ 0 & \kappa_{12}\zeta_{[a_1,a_2]} & \zeta_{[\delta_2,\mathfrak{q}]}\zeta_{[a_1,a_2]} & \beta_{h+1}\zeta_{[\delta_{h+1},a_2]} & \cdots & \beta_\ell\zeta_{[\delta_\ell,a_2]} & \zeta_{[\delta_{\ell+1},a_2]} & \cdots & \zeta_{[\delta_m,a_2]} & \vec{\mathbf{0}}\end{bmatrix}, \tag{4.44}$$

    where $\vec{\mathbf{0}}$ denotes the zero matrix of size $1\times(h-3)$.

  – If $h=2$, then φ can be rewritten as

$$\varphi=\begin{bmatrix}1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0\\ 0 & \kappa_{12}\zeta_{[a_1,a_2]} & \beta_{h+1}\zeta_{[\delta_{h+1},a_2]} & \cdots & \beta_\ell\zeta_{[\delta_m,a_2]} & \zeta_{[\delta_{\ell+1},a_2]} & \cdots & \zeta_{[\delta_m,a_2]}\end{bmatrix}. \tag{4.45}$$

- When $h=1$, the points of $D\backslash\{\delta_1\}$ are all on tails, so before applying blowups, there is no obvious choice of $\delta_2\in D\backslash\{\delta_1\}$ such that κ_{1i}, $i\ge 3$, should be compared with κ_{12}. In this case, we postpone the comparison of the κ_{1i} until $(\mathsf{r}_1\mathsf{p}_3)$ is complete.

We are ready to analyze the effects of the sequential blowups on $\mathcal{V}$. Since $D\cap F$ belongs to a non-separating bridge of F, $(\mathsf{r}_1\mathsf{p}_1)$ and $(\mathsf{r}_1\mathsf{p}_2)$ do not affect $\mathcal{V}$. Next, we construct the rooted

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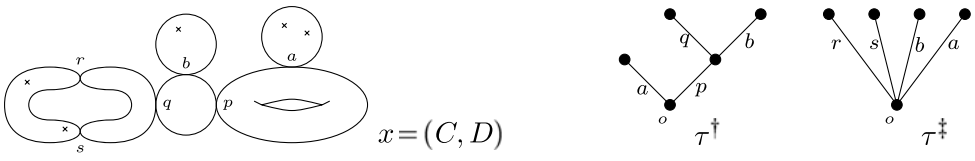


FIGURE 20. An example of $\tau^{\ddagger}$.

tree $\tau^{\ddagger}$ with which $(\mathsf{r}_1\mathsf{p}_3)$ is compatible on $\mathcal{V}$. Let

$$S'_\delta := \begin{cases} \{N_{[\delta]}\} & \text{if } \langle\delta\rangle \notin \mathrm{B}, \\ \{N_{[\delta]} \sqcup N_{[z_{\langle\delta\rangle},\mathfrak{p}]},\ N_{[\delta]} \sqcup N_{[z_{\langle\delta\rangle},\mathfrak{q}]}\} & \text{if } \langle\delta\rangle \in \mathrm{B}, \end{cases} \quad \delta \in D; \quad S' := \bigcup_{\delta\in D} S'_\delta.$$

In particular, we observe that $N_{[\delta_1,\mathfrak{p}]}$ and $N_{[\delta_2,\mathfrak{q}]}$ are inclusion-minimal in S' if $h \geq 2$, while $N_{[\delta_1,\mathfrak{p}]}$ and $N_{[\delta_1,\mathfrak{q}]}$ are inclusion-minimal in S' if $h = 1$.

We take the underlying set E of the proposed rooted tree $\tau^{\ddagger}$ to be the union of the inclusion-minimal elements of S', i.e.

$$E = \bigcup_{\text{inclusion-minimal } N'\in S'} N' \qquad (\subset N(C)).$$

Let C_1 be the maximal (w.r.t. inclusion) connected genus 1 subcurve of C that is disjoint from $D \cap F$. Then $E \subset C_1$. We define the relation $\preceq$ on E so that $e \preceq e'$ if and only if every connected subcurve C' of C_1 containing e and F_1 must contain e'. Intuitively, the two criteria above indicate that e' is 'closer' to F_1 than e on N', which shares the idea in the constructions of τ_1, τ_2 and $\tau^{\dagger}$ in Sections 4.5, 4.6 and 4.9, respectively. The relation $\preceq$ satisfies (4.5), thus the poset

$$\tau^{\ddagger} := (E, \prec) \tag{4.46}$$

is a rooted tree. The root-to-leaf paths of $\tau^{\ddagger}$ are exactly the inclusion-minimal elements of S'.

An example of $\tau^{\ddagger}$ is provided in Figure 20. Notice that $\tau^{\dagger}$ as in (4.37) is also defined in this case, and is involved in the proof of Section 4.10.4.

As described in Section 3.1, the blowup centers of $(\mathsf{r}_1\mathsf{p}_3)$ are locally given by

$$\overline{\mathfrak{M}}_{(3,k)} \cap \mathcal{V} = \bigcup_{\mathfrak{E}\in\Xi_k(\tau^{\ddagger})} \{\zeta_e = 0 : e \in \mathfrak{E}\}, \quad \text{where } \Xi_k(\tau^{\ddagger}) = \{\mathfrak{E} \in \Xi(\tau^{\ddagger}) : |\mathfrak{E}| = k+1\}, \quad k \geq 1.$$

By the stability of $\mathfrak{M}_2^{\mathrm{div}}$, we conclude that $(\mathsf{r}_1\mathsf{p}_3)$ is $\tau^{\ddagger}$-compatible on $\mathcal{V}$. Lemma 4.10 then implies that the pullback $\widetilde{\mathcal{V}}^{\mathsf{r}_1\mathsf{p}_3}$ of $\mathcal{V}$ is smooth.

After $(\mathsf{r}_1\mathsf{p}_3)$, we fix an arbitrary lift $y \in \widetilde{\mathcal{V}}^{\mathsf{r}_1\mathsf{p}_3}$ of $\widehat{x}$, as well as a small neighborhood $\mathcal{V}_y$ of y in $\widetilde{\mathcal{V}}^{\mathsf{r}_1\mathsf{p}_3}$. The RL sequence of y is denoted by $\overline{\mathbf{E}} = \{\mathfrak{E}_1, \ldots, \mathfrak{E}_{\mathbf{m}}\}$ as before.

By Definition 4.15, there exists at least one inclusion-minimal $N' \in S'$ such that $N' \subset \Delta_y$, hence there exists $\delta \in D$ such that $N' \in S'_\delta$. Mimicking (4.24), we set

$$S'(\Delta_y) := \{N' \in S' :\ N' \subset \Delta_y\}, \quad D(\Delta_y) := \{\delta \in D :\ S'_\delta \cap S'(\Delta_y) \neq \emptyset\}.$$

4.10.1 *Case C.1.* In this sub-case, we assume that $h \geq 3$ and F is inseparable, so we have

$$N_{[a_1,a_2]} = \emptyset \quad (\Longrightarrow \zeta_{[a_1,a_2]} = 1).$$

Moreover, $h \geq 3$ implies that $(\mathsf{r}_1\mathsf{p}_4)$ to $(\mathsf{r}_3\mathsf{p}_4)$ do not affect $\mathcal{V}_y$, hence the pullback of $\mathcal{V}_y$ to the final stack is smooth.

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It remains to show that the pullback of (4.41) is diagonalized after after $(\mathsf{r}_1\mathsf{p}_3)$. We divide the proof into three sub-cases as follows.

(1) If $D(\Delta_y)\subset F$, then there exists $i\in[h]$ such that at least one of $N_{[\delta_i,\mathfrak{p}]}$ and $N_{[\delta_i,\mathfrak{q}]}$, say $N_{[\delta_i,\mathfrak{p}]}$, is contained in Δ_y. By the assumption on δ_1, we then have that $N_{[\delta_1,\mathfrak{p}]}\subset\Delta_y$. Along with (4.43) and Proposition 4.17, this implies that, on $\mathcal{V}_y$, the pullbacks of κ_{12} and $\zeta_{[\delta_2,\mathfrak{q}]}$ respectively take the form

$$\widetilde{\kappa}_{12}=\left(\prod_{k\in[\mathbf{m}]}\check{\varepsilon}_k\right)\cdot(\widetilde{f}_2+\widetilde{g}_2\check{\zeta}_{[\delta_2,\mathfrak{q}]}),\quad \widetilde{\zeta}_{[\delta_2,\mathfrak{q}]}=\left(\prod_{k\in[\mathbf{m}]}\check{\varepsilon}_k\right)\cdot\check{\zeta}_{[\delta_2,\mathfrak{q}]},$$

where $\widetilde{f}_2$ and $\widetilde{g}_2$ are invertible functions and $\check{\zeta}_{[\delta_2,\mathfrak{q}]}$ denotes the proper transform of $\zeta_{[\delta_2,\mathfrak{q}]}$. Since $\mathcal{V}_y$ is assumed small, we conclude that either $\check{\zeta}_{[\delta_2,\mathfrak{q}]}$ or $(\widetilde{f}_2+\widetilde{g}_2\check{\zeta}_{[\delta_2,\mathfrak{q}]})$ is invertible. Moreover, for each $h<i\le\ell$, notice that

$$N_{[z_{\langle\delta_i\rangle},\mathfrak{p}]}\sqcup N_{[\delta_i,a_2]}=N_{[\delta_i]}\sqcup N_{[z_{\langle\delta_i\rangle},\mathfrak{p}]}\in S',\quad N_{[z_{\langle\delta_i\rangle},\mathfrak{q}]}\sqcup N_{[\delta_i,a_2]}=N_{[\delta_i]}\sqcup N_{[z_{\langle\delta_i\rangle},\mathfrak{q}]}\in S',$$

hence by (4.43) and Proposition 4.17, we see that $\prod_{k\in[\mathbf{m}]}\check{\varepsilon}_k$ divides the pullback of $\beta_i\zeta_{[\delta_i,a_2]}$. Similarly, for each $i>\ell$, notice that $N_{[\delta_i,a_2]}=N_{[\delta_i]}\in S'$, hence $\prod_{k\in[\mathbf{m}]}\check{\varepsilon}_k$ divides the pullback of $\zeta_{[\delta_i,a_2]}$. To summarize, we conclude that either $\widetilde{\kappa}_{12}$ or $\widetilde{\zeta}_{[\delta_2,\mathfrak{q}]}$ divides all the entries of the second row of the pullback of (4.44), and hence the pullback of φ is diagonalized after $(\mathsf{r}_1\mathsf{p}_3)$.

(2) If $D(\Delta_y)\not\subset F$, and $\langle\delta\rangle\in\mathrm{B}$ for all $\delta\in D(\Delta_y)$, then without loss of generality we assume

$$\delta_{h+1}\in D(\Delta_y)\quad\text{and}\quad N_{[z_{\langle\delta_{h+1}\rangle},\mathfrak{p}]}\subsetneq N_{[\delta_1,\mathfrak{p}]},$$

which implies that

$$N_{[\delta_{h+1}]}\sqcup N_{[z_{\langle\delta_{h+1}\rangle},\mathfrak{p}]}\in S'(\Delta_y).$$

Notice that $N_{[z_{\langle\delta_{h+1}\rangle},\mathfrak{p}]}\sqcup N_{[\delta_{h+1},a_2]}=N_{[z_{\langle\delta_{h+1}\rangle},\mathfrak{p}]}\sqcup N_{[\delta_{h+1}]}$. So by (4.43), we see that the pullback of $\beta_{h+1}\zeta_{[\delta_{h+1},a_2]}$ takes the form

$$\widetilde{\beta}_{h+1}\widetilde{\zeta}_{[\delta_{h+1},a_2]}=u_{h+1}\cdot\left(\prod_{k\in[\mathbf{m}]}\check{\varepsilon}_k\right),\quad\text{where } u_{h+1}\in\Gamma(\mathscr{O}^*_{\mathcal{V}_y}).$$

Meanwhile, since

$$N_{[\delta_1,\mathfrak{p}]},\quad N_{[\delta_2,\mathfrak{q}]},\quad N_{[z_{\langle\delta_i\rangle},\mathfrak{p}]}\sqcup N_{[\delta_i,a_2]}=N_{[z_{\langle\delta_i\rangle},\mathfrak{p}]}\sqcup N_{[\delta_i]},\quad h<i\le\ell,$$
$$N_{[z_{\langle\delta_i\rangle},\mathfrak{q}]}\sqcup N_{[\delta_i,a_2]}=N_{[z_{\langle\delta_i\rangle},\mathfrak{q}]}\sqcup N_{[\delta_i]},\quad h<i\le\ell,\quad\text{and}\quad N_{[\delta_j,a_2]}=N_{[\delta_j]},\quad \ell<j\le m$$

are all in S', we conclude from Proposition 4.17 that $\prod_{k\in[\mathbf{m}]}\check{\varepsilon}_k$, and consequently $\widetilde{\beta}_{h+1}\widetilde{\zeta}_{[\delta_{h+1},a_2]}$, divides all the entries of the second row of the pullback of (4.44), hence the pullback of φ is diagonalized after $(\mathsf{r}_1\mathsf{p}_3)$.

(3) Otherwise, there exists $\delta\in D(\Delta_y)$ such that $\langle\delta\rangle\notin\mathrm{B}$ (which implies $\delta\notin F$). Without loss of generality we assume

$$\delta_{\ell+1}\in D(\Delta_y).$$

Since $N_{[\delta_{\ell+1},a_2]}=N_{[\delta_{\ell+1}]}$, the pullback of $\zeta_{[\delta_{\ell+1},a_2]}$ takes the form

$$\widetilde{\zeta}_{[\delta_{\ell+1},a_2]}=u_{\ell+1}\cdot\left(\prod_{k\in[\mathbf{m}]}\check{\varepsilon}_k\right),\quad\text{where } u_{\ell+1}\in\Gamma(\mathscr{O}^*_{\mathcal{V}_y}).$$

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The same argument as in part (2) once again implies that the pullback of φ is diagonalized after $(\mathsf{r}_1\mathsf{p}_3)$.

In sum, the proof of Case C.1 is complete.

4.10.2 *Case C.2.* In this sub-case, we assume that $h=2$ and F is inseparable, so we still have

$$N_{[a_1,a_2]}=\emptyset \quad (\Longrightarrow \zeta_{[a_1,a_2]}=1).$$

Moreover, one can check directly that $(\mathsf{r}_1\mathsf{p}_4)$ to $(\mathsf{r}_3\mathsf{p}_2)$ and $(\mathsf{r}_3\mathsf{p}_4)$ do not affect $\mathcal{V}$.

Below, we divide the proof of Case C.2 into three sub-cases in the same way as in Case C.1. First, observe that the arguments of parts (2) and (3) of Case C.1 apply to Case C.2 verbatim, so after $(\mathsf{r}_1\mathsf{p}_3)$, $\mathcal{V}_y$ is smooth, on which the pullback of φ is already diagonalized. In addition, under the assumption of part (2) or (3), there exists $i\geq 3$ such that $N_{[\delta_i]}\subset\Delta_y$, so $(\mathsf{r}_3\mathsf{p}_3)$ does not affect $\mathcal{V}_y$ either, so the pullback of $\mathcal{V}_y$ to the final stack is isomorphic to itself, hence is smooth.

It remains to show Case C.2 when $D(\Delta_y)\subset F$. Without loss of generality we assume

$$N_{[\delta_1,\mathfrak{p}]}\subset\Delta_y.$$

By (4.43) and parts (3) and (4) of Lemma 4.16, we can write the pullback $\widetilde{\kappa}_{12}$ and the proper transform $\check{\kappa}_{12}$ of κ_{12} on $\mathcal{V}_y$ as

$$\widetilde{\kappa}_{12}=\left(\prod_{k\in[\mathbf{m}]}\check{\varepsilon}_k\right)\cdot\check{\kappa}_{12},$$

$$\check{\kappa}_{12}=f_2'\cdot\left(\prod_{e\in N_{[\delta_1,\mathfrak{p}]}}(1+c_e\xi_e)\right)+g_2'\cdot\left(\prod_{e\in N_{[\delta_2,\mathfrak{q}]}\cap\Delta_y}(1+c_e\xi_e)\right)\cdot\left(\prod_{e\in N_{[\delta_2,\mathfrak{q}]}\setminus\Delta_y}\check{\zeta}_e\right),$$

where f_2' and g_2' are invertible functions, the ξ_e are independent local parameters on $\mathcal{V}_y$ vanishing at y, and the c_e are constants satisfying $c_e=0$ if and only if $e\in\{\mathsf{e}_1,\ldots,\mathsf{e}_{\mathbf{m}}\}$, where $\mathsf{e}_k\in\Delta_{y,k}$ are as in part (1) of Lemma 4.16. So in particular, by taking e_1 to be the unique element of $N_{[\delta_1,\mathfrak{p}]}\cap\mathfrak{E}_{\mathbf{m}}$ and e_2 to be the unique element of $N_{[\delta_2,\mathfrak{q}]}\cap\mathfrak{E}_{\mathbf{m}}$, we conclude that c_{e_1} and c_{e_2} cannot be zero simultaneously.

- If $\check{\kappa}_{12}(y)\neq 0$, then $\check{\kappa}_{12}$ is invertible on $\mathcal{V}_y$ because $\mathcal{V}_y$ is small. Therefore, parallel to part (1) of Case C.1, we conclude that, on $\mathcal{V}_y$, among all the entries of of second row of the pullback of (4.45), the second entry divides the remaining entries, hence the pullback of φ is still diagonalized after $(\mathsf{r}_1\mathsf{p}_3)$. Moreover, $\check{\kappa}_{12}$ being invertible on $\mathcal{V}_y$ implies that the proper transforms of the blowup centers of $(\mathsf{r}_3\mathsf{p}_3)$ are disjoint from $\mathcal{V}_y$, hence $(\mathsf{r}_3\mathsf{p}_3)$ does not affect $\mathcal{V}_y$ either. In this way, we see that the pullback of $\mathcal{V}_y$ to the final stack is smooth.
- Assume $\check{\kappa}_{12}(y)=0$. Then, the above equation of $\check{\kappa}_{12}$ implies that $N_{[\delta_2,\mathfrak{q}]}\subset\Delta_y$, hence $\check{\kappa}_{12}$ is a local parameter on $\mathcal{V}_y$ that vanishes at y; moreover, $\check{\kappa}_{12}$ and $\check{\zeta}_e$, $e\in N(C)\setminus\Delta_y$ together form a subset of a system of local parameters on $\mathcal{V}_y$. To study the effect of $(\mathsf{r}_3\mathsf{p}_3)$ on $\mathcal{V}_y$, consider the sequential blowup of

  $$\pi':(\mathfrak{M}_2^{\mathrm{div}})'\longrightarrow\mathfrak{M}_2^{\mathrm{div}}$$

  along the proper transforms of $Z_1',Z_2',\ldots$, where Z_j' is the closed substack of $\mathfrak{M}_2^{\mathrm{div}}$ whose general points are pairs (C,D) such that C has j tails attached to its core F, and

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$|D\cap F|=2$. Recall that τ_2 is the rooted tree of (4.20) whose root-to-leaf paths are those inclusion-minimal $N_{[\delta_i]}$ among all $3\le i\le m$. One can then check directly that $\widehat{x}\in X^{\mathcal{V}}_{\tau_2}$ and π' is τ_2-compatible on $\mathcal{V}$. Corollary 4.28 thus implies that $\mathrm{PT}_y(\tau')$ is $\mathrm{PT}_y(\tau_2)$-compatible on $\mathcal{V}_y$.

We emphasize that, with $\mathcal{H}_k\subset\mathfrak{M}_2^{\mathrm{div}}$ as in Section 3.5, the sequential blowup along the proper transforms of $\mathcal{H}_k$, $k\ge1$, which serves as the 'prototype' of $(\mathsf{r}_3\mathsf{p}_3)$, is *not* $\mathrm{gr}(\tau_2)$-compatible on $\mathcal{V}$, because κ_{12}, as a sum of two products of local parameters in (4.43), is not always a local parameter on $\mathcal{V}$. Nonetheless, the assumptions of Case C.2, as well as $D(\Delta_y)\subset F$ imply that y lies in the blowup centers of $\mathrm{PT}_y(\pi')$. Taking $\check\kappa_{12}(y)=0$ into consideration, we conclude that

$$y\in X^{\mathcal{V}_y}_{\mathrm{gr}(\mathrm{PT}_y(\tau_2))},$$

where the grafted edge corresponds to $\check\kappa_{12}$. Moreover, notice that for every $j\ge1$, with $\check Z'_j$ denoting the proper transform of Z'_j, and $\mathcal{H}_j^{\mathsf{r}_3\mathsf{p}_2}\subset\widetilde{\mathfrak{M}}^{\mathrm{div},\mathsf{r}_3\mathsf{p}_2}$ denoting the proper transform of $\mathcal{H}_j$ as in (3.26), we have

$$\{\check\kappa_{12}=0\}\cap\check Z'_j\cap\mathcal{V}_y=\mathcal{H}_j^{\mathsf{r}_3\mathsf{p}_2}\cap\mathcal{V}_y.$$

Therefore, $(\mathsf{r}_3\mathsf{p}_3)$ is $\mathrm{gr}(\mathrm{PT}_y(\tau_2))$-compatible on $\mathcal{V}_y$, and hence the pullback of $\mathcal{V}_y$ to $\widetilde{\mathfrak{M}}^{\mathrm{div},\mathsf{r}_3\mathsf{p}_3}$, hence to the final stack, is smooth. Finally, we check that the pullback of (4.45) is diagonalized on the pullback of $\mathcal{V}_y$. In fact, on $\mathcal{V}_y$, the pullback of (4.45) takes the form

$$\begin{bmatrix}1&0\\0&\prod_{k\in[\mathbf{m}]}\check\varepsilon_k\end{bmatrix}\begin{bmatrix}1&0&0&\cdots&0\\0&\check\kappa_{12}&\prod_{e\in N_{[\delta_3]}\setminus\Delta_y}\check\zeta_e&\cdots&\prod_{e\in N_{[\delta_m]}\setminus\Delta_y}\check\zeta_e\end{bmatrix}. \tag{4.47}$$

By Proposition 4.29, we see that the root-to-leaf paths of $\mathrm{PT}_y(\tau_2)$ are exactly the inclusion-minimal elements of $\{N_{[\delta_i]}\setminus\Delta_y:3\le i\le m\}$. Taking into consideration that $(\mathsf{r}_3\mathsf{p}_3)$ is $\mathrm{gr}(\mathrm{PT}_y(\tau_2))$-compatible on $\mathcal{V}_y$, we conclude that the pullback of (4.47) is diagonalized after $(\mathsf{r}_3\mathsf{p}_3)$.

In sum, the proof of Case C.2 is complete.

4.10.3 *Case C.3.* In this sub-case, we assume that $h=1$ (i.e. $D\cap F=\{\delta_1\}$) and that F is inseparable, so we still have

$$N_{[a_1,a_2]}=\emptyset\quad(\Longrightarrow\zeta_{[a_1,a_2]}=1).$$

Moreover, one can check directly that $(\mathsf{r}_1\mathsf{p}_4)$ and (r_2) to $(\mathsf{r}_3\mathsf{p}_2)$ do not affect $\mathcal{V}$.

(a) If there exists $\delta\in D(\Delta_y)$ such that $\langle\delta\rangle\notin\mathrm{B}$, then the argument of part (3) of Case C.1 still applies to this situation verbatim.

(b) If $\langle\delta\rangle\in\mathrm{B}$ for all $\delta\in D(\Delta_y)$, and $D(\Delta_y)\not\subset F$, then without loss of generality we assume

$$\delta_2\in D(\Delta_y)\setminus F\quad\text{and}\quad N_{[z_{\langle\delta_2\rangle},\mathfrak{p}]}\subsetneq N_{[\delta_1,\mathfrak{p}]},$$

which implies that

$$N_{[\delta_2]}\sqcup N_{[z_{\langle\delta_2\rangle},\mathfrak{p}]}\in S'(\Delta_y),\quad\kappa_{12}=f'_2\zeta_{[z_{\langle\delta_2\rangle},\mathfrak{p}]}+g'_2\zeta_{[\delta_1,\mathfrak{q}]}\quad\text{for some } f'_2,g'_2\in\Gamma(\mathscr{O}^*_{\mathcal{V}}),$$

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and $(\mathsf{r}_1\mathsf{p}_4)$ to $(\mathsf{r}_3\mathsf{p}_2)$ and $(\mathsf{r}_3\mathsf{p}_4)$ do not affect (the pullback of) $\mathcal{V}_y$. Here, the expression for κ_{12} above follows from Proposition 2.22 (SH_3). As $\delta_2 \in D(\Delta_y)$ and F is inseparable, the pullback of the second entry of the second row of (4.41), i.e. $\widetilde{\kappa}_{12}\widetilde{\zeta}_{[\delta_2,a_2]}$, can be written as

$$\widetilde{\kappa}_{12}\widetilde{\zeta}_{[\delta_2,a_2]} = \widetilde{\kappa}_{12}\widetilde{\zeta}_{[\delta_2]} = u_2 \cdot \left(\prod_{k\in[\mathbf{m}]} \check{\varepsilon}_k\right) \cdot \left(f_2'' + g_2'' \cdot \widetilde{\zeta}_{[\delta_2]} \cdot \left(\prod_{e\in N_{[\delta_1,\mathfrak{q}]}\setminus\Delta_y} \check{\zeta}_e\right)\right),$$

where u_2, f_2'', and g_2'' are invertible functions on $\mathcal{V}_y$. Notice that δ_2 is on a tail, so $\widetilde{\zeta}_{[\delta_2]}$ vanishes at y. Taking into consideration that f_2'' is invertible and $\mathcal{V}_y$ is small, we see that $\widetilde{\kappa}_{12}\widetilde{\zeta}_{[\delta_2,a_2]}$ is equal to $\prod_{k\in[\mathbf{m}]}\check{\varepsilon}_k$ up to a unit. In particular, this implies that $\check{\kappa}_{12}$ is invertible on $\mathcal{V}_y$, hence $(\mathsf{r}_3\mathsf{p}_3)$ does not affect $\mathcal{V}_y$. Meanwhile, since

$$N_{[z_{\langle\delta_i\rangle},\mathfrak{p}]} \sqcup N_{[\delta_i,a_2]} = N_{[z_{\langle\delta_i\rangle},\mathfrak{p}]} \sqcup N_{[\delta_i]}, \quad 2 < i \le \ell,$$
$$N_{[z_{\langle\delta_i\rangle},\mathfrak{q}]} \sqcup N_{[\delta_i,a_2]} = N_{[z_{\langle\delta_i\rangle},\mathfrak{q}]} \sqcup N_{[\delta_i]}, \quad 2 < i \le \ell, \quad \text{and} \quad N_{[\delta_j,a_2]} = N_{[\delta_j]}, \quad \ell < j \le m$$

are all in S', we conclude from Proposition 4.17 that $\prod_{k\in[\mathbf{m}]}\check{\varepsilon}_k$, and consequently $\widetilde{\kappa}_{12}\widetilde{\zeta}_{[\delta_2,a_2]}$, divides all the entries of the second row of the pullback of (4.41), and hence the pullback of φ is diagonalized after $(\mathsf{r}_1\mathsf{p}_3)$.

(c) If $D(\Delta_y) \subset F$, i.e. $D(\Delta_y) = \{\delta_1\}$, then (4.41) pulls back to

$$\begin{bmatrix} 1 & 0 \\ 0 & \prod_{k\in[\mathbf{m}]}\check{\varepsilon}_k \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \check{\kappa}_{12}\prod_{e\in N_{[\delta_2]}\setminus\Delta_y}\check{\zeta}_e & \cdots & \check{\kappa}_{1m}\prod_{e\in N_{[\delta_m]}\setminus\Delta_y}\check{\zeta}_e \end{bmatrix},$$

where $N_{[\delta_i]}\setminus\Delta_y$, $i \ge 2$, are all nonempty. Observe that $(\mathsf{r}_1\mathsf{p}_4)$ does not affect $\mathcal{V}_y$ because F is assumed to be inseparable. In addition, $(\mathsf{r}_1\mathsf{p}_5)$ is $\mathrm{PT}_y(\tau_1)$-compatible on $\mathcal{V}_y$, where τ_1 is as in (4.23). By Proposition 4.29, the inclusion-minimal elements of $N_{[\delta_i]}\setminus\Delta_y$, $i \ge 2$, are exactly the root-to-leaf paths of $\mathrm{PT}_y(\tau_1)$. Comparing the above pullback of φ with (4.22), we conclude that the argument for Case A.5, i.e. the proof of Proposition 4.31, applies to the $D(\Delta_y) \subset F$ sub-case, as long as we can show that the functions $\check{\kappa}_{1i}$, $i \ge 2$, behave in the same way as κ_{1i}, $i \ge 2$, do in Case A.5. In fact, since $D(\Delta_y) = \{\delta_1\}$, without loss of generality we assume

$$N_{[\delta_1,\mathfrak{p}]} \subset \Delta_y.$$

Mimicking the argument that $\check{\kappa}_{12}$ is a local parameter in Section 4.10.2, we conclude from Proposition 2.22 (SH_3) that

- for every $2 \le i \le m$, the proper transform $\check{\kappa}_{1i}$ of κ_{1i} is either a unit or a local parameter on $\mathcal{V}_y$;
- $\check{\kappa}_{1i}(y)$ and $\check{\kappa}_{1j}(y)$ cannot be simultaneously zero whenever $\langle\delta_i\rangle \ne \langle\delta_j\rangle$; and
- if $\check{\kappa}_{1i}(y) = \check{\kappa}_{1j}(y) = 0$, then there exist invertible functions v_{ij} and u'_{ij} on $\mathcal{V}_y$ such that

$$\check{\kappa}_{1j} = v_{ij}\check{\kappa}_{1j} + u'_{ij}\prod_{e\in N_{[\delta_j\wedge\delta_k]}\setminus\Delta_y}\check{\zeta}_e,$$

where $N_{[\delta_j\wedge\delta_k]}$ is as in (2.11).

Consequently, the functions $\check{\kappa}_{1i}$, $i \ge 2$, behave in the same way as κ_{1i}, $i \ge 2$, do in Case A.5, thus the $D(\Delta_y) \subset F$ sub-case follows from the argument for Case A.5.

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4.10.4 *Case C.4.* In this sub-case, the underlying curve C of $\widehat{x}$ is separable, and after $(\mathsf{r}_1\mathsf{p}_3)$, on $\mathcal{V}_y$, (4.41) pulls back to

$$\widetilde{\varphi}^{\mathsf{r}_1\mathsf{p}_3} = \begin{bmatrix} 1 & 0 \\ 0 & \prod_{k\in[\mathbf{m}]} \check{\varepsilon}_k \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \check{\kappa}_{12} \prod_{e\in N_{[\delta_2,a_2]}\setminus\Delta_y} \check{\zeta}_e & \cdots & \check{\kappa}_{1m} \prod_{e\in N_{[\delta_m,a_2]}\setminus\Delta_y} \check{\zeta}_e \end{bmatrix}.$$

Recall that throughout Section 4.9, T_1 refers to the genus 1 inseparable component that is disjoint from the non-separating bridge B, hence disjoint from $D\cap F$. If there exists $\delta_i \in D(\Delta_y)$ such that $\langle\delta_i\rangle \in T_1$, then $N_{[\delta_i,a_2]}\setminus\Delta_y = N_{[\delta_i]}\setminus\Delta_y = \emptyset$. Moreover, by Proposition 2.17, we have that κ_{1i} is invertible on $\mathcal{V}$. Therefore, $\check{\kappa}_{1i}\prod_{e\in N_{[\delta_i,a_2]}\setminus\Delta_y}\check{\zeta}_e$ is a unit on $\mathcal{V}_y$, hence $\widetilde{\varphi}^{\mathsf{r}_1\mathsf{p}_3}$ is diagonalized. In addition, $(\mathsf{r}_1\mathsf{p}_4)$ to $(\mathsf{r}_3\mathsf{p}_4)$ do not affect $\mathcal{V}_y$.

It remains to verify the situation when $\langle\delta\rangle \in F\setminus T_1$ for every $\delta\in D(\Delta_y)$. In such a situation, we observe that $N_{[\delta_i,a_2]}\setminus\Delta_y$ is nonempty for every $i\in[m]$, and that $(\mathsf{r}_1\mathsf{p}_4)$ is $\mathrm{PT}_y(\tau^\dagger)$-compatible on $\mathcal{V}_y$, where the rooted tree $\tau^\dagger$ is as in (4.37). Moreover, by Proposition 4.29, the inclusion-minimal elements of $N_{[\delta_i,a_2]}\setminus\Delta_y$, $i\in[m]$, are exactly the root-to-leaf paths of $\mathrm{PT}_y(\tau^\dagger)$. Furthermore, mimicking the argument used in the $D(\Delta_y)\subset F$ sub-case of the proof of Case C.3, we see that for every δ_i satisfying the condition that $N_{[\delta_i,a_2]}\setminus\Delta_y$ is inclusion-minimal among all $2\le i\le m$, $\check{\kappa}_{1i}$ behaves in the same way as κ_{1i} does in Case B. Comparing the above $\widetilde{\varphi}^{\mathsf{r}_1\mathsf{p}_3}$ with (4.36), we thus conclude that the argument for Case B, i.e. the proof of Proposition 4.34, applies to the current situation.

To summarize, we have shown the following in Section 4.10.

Proposition 4.35. *Proposition 4.1 holds in Case C.*

4.11 Proof of Case D

In this subsection, we prove Proposition 4.1 in Case D.

Let $\mathcal{V}\to\mathfrak{M}_2^{\mathrm{div}}$ be a small affine smooth chart containing $\widehat{x}=(C,D)$. In Case D, we assume that x, the image of $\widehat{x}$ in $\mathfrak{M}_2^{\mathrm{wt}}$, lies in $\mathfrak{M}_{(2)}^{\mathrm{mn}}$. This implies that the core F of x contains two genus 1 inseparable components T_1 and T_2, as well as a (maximal) separating bridge B such that

$$D\cap \mathrm{B} = D\cap F\neq\emptyset.$$

Without loss of generality we assume that $a_1\in T_1$, $a_2\in T_2$, and there exists $1\le h\le m$ such that

$$D\cap F = \{\delta_1,\dots,\delta_h\}.$$

Furthermore, we assume

$$N_{[\delta_1,a_1]}\subset N_{[\delta_i,a_1]}\quad \forall\, i\in[h].$$

In addition, if $|D\cap F|\ge 2$ (i.e. $h\ge 2$), then we assume

$$N_{[\delta_2,a_2]}\subset N_{[\delta_i,a_2]}\quad \forall\, i\in[h].$$

Since $D\cap F\neq\emptyset$, $(\mathsf{r}_1\mathsf{p}_1)$ does not affect $\mathcal{V}$.

Next, we construct the rooted tree $\tau_\wedge$ with which $(\mathsf{r}_1\mathsf{p}_2)$ is compatible on $\mathcal{V}$. Let

$$S'' := \bigcup_{\delta\in D}\{N_{[\delta,a_1]}, N_{[\delta,a_2]}\}.$$

In particular, if $h\ge 2$, then $N_{[\delta_1,a_1]}$ and $N_{[\delta_2,a_2]}$ are inclusion-minimal; if $h=1$, then $N_{[\delta_1,a_1]}$ and $N_{[\delta_1,a_2]}$ are inclusion-minimal.

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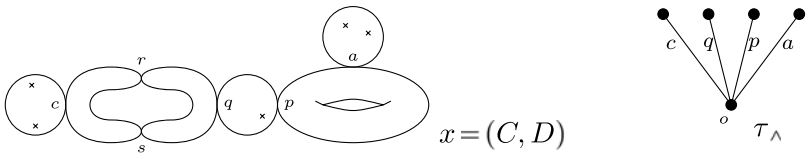


FIGURE 21. An example of $\tau_\wedge$.

We take the underlying set E of the proposed rooted tree $\tau_\wedge$ to be the union of the inclusion-minimal elements of S'', i.e.

$$E = \bigcup_{\text{inclusion-minimal } N'' \in S''} N'' \qquad (\subset N(C)).$$

One can check directly that E decomposes into two disjoint nonempty subsets:

$$E = E_1 \sqcup E_2,$$

such that for $s \in [2]$, $e \in E_s$ if and only if there exists $i \in [m]$ such that $e \in N_{[\delta_i, a_s]} \subset E$.

For $s \in [2]$, we define the relation $\preceq_s$ on E_s such that $e \preceq_s e'$ if and only if any connected subcurve of C containing T_s and e must contain e', which once again satisfies (4.5). In other words,

$$(E_1, \preceq_1),\ (E_2, \preceq_2) \in \mathbf{T}.$$

The tree orders $\preceq_1$ and $\preceq_2$ together determine a tree order $\preceq$ on E: $e \preceq e'$ if and only if either $e \preceq_1 e'$ or $e \preceq_2 e'$. In other words,

$$\tau_\wedge := (E, \preceq) \in \mathbf{T}. \tag{4.48}$$

The root-to-leaf paths of $\tau_\wedge$ are exactly the inclusion-minimal elements of S''.

An example of $\tau_\wedge$ is provided in Figure 21.

As described in Section 3.1, the blowup centers of $(\mathsf{r}_1\mathsf{p}_2)$ are locally given by

$$\overline{\mathfrak{M}}_{(2,k)} \cap \mathcal{V} = \bigcup_{\mathfrak{E} \in \Xi_k(\tau_\wedge)} \{\zeta_e = 0 : e \in \mathfrak{E}\}, \quad \text{where } \Xi_k(\tau_\wedge) = \{\mathfrak{E} \in \Xi(\tau_\wedge) : |\mathfrak{E}| = k+1\}, \quad k \geq 1.$$

By the stability of $\mathfrak{M}_2^{\mathrm{div}}$, we conclude that $(\mathsf{r}_1\mathsf{p}_2)$ is $\tau_\wedge$-compatible on $\mathcal{V}$. Lemma 4.10 then implies that the pullback $\widetilde{\mathcal{V}}^{\mathsf{r}_1\mathsf{p}_2}$ of $\mathcal{V}$ is smooth.

After $(\mathsf{r}_1\mathsf{p}_2)$, we fix an arbitrary lift $y \in \widetilde{\mathcal{V}}^{\mathsf{r}_1\mathsf{p}_2}$ of $\widehat{x}$, as well as a small neighborhood $\mathcal{V}_y$ of y in $\widetilde{\mathcal{V}}^{\mathsf{r}_1\mathsf{p}_2}$. The RL sequence of y is denoted by $\overline{\mathbf{E}} = \{\mathfrak{E}_1, \ldots, \mathfrak{E}_{\mathbf{m}}\}$ as before. The pullback $\widetilde{\varphi}^{\mathsf{r}_1\mathsf{p}_2}$ of the structural homomorphism φ to $\mathcal{V}_y$ can be written as

$$\left(\prod_{k\in[\mathbf{m}]} \check{\varepsilon}_k\right) \cdot \begin{bmatrix} c'_{11} \prod_{e \in N_{[\delta_1,a_1]} \setminus \Delta_y} \check{\zeta}_e & \cdots & c'_{1m} \prod_{e \in N_{[\delta_m,a_1]} \setminus \Delta_y} \check{\zeta}_e \\ c'_{21} \prod_{e \in N_{[\delta_1,a_2]} \setminus \Delta_y} \check{\zeta}_e & \cdots & c'_{2m} \prod_{e \in N_{[\delta_m,a_2]} \setminus \Delta_y} \check{\zeta}_e \end{bmatrix}, \tag{4.49}$$

where $\det \begin{bmatrix} c_{1i} & c_{1j} \\ c_{2i} & c_{2j} \end{bmatrix}$ is equal to $\check{\kappa}_{ij} \prod_{e \in N_{[\delta_i \wedge \delta_j]} \setminus \Delta_y} \check{\zeta}_e$ up to a unit.

By Definition 4.15, there exists at least one inclusion-minimal $N'' \in S''$ such that $N'' \subset \Delta_y$. We set

$$S''(\Delta_y) := \{N'' \in S'' :\ N'' \subset \Delta_y\}, \quad D(\Delta_y) := \{\delta \in D :\ \{N_{[\delta,a_1]}, N_{[\delta,a_2]}\} \cap S''(\Delta_y) \neq \emptyset\},$$

which are both nonempty.

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- If there exist distinct i and j such that $N_{[\delta_i,a_1]} \subset \Delta_y$, $N_{[\delta_j,a_2]} \subset \Delta_y$, and δ_i is not conjugate to δ_j (i.e. κ_{ij} is invertible on $\mathcal{V}$), then $(\mathsf{r}_1\mathsf{p}_3)$ to $(\mathsf{r}_3\mathsf{p}_4)$ do not affect $\mathcal{V}_y$, and the entries of the ith and jth columns of (4.49), as well as the corresponding 2×2 minor, are all invertible, hence $\widetilde{\varphi}^{\mathsf{r}_1\mathsf{p}_2}$ is diagonalizable on $\mathcal{V}_y$.
- If there exist distinct i and j such that $N_{[\delta_i,a_1]} \subset \Delta_y$ and $N_{[\delta_j,a_2]} \subset \Delta_y$, and δ_i is conjugate to δ_j whenever $N_{[\delta_i,a_1]} \subset \Delta_y$ and $N_{[\delta_j,a_2]} \subset \Delta_y$, then by Lemma 2.14, δ_i and δ_j must be on the same irreducible component of the separating bridge B, and there are no other points of D lying on F. From the assumptions on $D\cap \mathrm{B}$ in the proof of Case D as well as Proposition 2.17, we conclude that $D(\Delta_y) = \{\delta_1,\delta_2\} = D\cap \mathrm{B}$, and κ_{12} is a local parameter on $\mathcal{V}$ that vanishes at $\widehat{x}$. Consequently, after suitable row and column operations, (4.49) can be rewritten as
$$\left(\prod_{k\in[\mathbf{m}]} \check{\varepsilon}_k\right)\cdot \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \check{\kappa}_{12} & \prod_{e\in N_{[\delta_3]}\setminus\Delta_y} \check{\zeta}_e & \cdots & \prod_{e\in N_{[\delta_m]}\setminus\Delta_y} \check{\zeta}_e \end{bmatrix}.$$
Notice that $(\mathsf{r}_1\mathsf{p}_3)$ to $(\mathsf{r}_3\mathsf{p}_2)$ and $(\mathsf{r}_3\mathsf{p}_4)$ do not affect (the pullback of) $\mathcal{V}_y$, and $(\mathsf{r}_3\mathsf{p}_3)$ is $\mathrm{gr}(\mathrm{PT}_y(\tau_2))$-compatible on $\mathcal{V}_y$, where τ_2 is the rooted tree given by (4.20). Therefore, the pullback of $\mathcal{V}_y$ after $(\mathsf{r}_3\mathsf{p}_3)$ is smooth, hence the pullback of $\mathcal{V}_y$ after the entire blowup is smooth, on which the pullback of $\widetilde{\varphi}^{\mathsf{r}_1\mathsf{p}_2}$ is diagonalized.
- If there exists a unique $\delta = D$ such that $N_{[\delta,a_1]} \subset \Delta_y$ and $N_{[\delta,a_2]} \subset \Delta_y$, then $D\cap \mathrm{B} = \{\delta\}$, so $h=1$ and $\delta = \delta_1$. By Lemma 2.14, we further have that κ_{1i} is invertible on $\mathcal{V}$ for every $i\geq 2$ with $N_{[\delta_i,a_1]} \subset E$ or $N_{[\delta_i,a_2]} \subset E$. Consequently, after suitable row and column operations, (4.49) can be rewritten as
$$\left(\prod_{k\in[\mathbf{m}]} \check{\varepsilon}_k\right)\cdot \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \check{\kappa}_{12}\prod_{e\in N_{[\delta_2]}\setminus\Delta_y} \check{\zeta}_e & \cdots & \check{\kappa}_{1m}\prod_{e\in N_{[\delta_m]}\setminus\Delta_y} \check{\zeta}_e \end{bmatrix}.$$
Notice that in this situation, $(\mathsf{r}_1\mathsf{p}_3)$ and $(\mathsf{r}_1\mathsf{p}_4)$ do not affect $\mathcal{V}_y$, whereas $(\mathsf{r}_1\mathsf{p}_5)$ is $\mathrm{PT}_y(\tau_1)$-compatible on $\mathcal{V}_y$, where τ_1 is as in (4.23). Moreover, $\check{\kappa}_{1i}$, $i\geq 2$, behave in the same way as κ_{1i}, $i\geq 2$, in Case A.5, and hence the argument for Case A.5 (i.e. the proof of Proposition 4.31) applies to this situation.
- If one of E_1 and E_2, say E_1, satisfies $N''\subset E_1$ for all $N''\in S''(\Delta_y)$, then re-labeling δ_1 if necessary, after taking suitable row and column operations, we can rewrite (4.49) as
$$\left(\prod_{k\in[\mathbf{m}]} \check{\varepsilon}_k\right)\cdot \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \check{\kappa}_{12}\prod_{e\in N_{[\delta_2,a_2]}\setminus\Delta_y} \check{\zeta}_e & \cdots & \check{\kappa}_{1m}\prod_{e\in N_{[\delta_m,a_2]}\setminus\Delta_y} \check{\zeta}_e \end{bmatrix}.$$
We observe that $N_{[\delta_i,a_2]}\setminus\Delta_y$ is nonempty for every $i\in[m]$, and that $(\mathsf{r}_1\mathsf{p}_4)$ is $\mathrm{PT}_y(E_2,\preceq_2)$-compatible on $\mathcal{V}_y$. Moreover, the inclusion-minimal elements of $N_{[\delta_i,a_2]}\setminus\Delta_y$, $i\in[m]$, are exactly the root-to-leaf paths of $\mathrm{PT}_y(E_2,\preceq_2)$. Furthermore, mimicking the argument used in the $D(\Delta_y)\subset F$ sub-case of the proof of Case C.3, we see that for every δ_i satisfying the condition that $N_{[\delta_i,a_2]}\setminus\Delta_y$ is inclusion-minimal among all $2\leq i\leq m$, $\check{\kappa}_{1i}$ behaves in the same way as κ_{1i} in Case B. Comparing the above $\widetilde{\varphi}^{\mathsf{r}_1\mathsf{p}_2}$ with (4.36), we thus conclude that the argument for Case B, i.e. the proof of Proposition 4.34, applies to the current situation.

To summarize, we have shown the following in Section 4.11.

Proposition 4.36. *Proposition 4.1 holds in Case D.*

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4.12 Proof of Case E, Part I

In this subsection, we prove Proposition 4.1 in Case E, when the image of $\widehat{x}$ in $\mathfrak{M}_2^{\mathrm{wt}}$ belongs to $\mathfrak{M}_{(1)}^{\mathrm{mn}}$. This implies that

$$D \cap F = \emptyset,$$

i.e. all the points of D are on tails.

Let $\mathcal{V} \to \mathfrak{M}_2^{\mathrm{div}}$ be a small affine smooth chart containing $\widehat{x} = (C, D)$. By defining the subsets $D_{\text{i-m}} \subset D$ and $E \subset N(C)$ in exactly the same way as (4.19), and defining the relation $\preceq$ on E as $e \preceq e'$ if and only if any connected subcurve of C containing F and e must contain e', we see that

$$\tau_0 := (E, \preceq) \tag{4.50}$$

is a rooted tree whose root-to-leaf paths are exactly $N_{[\delta]}$, $\delta \in D_{\text{i-m}}$. Although the construction of τ_0 here is parallel to that of τ_2 in (4.20) and τ_1 in (4.23), we emphasize that the sets $D \backslash F$ containing $D_{\text{i-m}}$ are different for τ_0, τ_1, and τ_2. In Figure 16, if the two points of D on the core are removed, then τ_0 is equal to the rooted tree of Figure 16 (whereas τ_2 is not defined in this situation).

As described in Section 3.1, the blowup centers of $(\mathsf{r}_1\mathsf{p}_1)$ locally are given by

$$\overline{\mathfrak{M}}_{(1,k)} \cap \widetilde{\mathcal{V}} = \bigcup_{\mathfrak{E} \in \Xi_k(\tau_0)} \{\zeta_e = 0 \,\forall\, e \in \mathfrak{E}\}, \quad \text{where } \Xi_k(\tau_0) = \{\mathfrak{E} \in \Xi(\tau_0) : |\mathfrak{E}| = k\}. \tag{4.51}$$

One can thus check directly that $(\mathsf{r}_1\mathsf{p}_1)$ is τ_0-compatible on $\mathcal{V}$.

After $(\mathsf{r}_1\mathsf{p}_1)$, we denote by $\widetilde{\mathcal{V}}^{\mathsf{r}_1\mathsf{p}_1}$ the pullback of $\mathcal{V}$, and fix an arbitrary lift $y \in \widetilde{\mathcal{V}}^{\mathsf{r}_1\mathsf{p}_1}$ of $\widehat{x}$, as well as a small neighborhood $\mathcal{V}_y$ of y in $\widetilde{\mathcal{V}}^{\mathsf{r}_1\mathsf{p}_1}$. The RL sequence of y is denoted by $\overline{\mathbf{E}} = \{\mathfrak{E}_1, \ldots, \mathfrak{E}_{\mathbf{m}}\}$ as before. Parallel to (4.24), we set

$$D(\Delta_y) := \{\delta \in D :\, N_{[\delta]} \subset \Delta_y\}, \tag{4.52}$$

which determines a subset of the core F of C:

$$\langle \Delta_y \rangle := \{\langle \delta \rangle :\, \delta \in D(\Delta_y)\}.$$

Based on the topological type of F as well as the locations of the points of $D(\Delta_y)$, we divide the proof of Case E into the following sub-cases:

1. there does not exist any genus 1 inseparable component of F that is disjoint from $\langle \Delta_y \rangle$, and one of the following holds:
   a. F is separable;
   b. F is inseparable, and $|\langle \Delta_y \rangle| \geq 3$;
   c. F is inseparable, $|\langle \Delta_y \rangle| = 2$, and the elements of $\langle \Delta_y \rangle$ are not conjugate to each other;
   d. F is inseparable, $|\langle \Delta_y \rangle| = 2$, and the elements of $\langle \Delta_y \rangle$ are conjugate to each other;
   e. F is inseparable, $|\langle \Delta_y \rangle| = 1$, and the element of $\langle \Delta_y \rangle$ is not a Weierstrass point of F; or
   f. F is inseparable, $|\langle \Delta_y \rangle| = 1$, and the element of $\langle \Delta_y \rangle$ is a Weierstrass point of F;
2. there exists a unique genus 1 inseparable component of F that is disjoint from $\langle \Delta_y \rangle$, and there does not exist any (non-separating) bridge of F that contains $\langle \Delta_y \rangle$ as a subset;
3. there exists a unique non-separating bridge of F that contains $\langle \Delta_y \rangle$ as a subset; and
4. there exists a unique separating bridge of F that contains $\langle \Delta_y \rangle$ as a subset.

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The proof of Case E.1.a-c is parallel to that of Proposition 4.3. More precisely,

- in Case E.1.a, there exist two points of $D(\Delta_y)$, say δ_1 and δ_2, such that for $s=1,2$, $\langle\delta_s\rangle$ and a_s belong to the same genus 1 inseparable component of F; and
- in Case E.1.b and c, there exist two points of $D(\Delta_y)$, say δ_1 and δ_2, such that $\langle\delta_1\rangle$ and $\langle\delta_2\rangle$ are distinct and not conjugate to each other.

In either of the situations above, by Proposition 2.22 (SH_1) (and (SH_2) in the latter situation) as well as Proposition 4.17, the pullback of the structural homomorphism φ takes the following form on $\mathcal{V}_y$:

$$\left(\prod_{k\in[\mathbf{m}]}\check{\varepsilon}_k\right)\cdot\begin{bmatrix}1&0&0&\cdots&0\\0&1&0&\cdots&0\end{bmatrix},$$

i.e. it is diagonalized after $(\mathsf{r}_1\mathsf{p}_1)$. Moreover, the blowups starting from $(\mathsf{r}_1\mathsf{p}_2)$ do not affect $\mathcal{V}_y$, and hence the pullback of $\mathcal{V}_y$ in the final stack is smooth.

The remaining sub-cases of Case E are more complicated. We will verify them in later subsections.

4.13 First-order doubly-derived trees and the proof of Case E, Part II

In this subsection, we aim to show Case E.1.d, when there are two tails C_p and C_q of C, whose pivotal nodes are p and q, respectively, such that

$$D(\Delta_y)\subset C_p\sqcup C_q,\quad D_p(\Delta_y):=D(\Delta_y)\cap C_p\neq\emptyset,\quad D_q(\Delta_y):=D(\Delta_y)\cap C_q\neq\emptyset,$$

and p is conjugate to q on F. Without loss of generality we assume that there exist $1\le h\le\ell<s\le t\le m$ such that

$$D(\Delta_y)=\{\delta_1,\dots,\delta_h\}\sqcup\{\delta_{\ell+1},\dots,\delta_s\},\quad \langle\delta_1\rangle=\cdots=\langle\delta_\ell\rangle=p,\quad \langle\delta_{\ell+1}\rangle=\cdots=\langle\delta_t\rangle=q.$$

Moreover, we observe that the assumptions of Case E.1.d imply that $(\mathsf{r}_1\mathsf{p}_2)$ to (r_2) do not affect $\mathcal{V}_y$.

Since p and q are not on any non-separating bridge, by Proposition 2.22 (SH_3), we have

$$\kappa_{1,j}\text{ is a local parameter}\quad\forall\,\ell<j\le t,\quad \kappa_{1i}\in\Gamma(\mathscr{O}_{\mathcal{V}}^*)\quad\forall\,1<i\le\ell.$$

Hence by Proposition 2.22 (SH_2), after suitable elementary row and column operations, the pullback $\widetilde{\varphi}^{\mathsf{r}_1\mathsf{p}_1}$ of the structural homomorphism φ takes the following form on $\mathcal{V}_y$:

$$\Big(\prod_{k\in[\mathbf{m}]}\check{\varepsilon}_k\Big)\Big[\,0\quad\underbrace{\eta_2\quad\cdots\quad\eta_\ell}_{\text{DNE if }\ell=1}\quad\check{\kappa}_{1,\ell+1}\quad\underbrace{\eta_{\ell+2}\quad\cdots\quad\eta_t}_{\text{DNE if }t=\ell+1}\quad\eta_{t+1}\quad\cdots\quad\eta_m\,\Big],$$

where

$$\eta_i':=\begin{cases}\big(\prod_{k\in[\mathbf{m}]_{[\delta_1\wedge\delta_i]}}\check{\varepsilon}_k\big)\big(\prod_{e\in N_{[\delta_i]}\setminus\Delta_y}\check{\zeta}_e\big) & \text{if } 1<i\le\ell,\\ \big(\prod_{k\in[\mathbf{m}]_{[\delta_{\ell+1}\wedge\delta_i]}}\check{\varepsilon}_k\big)\big(\prod_{e\in N_{[\delta_i]}\setminus\Delta_y}\check{\zeta}_e\big) & \text{if } \ell+1<i\le t,\\ \prod_{e\in N_{[\delta_i]}\setminus\Delta_y}\check{\zeta}_e & \text{if } t<i;\end{cases}\qquad [\mathbf{m}]_{[\delta_i\wedge\delta_j]}\text{ are as in (4.25).}$$

In a parallel to Lemma 4.22, we see that there exist $1\le j\le h$ and $\ell+1\le r\le s$ such that

$$N_{[\wedge D_p(\Delta_y)]}:=\bigcap_{\delta\in D_p(\Delta_y)}N_{[\delta]}=N_{[\delta_1\wedge\delta_j]},\quad N_{[\wedge D_q(\Delta_y)]}:=\bigcap_{\delta\in D_q(\Delta_y)}N_{[\delta]}=N_{[\delta_{\ell+1}\wedge\delta_r]}.\tag{4.53}$$

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Let

$$[\mathbf{m}]_{[\wedge_2 D(\Delta_y)]} := \{k \in [\mathbf{m}] : N_{[\wedge D_p(\Delta_y)]} \cap \mathfrak{E}_k \neq \emptyset,\ N_{[\wedge D_q(\Delta_y)]} \cap \mathfrak{E}_k \neq \emptyset\},$$
$$[\mathbf{m}]_{[\delta_i \wedge_2 D(\Delta_y)]} := \begin{cases} \{k \in [\mathbf{m}]_{[\wedge D(\Delta_y)]} : N_{[\delta_i]} \cap N_{[\wedge D_p(\Delta_y)]} \cap \mathfrak{E}_k \neq \emptyset\} & \text{if } 1 \le i \le \ell, \\ \{k \in [\mathbf{m}]_{[\wedge D(\Delta_y)]} : N_{[\delta_i]} \cap N_{[\wedge D_q(\Delta_y)]} \cap \mathfrak{E}_k \neq \emptyset\} & \text{if } \ell+1 \le i \le t, \\ \emptyset & \text{if } t+1 \le i \le m. \end{cases} \tag{4.54}$$

Here, the subscript 2 of $\wedge_2$ indicates that $D(\Delta_y)$ splits into two subsets $D_p(\Delta_y)$ and $D_q(\Delta_y)$, and '$\wedge$' is taken within each of the two subsets. Then, we obtain the following analogue of (4.28): under suitable elementary column operations, the second row of $\widetilde{\varphi}^{\mathsf{r}_1\mathsf{p}_1}$ can be written as

$$\Big(\prod_{k\in[\mathbf{m}]} \check{\varepsilon}_k\Big)\Big[\,0 \quad \underbrace{\eta_2 \quad \cdots \quad \eta_\ell}_{\text{DNE if } \ell=1} \quad \check{\kappa}_{1,\ell+1} \quad \underbrace{\eta_{\ell+2} \quad \cdots \quad \eta_t}_{\text{DNE if } t=\ell+1} \quad \eta_{t+1} \quad \cdots \quad \eta_m\,\Big], \tag{4.55}$$

where

$$\eta_i := \Bigg(\prod_{k\in[\mathbf{m}]_{[\delta_i\wedge_2 D(\Delta_y)]}} \check{\varepsilon}_k\Bigg)\Bigg(\prod_{e\in N_{[\delta_i]}\setminus\Delta_y} \check{\zeta}_e\Bigg).$$

Below, we divide the proof into two sub-cases, depending on the cardinal of $D(\Delta_y)$.

4.13.1 $|D(\Delta_y)| \ge 3$. In this situation, we have $h + (s-\ell) \ge 3$, i.e. $\{\delta_1, \delta_{\ell+1}\}$ is a strict subset of $D(\Delta_y)$. Then (4.53) implies that there exists $i_0 \in [t]\setminus\{1, \ell+1\}$ such that

$$[\mathbf{m}]_{[\delta_{i_0}\wedge_2 D(\Delta_y)]} = [\mathbf{m}]_{[\wedge_2 D(\Delta_y)]}.$$

The corresponding term η_{i_0} in (4.55) can thus be written as $\eta_{i_0} = \prod_{k\in[\mathbf{m}]_{[\wedge_2 D(\Delta_y)]}} \check{\varepsilon}_k$.

To see how $(\mathsf{r}_3\mathsf{p}_1)$ affects $\mathcal{V}_y$, we mimic the construction of the first-order derived tree ϱ_y as in (4.29) and set

$$D_{y,\text{i-m}} := \{\delta \in D : [\mathbf{m}]_{[\delta_i\wedge_2 D(\Delta_y)]} \sqcup (N_{[\delta_i]}\setminus\Delta_y) \text{ is inclusion-minimal among all } i \in [m]\},$$
$$E_y := \bigcup_{\delta\in D_{y,\text{i-m}}} ([\mathbf{m}]_{[\delta\wedge_2 D(\Delta_y)]} \sqcup (N_{[\delta]}\setminus\Delta_y)).$$

On E_y, consider the partial order $\preceq_y$ determined by

- for every $k, k' \in [\mathbf{m}]_{[\wedge_2 D(\Delta_y)]}$, $k \prec_y k'$ if and only if $k > k'$;
- for every $e, e' \in \bigcup_{\delta\in D_{y,\text{i-m}}} (N_{[\delta]}\setminus\Delta_y)$, $e \preceq_y e'$ if and only if every connected subcurve of C containing F and e must contain e'; and
- for every $k \in [\mathbf{m}]_{[\wedge_2 D(\Delta_y)]}$ and $e \in \bigcup_{\delta\in D_{y,\text{i-m}}} (N_{[\delta]}\setminus\Delta_y)$ $(\subset \tau_0)$,

$$e \prec_y k \quad\Longleftrightarrow\quad e \in (N_{[\wedge D_p(\Delta_y)]} \cap \mathfrak{E}_k)^{\prec} \sqcup (N_{[\wedge D_q(\Delta_y)]} \cap \mathfrak{E}_k)^{\prec} \text{ in } \tau_0.$$

One can check directly that $\preceq_y$ is a tree order on E_y as per (4.5). We call the rooted tree

$$\upsilon_y := (E_y, \preceq_y) \tag{4.56}$$

a first-order doubly-derived tree.

In Figure 22, we provide an illustration of the notion of first-order doubly-derived trees. Here, τ is the same rooted tree as in Figure 18. Given $\widehat{x} \in X_\tau^{\mathcal{V}}$, consider two lifts y and z of $\widehat{x}$, sharing

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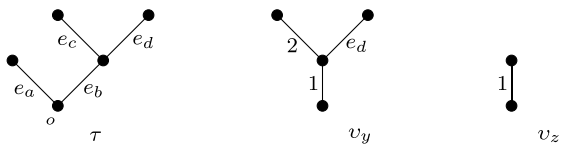


FIGURE 22. First-order doubly-derived trees.

the same RLS $\overline{\mathbf{E}} = \{\{e_a, e_b\}, \{e_a, e_c, e_d\}\}$ and satisfying

$$\Delta_y = \{e_a, e_b, e_c\}, \quad \Delta_z = \{e_a, e_b, e_c, e_d\}.$$

The corresponding doubly-derived trees υ_y and υ_z are respectively illustrated in Figure 22.

Notice that the root-to-leaf paths of υ_y are those $[\mathbf{m}]_{[\delta \wedge_2 D(\Delta_y)]} \sqcup (N_{[\delta]} \backslash \Delta_y)$ with $\delta \in D_{y,\text{i-m}}$. In particular, $[\mathbf{m}]_{[\wedge_2 D(\Delta_y)]}$ gives a root-to-leaf path of υ_y; in other words, every transverse section $\mathfrak{F}$ of υ_y contains a unique element of $[\mathbf{m}]_{[\wedge_2 D(\Delta_y)]}$, denoted by $k_{\mathfrak{F}}$.

Recall that (the prototypes of) the blowup centers $K_{k,k'}$, $2 \le k \le k'$, of $(\mathsf{r}_3\mathsf{p}_1)$ are described in Section 3.3. One can then check directly that

$$\check{K}_{k,k'} \cap \mathcal{V}_y = \bigcup_{\mathfrak{F} \in \Xi(\upsilon_y) \text{ s.t. } |\mathfrak{E}_{k_{\mathfrak{F}}}| = k,\, |\mathfrak{F}| = k'-1} \{\check{\kappa}_{1,\ell+1} = 0;\ \check{\varepsilon}_{k_{\mathfrak{F}}} = 0;\ \check{\zeta}_e = 0 \,\forall\, e \in \mathfrak{F} \backslash \{k_{\mathfrak{F}}\}\}, \quad 2 \le k \le k'.$$

Following the same argument as for (4.30), we see that the above equation gives a partition of $\Xi(\upsilon_y)$, hence of $\Xi(\mathrm{gr}(\upsilon_y))$, satisfying Definition 4.9. In other words, $(\mathsf{r}_3\mathsf{p}_1)$ is $\mathrm{gr}(\upsilon_y)$-compatible on $\mathcal{V}_y$, where the grafted edge corresponds to the local parameter $\check{\kappa}_{1,\ell+1}$. Comparing the root-to-leaf paths of $\mathrm{gr}(\upsilon_y)$ and the terms of (4.55), we conclude from Proposition 4.17 that the pullback of $\widetilde{\varphi}^{\mathsf{r}_1\mathsf{p}_1}$ is diagonalized after $(\mathsf{r}_3\mathsf{p}_1)$. Moreover, by Lemma 4.10, the pullback of $\mathcal{V}_y$ is smooth. Finally, notice that $(\mathsf{r}_3\mathsf{p}_2)$ to $(\mathsf{r}_3\mathsf{p}_4)$ do not affect the pullback of $\mathcal{V}_y$, and hence the proof for the $h + (s - \ell) \ge 3$ situation is complete.

4.13.2 $|D(\Delta_y)| = 2$. In this situation, we have $h + (s - \ell) = 2$, i.e. $D_p(\Delta_y) = \{\delta_1\}$ and $D_q(\Delta_y) = \{\delta_{\ell+1}\}$. Then the above construction of the first-order doubly-derived tree υ_y still applies to this situation verbatim, and $(\mathsf{r}_3\mathsf{p}_1)$ is still $\mathrm{gr}(\upsilon_y)$-compatible on $\mathcal{V}_y$, so the pullback of $\mathcal{V}_y$ is still smooth. However, on the one hand, $[\mathbf{m}]_{[\wedge_2 D(\Delta_y)]}$ is a root-to-leaf path of υ_y; on the other hand, for every $i \in [m] \backslash \{1, \ell+1\}$, the term η_i in (4.55) contains factors that are not labeled by $[\mathbf{m}]_{[\wedge_2 D(\Delta_y)]}$ because $N_{[\delta_i]} \backslash \Delta_y$ is nonempty. Thus, the pullback of $\widetilde{\varphi}^{\mathsf{r}_1\mathsf{p}_1}$ is possibly not diagonalized after $(\mathsf{r}_3\mathsf{p}_1)$.

More precisely, let z be an arbitrary lift of y after $(\mathsf{r}_3\mathsf{p}_1)$. From (4.55), we see that the pullback of $\widetilde{\varphi}^{\mathsf{r}_1\mathsf{p}_1}$ is not diagonalized if and only if $[\mathbf{m}]_{[\wedge_2 D(\Delta_y)]} \subset \Delta_z$ and $(N_{[\delta_i]} \backslash \Delta_y) \not\subset \Delta_z$ for any $i \in [m] \backslash \{1, \ell+1\}$.

Nonetheless, since $|D(\Delta_y)| = 2$ and $\kappa_{1,\ell+1}(\widehat{x}) = 0$, we define the rooted tree τ_2 in a parallel way to (4.20), but with two modifications:

- each $N_{[\delta]}$ should be replaced by $N_{[\delta]} \backslash \Delta_y$, and
- δ_2 should be replaced by $\delta_{\ell+1}$.

One can then check directly that $(\mathsf{r}_3\mathsf{p}_2)$ does not affect the pullback of $\mathcal{V}_y$, whereas the prototype of $(\mathsf{r}_3\mathsf{p}_3)$, i.e. the blowup of $\mathfrak{M}_2^{\mathrm{div}} \times_{\mathfrak{M}_2^{\mathrm{wt}}} \widetilde{\mathfrak{M}}^{\mathsf{r}_2}$ successively along the proper transforms of $\mathcal{H}_k$, $k \ge 0$, is $\mathrm{gr}(\tau_2)$-compatible on $\mathcal{V}_y$. Hence by Corollary 4.28, $(\mathsf{r}_3\mathsf{p}_3)$ is $\mathrm{PT}_z(\mathrm{gr}(\tau_2))$-compatible,

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or equivalently $\mathrm{gr}(\mathrm{PT}_z(\tau_2))$-compatible, on a small (smooth) neighborhood $\mathcal{V}_z$ of z. In particular, this implies that $\mathcal{V}_z$ is smooth. Moreover, the root-to-leaf paths of τ_2 are those inclusion-minimal $N_{[\delta_i]}\backslash(\Delta_y \sqcup \Delta_z)$, along with the grafted edge corresponding to $\check{\kappa}_{1,\ell+1}$. Therefore, the pullback of $\widetilde{\varphi}^{\mathsf{r}_3\mathsf{p}_1}$ becomes diagonalized on $\mathcal{V}_z$. Finally, notice that $(\mathsf{r}_3\mathsf{p}_4)$ does not affect $\mathcal{V}_z$, so the proof of the $h+(s-\ell)=2$ situation is complete.

4.14 Proof of Case E, Part III

In this subsection, we aim to show Case E.1.e, when $D(\Delta_y)$ is contained in a unique tail whose pivotal node is not a Weierstrass point. Without loss of generality we assume there exist $1 \le h \le \ell \le m$ such that

$$D(\Delta_y)=\{\delta_1,\dots,\delta_h\},\quad \langle\delta_i\rangle=\langle\delta_1\rangle\quad \forall\, 1\le i\le \ell,\quad \langle\delta_i\rangle\neq\langle\delta_1\rangle\quad \forall\, \ell<i\le m.$$

Since $\langle\delta_1\rangle$ is not a Weierstrass point of F, by Proposition 2.22 (SH_2) and the argument for (4.28), after suitable elementary row and column operations, the pullback $\widetilde{\varphi}^{\mathsf{r}_1\mathsf{p}_1}$ of the structural homomorphism φ takes the following form on $\mathcal{V}_y$:

$$\Pi^{\mathsf{r}_1\mathsf{p}_1}\cdot\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0\\ 0 & \underbrace{\eta_2 & \cdots & \eta_\ell}_{\text{DNE if } \ell=1} & \check{\kappa}_{1,\ell+1}\eta_{\ell+1} & \cdots & \check{\kappa}_{1m}\eta_m \end{bmatrix}, \tag{4.57}$$

where

$$\Pi^{\mathsf{r}_1\mathsf{p}_1}:=\prod_{k\in[\mathbf{m}]}\check{\varepsilon}_k,\quad \eta_i:=\left(\prod_{k\in[\mathbf{m}]_{[\delta_i\wedge D(\Delta_y)]}}\check{\varepsilon}_k\right)\left(\prod_{e\in N_{[\delta_i]}\backslash\Delta_y}\check{\zeta}_e\right),$$

and the $[\mathbf{m}]_{[\delta_i\wedge D(\Delta_y)]}$ are as in (4.27). Moreover, when $h\ge 2$, there exists $2\le i_0\le h$ such that $[\mathbf{m}]_{[\delta_{i_0}\wedge D(\Delta_y)]}=[\mathbf{m}]_{[\wedge D(\Delta_y)]}$.

The assumption of Case E.1.e implies that $(\mathsf{r}_1\mathsf{p}_2)$ to $(\mathsf{r}_1\mathsf{p}_4)$ and $(\mathsf{r}_3\mathsf{p}_2)$ do not affect $\mathcal{V}_y$. Depending on whether $D(\Delta_y)$ is a singleton (i.e. whether $h=1$), $(\mathsf{r}_1\mathsf{p}_5)$, $(\mathsf{r}_3\mathsf{p}_3)$, and $(\mathsf{r}_3\mathsf{p}_4)$ may or may not be relevant, so we divide the proof of Case E.1.e into two sub-cases as follows.

4.14.1 $|D(\Delta_y)|\ge 2$. In this situation, $(\mathsf{r}_1\mathsf{p}_5)$, $(\mathsf{r}_3\mathsf{p}_3)$, and $(\mathsf{r}_3\mathsf{p}_4)$ do not affect $\mathcal{V}_y$.

To see how (r_2) affects $\mathcal{V}_y$, we break it into 'phases'. More precisely, for each $j\ge 1$, we denote by π'_j the blowup of $\mathfrak{M}_2^{\mathrm{div}}\times_{\mathfrak{M}_2^{\mathrm{wt}}}\widetilde{\mathfrak{M}}^{\mathsf{r}_1}$ successively along the proper transforms of

$$X^{\mathrm{div}}_{j,j'}:=\mathfrak{M}_2^{\mathrm{div}}\times_{\mathfrak{M}_2^{\mathrm{wt}}}X_{j,j'},\quad j'=j,\,j+1,\,\cdots.$$

As mentioned in Section 3.2, the stability of $\mathfrak{M}_2^{\mathrm{wt}}$ implies that there exists a finite number $j_0\ge 1$ such that (r_2) is equivalent to exerting π'_{j_0} to $\mathcal{V}_y$, then exerting $\mathrm{PT}_{\pi'_{j_0}}(\pi'_{j_0-1})$ (see Definition 4.26) to the pullback of $\mathcal{V}_y$, and so forth. Below, we find the rooted trees with which these π'_j are compatible on $\mathcal{V}_y$.

Consider the first-order derived tree ϱ_y given in (4.29), which can be defined verbatim in the current situation. (Notice that $D\backslash F=D$ here, while $D\backslash F=D\backslash\{\delta_1\}$ in (4.29).) The root-to-leaf paths of ϱ_y are exactly the inclusion-minimal elements of

$$\{[\mathbf{m}]_{[\delta_i\wedge D(\Delta_y)]}\sqcup(N_{[\delta_i]}\backslash\Delta_y):\, i\in[m]\}.$$

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For each $k \in [\mathbf{m}]_{[\wedge D(\Delta_y)]}$, we denote by $\varrho_{y,k}$ the *subposet* of ϱ_y determined by

$$\varrho_{y,k} := \bigcup_{\mathfrak{F}\in\Xi(\varrho_y)\text{ s.t. }k\in\mathfrak{F}} \mathfrak{F} \quad \subset \varrho_y.$$

It is straightforward that each $\varrho_{y,k}$ is a rooted tree as per Definition 4.6 containing the singleton $\{k\}$ as a root-to-leaf path. Below, we show that $\pi'_{|\mathfrak{E}_k|}$ is $\varrho_{y,k}$-compatible on $\mathcal{V}_y$.

In fact, one can check directly that for each $j \geq 1$, the following holds.

- If there exists $k \in [\mathbf{m}]_{[\wedge D(\Delta_y)]}$ such that $|\mathfrak{E}_k| = j$ (recall that $\mathfrak{E}_k \in \overline{\mathbf{E}}$ is a transverse section of τ_0), then the stability of $\mathfrak{M}_2^{\mathrm{wt}}$ implies that this k is unique. We denote it by k_j. In this situation, the proper transform of $X^{\mathrm{div}}_{j,j'}$ takes the following form on $\mathcal{V}_y$:

$$\check{X}^{\mathrm{div},\mathsf{r}_1\mathsf{p}_1}_{j,j'} \cap \mathcal{V}_y = \bigcup_{\mathfrak{F}\in\Xi(\varrho_{y,k_j})\text{ s.t. }|\mathfrak{F}|=j'} \{\check{\varepsilon}_{k_j} = 0;\ \check{\zeta}_e = 0\ \forall\, e \in \mathfrak{F}\backslash\{k_j\}\}.$$

  Here, recall that $\{\check{\varepsilon}_{k_j} = 0\}$ is the exceptional divisor on $\mathcal{V}_y$ determined by $\mathfrak{E}_{k_j}$; see (4.11) and (4.12). Once again, by the stability of $\mathfrak{M}_2^{\mathrm{wt}}$, π'_j is ϱ_{y,k_j}-compatible on $\mathcal{V}_y$.
- Otherwise, we have

$$\check{X}^{\mathrm{div},\mathsf{r}_1\mathsf{p}_1}_{j,j'} \cap \mathcal{V}_y = \emptyset,$$

  hence π'_j does not affect $\mathcal{V}_y$.

By Lemma 4.10, the pullback of $\mathcal{V}_y$ becomes smooth after (r_2).

Next, we study the pullback $\widetilde{\varphi}^{\mathsf{r}_2}$ of φ after (r_2). For each $k \in [\mathbf{m}]_{[\wedge D(\Delta_y)]}$, we fix a lift y_k of y after exerting the proper transform of $\pi'_{|\mathfrak{E}_k|}$, satisfying the condition that the image of y_k under the proper transform of $\pi'_{|\mathfrak{E}_k|}$ is y_{k+1}; here, when $k = \max([\mathbf{m}]_{[\wedge D(\Delta_y)]})$, we set $y_{k+1} := y$. In particular,

$$y_1 \in \widetilde{\mathfrak{M}}_2^{\mathrm{div}}.$$

The RLS of each y_k is denoted by $\overline{\mathbf{E}}_k = \{\mathfrak{E}_{k,1}, \ldots, \mathfrak{E}_{k,\mathbf{m}_k}\}$. It is noteworthy that some $\mathbf{m}_k$ can be 0; i.e. the proper transform of $\pi'_{|\mathfrak{E}_k|}$ does not affect a neighborhood of y_k. Near the final lift y_1, the local parameters corresponding to the exceptional divisors obtained in (r_2) are denoted by $\check{\varepsilon}_{k,k'}$, $k \in [\mathbf{m}]_{[\wedge D(\Delta_y)]}$, $k' \in [\mathbf{m}_k]$. Let

$$\Pi^{\mathsf{r}_2} := \prod_{k\in[\mathbf{m}]_{[\wedge D(\Delta_y)]}} \prod_{k'\in[\mathbf{m}_k]} \check{\varepsilon}_{k,k'}. \tag{4.58}$$

The construction of $\varrho_{y,k}$ implies that

$$\Xi(\varrho_y) = \bigsqcup_{k\in[\mathbf{m}]_{[\wedge D(\Delta_y)]}} \Xi(\varrho_{y,k}). \tag{4.59}$$

So, the intersection of every root-to-leaf path of ϱ_y and every $\mathfrak{E}_{k,k'}$ is a singleton. Therefore, for every $2 \leq i \leq m$, the pullback $\widetilde{\eta}_i^{\mathsf{r}_2}$ of η_i after (r_2) satisfies

$$\Pi^{\mathsf{r}_2} \,|\, \widetilde{\eta}_i^{\mathsf{r}_2} \quad \text{on a neighborhood of } y_1. \tag{4.60}$$

Along with Lemma 4.27, (4.59) also implies the existence of a root-to-leaf path $\wp$ of ϱ_y satisfying

$$\wp \subset \Delta^{\mathsf{r}_2}_{y_1} := \bigsqcup_{k\in[\mathbf{m}]_{[\wedge D(\Delta_y)]}} \Delta_{y_k}, \tag{4.61}$$

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where the Δ_{y_k} are as in Definition 4.15. In other words, there exists $2 \le i_1 \le m$ such that

$$\widetilde{\eta}_{i_1}^{\mathsf{r}_2}/\Pi^{\mathsf{r}_2} \text{ is invertible on a neighborhood of } y_1. \tag{4.62}$$

- If there exists a root-to-leaf path $\wp$ of ϱ_y satisfying (4.61) as well as $\wp \cap [\mathbf{m}] \neq \emptyset$, then the index i_1 above can be taken in $[\ell]$, i.e. $\langle \delta_{i_1} \rangle = \langle \delta_1 \rangle$. So, by (4.57), (4.60) and (4.62), $\widetilde{\varphi}^{\mathsf{r}_2}$ is diagonalized after (r_2).
- If every root-to-leaf path $\wp$ of ϱ_y satisfying (4.61) is disjoint from $[\mathbf{m}]$, then we set

$$D(\Delta_{y_1}^{\mathsf{r}_2}) := \{\delta_{i_1} : 2 \le i_1 \le m,\ i_1 \text{ satisfies (4.62)}\}. \tag{4.63}$$

  For every $\delta_{i_1} \in D(\Delta_{y_1}^{\mathsf{r}_2})$, we have $\langle \delta_{i_1} \rangle \neq \langle \delta_1 \rangle$. We have the following two possibilities.
  - If there exists $\delta_{\ell'} \in D(\Delta_{y_1}^{\mathsf{r}_2})$ such that $\langle \delta_{\ell'} \rangle$ is not conjugate to $\langle \delta_1 \rangle$ on F, then $\check{\kappa}_{1,\ell'}$ is invertible on $\mathcal{V}_y$. By (4.57), (4.60) and (4.62), $\widetilde{\varphi}^{\mathsf{r}_2}$ is diagonalized after (r_2).
  - If $\langle \delta_{\ell'} \rangle$ is conjugate to $\langle \delta_1 \rangle$ on F for all $\delta_{\ell'} \in D(\Delta_{y_1}^{\mathsf{r}_2})$, then the assumption of Case E.1 implies that these $\delta_{\ell'}$ are on the same tail C'' whose pivotal node is distinct from and conjugate to $\langle \delta_1 \rangle$; in addition, $\check{\kappa}_{1,\ell+1}$ is a local parameter vanishing at y_1. Notice that we can rewrite the notion $[\mathbf{m}]_{[\wedge_2 D(\Delta_y)]}$ of (4.54) in the following equivalent way:

$$[\mathbf{m}]_{[\wedge_2 D(\Delta_y)]} := \left\{ k \in [\mathbf{m}] : \left| \left(\bigcup_{\delta \in D(\Delta_y)} N_{[\delta]} \right) \cap \mathfrak{E}_k \right| = 2 \right\},$$

  which gives rise to the same partition $D(\Delta_y) = D_p(\Delta_y) \sqcup D_q(\Delta_y)$. Then, we can define $[\mathbf{m}]_{[\delta_i \wedge_2 D(\Delta_y)]}$, $i \in [m]$, likewise. Applying the same idea to the current situation, we set

$$[\mathbf{m}]_{[\wedge_2 D(\Delta_y, \Delta_{y_1}^{\mathsf{r}_2})]} := \left\{ k \in [\mathbf{m}] : \left| \left(\bigcup_{\delta \in D(\Delta_y) \sqcup D(\Delta_{y_1}^{\mathsf{r}_2})} N_{[\delta]} \right) \cap \mathfrak{E}_k \right| = 2 \right\}, \tag{4.64}$$

  and define $[\mathbf{m}]_{[\delta_i \wedge_2 D(\Delta_y, \Delta_{y_1}^{\mathsf{r}_2})]}$, $i \in [m]$, analogously. Then, by Proposition 2.22 (SH_3), under suitable elementary column operations, the second row of $\widetilde{\varphi}^{\mathsf{r}_2}$ can be written as

$$\Pi^{\mathsf{r}_1 \mathsf{p}_1} \cdot \Pi^{\mathsf{r}_2} \cdot \begin{bmatrix} 0 & \check{\eta}_2 & \cdots & \check{\eta}_\ell & \check{\kappa}_{1,\ell+1} & \check{\eta}_{\ell+2} & \cdots & \check{\eta}_m \end{bmatrix},$$
$$\text{where} \quad \check{\eta}_i := \left(\prod_{k \in [\mathbf{m}]_{[\delta_i \wedge_2 D(\Delta_y, \Delta_{y_1}^{\mathsf{r}_2})]}} \check{\varepsilon}_k \right) \left(\prod_{e \in N_{[\delta_i]} \setminus (\Delta_y \sqcup \Delta_{y_1}^{\mathsf{r}_2})} \check{\zeta}_e \right).$$

  Comparing the above expression of $\widetilde{\varphi}^{\mathsf{r}_2}$ with (4.55), we see that the construction of Section 4.13.1 can be applied in a parallel manner, which implies that the pullback of φ^{r_2} becomes diagonalized, and the pullback of a neighborhood of y_1 is smooth, after $(\mathsf{r}_3 \mathsf{p}_1)$.

4.14.2 $|D(\Delta_y)| = 1$. In this situation, $(\mathsf{r}_1 \mathsf{p}_5)$ is involved. By defining the subsets $D_{\text{i-m},y} \subset D$ and $E_y \subset N(C)$ in exactly the same way as $D_{\text{i-m}}$ and E in (4.19), and defining the relation $\preceq_y$ on E_y given by $e \preceq_y e'$ if and only if any connected subcurve of C containing F and e must contain e', we see that

$$\tau_{1,y} := (E_y, \preceq_y)$$

is a rooted tree, whose root-to-leaf paths are exactly $N_{[\delta]} \setminus \Delta_y$, $\delta \in D_{\text{i-m},y}$. One can then check directly that $(\mathsf{r}_1 \mathsf{p}_5)$ is $\tau_{1,y}$-compatible on $\mathcal{V}_y$. After $(\mathsf{r}_1 \mathsf{p}_5)$, we fix a lift z of y in $\widetilde{\mathfrak{M}}^{\mathsf{r}_1 \mathsf{p}_5}$ and define $D(\Delta_z)$ to be the analogue of (4.24).

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- If there exists $\delta \in D(\Delta_z)$ such that $\langle\delta\rangle \neq \langle\delta_1\rangle$, then the proof of Case A.5 can be applied to this situation in a parallel way; cf. Sections 4.6–4.8.
- If $\langle\delta\rangle = \langle\delta_1\rangle$ for all $\delta \in D(\Delta_z)$, then the argument in Section 4.14.1 can be applied to this situation in a parallel way.

4.15 Second-order derived trees and the proof of Case E, Part IV

In this subsection, we aim to show Case E.1.f, when $D(\Delta_y)$ is contained in a unique tail whose pivotal node is a Weierstrass point. Without loss of generality we assume that there exists $1 \leq \ell \leq m$ such that

$$D(\Delta_y) \subset \{\delta_1, \dots, \delta_\ell\}, \quad \langle\delta_i\rangle = \langle\delta_1\rangle \quad \forall\, 1 \leq i \leq \ell, \quad \langle\delta_i\rangle \neq \langle\delta_1\rangle \quad \forall\, \ell < i \leq m.$$

Since $\langle\delta_1\rangle$ is a Weierstrass point of F, we have $\kappa_{1i}(\widetilde{x}) = 0$ for all $i \in [\ell]$, whereas κ_{1j}, $\ell < j \leq m$, are all invertible. By Proposition 2.22 (SH_2) and the argument for (4.28), after suitable elementary row and column operations, the pullback $\widetilde{\varphi}^{\mathsf{r}_1\mathsf{p}_1}$ of the structural homomorphism φ takes the following form on $\mathcal{V}_y$:

$$\widetilde{\varphi}^{\mathsf{r}_1\mathsf{p}_1} = \Pi^{\mathsf{r}_1\mathsf{p}_1} \cdot \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \underbrace{\check{\kappa}_{12}\eta'_2 & \cdots & \check{\kappa}_{1\ell}\eta'_\ell}_{\text{DNE if } \ell = 1} & \eta'_{\ell+1} & \cdots & \eta'_m \end{bmatrix}, \tag{4.65}$$

where

$$\Pi^{\mathsf{r}_1\mathsf{p}_1} := \prod_{k \in [\mathbf{m}]} \check{\varepsilon}_k, \quad \eta'_i := \left(\prod_{k \in [\mathbf{m}]_{[\delta_1 \wedge \delta_i]}} \check{\varepsilon}_k \right) \left(\prod_{e \in N_{[\delta_i]} \setminus \Delta_y} \check{\zeta}_e \right),$$

and the $[\mathbf{m}]_{[\delta_1 \wedge \delta_i]}$ are as in (4.25).

The assumption of Case E.1.f implies that $(\mathsf{r}_1\mathsf{p}_2)$ to $(\mathsf{r}_1\mathsf{p}_4)$ do not affect $\mathcal{V}_y$. Depending on the cardinal of $D(\Delta_y)$, $(\mathsf{r}_1\mathsf{p}_5)$, (r_2), $(\mathsf{r}_3\mathsf{p}_1)$, $(\mathsf{r}_3\mathsf{p}_3)$, and $(\mathsf{r}_3\mathsf{p}_4)$ may or may not be relevant, so we divide the proof of Case E.1.f into the following sub-cases.

4.15.1 $|D(\Delta_y)| \geq 2$. In this situation, we have $h \geq 2$, and hence $(\mathsf{r}_1\mathsf{p}_5)$ and $(\mathsf{r}_3\mathsf{p}_4)$ do not affect $\mathcal{V}_y$.

As shown in Section 4.15.1, the pullback of $\mathcal{V}_y$ is smooth after (r_2). In addition, we obtain a sequence y_k, $k \in [\mathbf{m}]_{[\wedge D(\Delta_y)]}$, of the lifts of y, with $y_1 \in \widetilde{\mathfrak{M}}_2^{\mathrm{div}}$. For Π^{r_2} as in (4.58), the analogue of (4.60) is still valid in the current case, with each $\widetilde{\eta}_i^{\mathsf{r}_2}$ replaced with the pullback of η'_i. Let $\Delta_{y_1}^{\mathsf{r}_2}$ and $D(\Delta_{y_1}^{\mathsf{r}_2})$ be as in (4.61) and (4.63), respectively, and $[\mathbf{m}]_{[\wedge_2 D(\Delta_y, \Delta_{y_1}^{\mathsf{r}_2})]}$ be as in (4.64).

- If there exists $\ell < j \leq m$ such that $\delta_j \in D(\Delta_{y_1}^{\mathsf{r}_2})$, then the pullback of η'_i is equal to Π^{r_2} (up to a unit) locally near y_1. Therefore, $\widetilde{\varphi}^{\mathsf{r}_2}$ is diagonalized.
- If $D(\Delta_{y_1}^{\mathsf{r}_2}) \subset \{\delta_1, \dots, \delta_\ell\}$, then the following analogue of Lemma 4.22 holds: there exists $2 \leq i \leq h$, say $i = 2$, such that

$$N_{[\delta_1 \wedge \delta_2]} = \bigcap_{\delta \in D(\Delta_y) \sqcup D(\Delta_{y_1}^{\mathsf{r}_2})} N_{[\delta]}.$$

Applying Proposition 2.22 (SH_3), we see that, after suitable elementary row and column operations, the second row of the pullback $\widetilde{\varphi}^{\mathsf{r}_2}$ of (4.65) takes the following form near y_1:

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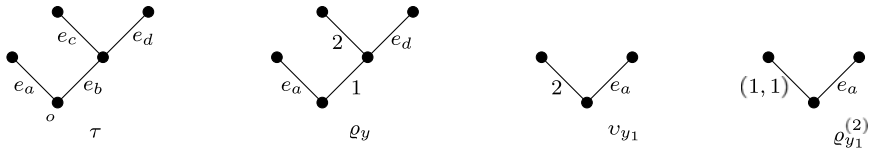


FIGURE 23. First-order doubly-derived and second-order derived trees.

$$\Pi^{\mathsf{r}_1\mathsf{p}_1}\cdot\Pi^{\mathsf{r}_2}\cdot\begin{bmatrix}0 & \check{\kappa}_{12} & \eta_3'' & \cdots & \eta_m''\end{bmatrix},\quad\text{where}$$

$$\eta_i'':=\left(\prod_{k\in[\mathbf{m}]_{[\delta_i\wedge D(\Delta_y,\Delta^{\mathsf{r}_2}_{y_1})]}}\ \prod_{k'\in[\mathbf{m}_k]}\check{\varepsilon}_{k,k'}\right)\left(\prod_{k\in[\mathbf{m}]_{[\delta_i\wedge_2 D(\Delta_y,\Delta^{\mathsf{r}_2}_{y_1})]}}\check{\varepsilon}_k\right)\left(\prod_{e\in N_{[\delta_i]}\setminus(\Delta_y\sqcup\Delta^{\mathsf{r}_2}_{y_1})}\check{\zeta}_e\right),$$

where

$$[\mathbf{m}]_{[\wedge D(\Delta_y,\Delta^{\mathsf{r}_2}_{y_1})]}:=\left\{k\in[\mathbf{m}]:\left|\left(\bigcup_{\delta\in D(\Delta_y)\sqcup D(\Delta^{\mathsf{r}_2}_{y_1})}N_{[\delta]}\right)\cap\mathfrak{E}_k\right|=1\right\},$$

$[\mathbf{m}]_{[\delta_i\wedge D(\Delta_y,\Delta^{\mathsf{r}_2}_{y_1})]}$ are analogous to (4.54), and $\mathbf{m}_k$ are as in Section 4.14.1.
To see how $(\mathsf{r}_3\mathsf{p}_1)$ and $(\mathsf{r}_3\mathsf{p}_2)$ affect a neighborhood of y_1, we first observe that the inclusion-minimal elements of the following set

$$\{[\mathbf{m}]_{[\delta_i\wedge_2 D(\Delta_y,\Delta^{\mathsf{r}_2}_{y_1})]}\sqcup(N_{[\delta_i]}\setminus(\Delta_y\sqcup\Delta^{\mathsf{r}_2}_{y_1}))\,:\,3\le i\le m\}$$

are exactly the root-to-leaf paths of the rooted tree υ_{y_1}, and that $(\mathsf{r}_3\mathsf{p}_1)$ is $\mathrm{gr}(\upsilon_{y_1})$-compatible near y_1, where the grafted edge corresponds to the local parameter $\check{\kappa}_{12}$. Next, we mimic the first-order derived tree ϱ_y of (4.29) and construct the **second-order derived tree** $\varrho^{(2)}_{y_1}$ at y_1. More precisely, let E'_{y_1} be the union of the inclusion-minimal elements of the following set

$$\{\{(k,k'):k\in[\mathbf{m}]_{[\delta_i\wedge D(\Delta_y,\Delta^{\mathsf{r}_2}_{y_1})]},\ k'\in[\mathbf{m}_k]\}\sqcup(N_{[\delta_i]}\setminus(\Delta_y\sqcup\Delta^{\mathsf{r}_2}_{y_1}))\,:\,3\le i\le m\}.\tag{4.66}$$

This is the analogue of E_y in (4.29). Then, we endow E'_{y_1} with the partial order $\prec'_{y_1}$ such that:

- for every $(k,k'),(h,h')\in E'_{y_1}$, we have $(k,k')\prec'_{y_1}(h,h')$ if and only if either $k>h$, or $k=h$ and $k'>h'$;
- for every $e,e'\in E'_{y_1}\cap N(C)$, we have $e\preceq_y e'$ if and only if any connected subcurve of C containing F and e must contain e'; and
- for every (k,k') and e in E'_{y_1}, $e\prec'_{y_1}(k,k')$ if and only if $e\in([\mathbf{m}_k]\cap\mathfrak{E}_{k,k'})^{\prec_y}$ in $\varrho_{y,k}$, where $\varrho_{y,k}$ is as in Section 4.14.1.

One can check directly that $\prec'_{y_1}$ is a tree order as per (4.5). Then, we set

$$\varrho^{(2)}_{y_1}:=(E'_{y_1},\ \prec'_{y_1}).$$

In Figure 23, we provide an example illustrating both first-order doubly-derived and second-order derived trees. Here, τ is the same rooted tree as in Figure 18. Given $\widehat{x}\in X^{\mathcal{V}}_{\tau}$, consider a lift y of $\widehat{x}$ after $(\mathsf{r}_1\mathsf{p}_1)$, whose RLS and dominant edges are given by satisfying

$$\overline{\mathbf{E}}=\{\{e_a,e_b\},\{e_a,e_c,e_d\}\},\quad \Delta_y=\{e_b,e_c\}.$$

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The first-order derived tree ϱ_y is the same as Graph (5) of Figure 18. Let y_1 be a lift of y after (r_2), satisfying

$$\Delta_{y_1}^{\mathsf{r}_2} = \{e_d, 1\}.$$

The corresponding doubly-derived tree υ_{y_1} and second-order derived tree $\varrho_{y_1}^{(2)}$ are illustrated in Figure 23. Observe that the root-to-leaf paths of $\varrho_{y_1}^{(2)}$ are exactly the inclusion-minimal elements of (4.66). In addition, recall that in Section 3.4, (the prototypes of) the blowup centers of $(\mathsf{r}_3\mathsf{p}_2)$ are denoted by $W_{k,k',k''}$, $1 \le k \le k' \le k''$. One can also check directly that, near y_1, the proper transform of each $W_{k,k',k''}$ takes the form

$$\bigcup_{\mathfrak{F}\in\Xi(\varrho_{y_1}^{\mathsf{r}_2})\text{ s.t. }|\mathfrak{E}_{k_{\mathfrak{F}}}|=k,\,|\mathfrak{F}|=k''} \{\check{\kappa}_{12}=0;\ \check{\varepsilon}_{k_{\mathfrak{F}},k'}=0;\ \check{\zeta}_e=0\,\forall\, e\in\mathfrak{F}\backslash\{(k_{\mathfrak{F}},k')\}\},\quad 1\le k\le k'\le k''.$$

Following the same argument as for (4.30), we see that the above equation gives a partition of $\Xi(\varrho_{y_1}^{\mathsf{r}_2})$, hence of $\Xi(\mathrm{gr}(\varrho_{y_1}^{\mathsf{r}_2}))$, satisfying Definition 4.9. In other words, the prototype of $(\mathsf{r}_3\mathsf{p}_2)$ is $\mathrm{gr}(\varrho_{y_1}^{\mathsf{r}_2})$-compatible near y_1, where the grafted edge corresponds to the local parameter $\check{\kappa}_{12}$. Consequently, for any lift z of y_1 after $(\mathsf{r}_3\mathsf{p}_1)$, Corollary 4.28 ensures that $(\mathsf{r}_3\mathsf{p}_2)$ is $\mathrm{PT}_z(\mathrm{gr}(\varrho_{y_1}^{\mathsf{r}_2}))$-compatible near z. From the above two paragraphs, we see that the pullback of a neighborhood of y_1 is smooth after $(\mathsf{r}_3\mathsf{p}_2)$. Taking into consideration the above equations for η_i'', we conclude that the pullback of $\widetilde{\varrho}^{\mathsf{r}_2}$ is diagonalized after $(\mathsf{r}_3\mathsf{p}_2)$.

4.15.2 $|D(\Delta_y)| = 1$. In this situation, $(\mathsf{r}_1\mathsf{p}_5)$, which is locally $\tau_{y,1}$-compatible on $\mathcal{V}_y$, is involved. Here, $\tau_{y,1}$ is the analogue of τ_1 in (4.23), with $N_{[\delta]}$ replaced by $N_{[\delta]}\backslash\Delta_y$. After $(\mathsf{r}_1\mathsf{p}_5)$, we fix an arbitrary lift z of y and define $D(\Delta_z)$ in the same way as (4.24).

If there exists $\ell+1 \le i \le m$ such that $\delta_i \in D(\Delta_z)$, then from (4.65) we conclude that the pullback of φ is diagonalized on a neighborhood $\mathcal{V}_z$ of z, and the blowups starting from (r_2) do not affect $\mathcal{V}_z$.

Otherwise, the points of $D(\Delta_z)$ are all on the tail that contains $\{\delta_1\}$ $(= D(\Delta_y))$. The argument is then parallel to Section 4.15.1. We omit further details.

4.16 Proof of Case E, Part V

In this subsection, we aim to show Case E.2, Case E.3, and Case E.4.

4.16.1 *Proof of Case E.2.* In this case, we write

$$F = F_1 \cup \mathrm{B} \cup F_2$$

as in the beginning of Section 4.9, and assume that $a_1 \in F_1$, $a_2 \in F_2$, and the pivotal nodes of the tails containing points of $D(\Delta_y)$ are on $F_1 \cup \mathrm{B}$. One can check directly that $(\mathsf{r}_1\mathsf{p}_2)$ and $(\mathsf{r}_1\mathsf{p}_3)$ do not affect $\mathcal{V}_y$, but $(\mathsf{r}_1\mathsf{p}_4)$, which is locally $\tau_y^\dagger$-compatible on $\mathcal{V}_y$, is involved. Here, $\tau_y^\dagger$ is the analogue of $\tau^\dagger$ in (4.37), with $N_{[\delta,a_2]}$ replaced by $N_{[\delta,a_2]}\backslash\Delta_y$. After $(\mathsf{r}_1\mathsf{p}_4)$, we fix an arbitrary lift z of y and define $D(\Delta_z)$ in the same way as in Section 4.9.

If there exists $\delta \in D(\Delta_z)$ such that the pivotal node of the tail containing δ is on F_2, then we can mimic the argument of Section 4.9 and conclude that the pullback of φ is diagonalized on a neighborhood $\mathcal{V}_z$ of z, and that the blowups starting from $(\mathsf{r}_1\mathsf{p}_5)$ do not affect $\mathcal{V}_z$.

Otherwise, the pivotal nodes of the tails containing points of $D(\Delta_z)$ are on $F_1 \cup \mathrm{B}$ as well. The argument is then parallel to the proof of Case E.1. We omit further details.

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4.16.2 *Proof of Case E.3.* In this case, we write

$$F = F_1 \cup \mathrm{B}$$

as in the beginning of Section 4.10, and assume that the pivotal nodes of the tails containing points of $D(\Delta_y)$ are on B. One can check directly that $(\mathsf{r}_1\mathsf{p}_2)$ does not affect $\mathcal{V}_y$, but $(\mathsf{r}_1\mathsf{p}_3)$, which is locally $\tau_y^\ddagger$-compatible on $\mathcal{V}_y$, is involved. Here, $\tau_y^\ddagger$ is the analogue of $\tau^\ddagger$ in (4.46), with N' replaced by $N'\backslash\Delta_y$. After $(\mathsf{r}_1\mathsf{p}_3)$, we fix an arbitrary lift z of y and define $D(\Delta_z)$ in the same way as in Section 4.10.

Depending on the position of the points of $D(\Delta_z)$, we have one of the following:

- the pullback of φ is diagonalized on a neighborhood $\mathcal{V}_z$ of z, and the blowups starting from $(\mathsf{r}_1\mathsf{p}_4)$ do not affect $\mathcal{V}_z$; or
- the argument is parallel to the proof of Case E.2; or
- the argument is parallel to the proof of Case E.1.

We omit further details.

4.16.3 *Proof of Case E.4.* In this case, we write

$$F = T_1 \cup \mathrm{B} \cup T_2$$

as in the beginning of Section 4.11, and assume that $a_1 \in F_1$, $a_2 \in F_2$, and the pivotal nodes of the tails containing points of $D(\Delta_y)$ are on B. One can check directly that $(\mathsf{r}_1\mathsf{p}_3)$ does not affect $\mathcal{V}_y$, but $(\mathsf{r}_1\mathsf{p}_2)$, which is locally $\tau_{y,\wedge}$-compatible on $\mathcal{V}_y$, is involved. Here, $\tau_{y,\wedge}$ is the analogue of $\tau_\wedge$ in (4.48), with N'' replaced by $N''\backslash\Delta_y$. After $(\mathsf{r}_1\mathsf{p}_2)$, we fix an arbitrary lift z of y and define $D(\Delta_z)$ in the same way as in Section 4.10.

Depending on the position of the points of $D(\Delta_z)$, we have one of the following:

- the pullback of φ is diagonalized on a neighborhood $\mathcal{V}_z$ of z, and the blowups starting from $(\mathsf{r}_1\mathsf{p}_3)$ do not affect $\mathcal{V}_z$; or
- the argument is parallel to the proof of Case E.2; or
- the argument is parallel to the proof of Case E.1.

We omit further details.

To summarize, in Sections 4.12 to 4.16 we have shown the following.

Proposition 4.37. *Proposition 4.1 holds in Case E.*

5. Resolutions of $\overline{M}_2(\mathbb{P}^n, d)$

In this section, we construct the stacks $\widetilde{M}_2(\mathbb{P}^n, d)$ and $\widetilde{M}_2(\mathbb{P}^n, d; k)$, $k \geq 2$, that are proposed in Section 1, and prove Theorems 1.1 and 1.4 as well as Corollaries 1.2 and 1.3 of Section 1.

5.1 The construction of $\widetilde{M}_2(\mathbb{P}^n, d)$

We continue with the notation as in Section 2.1 and the previous sections. Let $\widetilde{\mathcal{V}} = \mathcal{V} \times_{\mathfrak{M}_2^{\mathrm{div}}} \widetilde{\mathfrak{M}}_2^{\mathrm{div}}$. Then

$$\mathscr{E}_{\widetilde{\mathcal{V}}} = \mathscr{E}_{\mathcal{V}} \times_{\mathcal{V}} \widetilde{\mathcal{V}}$$

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is the total space of the pullback bundle $\beta^*\rho_*\mathscr{L}(\mathcal{A})^{\oplus n}$, where $\beta:\widetilde{\mathcal{V}}\to\mathcal{V}$ denotes the projection. The tautological restriction homomorphism 'rest' of (2.6) pulls back to give rise to

$$\widetilde{\mathrm{rest}}:\beta^*\rho_*\mathscr{L}(\mathcal{A})^{\oplus n}\longrightarrow\beta^*\rho_*(\mathscr{L}(\mathcal{A})^{\oplus n}|_{\mathcal{A}}). \tag{5.1}$$

The section Φ of (2.7) pulls back to give rise to

$$\tilde{\Phi}\in\Gamma(\mathscr{E}_{\widetilde{\mathcal{V}}},\ \tilde{p}^*\beta^*\rho_*(\mathscr{L}(\mathcal{A})^{\oplus n}|_{\mathcal{A}})), \tag{5.2}$$

where $\tilde{p}:\mathscr{E}_{\widetilde{\mathcal{V}}}\to\widetilde{\mathcal{V}}$ denotes the projection. Given small open $U\subset\overline{M}_2(\mathbb{P}^n,d)$ and $H\in\mathbf{H}_U$ as in (1.1), the morphism $f_{U,H}:U\to\mathcal{V}\,(\to\mathfrak{M}_2^{\mathrm{div}})$ determines

$$\widetilde{\mathcal{U}}=\widetilde{\mathcal{U}}_{U,H}:=\widetilde{\mathcal{V}}\times_{\mathcal{V};f_{U,H}}U. \tag{5.3}$$

The immersion $\mathcal{U}\to\mathscr{E}_{\mathcal{V}}$ of Theorem 2.5 then naturally induces an open immersion

$$\widetilde{\mathcal{U}}\longrightarrow(\widetilde{\Phi}=0)\subset\mathscr{E}_{\widetilde{\mathcal{V}}}.$$

Lemma 5.1. *With notation as above, $\widetilde{\mathcal{U}}$ has smooth irreducible components and normal crossing singularities.*

Proof. We trivialize $\beta^*\rho_*\mathscr{M}(\mathcal{D})$ and $\beta^*\rho_*\mathscr{O}_{\mathcal{A}}(\mathcal{A})$ so that

$$\mathscr{E}_{\widetilde{\mathcal{V}}}\cong\widetilde{\mathcal{V}}\times(\mathbb{A}^{d+1})^n.$$

By Proposition 4.1, $\widetilde{\mathcal{V}}$ can be covered by $\{\widetilde{\mathcal{V}}'\}$ so that on each $\widetilde{\mathcal{V}}'\subset\widetilde{\mathcal{V}}$, the homomorphism

$$\beta^*\varphi:\beta^*\rho_*\mathscr{M}(\mathcal{D})\longrightarrow\beta^*\rho_*\mathscr{O}_{\mathcal{A}}(\mathcal{A})$$

can be diagonalized as

$$\begin{bmatrix}z_1&0&0&\cdots&0\\0&z_2&0&\cdots&0\end{bmatrix}\quad\text{with}\quad z_1|z_2\in\Gamma(\mathscr{O}_{\widetilde{\mathcal{V}}'}),$$

where z_1 and z_2 are both products of some smooth parameters corresponding to (the proper transforms of) the exceptional divisors.

Let $w_j^i\in\mathbb{A}^1$ for all $1\le i\le n$ and $0\le j\le d$. The equation $\widetilde{\Phi}=0$ is then equivalent to the following equations

$$z_1w_1^i=0,\quad z_2w_2^i=0,\quad 1\le i\le n. \tag{5.4}$$

Thus, $\widetilde{\mathcal{U}}$ has normal crossing singularities.

Since the main component of $\widetilde{\Phi}^{-1}(0)\cap\widetilde{\mathcal{V}}'$ consists of general points with non-vanishing z_1 and z_2, we conclude from (5.4) that the main component is defined by

$$w_1^i=w_2^i=0,\quad 1\le i\le n.$$

Thus, the main component of $\widetilde{\Phi}^{-1}(0)\cap\widetilde{\mathcal{V}}'$ is smooth.

Each of the remaining irreducible components X of $\widetilde{\Phi}^{-1}(0)\cap\widetilde{\mathcal{V}}'$ is contained in the proper transform of an exceptional divisor, so locally there exists a unique local parameter ξ (whose zero locus locally defines the proper transform of the given exceptional divisor) such that $X\subset\{\xi=0\}(\subset\mathscr{E}_{\widetilde{\mathcal{V}}})$. This ξ is either a factor of z_1 (hence z_2), or a factor of z_2 but not z_1. In the former case, (5.4) implies that X is simply given by $\xi=0$; there are no constraints on w_1^i or w_2^i. In the latter case, (5.4) also implies that X is given by

$$\xi=w_1^i=0,\quad 1\le i\le n.$$

Either way, X is smooth. □

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Next, we show that all the $\widetilde{\mathcal{U}}_{U,H}$ as in (5.3) glue to form the desired $\widetilde{M}_2(\mathbb{P}^n,d)$. Consider arbitrary small open $U_s/\mathcal{V}_s$, $s=1,2$, of $\overline{M}_2(\mathbb{P}^n,d)$, along with the morphism $f_{U_s,H_s}:U_s\to\mathcal{V}_s$, as in Section 2.1, satisfying $U_1\cap U_2\neq\emptyset$. For each step $\ell:=\mathsf{r}_i\mathsf{p}_j\mathsf{s}_t$ of the sequential blowup of $\mathfrak{M}_2^{\mathrm{div}}$, we denote by $\ell+1$ the step immediately after ℓ, and by $\widetilde{\mathfrak{M}}_2^{\mathrm{div},\ell}$ the corresponding blowup stack. Let $\widetilde{\mathcal{V}}_s^\ell=\mathcal{V}_s\times_{\mathfrak{M}_2^{\mathrm{div}}}\widetilde{\mathfrak{M}}_2^{\mathrm{div},\ell}$, $s=1,2$. In addition, we set $\widetilde{\mathfrak{M}}_2^{\mathrm{div},0}:=\widetilde{\mathfrak{M}}_2^{\mathrm{div}}$ and $\widetilde{\mathcal{V}}_s^0:=\mathcal{V}_s$.

LEMMA 5.2. *With notation as above, there exists an isomorphism*

$$(\widetilde{\mathcal{V}}_1^\ell\times_{\mathcal{V}_1;f_{U_1,H_1}}(U_1\cap U_2))/(U_1\cap U_2)\longrightarrow(\widetilde{\mathcal{V}}_2^\ell\times_{\mathcal{V}_2;f_{U_2,H_2}}(U_1\cap U_2))/(U_1\cap U_2)$$

between the above two blowups of $U_1\cap U_2$.

The statement of Lemma 5.2 would be trivial if there existed $H\in\check{\mathbb{P}}^n$ lying in the common intersection of $\mathbf{H}_U$ (cf. (1.1)) for all small $U\subset\overline{M}_2(\mathbb{P}^n,d)$, which would give rise to a natural global morphism from $\overline{M}_2(\mathbb{P}^n,d)$ to $\mathfrak{M}_2^{\mathrm{div}}$. Since such H does not exist, we introduce Lemma 5.2 to ensure that $\widetilde{\mathcal{U}}_{U,H}$ glue.

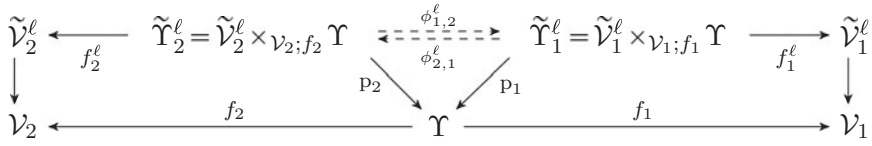
Proof of Lemma 5.2. For conciseness, in this proof, we write

$$\Upsilon=U_1\cap U_2,\quad \widetilde{\Upsilon}_s^\ell=\widetilde{\mathcal{V}}_s^\ell\times_{\mathcal{V}_s;f_{U_s,H_s}}\Upsilon,\quad f_s=f_{U_s,H_s}|_\Upsilon:\Upsilon\longrightarrow\mathcal{V}_s,\quad s=1,2.$$

In addition, we denote by

$$\mathrm{p}_s:\widetilde{\Upsilon}_s^\ell\longrightarrow\Upsilon\quad\text{and}\quad \mathrm{f}_s^\ell:\widetilde{\Upsilon}_s^\ell\longrightarrow\widetilde{\mathcal{V}}_s^\ell,\quad s=1,2,$$

the natural morphisms. We aim to show by induction over ℓ that there exist isomorphisms $\phi_{21}^\ell:\widetilde{\Upsilon}_1^\ell\to\widetilde{\Upsilon}_2^\ell$ and $\phi_{12}^\ell:\widetilde{\Upsilon}_2^\ell\to\widetilde{\Upsilon}_1^\ell$, fitting in the Cartesian diagram:



The base case ($\ell=0$) is when no blowup is exerted on U, which is trivial.

Assume the above diagram holds for the unique isomorphisms ϕ_{21}^ℓ and ϕ_{12}^ℓ after the ℓth step. For $s=1,2$

$$Z_s\subset\widetilde{\mathcal{V}}_s^\ell\quad\text{and}\quad \mathrm{q}_s:\widetilde{\Upsilon}_s^{\ell+1}=\widetilde{\Upsilon}_s^\ell\times_{\widetilde{\mathcal{V}}_s^\ell}\widetilde{\mathcal{V}}_s^{\ell+1}\longrightarrow\widetilde{\Upsilon}_s^\ell$$

be the center of the blowup $\widetilde{\mathcal{V}}_s^{\ell+1}/\widetilde{\mathcal{V}}_s^\ell$ and the natural morphism, respectively.

Notice that for every node e of C, the node-smoothing parameters $\zeta_e^{\mathcal{V}_1}$ and $\zeta_e^{\mathcal{V}_2}$ satisfy

$$\zeta_e^{\mathcal{V}_1}\circ f_1=0\quad\Longleftrightarrow\quad\zeta_e^{\mathcal{V}_2}\circ f_2=0\quad\text{on }\Upsilon.\tag{5.5}$$

In other words, $\zeta_e^{\mathcal{V}_1}\circ f_1$ and $\zeta_e^{\mathcal{V}_2}\circ f_2$ differ by a (multiplicative) unit on Υ.

Moreover, if the $(\ell+1)$th step is in (r_3), and the blowups $\widetilde{\mathcal{V}}_s^{\ell+1}/\widetilde{\mathcal{V}}_s^\ell$, $s=1,2$, are not trivial, then for every $z_s\in\Upsilon_s^\ell$ whose image in $\widetilde{\mathcal{V}}_s^\ell$ is contained in Z_s, by writing $(C,u)=\mathrm{p}_s(\mathrm{z}_s)$, we have $\deg(u|_F)\le2$, where F denotes the core of C. Let

$$\Upsilon':=\{(C,u)\in\Upsilon:\deg(u|_F)\le2\}\quad\subset\Upsilon.$$

This implies

$$(\mathrm{f}_s^\ell)^{-1}(Z_s)\subset\mathrm{p}_s^{-1}(\Upsilon'),\quad s=1,2.$$

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Note that for every $(C,u)\in\Upsilon'$ and irreducible component Σ of C, the divisors $u^{-1}(H_1)\cap\Sigma$ and $u^{-1}(H_2)\cap\Sigma$ are linearly equivalent, hence

$$\kappa_{ij}\circ f_1=0 \quad\Longleftrightarrow\quad \kappa_{ij}\circ f_2=0 \quad \text{on } \Upsilon'. \tag{5.6}$$

In other words, $\kappa_{ij}\circ f_1$ and $\kappa_{ij}\circ f_2$ differ by a (multiplicative) unit on Υ'.

Given $(C,u)\in\Upsilon$, let $\widehat{x}_1$ and $\widehat{x}_2$ be its images in $\mathfrak{M}_2^{\mathrm{div}}$ under f_1 and f_2, respectively; i.e. $\widehat{x}_1=(C,u^{-1}(H_1))$ and $\widehat{x}_2=(C,u^{-1}(H_2))$. As shown in Section 4, near the respective pullbacks of $\widehat{x}_1$ and $\widehat{x}_2$, the blowup centers of any step ℓ' of the sequential blowups are locally given by the *same* equations

$$\{\gamma_{\ell';i}=0 : i\in I_{\ell'}\}, \tag{5.7}$$

where $I_{\ell'}$ is an index set and each $\gamma_{\ell';i}$ is a quotient of monomials of the pullbacks of $\zeta_e^{\mathcal{V}_s}$ and/or κ_{ij}. Hence by (5.5) and (5.6), we have

$$(f_1^\ell\circ\phi_{1,2}^\ell)^{-1}(Z_1)=(f_2^\ell)^{-1}(Z_2). \tag{5.8}$$

For $s=1,2$, let N_s^ℓ be the union of all the irreducible components of $\widetilde{\Upsilon}_s^\ell$ whose images in $\widetilde{\mathcal{V}}_s^\ell$ under f_s^ℓ are contained in Z_s, and M_s^ℓ be the union of the remaining irreducible components of $\widetilde{\Upsilon}_s^\ell$. Then, we have

$$\widetilde{\Upsilon}_s^\ell=\mathrm{M}_s^\ell\cup\mathrm{N}_s^\ell,\quad \phi_{1,2}^\ell(\mathrm{M}_2^\ell)=\mathrm{M}_1^\ell,\quad \phi_{1,2}^\ell(\mathrm{N}_2^\ell)=\mathrm{N}_1^\ell.$$

We then set

$$\widetilde{\mathrm{M}}_s^{\ell+1}=\mathrm{q}_s^{-1}(\mathrm{M}_s^\ell),\quad \widetilde{\mathrm{N}}_s^{\ell+1}=\mathrm{q}_s^{-1}(\mathrm{N}_s^\ell).$$

By (5.8), we have

$$(f_1^\ell\circ\phi_{1,2}^\ell\circ\mathrm{q}_2)^{-1}(\mathrm{Z}_1)=(\mathrm{f}_2^\ell\circ\mathrm{q}_2)^{-1}(\mathrm{Z}_2).$$

Observe that the intersection of $(f_2^\ell\circ\mathrm{q}_2)^{-1}(\mathrm{Z}_2)$ and $\widetilde{\mathrm{M}}_2^{\ell+1}$ is a Cartier divisor of $\widetilde{\mathrm{M}}_2^{\ell+1}$, because the pullback of Z_2 to $\widetilde{\mathcal{V}}_2^{\ell+1}$ is the exceptional divisor, and the image of every irreducible component of M_2^ℓ under f_2^ℓ is not contained in Z_2. Therefore, the intersection of $(f_1^\ell\circ\phi_{1,2}^\ell\circ\mathrm{q}_2)^{-1}(\mathrm{Z}_1)$ and $\widetilde{\mathrm{M}}_2^{\ell+1}$ is a Cartier divisor of $\widetilde{\mathrm{M}}_2^{\ell+1}$. The universality of blowing up thus induces a unique morphism

$$f_{1,2}^{\ell+1}:\widetilde{\mathrm{M}}_2^{\ell+1}\longrightarrow\widetilde{\mathcal{V}}_1^{\ell+1} \tag{5.9}$$

lifting the restriction to $\widetilde{\mathrm{M}}_2^{\ell+1}$ of the morphism $f_1^\ell\circ\phi_{1,2}^\ell\circ\mathrm{q}_2$.

To extend $f_{1,2}^{\ell+1}$ to the entire $\widetilde{\Upsilon}_2^{\ell+1}$, note that

$$\widetilde{\mathrm{N}}_s^{\ell+1}=\mathrm{N}_s^\ell\times_{f_s^\ell,Z_s}\mathcal{E}_s^{\ell+1},\quad s=1,2,$$

where $\mathcal{E}_s^{\ell+1}$ denotes the exceptional divisor of $\widetilde{\mathcal{V}}_s^{\ell+1}/\widetilde{\mathcal{V}}_s^\ell$. Therefore, (5.5), (5.6), and the sentence containing (5.7) naturally induce an isomorphism

$$\phi_{1,2;\mathrm{N}}^{\ell+1}:\widetilde{\mathrm{N}}_2^{\ell+1}\xrightarrow{\sim}\widetilde{\mathrm{N}}_1^{\ell+1},$$

satisfying the condition that $f_1^{\ell+1}\circ\phi_{1,2;\mathrm{N}}^{\ell+1}$ matches (5.9) on $\widetilde{\mathrm{M}}_2^{\ell+1}\cap\widetilde{\mathrm{N}}_2^{\ell+1}$. In other words, we obtain

$$f_{1,2}^{\ell+1}:\widetilde{\Upsilon}_2^{\ell+1}\longrightarrow\widetilde{\mathcal{V}}_1^{\ell+1}$$

extending (5.9), whose restriction to $\widetilde{\mathrm{N}}_2^{\ell+1}$ equals $f_1^{\ell+1}\circ\phi_{1,2;\mathrm{N}}^{\ell+1}$.

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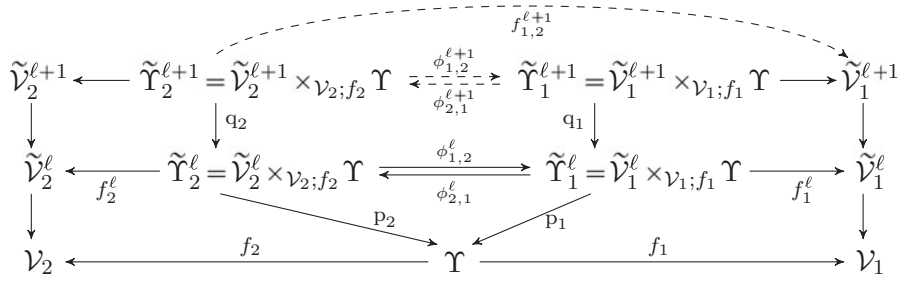
Then, the universality of pullbacks induces a unique morphism

$$\phi_{1,2}^{\ell+1} : \widetilde{\Upsilon}_2^{\ell+1} \longrightarrow \widetilde{\Upsilon}_1^{\ell+1}$$

that $f_{1,2}^{\ell+1}$ and $\phi_{1,2}^{\ell} \circ \mathrm{q}_2$ respectively factor through. By symmetry, we analogously obtain a unique morphism

$$\phi_{2,1}^{\ell+1} : \widetilde{\Upsilon}_1^{\ell+1} \longrightarrow \widetilde{\Upsilon}_2^{\ell+1}$$

that $\phi_{2,1}^{\ell} \circ \mathrm{q}_1$ factors through. In sum, we have the following commutative diagram:



Notice that the composition $\phi_{2,1}^{\ell+1} \circ \phi_{1,2}^{\ell+1}$ restricts to the identity morphism on $\widetilde{\mathrm{M}}_2^{\ell+1} \backslash (f_2^{\ell} \circ \mathrm{q}_2)^{-1}(Z_2)$, which is open and dense in $\widetilde{\mathrm{M}}_2^{\ell+1}$, hence

$$\phi_{1,2}^{\ell}|_{\widetilde{\mathrm{M}}_2^{\ell+1}} \quad \text{and} \quad \phi_{2,1}^{\ell}|_{\widetilde{\mathrm{M}}_1^{\ell+1}}$$

are inverse to each other. As

$$\phi_{1,2}^{\ell}|_{\widetilde{\mathrm{N}}_2^{\ell+1}} \quad \text{and} \quad \phi_{2,1}^{\ell}|_{\widetilde{\mathrm{N}}_1^{\ell+1}}$$

are, respectively, $\phi_{1,2;\mathrm{N}}^{\ell}$ and its inverse, we conclude that $\phi_{1,2}^{\ell}$ and $\phi_{2,1}^{\ell}$ are inverse to each other. This completes the induction. □

Lemma 5.2 implies that we can glue $\widetilde{\mathcal{U}}_{U,H}$ for all small open $U \subset \overline{M}_2(\mathbb{P}^n, d)$ and all $H \in \mathbf{H}_U$ together. The resulting Deligne–Mumford stack is denoted by

$$\widetilde{M}_2(\mathbb{P}^n, d).$$

The corollary below follows directly from Lemma 5.2.

COROLLARY 5.3. *For every small open $U \subset \overline{M}_2(\mathbb{P}^n, d)$ and every $H \in \mathbf{H}_U$, there exists a commutative diagram*

$$\begin{array}{ccc} \widetilde{\mathcal{U}}_{U,H} & \xrightarrow{\sim} & U \times_{\overline{M}_2(\mathbb{P}^n,d)} \widetilde{M}_2(\mathbb{P}^n, d) \\ & \searrow \quad \swarrow & \\ & U & \end{array}$$

where the horizontal arrow is an isomorphism of stacks over U, and the other two arrows are natural morphisms.

The stack $\widetilde{M}_2(\mathbb{P}^n, d)$ enjoys the following properties. Recall that the main component of $\widetilde{M}_2(\mathbb{P}^n, d)$ is denoted by $\widetilde{M}_2(\mathbb{P}^n, d)^{\mathrm{mc}}$.

THEOREM 5.4. *$\widetilde{M}_2(\mathbb{P}^n, d)$ has smooth irreducible components and normal crossing singularities. Moreover, for $d > 2$, $\widetilde{M}_2(\mathbb{P}^n, d)^{\mathrm{mc}}$ is of the expected dimension.*

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Proof. The first statement follows from Lemma 5.1. As for the dimension, note that $\dim \mathscr{E}_{\widetilde{\mathcal{V}}} = 3+d+(d+1)n$. Hence the dimension of the main component of $\widetilde{\mathcal{U}}$ is

$$3+d+(d+1)n-2n = d(n+1)-n+3$$

which is the virtual dimension of $\overline{M}_2(\mathbb{P}^n, d)$. □

When $d=2$, the interior of $\widetilde{M}_2(\mathbb{P}^n,2)^{\mathrm{mc}}$ consists of stable maps that are double covers of smooth rational curves, which are smooth and of dimension $2n+4$. Since the virtual dimension of $\overline{M}_2(\mathbb{P}^n,2)$ is $n+5$, the main component is of the wrong dimension unless $n=1$. The entire moduli can be treated by hand.

5.2 The desingularization of $\boldsymbol{\pi_*\mathfrak{f}^*\mathscr{O}_{\mathbb{P}^n}(k)}$

In this subsection, we study the pullback of the derived object $\mathbf{R}\pi_*\mathfrak{f}^*\mathscr{O}_{\mathbb{P}^n}(k)$ to $\widetilde{M}_2(\mathbb{P}^n,d)$. Recall that $(\pi,\mathfrak{f})$ is the universal family of $\overline{M}_2(\mathbb{P}^n,d)$; see (1.3).

Proposition 5.5. *For every $k\in\mathbb{Z}_{>0}$, the pullback of $\mathbf{R}\pi_*\mathfrak{f}^*\mathscr{O}_{\mathbb{P}^n}(k)$ to $\widetilde{M}_2(\mathbb{P}^n,d)^{\mathrm{mc}}$ is locally diagonalizable. Moreover, for $k=1$, the pullback of $\mathbf{R}\pi_*\mathfrak{f}^*\mathscr{O}_{\mathbb{P}^n}(1)$ to the entire $\widetilde{M}_2(\mathbb{P}^n,d)$ is locally diagonalizable.*

Proof. Given $k\in\mathbb{Z}_{>0}$ and $z=[C_0,u_0]\in\overline{M}_2(\mathbb{P}^n,d)$, we can choose a sufficiently small open $U\subset\overline{M}_2(\mathbb{P}^n,d)$ containing z, a hyperplane $H\in\check{P}^n$, and a degree k morphism $\psi_k:\mathbb{P}^n\longrightarrow\mathbb{P}^n$ such that

$$(C_0, u_0^{-1}\circ\psi_k^{-1}(H))\in\mathfrak{M}_2^{\mathrm{div}}.$$

Given a small chart $\mathcal{V}_{kd}\longrightarrow\mathfrak{M}_2^{\mathrm{div}}$ containing the above point, since U is assumed to be sufficiently small, we see that

$$f_k = f_{U,H,\psi_k}: U\longrightarrow\mathcal{V}_{kd},\quad [C,u]\mapsto(C,u^{-1}\circ\psi_k^{-1}(H)), \tag{5.10}$$

is well defined.

By Proposition 4.1 and Lemma 2.4, we see that the pullback of $\mathbf{R}\rho_*\mathscr{O}_{\mathcal{C}}(\mathcal{D})$ to the blowup $\widetilde{\mathcal{V}}_{kd}$ of $\mathcal{V}_{kd}$ is locally diagonalizable. Therefore, the pullback of $\mathbf{R}\pi_*\mathfrak{f}^*\mathscr{O}_{\mathbb{P}^n}(k)$ is locally diagonalizable on

$$\widetilde{\mathcal{U}}_{kd} := U\times_{f_k;\mathcal{V}_{kd}}\widetilde{\mathcal{V}}_{kd}. \tag{5.11}$$

When $k=1$, by setting ψ_1 to be the identity morphism, we see that f_1 is equal to $f_{U,H}$ of (1.2), and hence that $\widetilde{\mathcal{U}}_d=\widetilde{\mathcal{U}}_{U,H}$ is an open neighborhood of the preimage of z in $\widetilde{M}_2(\mathbb{P}^n,d)$. Since z is arbitrary, we obtain the latter statement of Proposition 5.5.

To justify the former statement of Proposition 5.5, we assume $k\geq 2$. Following the notation in the proof of Lemma 5.2, for each step $\ell:=\mathsf{r}_i\mathsf{p}_j\mathsf{s}_t$ of the sequential blowups of $\mathfrak{M}_2^{\mathrm{div}}$, we denote by $\ell-1$ the step immediately before ℓ, and by $\widetilde{\mathfrak{M}}_2^{\mathrm{div},\ell}$ the corresponding blowup stack. In addition, we set

$$\widetilde{\mathfrak{M}}_2^{\mathrm{div},0}:=\widetilde{\mathfrak{M}}_2^{\mathrm{div}},\quad \widetilde{\mathcal{V}}_{kd}^{\ell}:=\mathcal{V}_{kd}\times_{\mathfrak{M}_2^{\mathrm{div}}}\widetilde{\mathfrak{M}}_2^{\mathrm{div},\ell},\quad \widetilde{\mathcal{U}}_{kd}^{\ell}:=U\times_{f_k;\mathcal{V}_{kd}}\widetilde{\mathcal{V}}_{kd}^{\ell}.$$

Similarly, by writing $\ell=\mathsf{r}_i$ (respectively, $\ell=\mathsf{r}_i\mathsf{p}_j$), we consider ℓ as the last step of (r_i) (respectively, $(\mathsf{r}_i\mathsf{p}_j)$) that affects the local chart. Moreover, we denote by

$$\widetilde{\mathcal{U}}_{kd}^{\ell,\mathrm{mc}}\subset\widetilde{\mathcal{U}}_{kd}^{\ell}$$

the main component of $\widetilde{\mathcal{U}}_{kd}^{\ell}$, i.e. the irreducible component whose general points have smooth domain curves. Since the interior of the main components remains unchanged during the

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sequential blowups, we have

$$\mathrm{p}_k^{\ell-1,\ell}(\widetilde{\mathcal{U}}_{kd}^{\ell,\mathrm{mc}}) \subset \widetilde{\mathcal{U}}_{kd}^{\ell-1,\mathrm{mc}} \tag{5.12}$$

for any step ℓ, where $\mathrm{p}_k^{\ell-1,\ell}: \widetilde{\mathcal{U}}_{kd}^{\ell} \longrightarrow \widetilde{\mathcal{U}}_{kd}^{\ell-1}$ is the natural morphism.

For every step ℓ in $(\mathsf{r}_1\mathsf{p}_1)$-$(\mathsf{r}_1\mathsf{p}_4)$, we claim there exists an isomorphism $\phi_k^\ell: \widetilde{\mathcal{U}}_d^\ell \longrightarrow \widetilde{\mathcal{U}}_{kd}^\ell$, fitting in the following Cartesian diagram:

$$\begin{array}{ccccccc}
\widetilde{\mathcal{V}}_d^\ell & \xleftarrow{f_1^\ell} & \widetilde{\mathcal{U}}_d^\ell = \mathcal{U}\times_{f_1;\mathcal{V}_d}\widetilde{\mathcal{V}}_d^\ell & \overset{\phi_k^\ell}{\dashrightarrow} & \widetilde{\mathcal{U}}_{kd}^\ell = \mathcal{U}\times_{f_k;\mathcal{V}_{kd}}\widetilde{\mathcal{V}}_{kd}^\ell & \xrightarrow{f_k^\ell} & \widetilde{\mathcal{V}}_{kd}^\ell \\
\downarrow & & \searrow^{\mathrm{p}_1^\ell} \quad \swarrow_{\mathrm{p}_k^\ell} & & & & \downarrow \\
\mathcal{V}_d & \xleftarrow{f_1} & U & & \xrightarrow{f_k} & & \mathcal{V}_{kd}
\end{array}$$

where f_1^ℓ, p_1^ℓ, f_k^ℓ, and p_k^ℓ are, respectively, the natural morphisms.

The above claim follows from an inductive argument (over ℓ) similar to the proof of Lemma 5.2. The key point is as follows. For every $[C,u]\in U$, let (C,D_{kd}) and (C,D_d) be its images in $\mathfrak{M}_2^{\mathrm{div}}$ under f_k and f_1, respectively. On the one hand, the difference between D_{kd} and D_d is that on each irreducible component Σ of C, we have

$$|D_{kd}\cap\Sigma| = k\,|D_d\cap\Sigma|. \tag{5.13}$$

On the other hand, notice that the blowup centers in $(\mathsf{r}_1\mathsf{p}_1)$ to $(\mathsf{r}_1\mathsf{p}_4)$ are all determined by whether the weight of an irreducible component of a curve is zero or positive, regardless of the precise value of the weight. Therefore, for each step ℓ in $(\mathsf{r}_1\mathsf{p}_1)$ to $(\mathsf{r}_1\mathsf{p}_4)$, the pullbacks of the blowup centers via, respectively, $f_k^{\ell-1}\circ\phi_k^{\ell-1}$ and $f_1^{\ell-1}$ are identical on $\widetilde{\mathcal{U}}_d^{\ell-1}$. The remainder of the inductive argument utilizes the universalities of blowups and pullbacks, which is parallel to the proof of Lemma 5.2, hence is omitted.

Depending on the location of $z\in\overline{M}_2(\mathbb{P}^n,d)$, the blowups in $(\mathsf{r}_1\mathsf{p}_5)$ may affect $\widetilde{\mathcal{U}}_d^{\mathsf{r}_1\mathsf{p}_4}$, but they cannot affect $\widetilde{\mathcal{U}}_{kd}^{\mathsf{r}_1\mathsf{p}_4}$ because of (5.13). Although this still leads to a morphism $\widetilde{\mathcal{U}}_d^{\mathsf{r}_1\mathsf{p}_5}/U \longrightarrow \widetilde{\mathcal{U}}_{kd}^{\mathsf{r}_1\mathsf{p}_5}/U$, the blowup centers of (r_2) do not always match, so there may not exist any morphism of $\widetilde{\mathcal{U}}_d^{\mathsf{r}_2}/U$ to $\widetilde{\mathcal{U}}_{kd}^{\mathsf{r}_2}/U$. Nonetheless, we will show that $(\mathsf{r}_1\mathsf{p}_5)$ does not affect the main component of $\widetilde{\mathcal{U}}_d^{\mathsf{r}_1\mathsf{p}_4}$, as follows.

For every

$$\widetilde{z}\in\widetilde{\mathcal{U}}_d^{\mathsf{r}_1\mathsf{p}_4},$$

we denote by $y^{\mathsf{r}_1\mathsf{p}_j}$, $1\le j\le 4$, its images in $\widetilde{\mathcal{V}}_d^{\mathsf{r}_1\mathsf{p}_j}$, respectively. If $y^{\mathsf{r}_1\mathsf{p}_4}$ lies in some blowup centers of $(\mathsf{r}_1\mathsf{p}_5)$, then there are only two possibilities:

- either the core F of C does not contain any point of D_d, and there exists a unique point of D_d, say δ_1, such that $D(\Delta_{y^{\mathsf{r}_1\mathsf{p}_1}})$ as in (4.52) is equal to $\{\delta_1\}$, and for every $\delta\in D_d\backslash\{\delta_1\}$,
$$N_{[\delta]}\not\subset \Delta_{y^{\mathsf{r}_1\mathsf{p}_1}}\sqcup\Delta_{y^{\mathsf{r}_1\mathsf{p}_2}}\sqcup\Delta_{y^{\mathsf{r}_1\mathsf{p}_3}}\sqcup\Delta_{y^{\mathsf{r}_1\mathsf{p}_4}}; \tag{5.14}$$
- or $D_d\cap F$ is a singleton, whose unique element, denoted by δ_1, belongs to a separating bridge of F, and (5.14) holds for every $\delta\in D_d\backslash\{\delta_1\}$.

In either situation, the homomorphism φ, given by the $2\times d$ matrix $\big[c_{si}\zeta_{[\delta_i,a_s]}\big]_{1\le s\le 2,1\le i\le d}$ as in Proposition 2.10, pulls back to the following form on $\widetilde{\mathcal{V}}_d^{\mathsf{r}_1\mathsf{p}_4}$:

$$\begin{bmatrix}\xi_1 & 0\\ 0 & \xi_2\end{bmatrix}\begin{bmatrix}\widetilde{c}_{11} & g_{12} & \cdots & g_{1d}\\ \widetilde{c}_{21} & g_{22} & \cdots & g_{2d}\end{bmatrix},$$

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where $\widetilde{c}_{s1}$ are invertible, ξ_s are the products of certain local parameters corresponding to (the proper transforms of) the exceptional divisors obtained prior to $(\mathsf{r}_1\mathsf{p}_5)$, and g_{si} are regular functions satisfying

$$g_{si}(y^{\mathsf{r}_1\mathsf{p}_4}) = 0. \tag{5.15}$$

Here, (5.15) follows from (5.14) and the fact that $\zeta_{[\delta_i,a_s]}$ is divisible by $\zeta_{[\delta_i]}$.

As in the proof of Lemma 5.1, the pullbacks of $\rho_*\mathscr{M}(\mathcal{D}_i)$, $1 \le i \le d$, as well as $\rho_*\mathscr{O}_{\mathcal{A}}(\mathcal{A})$ can be trivialized so that

$$\mathscr{E}_{\mathcal{V}_d} \times_{\mathcal{V}_d} \widetilde{\mathcal{V}}_d^{\mathsf{r}_1\mathsf{p}_4} \cong \widetilde{\mathcal{V}}_d^{\mathsf{r}_1\mathsf{p}_4} \times (\mathbb{A}^{d+1})^n.$$

The main component $\widetilde{\mathcal{U}}_d^{\mathsf{r}_1\mathsf{p}_4,\mathrm{mc}}$ of $\widetilde{\mathcal{U}}_d^{\mathsf{r}_1\mathsf{p}_4}$ satisfies the following local equations:

$$\widetilde{c}_{s1} w_1^h + \sum_{i=2}^{d} g_{si} w_i^h = 0, \quad 1 \le s \le 2, \quad 1 \le h \le n, \tag{5.16}$$

where each $w_i^h \in \mathbb{A}^1$ is a free variable corresponding to the trivialization of the pullback of $\rho_*\mathscr{M}(\mathcal{D}_i)$.

Suppose $\widetilde{z} \in \widetilde{\mathcal{U}}_d^{\mathsf{r}_1\mathsf{p}_4,\mathrm{mc}}$, while $y^{\mathsf{r}_1\mathsf{p}_4}$ lies in some blowup centers of $(\mathsf{r}_1\mathsf{p}_5)$. Then, by (5.15) and (5.16), $\widetilde{z}$ satisfies

$$w_1^1 = w_1^2 = \cdots = w_1^n = 0.$$

However, the above equalities imply that the restriction of u to the irreducible component $\Sigma \subset C$ containing δ_1 is constant, because Σ does not contain any other point of D_d. This contradicts the fact that $\deg(u|_\Sigma) = 1$ (because $D_d \cap \Sigma = \{\delta_1\}$). Therefore, the image of $\widetilde{\mathcal{U}}_d^{\mathsf{r}_1\mathsf{p}_4,\mathrm{mc}}$ in $\widetilde{\mathcal{V}}_d^{\mathsf{r}_1\mathsf{p}_4}$ misses the blowup centers of $(\mathsf{r}_1\mathsf{p}_5)$, i.e. $\widetilde{\mathcal{U}}_d^{\mathsf{r}_1\mathsf{p}_5,\mathrm{mc}}/U$ is isomorphic to $\widetilde{\mathcal{U}}_d^{\mathsf{r}_1\mathsf{p}_4,\mathrm{mc}}/U$.

Since $\widetilde{\mathcal{U}}_{kd}^{\mathsf{r}_1\mathsf{p}_5,\mathrm{mc}}/U$ is isomorphic to $\widetilde{\mathcal{U}}_{kd}^{\mathsf{r}_1\mathsf{p}_4,\mathrm{mc}}/U$, the isomorphism $\phi_k^{\mathsf{r}_1\mathsf{p}_4}$ induces an isomorphism

$$\phi_k^{\mathsf{r}_1\mathsf{p}_5,\mathrm{mc}} : \widetilde{\mathcal{U}}_d^{\mathsf{r}_1\mathsf{p}_5,\mathrm{mc}}/U \longrightarrow \widetilde{\mathcal{U}}_{kd}^{\mathsf{r}_1\mathsf{p}_5,\mathrm{mc}}/U.$$

Moreover, just like in the first four phases of (r_1), the blowups of (r_2) to $(\mathsf{r}_3\mathsf{p}_2)$ have the same effect on $\widetilde{\mathcal{U}}_d^{\mathsf{r}_1\mathsf{p}_5,\mathrm{mc}}$ and $\widetilde{\mathcal{U}}_{kd}^{\mathsf{r}_1\mathsf{p}_5,\mathrm{mc}}$ because of (5.13). Taking (5.12) into consideration, we obtain an isomorphism

$$\phi_k^{\mathsf{r}_3\mathsf{p}_2,\mathrm{mc}} : \widetilde{\mathcal{U}}_d^{\mathsf{r}_3\mathsf{p}_2,\mathrm{mc}}/U \longrightarrow \widetilde{\mathcal{U}}_{kd}^{\mathsf{r}_3\mathsf{p}_2,\mathrm{mc}}/U.$$

Finally, note that $(\mathsf{r}_3\mathsf{p}_3)$ does not affect $\widetilde{\mathcal{U}}_{kd}^{\mathsf{r}_3\mathsf{p}_2,\mathrm{mc}}$, for otherwise $k = 2$, and $\widetilde{\mathcal{U}}_d^{\mathsf{r}_3\mathsf{p}_2,\mathrm{mc}}$ contains points whose images in $\widetilde{\mathcal{U}}_d^{\mathsf{r}_1\mathsf{p}_4,\mathrm{mc}}$ belong to some blowup centers of $(\mathsf{r}_1\mathsf{p}_5)$, which we have shown is impossible. In addition, $(\mathsf{r}_3\mathsf{p}_4)$ does not affect $\widetilde{\mathcal{U}}_{kd}^{\mathsf{r}_3\mathsf{p}_2,\mathrm{mc}}$ either, because we have shown that $(\mathsf{r}_1\mathsf{p}_5)$ does not affect $\widetilde{\mathcal{U}}_{kd}^{\mathsf{r}_1\mathsf{p}_4}$. In other words, $\widetilde{\mathcal{U}}_{kd}^{\mathsf{r}_3\mathsf{p}_4,\mathrm{mc}}/U$ is isomorphic to $\widetilde{\mathcal{U}}_{kd}^{\mathsf{r}_3\mathsf{p}_2,\mathrm{mc}}/U$. After all the sequential blowups terminate, we thus obtain a morphism

$$\phi_k^{\mathrm{mc}} : (U \times_{\overline{M}_2(\mathbb{P}^n,d)} \widetilde{M}_2(\mathbb{P}^n, d)^{\mathrm{mc}})/U \longrightarrow \widetilde{\mathcal{U}}_{kd}/U,$$

given by the composition of the following:

$$(U \times_{\overline{M}_2(\mathbb{P}^n,d)} \widetilde{M}_2(\mathbb{P}^n, d)^{\mathrm{mc}}) \xrightarrow{\sim} \widetilde{\mathcal{U}}_d^{\mathsf{r}_3\mathsf{p}_4,\mathrm{mc}} \longrightarrow \widetilde{\mathcal{U}}_d^{\mathsf{r}_3\mathsf{p}_2,\mathrm{mc}} \xrightarrow{\sim} \widetilde{\mathcal{U}}_{kd}^{\mathsf{r}_3\mathsf{p}_2,\mathrm{mc}} \xrightarrow{\sim} \widetilde{\mathcal{U}}_{kd}^{\mathsf{r}_3\mathsf{p}_4,\mathrm{mc}} \hookrightarrow \widetilde{\mathcal{U}}_{kd},$$

in which the first isomorphism is as in Corollary 5.3, and the second isomorphism is $\phi_k^{\mathsf{r}_3\mathsf{p}_2,\mathrm{mc}}$. As shown in the sentence containing (5.11), the pullback of $\mathbf{R}\pi_*\mathfrak{f}^*\mathscr{O}_{\mathbb{P}^n}(k)$ is locally diagonalizable on $\widetilde{\mathcal{U}}_{kd}$, hence is on $U \times_{\overline{M}_2(\mathbb{P}^n,d)} \widetilde{M}_2(\mathbb{P}^n, d)^{\mathrm{mc}}$. Since U is arbitrary, the proof of Proposition 5.5 is complete. □

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Corollary 5.6. *Let* $(\widetilde{\pi}^{\mathrm{mc}}, \widetilde{\mathfrak{f}}^{\mathrm{mc}})$ *be the pullback to* $\widetilde{M}_2(\mathbb{P}^n, d)^{\mathrm{mc}}$ *of the universal family* $(\pi, \mathfrak{f})$. *Then the direct image sheaf* $(\widetilde{\pi}^{\mathrm{mc}})_*(\widetilde{\mathfrak{f}}^{\mathrm{mc}})^*\mathscr{O}_{\mathbb{P}^n}(k)$ *is locally free for all* $k \geq 1$. *It is of rank* $kd-1$ *if* $d>2$.

Proof. Given small open $U/\mathcal{V}$ of $\overline{M}_2(\mathbb{P}^n, d)$ and $H \in \mathbf{H}_U$, let $\widetilde{\mathcal{U}} = \widetilde{\mathcal{U}}_{U,H}$ be as in (5.3). Recall that $\mathcal{X}/U$ is the universal family over U, $(\mathcal{C}, \mathcal{D})$ is the universal family over $\mathcal{V}$, and $\mathcal{A} \subset \mathcal{C}$ is the auxiliary divisor as in (2.2). We set

$$\widetilde{\mathcal{X}}' = \mathcal{X} \times_U \widetilde{\mathcal{U}} = \mathcal{C} \times_{\mathcal{V}} \widetilde{\mathcal{U}}, \quad \widetilde{\mathcal{D}}' = \mathcal{D} \times_U \widetilde{\mathcal{U}} = \mathcal{D} \times_{\mathcal{V}} \widetilde{\mathcal{U}},$$

where the second equality in each set of identities follows from the universality of $\mathfrak{M}_2^{\mathrm{div}}$. Let $\widetilde{\mathcal{A}}' = \mathcal{A} \times_{\mathcal{V}} \widetilde{\mathcal{U}}$ and $\widetilde{\mathscr{L}}' = \mathscr{O}_{\widetilde{\mathcal{X}}'}(\widetilde{\mathcal{D}}')$.

The same argument as is used in the proof of [HL10, Theorem 2.11], particularly the sentence containing [HL10, (5.22)], applies to the current situation verbatim. More precisely, the Cartesian diagram

$$\begin{array}{ccc} \widetilde{\mathcal{X}}' & \xrightarrow{\tilde{\beta}} & \mathcal{C} \\ {\scriptstyle \tilde{\pi}'}\downarrow & & \downarrow{\scriptstyle \rho} \\ \widetilde{\mathcal{U}} & \xrightarrow{\beta} & \mathcal{V} \end{array}$$

along with the cohomology and base change theorem, gives the following Cartesian diagram.

$$\begin{array}{ccc} \tilde{\pi}'_*\widetilde{\mathscr{L}}'(\widetilde{\mathcal{A}}') & \xrightarrow{\tilde{\varphi}} & \tilde{\pi}'_*\widetilde{\mathscr{O}}'_{\widetilde{\mathcal{A}}'}(\widetilde{\mathcal{A}}') \\ \wr\downarrow & & \wr\downarrow \\ \beta^*\rho_*\mathscr{L}(\mathcal{A}) & \xrightarrow{\beta^*\varphi} & \beta^*\rho_*\mathscr{O}_{\mathcal{A}}(\mathcal{A}) \end{array}$$

In the diagram above, the two vertical arrows are isomorphisms, and $\tilde{\varphi}$ is the evaluation homomorphism. Thus, investigating $\tilde{\varphi}$ is equivalent to investigating the pullback of φ to $\widetilde{\mathcal{U}}$. Therefore, in order to show that $(\widetilde{\pi}^{\mathrm{mc}})_*(\widetilde{\mathfrak{f}}^{\mathrm{mc}})^*\mathscr{O}_{\mathbb{P}^n}(k)$ is locally free, it suffices to show that the pullback of $\pi_*\mathfrak{f}^*\mathscr{O}_{\mathbb{P}^n}(k)$ to $\widetilde{M}_2(\mathbb{P}^n, d)^{\mathrm{mc}}$ is locally free, which follows immediately from Propositions 5.5 and 2.8.

As for the rank, when $d>2$, notice that near a general point of $\overline{M}_2(\mathbb{P}^n, d)^{\mathrm{mc}}$, the matrix of φ, whose size is $2 \times kd$, is of rank 2. Thus, $\ker\varphi$ is of rank $kd-2$. So, by Lemma 2.4, $\tilde{\pi}'_*\widetilde{\mathscr{L}}'$ is of rank $kd-1$. □

5.3 The construction of $\widetilde{M}_2(\mathbb{P}^n, d; k)$

For every $k \in \mathbb{Z}_{>0}$, following the same argument as in Lemma 5.2, we observe that the pullbacks $\widetilde{\mathcal{U}}_{kd}$ as in (5.11) can also be glued together to form a Deligne–Mumford stack, denoted by

$$\widetilde{M}_2(\mathbb{P}^n, d; k).$$

For $k>1$, $\widetilde{M}_2(\mathbb{P}^n, d; k)$ is generally not as smooth as $\widetilde{M}_2(\mathbb{P}^n, d; 1) = \widetilde{M}_2(\mathbb{P}^n, d)$; nonetheless, as far as the local diagonalization of $\mathbf{R}\pi_*\mathfrak{f}^*\mathscr{O}_{\mathbb{P}^n}(k)$ is concerned, $\widetilde{M}_2(\mathbb{P}^n, d; k)$ behaves more nicely than $\widetilde{M}_2(\mathbb{P}^n, d)$.

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Proposition 5.7. *For every $k\in\mathbb{Z}_{>0}$, there exist a Deligne–Mumford stack $\widetilde{M}_2(\mathbb{P}^n,d;k)$ and a morphism $\widetilde{M}_2(\mathbb{P}^n,d;k)\longrightarrow\overline{M}_2(\mathbb{P}^n,d)$ such that for every small open $U\subset\overline{M}_2(\mathbb{P}^n,d)$, $H\in\check{\mathbb{P}}^n$, and degree k morphism $\psi_k:\mathbb{P}^n\longrightarrow\mathbb{P}^n$ satisfying the condition that f_{U,H,ψ_k} as in (5.10) is well defined, we have the following commutative diagram.*

$$\begin{array}{ccc} U\times_{f_{U,H,\psi_k},\mathfrak{M}_2^{\mathrm{div}}}\widetilde{\mathfrak{M}}_2^{\mathrm{div}} & \xrightarrow{\ \sim\ } & U\times_{\overline{M}_2(\mathbb{P}^n,d)}\widetilde{M}_2(\mathbb{P}^n,d;k) \\ & \searrow \quad \swarrow & \\ & U & \end{array}$$

In the diagram above, the horizontal arrow is an isomorphism (of stacks over U), and the other two arrows are natural morphisms.

Moreover, the pullback of the derived object $\mathbf{R}\pi_*\mathfrak{f}^*\mathscr{O}_{\mathbb{P}^n}(k)$ to $\widetilde{M}_2(\mathbb{P}^n,d;k)$ becomes locally diagonalizable.

Furthermore, for every irreducible component N of $\widetilde{M}_2(\mathbb{P}^n,d;k)$, with $(\widetilde{\pi}_N,\widetilde{\mathfrak{f}}_N)$ denoting the pullback to N of the universal family (1.3), the direct image sheaf $(\widetilde{\pi}_N)_*(\widetilde{\mathfrak{f}}_N)^*\mathscr{O}_{\mathbb{P}^n}(k)$ is locally free. It is of rank $kd-1$ if $d>2$ and the general points of N have smooth domain curves.

Proof. The existence of $\widetilde{M}_2(\mathbb{P}^n,d;k)$ follows from a line-by-line repetition of the proof of Lemma 5.2, and hence we omit the detail. The local diagonalizability of the pullback of $\mathbf{R}\pi_*\mathfrak{f}^*\mathscr{O}_{\mathbb{P}^n}(k)$ to $\widetilde{M}_2(\mathbb{P}^n,d;k)$ follows directly from Proposition 4.1. The proof of the remaining statements is parallel to the proof of Corollary 5.6, and hence is omitted. □

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Conflicts of interest

None.

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