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Wave dispersion derived from the square-root Klein–Gordon–Poisson system

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Abstract. Recently, there has been great interest around quantum relativistic models for plasmas. In particular, striking advances have been obtained by means of the Klein–Gordon–Maxwell system, which provides a first-order approach to the relativistic regimes of quantum plasmas. The Klein–Gordon–Maxwell system provides a reliable model as long as the plasma spin dynamics is not a fundamental aspect, to be addressed using more refined (and heavier) models involving the Pauli–Schrödinger or Dirac equations. In this work, a further simplification is considered, tracing back to the early days of relativistic quantum theory. Namely, we revisit the square-root Klein–Gordon–Poisson system, where the positive branch of the relativistic energy–momentum relation is mapped to a quantum wave equation. The associated linear wave propagation is analyzed and compared with the results in the literature. We determine physical parameters where the simultaneous quantum and relativistic effects can be noticeable in weakly coupled electrostatic plasmas.

1. Introduction

Recently, there has been much interest in the analysis of relativistic effects in quantum plasmas, due to potential applications in the realm of multi-petawatt lasers and dense astrophysical objects like white dwarfs and neutron stars. Promising new models have appeared, like in weakly relativistic expansions including spin dynamic effects (Asenjo et al. 2012), hydrodynamic equations derived from covariant quantum kinetics (Zhu and Ji 2010), collective Klein–Gordon–Maxwell models (Eliasson and Shukla 2012), quantum relativistic multi-stream approaches (Haas et al. 2012a, b), quantum kinetic theories derived from the Klein–Gordon–Maxwell system (Mendonça 2011) as well as the quantum plasmadynamics approach (Melrose 2008). There is also a recent study on Landau damping in relativistic quantum kinetics of degenerate plasma (Zhu and Ji 2012).

In the present communication, we revisit the semi-relativistic quantum kinetic model introduced in Aki et al. (2008), which originates from the simplest possible quantum relativistic theory, based on the square-root Klein–Gordon equation (Bjorken and Drell 1964). Here we explore the implications of such a model, with respect to plasma behavior, i.e. wave propagation. We are aware of the intrinsic limitations of the square-root Klein–Gordon approach, to be detailed in the continuation. However, since it is the simplest approach, it is worth discussing in some detail, as a benchmark for more sophisticated analysis. The resulting robust modeling, not too expensive analytically or numerically, can then produce useful insights into complex quantum relativistic phenomena. Also, the square-root Klein–Gordon equation has had applications e.g. in the study of subbarrier relativistic effects in heavy nucleons penetration coefficients (Anchishkin 1997).

This work is organized as follows. In Sec. 2, we rederive the quantum kinetic theory associated with the square-root Klein–Gordon–Poisson system, as already done in Aki et al. (2008). A distinctive feature of our procedure is that here the evolution equation satisfied by the non-covariant Wigner function is set into a manifestly Schrödinger-like form. In Sec. 3, the dispersion relation for linear electrostatic waves is obtained. A new quantum-modified relativistic factor is then identified. An application is devised for a mono-energetic beam equilibrium. Physically relevant parameters are found, where the joint quantum and relativistic effects significantly alter the wave propagation. In addition, the underlying validity conditions and limitations of the model are considered. Section 4 compares the results from our simplified model with more accurate approaches existing in the literature. Section 5 has the concluding remarks.

2. Square-root Klein–Gordon–Poisson system and Wigner transform

Our starting point is the Hamiltonian

$$H = m c^2 \left(1 + \frac{p^2}{m^2 c^2}\right)^{1/2} - e\,\phi, \tag{2.1}$$

where $-e$ and m are, respectively, the electron charge and rest mass, c is the speed of light, $\mathbf{p}$ is the electron momentum (with $p = |\mathbf{p}|$) and $\phi = \phi(\mathbf{r}, t)$ is the electrostatic potential at position $\mathbf{r}$ and time t. Equation (2.1)

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is consistent with the energy–momentum relation

$$(H + e\phi)^2 = p^2c^2 + m^2c^4, \tag{2.2}$$

disregarding the negative energy solutions associated with the positron sector of Hilbert space. Assuming $i\hbar\partial\psi/\partial t = H\psi$ and applying the quantization rules $\mathbf{p} \to -i\hbar\nabla$ and $E \to i\hbar\partial/\partial t$, one obtains

$$i\hbar\frac{\partial\psi}{\partial t} = \left[mc^2\left(1 - \frac{\hbar^2\nabla^2}{m^2c^2}\right)^{1/2} - e\phi\right]\psi, \tag{2.3}$$

where $\hbar$ is Planck's constant divided by 2π and $\psi = \psi(\mathbf{r}, t)$ is the wave function.

The square-root Klein–Gordon equation (2.3) was proposed in the early days of relativistic quantum mechanics (Bjorken and Drell 1964), but soon abandoned in favor of the Klein–Gordon and Dirac equations, for particles with spin zero and one-half, respectively. The main drawback with (2.3) is that time and space appear in a non-symmetric way in it, spoiling manifest covariance. Moreover, the square-root operator poses analytic difficulties, since it needs to be implemented using Fourier-transformed variables along with an infinite power-series expansion. In other words, one is obliged to deal with a non-local operator. The introduction of electromagnetic fields in a relativistic invariant manner is hence apparently an unfeasible task within the framework of the square-root Klein–Gordon equation. Indeed, using the minimal coupling scheme one would deal with infinite powers of the quantity $\mathbf{p} + e\mathbf{A}$, where $\mathbf{A} = \mathbf{A}(\mathbf{r}, t)$ is the vector potential. Although in principle solvable up to arbitrary desired order, this implies a troublesome problem due to non-commutativeness of momentum and position. In addition, one can still wonder about alternative schemes to interpret the square-root operator in the presence of magnetic fields (Gill and Zachary 2005), at the price of some lost of simplicity.

Evidently, (2.1) does not take into account retardation effects nor space-time curvature, besides magnetic fields or spin. Hence, we are dealing with a semi-relativistic model. Not surprisingly, similar semi-relativistic models are popular in the treatment of boson stars (Michelangeli and Schlein 2012), where N-body bosons systems interact gravitationally, without the need of general relativity nor magnetic forces to be included. The same modeling but also for fermionic systems was treated in Lieb and Thirring (1984) and Fröhlich and Lenzmann (2007), in connection with the analysis of the gravitational collapse of white dwarfs and the related Chandrasekhar limit. Using the same reasoning, the Hamiltonian (2.1) can be seriously considered as the starting point for electrostatic relativistic quantum plasmas – hence excluding any magnetic interaction. Moreover, we will observe that the practical implementation of the square-root operator in the context of quantum kinetic theory poses no extra difficulties, in comparison with the already existing non-local potential term in the quantum Vlasov equation. Since quantum plasma theory is usually more concerned with kinetic formulations than the wave equations themselves, in this respect one sees no profound reason to discard the semi-relativistic formulation. Finally, note that even within the electron sector of Hilbert space and in the electrostatic approximation one has an extra $\propto \nabla \cdot \mathbf{E}$ Darwin term in the weakly relativistic approximation to the Dirac equation (Zawadzki 2005). This term, related to *Zitterbewegung* and the associated enlarged expected value of the quadratic displacement, can be shown to be of higher order and will be ignored in what follows.

After all, the motivation in considering the square-root Klein–Gordon model is in its simplicity. Indeed, in the case of electrostatic quantum plasma, it offers the simplest conceivable method to incorporate relativistic effects. In particular, the corresponding kinetic theory can be constructed from a non-covariant Wigner function, in contrast to the Klein–Gordon case which deserves a Wigner transform in both space and time (Mendonça 2011). This follows since (2.3) clearly breaks down the symmetry between space and time variables. From one side, this is a further limitation, since any relativistic theory should be preferably covariant. From the other side, it is conceptually useful to have a relativistic model which can be directly comparable to the non-relativistic version (the Wigner–Poisson system) in quantum plasmas. In this way, we have a benchmark model, allowing a zeroth-order understanding of relativistic phenomena in quantum plasma[†].

Equation (2.3) is coupled with Poisson's equation

$$\nabla^2\phi = \frac{e}{\varepsilon_0}(|\psi|^2 - n_0), \tag{2.4}$$

where ε_0 is the vacuum permittivity. We suppose a jellium plasma with uniform background ionic density n_0 to ensure global charge neutrality. The system (2.3)–(2.4) can be entitled the square-root Klein–Gordon–Poisson system. The normalization $\int d\mathbf{r}|\psi|^2 = N =$ number of particles of the system is used.

It should be noted that a more general version can be encountered in the literature, allowing for mixed states (Aki et al. 2008). In this case, one has a countable set of square-root Klein–Gordon equations for the quantum statistical ensemble, with self-consistent scalar field mediated by Poisson's equation. Our attempt here is to address the plasma aspects of the model, specially the wave dispersion analysis. For these issues, a pure state ensemble (a single wave function) ψ is sufficient, at least to begin with.

It is useful to write our model in terms of a kinetic theory, so as to provide a relativistic version of the Wigner–Poisson system. Following (Aki et al. 2008), we Wigner transform (2.3) by means of the non-covariant

[†] In Mendonça (2011), the Klein–Gordon–Maxwell system was referred to as a 'first- order solution for a very difficult problem'. In this sense, the square-root version may be named a 'zeroth-order solution'.

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Wigner function (Wigner 1932):

$$f(\mathbf{p},\mathbf{r}) = \frac{1}{(2\pi\hbar)^3}\int d\mathbf{r}'\,\psi^*\left(\mathbf{r}+\frac{\mathbf{r}'}{2}\right) \times \exp\left(\frac{i\,\mathbf{p}\cdot\mathbf{r}'}{\hbar}\right)\psi\left(\mathbf{r}-\frac{\mathbf{r}'}{2}\right). \quad (2.5)$$

In (2.5) and henceforth, for the sake of notation, the explicit time dependence is omitted unless necessary.

For completeness, we rederive the evolution equation satisfied by the Wigner function (Aki et al. 2008). Consider the Fourier transform in space along with its inverse

$$\hat{\psi}(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}}\int d\mathbf{r}\,\psi(\mathbf{r})\,e^{-i\,\mathbf{k}\cdot\mathbf{r}},$$
$$\psi(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}}\int d\mathbf{k}\,\hat{\psi}(\mathbf{k})\,e^{i\,\mathbf{k}\cdot\mathbf{r}}, \quad (2.6)$$

with associated wave vector $\mathbf{k}$. One finds

$$f(\mathbf{p},\mathbf{r}) = \frac{1}{(2\pi\hbar)^3}\int d\mathbf{k}\,\hat{\psi}^*\left(\frac{\mathbf{p}}{\hbar}+\frac{\mathbf{k}}{2}\right)e^{-i\,\mathbf{k}\cdot\mathbf{r}}\hat{\psi}\left(\frac{\mathbf{p}}{\hbar}-\frac{\mathbf{k}}{2}\right). \quad (2.7)$$

Momentarily, only the kinetic energy term will be considered since the scalar potential contribution to $\partial f/\partial t$ is given in terms of a well-known pseudo-differential operator (Haas 2011). In this context, one gets

$$i\,\hbar\frac{\partial\hat{\psi}}{\partial t} = m\,c^2\left(1+\frac{\hbar^2k^2}{m^2c^2}\right)^{1/2}\hat{\psi},$$
$$i\,\hbar\frac{\partial\hat{\psi}^*}{\partial t} = -m\,c^2\left(1+\frac{\hbar^2k^2}{m^2c^2}\right)^{1/2}\hat{\psi}^*, \quad k=|\mathbf{k}|. \quad (2.8)$$

Note that (2.8) provides a way to interpret the square-root operator, upon its action on dual space. The inverse Fourier transform is an integral – in this sense the square-root operator acts in a non-local way.

Equations (2.7) and (2.8) yield

$$\frac{\partial f}{\partial t} = -\frac{m\,c^2}{i\,\hbar(2\pi\hbar)^3}\int d\mathbf{k}\,\hat{\psi}^*\left(\frac{\mathbf{p}}{\hbar}+\frac{\mathbf{k}}{2}\right)e^{-i\,\mathbf{k}\cdot\mathbf{r}}\hat{\psi}\left(\frac{\mathbf{p}}{\hbar}-\frac{\mathbf{k}}{2}\right) \times\left[\left(1+\frac{1}{m^2c^2}\left(\mathbf{p}+\frac{\hbar\,\mathbf{k}}{2}\right)^2\right)^{1/2} -\left(1+\frac{1}{m^2c^2}\left(\mathbf{p}-\frac{\hbar\,\mathbf{k}}{2}\right)^2\right)^{1/2}\right]. \quad (2.9)$$

To proceed, consider the identity

$$\hat{\psi}^*\left(\frac{\mathbf{p}}{\hbar}+\frac{\mathbf{k}}{2}\right)\hat{\psi}\left(\frac{\mathbf{p}}{\hbar}-\frac{\mathbf{k}}{2}\right) = \hbar^3\int d\mathbf{r}f(\mathbf{p},\mathbf{r})\,e^{i\,\mathbf{k}\cdot\mathbf{r}} \quad (2.10)$$

following from (2.7). From (2.9) and (2.10) and restoring the scalar potential term, we finally obtain the Schrödinger-like equation

$$i\hbar\frac{\partial f}{\partial t} = \mathscr{H}[f], \quad (2.11)$$

where $\mathscr{H}[f]$ is a functional defined by

$$\mathscr{H}[f] = m\,c^2\int\frac{d\mathbf{p}'\,d\mathbf{r}'}{(2\pi\hbar)^3}f(\mathbf{p},\mathbf{r}')\exp\left[\frac{i\,\mathbf{p}'\cdot(\mathbf{r}-\mathbf{r}')}{\hbar}\right] \times\left[\left(1+\frac{1}{m^2c^2}\left(\mathbf{p}+\frac{\mathbf{p}'}{2}\right)^2\right)^{1/2} -\left(1+\frac{1}{m^2c^2}\left(\mathbf{p}-\frac{\mathbf{p}'}{2}\right)^2\right)^{1/2}\right] + e\int\frac{d\mathbf{p}'\,d\mathbf{r}'}{(2\pi\hbar)^3}f(\mathbf{p}',\mathbf{r})\exp\left[\frac{i\,(\mathbf{p}-\mathbf{p}')\cdot\mathbf{r}'}{\hbar}\right] \times\left[\phi\left(\mathbf{r}+\frac{\mathbf{r}'}{2}\right)-\phi\left(\mathbf{r}-\frac{\mathbf{r}'}{2}\right)\right]. \quad (2.12)$$

A very detailed derivation of the non-local potential energy term can be found e.g. in Haas (2011). The distinctive feature of the relativistic extension is the extra non-locality also appearing in the kinetic energy term. Coupling with Poisson's equation in the form

$$\nabla^2\phi = \frac{e}{\varepsilon_0}\left(\int d\mathbf{p}f(\mathbf{p},\mathbf{r})-n_0\right) \quad (2.13)$$

gives a quantum relativistic generalization of the Vlasov–Poisson system (Aki et al. 2008).

In the formal non-quantum ($\hbar\to 0$) limit, it is easy to obtain

$$\frac{\partial f}{\partial t}+\frac{\mathbf{p}\cdot\nabla f}{\gamma m}+e\nabla\phi\cdot\frac{\partial f}{\partial\mathbf{p}} = 0, \quad (2.14)$$

which is the relativistic Vlasov equation, where $\gamma = [1+p^2/(m^2c^2)]^{1/2}$. Here, evidently '$\hbar\to 0$' is just a shorthand notation, denoting the appropriate limit when quantum diffraction is negligible. On the same footing, the formal non-relativistic ($1/c\to 0$) limit yields

$$\frac{\partial f}{\partial t}+\frac{\mathbf{p}\cdot\nabla f}{m}-\frac{e}{i\hbar}\int\frac{d\mathbf{p}'\,d\mathbf{r}'}{(2\pi\hbar)^3}f(\mathbf{p}',\mathbf{r})\exp\left[\frac{i\,(\mathbf{p}-\mathbf{p}')\cdot\mathbf{r}'}{\hbar}\right] \times\left[\phi\left(\mathbf{r}+\frac{\mathbf{r}'}{2}\right)-\phi\left(\mathbf{r}-\frac{\mathbf{r}'}{2}\right)\right] = 0, \quad (2.15)$$

which is the well-known non-relativistic quantum Vlasov equation.

Observe that starting with the Klein–Gordon equation, one arrives at an equation whose non-relativistic limit is *not* the usual quantum Vlasov equation (see (11) of Mendonça 2011). The reason is that in a covariant formalism the time is also Wigner transformed. It is useful to have an alternative tool allowing a more straightforward comparison with the well-known non-relativistic theory. Furthermore, one may argue that the extra non-local contribution in (2.12) poses no fundamental difficulties, since it can be handled basically in the same way as the potential energy term. It is interesting to note that the relativistic quantum Vlasov equation satisfied by the non-covariant Wigner equation shows a perfect symmetry between momentum and coordinate

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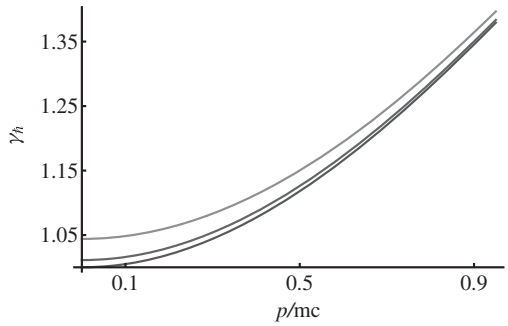


Figure 1. Quantum-modified relativistic factor from (3.2), assuming $\mathbf{k} \parallel \mathbf{p}$, with $\hbar k = 0$ (bottom curve), $\hbar k = 0.3mc$ (middle curve) and $\hbar k = 0.6mc$ (top curve).

variables, in the same sense as $\mathscr{H}[f]$ has both the kinetic and potential terms expressed as integrals.

3. Wave dispersion

The main concern of the present communication is about the linear wave propagation implied by the system (2.11) and (2.13). Linearizing around the equilibrium $f = f_0(\mathbf{p}), \phi = 0$, assuming plane wave perturbations proportional to $\exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)]$ and following the standard procedure, one finds the dielectric function

$$\varepsilon(\omega,\mathbf{k}) = 1 - \frac{m\omega_p^2}{n_0\hbar k^2}\int d\mathbf{p}\,\frac{f_0(\mathbf{p}+\hbar\mathbf{k}/2) - f_0(\mathbf{p}-\hbar\mathbf{k}/2)}{\mathbf{k}\cdot\mathbf{p}/(\gamma_\hbar m) - \omega} = 0, \tag{3.1}$$

where $\omega_p = [n_0e^2/(m\varepsilon_0)]^{1/2}$ is the plasma frequency and where

$$\gamma_\hbar = \frac{\hbar\,\mathbf{k}\cdot\mathbf{p}}{m^2c^2}\left[\left(1 + \frac{1}{m^2c^2}\left(\mathbf{p} + \frac{\hbar\mathbf{k}}{2}\right)^2\right)^{1/2} - \left(1 + \frac{1}{m^2c^2}\left(\mathbf{p} - \frac{\hbar\mathbf{k}}{2}\right)^2\right)^{1/2}\right]^{-1} \tag{3.2}$$

is a quantum-modified relativistic factor (which is different from the one in Aki et al. 2008). The semi-quantum and semi-relativistic expansions are, respectively,

$$\gamma_\hbar = \gamma + \frac{\hbar^2}{8\,m^2c^2\gamma^3}\left(k^2 + \frac{|\mathbf{k}\times\mathbf{p}|^2}{m^2c^2}\right) + O(\hbar^4) \tag{3.3}$$

and

$$\gamma_\hbar = 1 + \frac{p^2 + \hbar^2k^2/4}{2\,m^2c^2} + O(1/c^4). \tag{3.4}$$

We see that the contribution of the quantum recoil is for a larger $\gamma_\hbar$ and consequently in favor of a larger relativistic mass, as confirmed in Fig. 1.

Expanding up to the leading relativistic correction the result is

$$\varepsilon(\omega,\mathbf{k}) = 1 - \frac{m\omega_p^2}{n_0\hbar k^2}\int d\mathbf{p}\left[\frac{f_0(\mathbf{p}+\hbar\mathbf{k}/2) - f_0(\mathbf{p}-\hbar\mathbf{k}/2)}{\mathbf{k}\cdot\mathbf{p}/m - \omega}\right] \times\left[1 + \frac{\mathbf{k}\cdot\mathbf{p}\,(p^2+\hbar^2k^2/4)}{2\,m^3c^2(\mathbf{k}\cdot\mathbf{p}/m - \omega)}\right]. \tag{3.5}$$

In (3.5), the $O(1/c^4)$ contribution is disregarded. In practice, the weakly relativistic approximation may be quite useful for calculations since the resonances in (3.1) are not simple poles. The same applies to the purely relativistic and non-quantum case too, but the quantum situation is worse due to the finite difference and nonlinear form of $\gamma_\hbar$.

In this work, we apply a further approximation, namely the simultaneous weakly quantum and weakly relativistic version of the dielectric function. This amounts to keeping only the leading quantum recoil correction in (3.5) (also disregarding the joint quantum and relativistic perturbations which are $O(\hbar^2/c^2)$). Assuming decaying boundary conditions and integrating by parts, the result is

$$\varepsilon(\omega,\mathbf{k}) = 1 - \frac{\omega_p^2}{n_0}\int\frac{d\mathbf{p}f_0(\mathbf{p})}{(\omega - \mathbf{k}\cdot\mathbf{p}/m)^2} - \frac{\hbar^2k^4\omega_p^2}{4m^2n_0}\int\frac{d\mathbf{p}f_0(\mathbf{p})}{(\omega - \mathbf{k}\cdot\mathbf{p}/m)^4} - \frac{\omega_p^2}{2n_0m^2c^2k^2}\int\frac{d\mathbf{p}f_0(\mathbf{p})}{(\omega - \mathbf{k}\cdot\mathbf{p}/m)^3} \times\left[2(\mathbf{k}\cdot\mathbf{p})^2\left(\frac{\mathbf{k}\cdot\mathbf{p}}{m} - \omega\right) - p^2k^2\left(\frac{\mathbf{k}\cdot\mathbf{p}}{m} + \omega\right)\right]. \tag{3.6}$$

Consider the simplest possible case, a mono-energetic beam,

$$f_0(\mathbf{p}) = n_0\,\delta(\mathbf{p} - \mathbf{p}_0), \tag{3.7}$$

where for definiteness we take $\mathbf{p}_0 \parallel \mathbf{k}$. Observe that this is a quantum-mechanically acceptable equilibrium, associated with infinite accuracy in momentum space and complete delocalization in position, corresponding to the plane wave $\psi = \sqrt{n_0}\exp(i\mathbf{p}_0\cdot\mathbf{r}/\hbar)$.

Inserting in (3.6) and solving recursively the dispersion relation up to $O(\hbar^2)$ and $O(1/c^2)$ results in

$$\omega \simeq \omega_p\left(1 - \frac{3p_0^2}{4m^2c^2}\right) + \frac{p_0k}{m}\left(1 - \frac{p_0^2}{2m^2c^2}\right) + \frac{\hbar^2k^4}{8m^2\omega_p}, \tag{3.8}$$

which can be checked to be consistent either with the exact non-relativistic

$$\left(\omega - \frac{p_0k}{m}\right)^2 = \omega_p^2 + \frac{\hbar^2k^4}{4m^2} \quad (1/c \to 0) \tag{3.9}$$

or exact non-quantum

$$\left(\omega - \frac{p_0k}{\gamma m}\right)^2 = \frac{\omega_p^2}{\gamma^3} \quad (\hbar \to 0) \tag{3.10}$$

dispersion relations.

To proceed, we consider the case where the Doppler correction is much larger than the plasma frequency,

$$\frac{p_0k}{m} \gg \omega_p. \tag{3.11}$$

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The motivation for this assumption is to pay attention to the cases where the relativistic correction is significant. Hence, (3.8) reduces to

$$\omega \simeq \frac{p_0 k}{m}\left(1-\frac{p_0^2}{2m^2c^2}\right)+\frac{\hbar^2k^4}{8m^2\omega_p}. \tag{3.12}$$

From this simple expression, it is evident that quantum diffraction acts in the opposite direction of the relativistic effects, which is in accordance with recent findings on quantum corrected wake field acceleration (Zhu and Ji 2010). Moreover, for large enough wavenumbers such that

$$\frac{\hbar^3k^3}{p_0^3}=\frac{4\hbar\omega_p}{mc^2}, \tag{3.13}$$

the quantum and relativistic corrections cancel out. Let us examine the implications of this result in concrete examples. First of all, we explicitly write the assumptions leading to (3.13): (a) weakly relativistic and quantum effects; (b) large Doppler shift (3.11); (c) cold beam and weak coupling approximations; (d) no QED effects like pair creation, or $\hbar\omega \ll 2mc^2$. One further limitation is due to the neglect of the return current and the associated magnetic field generation. This approximation can be justified for a sufficiently small beam cross-section, as discussed in detail in Bludman et al. (1960).

Condition (c) is equivalent to

$$K \gg \max\{\kappa_B T, E_F\} \gg U, \tag{3.14}$$

where κ_B is the Boltzmann constant, T is the bulk temperature and

$$K = mc^2(\gamma-1), \quad E_F = \left(m^2c^4+\hbar^2c^2(3\pi^2n_0)^{2/3}\right)^{1/2}-mc^2, \tag{3.15}$$

$$U = \frac{e^2n_0^{1/3}}{4\pi\varepsilon_0}$$

are, respectively, the beam kinetic energy, the Fermi energy (both in their relativistic form) and a measure of the electrostatic energy. The inequalities (3.14) allow us to neglect both thermal and collisional contributions. From now on, assume a beam with speed $v_0 = 0,4c$, which is weakly relativistic since then $p_0^2/(2m^2c^2) = 0,1$ (or 10% relativistic correction in the frequency), where $p_0 = \gamma m v_0$. Equivalently, the beam's kinetic energy is set as $K = 46,7\,\mathrm{keV}$.

We can consider first a non-degenerate plasma with $n_0 = 10^{24}\,\mathrm{m}^{-3}$. In this case, (3.13) gives $k = 7,5\times 10^9\,\mathrm{m}^{-1}$ and a wavelength $\lambda = 2\pi/k = 8,4\times 10^{-10}\,\mathrm{m}$. Such a plasma is indeed non-degenerate, because (3.14) implies $46,7\,\mathrm{keV} \gg \kappa_B T \gg 0,1\,\mathrm{eV}$, which always exceeds by far the Fermi energy $E_F = 3,7\,\mathrm{meV}$. Moreover, in such conditions the large Doppler and negligible QED assumptions are fairly well satisfied. From (3.12), one finds $\omega = 9,8\times 10^{17}\,\mathrm{Hz}$.

Secondly, consider a large density $n_0 = 10^{32}\,\mathrm{m}^{-3}$, which can be attained in intense laser compression experiments. From (3.13), one gets $k = 1,6\times 10^{11}\,\mathrm{m}^{-1}$ as the wavenumber for which quantum diffraction equals the relativistic correction, or $\lambda = 3,9\times 10^{-11}\mathrm{m}$ (in the hard X-ray regime). For temperatures smaller than $E_F = 787,0\,\mathrm{eV}$ such a plasma is degenerate. Moreover, one finds $E_F/K = 0,02$ and $U/E_F = 0,09$, in agreement with the cold beam and weak coupling hypothesis. Finally, the large Doppler shift and negligible QED effects are also fairly well verified. From (3.12), one finds $\omega = 2,1\times 10^{19}\,\mathrm{Hz}$.

4. Comparison with more accurate models

It is possible to make a detailed comparison with the existing results in the literature (Kowalenko et al. 1985; Melrose 2008) for a fully degenerate relativistic Bose gas, assuming

$$f_0 = n_0\delta(\mathbf{p}). \tag{4.1}$$

In this case where there is no streaming particles, the dispersion relation (3.1) can be exactly solved (with no semi-quantum nor semi-relativistic assumptions as made before), with the result

$$\omega^2 = \frac{\omega_p^2}{\tilde{\gamma}}+\frac{\hbar^2k^4}{4m^2\tilde{\gamma}^2}, \tag{4.2}$$

where

$$\tilde{\gamma} = \frac{\hbar^2k^2}{2m^2c^2}\left[\left(1+\frac{\hbar^2k^2}{m^2c^2}\right)^{1/2}-1\right]^{-1}. \tag{4.3}$$

In the semi-quantum and semi-relativistic limit, one has $\tilde{\gamma} = 1$ and we regain the well-known result (3.9) as well as (3.8), both with $p_0 = 0$.

To make an easier comparison with the pre-existing results, from now on we set units so that $\hbar = c = 1$. Equation (4.2) then reads

$$\omega^2 = 2m^2+k^2-\frac{2m^2\omega_p^2}{k^2}+2m^2\left(\frac{\omega_p^2}{k^2}-1\right)\left(1+\frac{k^2}{m^2}\right)^{1/2}$$

$$= \omega_p^2-\frac{\omega_p^2k^2}{4m^2}+\frac{(\omega_p^2+2m^2)k^4}{8m^4}+O(k^6), \tag{4.4}$$

while the result for the plasmon mode arising from more elaborate models in terms of the second-quantized Hamiltonian for an assembly of charged bosons and anti-bosons using the Feshbach–Villars representation (Kowalenko et al. 1985; Melrose 2008) is

$$\omega^2 = 2m^2+k^2+\frac{1}{2}\left[\omega_p^2-\left((\omega_p^2-4m^2)^2+16m^2k^2\right)^{1/2}\right]$$

$$= \omega_p^2-\frac{\omega_p^2k^2}{4m^2-\omega_p^2}+\frac{16m^4k^4}{(4m^2-\omega_p^2)^3}+O(k^6), \tag{4.5}$$

where a long wavelength expansion was performed, retaining only up to $O(k^4)$ terms with k a suitable non-dimensional wave vector. We ignore the anti-boson sector since our model is not concerned with pair creation effects. Further simplifying the last expression using $4m^2 \gg \omega_p^2$, in accordance with the non-pair-creation

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assumption, one finds

$$\omega^2 = \omega_p^2 - \frac{\omega_p^2 k^2}{4m^2} + \frac{(3\omega_p^2/2 + 2m^2)k^4}{8m^4} + O(k^6), \quad (4.6)$$

quite consistent with (4.4) except for a numerical factor $3/2$ instead of unity in a coefficient of the $\propto k^4$ term. After all, it comes as no surprise to have small discrepancies due to the several drastic assumed hypothesis. It is clear that in the presence of pair creation one has to employ a more detailed analysis. Similar conclusions are obtained for the very small wavenumber region.

5. Conclusion

In this work, we analyzed the electrostatic waves following a semi-relativistic Wigner–Poisson system given by the Schrödinger-like equation (2.11) and Poisson's equation (2.13). A linear dispersion relation containing a quantum-modified relativistic factor was deduced. Afterward, the case of a mono-energetic beam was discussed and sample physical parameters allowing for jointly significant relativistic and quantum effects were identified. From the last section, one concludes that the quantum diffraction effects are more easily measurable in non-degenerate plasmas, since they correspond to wavelengths on the nanoscale in this case. For larger densities, we found that the relativistic correction is dominant except for extremely small spatial scales.

In spite of the overall simplification which limits the applicability of the square-root Klein–Gordon approach, it is nevertheless an useful method for direct comparison with the known results for traditional plasma. A further limitation which is worth commenting is that the integro-differential equation (2.11) for the Wigner function makes it hardly accessible to nonlinear analysis, except perhaps by means of numerical simulation. This is a common disadvantage in any kinetic theory.

We expect that the parameter analysis of the last section can inspire similar approaches to serious investigation of the relevance (or irrelevance) of quantum diffraction effects in specific plasma problems. The study of more structured equilibria than a single beam is also recommendable.

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