Ergod. Th. & Dynam. Sys. (2013), **33**, 168–182 doi:10.1017/S0143385711000939
© *Cambridge University Press, 2012*

Multidimensional expanding maps with singularities: a pedestrian approach

CARLANGELO LIVERANI

Dipartimento di Matematica, II Università di Roma (Tor Vergata), Via della Ricerca Scientifica, 00133 Roma, Italy
(e-mail: liverani@mat.uniroma2.it)

(*Received* 26 *March* 2011 *and accepted in revised form* 16 *September* 2011)

Abstract. I provide a proof of the existence of absolutely continuous invariant measures (and study their statistical properties) for multidimensional piecewise expanding systems with not necessarily bounded derivative or distortion. The proof uses basic properties of multidimensional BV functions (the space of functions of bounded variations).

1. *Introduction*

Lately, several authors attempted to treat billiards directly by transfer operator methods. While the first attempts can be traced, at least, as far back as 1995†, most of the related work is quite recent and is motivated by the anisotropic Banach space approach started in [**8**]. Such an approach has been very successful in the smooth case [**2, 6, 7, 9, 19, 20, 29–31**] but its extension to the piecewise smooth case is still unsatisfactory in spite of encouraging recent progress, see [**3–5, 15**]. The hopes motivating such a program are to obtain simpler, stronger and more flexible proofs for, e.g., decay of correlations, statistical stability and limit laws.

Given such a situation, it is natural to reconsider the piecewise expanding case. The payoff of this revisitation is twofold. On the one hand, I generalize the existing results to the case in which the derivative of the map is unbounded and there may be countably many domains of smoothness. On the other hand, I obtain a more streamlined and elementary proof. Hence, this paper both contains new results and is a useful warm up for the hyperbolic setting.

More precisely, I show that the blow up of the derivative can be, at least in some cases, easily handled without resorting to the 'homogeneity strips' strategy usually employed in billiards. The method presented here is the higher dimensional analogue of the one introduced in [**34**]. I work in the piecewise $\mathcal{C}^2$ setting. One could extend the theory to the $\mathcal{C}^{1+\alpha}$ case, but would have to use a generalization of BV spaces that would considerably cloud the argument (but see [**36**] for a possible alternative).

† See [**27**], which was in fact applicable to billiards by introducing the usual homogeneity strips strategy, even though such an extension was not present in the published version.

https://doi.org/10.1017/S0143385711000939 Published online by Cambridge University Press

I use the space of functions of bounded variation (unlike some more unconventional spaces as in [**35**]) and very little of the theory of bounded variation functions (e.g. no computations based on the traces of a BV function are used, contrary to most of the literature starting from [**17, 22**] to the more recent [**13, 14**]).

2. *The class of maps*

Let us consider classes of maps $T : \Omega \to \overline{\Omega}$, $\Omega = \mathring{\Omega} \subset [0, 1]^d$, satisfying specified subsets of the following conditions†. Let m be the Lebesgue measure.

(0) For each $n \in \mathbb{N}$, $m(T^{-n}\partial\Omega) = 0$.

(1) There exists a countable collection of disjoint connected open sets $\{\Omega_k\}$, $\bigcup_k \Omega_k = \Omega$‡, such that T is $\mathcal{C}^2$, with $\mathcal{C}^1$ inverse, when restricted to each Ω_k.

(2) For all $k \in \mathbb{N}$, $T|_{\Omega_k}$ and $(DT)^{-1}|_{\Omega_k}$ have a continuous extension to $\overline{\Omega}_k$§.

(3) For each $j \in \{1, \dots, d\}$¶, $x_{\neq j} \in \mathbb{R}^{d-1}$ and $k \in \mathbb{N}$,

$$\Omega_k \cap \{(x_{\neq j}, y)\}_{y\in\mathbb{R}} = \bigcup_\ell \{x_{\neq j}\} \times I_{k,\ell}(x_{\neq j}),$$

where the $I_{k,\ell}(x_{\neq j}) \subset \mathbb{R}$ are, possibly countably many, disjoint open intervals‖. In addition, there exist $\delta > 0$ and $\lambda > 1$ such that**

$$\sup_{x\in[0,1]^d} \sup_j \sum_{\{k,\ell:[x_j-\delta,x_j+\delta]\cap I_{k,\ell}(x_{\neq j})\neq\emptyset\}} \|\mathbb{1}_{I_{k,\ell}(x_{\neq j})}(\cdot)[(DT)^{-1}](x_{\neq j}, \cdot)\|_{L^\infty} \le \lambda^{-1},$$

where $\mathbb{1}_A$ stands for the indicator function of the set A.

(4) For each $k \in \mathbb{N}$, let

$$\Omega_k^\varepsilon = \{x \in \Omega_k : \operatorname{dist}(x, \partial\Omega_k) \ge \varepsilon\} \quad \text{and} \quad \partial^\varepsilon\Omega_k = \Omega_k \backslash \Omega_k^\varepsilon.$$

Then, for each $j \in \{1, \dots, d\}$,

$$\lim_{\varepsilon\to 0} \sup_{x_{\neq j}} \sum_k \int_0^1 \|\partial_{x_j}[(DT)^{-1}](x_{\neq j}, y)\|_\infty \mathbb{1}_{\partial^\varepsilon\Omega_k}(x_{\neq j}, y)\, dy = 0.$$

(5) For each $i, j, k \in \{1, \dots, d\}$, $(DT)_{kj}[(DT)^{-1}]_{ji} \in L^1(\Omega, m)$.

† If a given T does not satisfy the conditions below, they may still hold for some power of T.

‡ Note that, with the present definition, Ω is necessarily disconnected unless $\{\Omega_k\}$ consists of a single element. Moreover, one can check that $\overline{\bigcup_k \partial\Omega_k} = \partial\Omega$.

§ I will use the notation $(D_xT)^{-1}$ or $[(DT)^{-1}](x)$ for the inverse of the matrix DT computed at the point x.

¶ From now on, the notation $x_{\neq j}$ stands for the vector $(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_d)$ and $(x_{\neq j}, y)$ stands for $(x_1, \dots, x_{j-1}, y, x_{j+1}, \dots, x_d)$.

‖ Note that, for some $x_{\neq j}$, the set $\{I_{k,\ell}(x_{\neq j})\}$ may be empty.

** The following could be replaced by a weaker condition in which one takes an inf on all the possible orthonormal bases. This is due to the fact that in the definition (2.1) below there appears the divergence, which is rotationally invariant. I avoid this generalization to favor readability. In particular, to implement such a strategy I would be forced to use a rotation-invariant norm, such as the Euclidean one, instead of $\|\varphi(x)\|_\infty = \max_{i\in\{1,\dots,d\}} |\varphi_i(x)|$, $\|A(x)\|_\infty = \max_i \sum_j |A(x)_{i,j}|$, $\|\varphi\|_{L^\infty} = \sup_x \|\varphi(x)\|_\infty$ and $\|A\|_{L^\infty} = \sup_x \|A(x)\|_\infty$, which I find more convenient here.

https://doi.org/10.1017/S0143385711000939 Published online by Cambridge University Press

(6) There exist $C > 0$, $\alpha \in (0, 1]$ and $a \geq 1$ such that, for each $x \in \Omega$ and $\varepsilon > 0$, we have†

$$m\left(\bigcup_k \partial^\varepsilon \Omega_k\right) \leq C^d \varepsilon^\alpha; \quad \|\nabla \det(D_x T)\|_\infty \leq C d(x, \partial\Omega)^{-a}.$$

Remark 2.1. The above conditions are not all independent; I introduced some partial repetitions to make them more transparent. The zeroth condition simply ensures that the dynamics is well-defined Lebesgue almost surely. In the case $d = 1$, the first two conditions reduce to the usual definition of piecewise expanding maps. Condition three reduces, in one dimension and when the partition is finite, to the standard condition that the expansion be larger than two‡. In higher dimensions, for a finite partition, condition three is implied by the condition that the boundaries of the Ω_k are transversal to the coordinate axes and that the maximal number of the $\overline{\Omega}_k$ covering a point is strictly bounded by the minimal expansion ($\|(DT)^{-1}\|_{L^\infty}^{-1}$). In one dimension, condition four simply says that the inverse of the derivative must be locally in $W^{1,1}$ (i.e. its derivative must be in L^1); this is essentially Rychlik's condition [**34**]. In higher dimensions, assuming a finite partition with transversal boundaries, condition four is implied by a uniform integrability condition for the functions $\|\partial_{x_j}[(DT)^{-1}](x_{\neq j}, \cdot)\|_\infty$§. Condition five trivializes in one dimension and is fairly weak in any dimension. Finally, the last condition is not needed if $d = 1$ (see Theorem 1).

As in the one-dimensional case, we will work with the BV and the L^1 norms, but BV needs a word of explanation. For each $h \in L^1(\mathbb{R}^d, m)$, define¶

$$\|h\|_{BV(\mathbb{R}^d)} := \sup_{\substack{\varphi \in C_0^1(\mathbb{R}^d, \mathbb{R}^d) \\ \|\varphi\|_{L^\infty} \leq 1}} \int_{\mathbb{R}^d} h \cdot \operatorname{div} \varphi. \tag{2.1}$$

As usual,

$$BV(\mathbb{R}^d) = \{h \in L^1(\mathbb{R}^d, m) : \|h\|_{BV(\mathbb{R}^d)} < \infty\}.$$

Next,

$$\|h\|_{BV(\Omega)} := \|\mathbb{1}_\Omega h\|_{BV(\mathbb{R}^d)} \tag{2.2}$$

and $BV(\Omega) = \{h \in BV(\mathbb{R}^d) : \operatorname{supp} h \subset \Omega\}$. From now on, I will use BV to designate $BV(\Omega)$, if no confusion arises‖. For the reader's convenience, I collect in Appendix A the relevant properties of BV functions.

† One could even allow an exponential blow up at singularities, provided the rate is taken small enough. It suffices to truncate the restricted transfer operators (defined in §4) at a polynomial distance from the singularity rather than at an exponential distance. I choose not to pursue this generalization because on the one hand it is obvious and on the other hand it would only make the argument less transparent.

‡ In fact, it is slightly weaker insofar as it requires that the sum of the inverse of the minimal expansion of any two consecutive intervals is less than one. At any rate, in one dimension the larger than two condition can always be satisfied, for some power of T, provided the expansion is larger than one. This is not true in higher dimensions (see [**11, 37**]) although it can be achieved in special cases (see [**10, 12, 38**]).

§ I doubt that having only the inverse of the derivative locally in $W^{1,1}$ suffices to prove Theorem 1.

¶ By C_0^r, I mean the vector space of r-times differentiable functions with compact support. Here and in the following I will not specify the variable of integration if no confusion arises. In any case, all the integrations are with respect to the Lebesgue measure unless otherwise stated.

‖ Note that $BV(\Omega)$ is a closed vector space under the $BV(\mathbb{R}^d)$ norm and that (2.2) is a norm in $BV(\Omega)$. Also, for $h \in L^1(\Omega, m)$, I will often write $h \in BV(\Omega)$ to mean $\mathbb{1}_\Omega h \in BV(\Omega)$, i.e. the functions are extended to be zero on Ω^c.

https://doi.org/10.1017/S0143385711000939 Published online by Cambridge University Press

Remark 2.2. Note that the alternative to define the norm in (2.2) for each $h \in L^1(\Omega)$ directly by (2.1) with the sup restricted to the φ with support contained in Ω would not work (in particular, Lemma 3.1 may fail; see [**33**] for a counterexample).

We are now ready to state precisely the result proved in this paper.

THEOREM 1. *Any map satisfying assumptions (0)–(5) has (at most finitely many) invariant measures absolutely continuous with respect to Lebesgue. Such measures have densities that belong to BV and their supports yield the ergodic decomposition of the Lebesgue measure*†. *If there exists only one absolutely continuous invariant measure μ, with density h_*, then the following alternative holds: either the dynamical system (Ω, T, μ) is not mixing or there exist $\sigma \in (\lambda^{-1}, 1)$ and $C > 0$ such that, for all $h \in BV(\Omega)$ and $f \in L^d(\Omega, m)$‡,*

$$\left| \int_\Omega f \circ T^n h - \int_\Omega h \cdot \int_\Omega h_* f \right| \le C \|h\|_{BV} \|f\|_{L^d} \sigma^n.$$

If the first alternative holds true, then there exist $N \in \mathbb{N}$ and disjoint sets $\{\widetilde{\Omega}_j\}_{j=1}^N$, $\Omega = \bigcup_j \widetilde{\Omega}_j$ Lebesgue almost surely, such that $T\widetilde{\Omega}_j = \widetilde{\Omega}_{j+1}$ for $j < N$ and $T\widetilde{\Omega}_N = \widetilde{\Omega}_1$. Moreover, for each $j \le N$, $(T^N, \widetilde{\Omega}_j, \mu)$ is an exponentially mixing dynamical system.

*Finally, if $d = 1$ or if the map satisfies also assumption (6), then there exists $\varepsilon_* > 0$ such that for each absolutely continuous invariant measure the associated density is uniformly positive in an open ball of size ε_*. In particular, the ergodic components of the Lebesgue measure are open (*mod 0*).*

Remark 2.3. Note that the above theorem is constructive since ε_* can be computed by looking at the proof and keeping track of the various constants§. This provides a bound on the number of ergodic components. One can then prove that there is only one ergodic component by finding an ε_*-dense orbit. Moreover, if there is only one invariant measure and two periodic orbits (away from $\partial\Omega$) that have coprime periods, then the measure is mixing.

Remark 2.4. It is important to remark that many other results can be obtained once Theorem 1 (or rather the spectral picture, based on Lemma 3.1, from which the theorem follows) is established. For example, approximation results (extending the ones in [**32**]) follow by applying the theorems in [**25, 26**]. Also, various limit theorems (in particular the central limit theorem as in [**28**]) can be obtained by using, e.g., the results in [**18**]. Finally, the interested reader can easily extend the present arguments to more general transfer operators by imposing appropriate conditions on the potentials, in the spirit of [**34**].

2.1. *An example.* Here is an example (chosen more or less at random). Let $\Omega \subset (0, 1)^2$ be given by the smoothness domains of the map

$$T(x, y) = (a\sqrt{x}, by + \sqrt{x}) - \lfloor (a\sqrt{x}, by + \sqrt{x}) \rfloor,$$

† That is, all the indecomposable invariant sets (mod 0) of positive Lebesgue measure.

‡ To be precise, $f \circ T^n$ is defined only on $\bigcap_{n\in\mathbb{N}} T^{-n}\Omega \subset \Omega$, which differs from Ω for a zero measure set by hypothesis (0). Since it is irrelevant, I will not comment on this any further.

§ I choose not to do this in favor of readability.

https://doi.org/10.1017/S0143385711000939 Published online by Cambridge University Press

where $a, b > 6$ and $\lfloor c \rfloor$ is the integer part of c. Then

$$(DT)^{-1} = \begin{pmatrix} \frac{2\sqrt{x}}{a} & 0 \\ -\frac{1}{ab} & \frac{1}{b} \end{pmatrix}.$$

It is easy to check that the above map satisfies all the conditions of Theorem 1.

Proof of Theorem 1. As usual, one first defines the transfer operator and then proves a Lasota–Yorke inequality. More precisely, for each $x \in \Omega$,

$$\mathcal{L}h(x) := \sum_{y \in T^{-1}(x)} |\det D_y T|^{-1} h(y). \tag{2.3}$$

Then, for each $h \in L^1(\mathbb{R}^d, m)$ and $\varphi \in L^\infty(\mathbb{R}^d, m)$, we have†

$$\int_\Omega h \cdot \varphi \circ T = \int_\Omega \varphi \cdot \mathcal{L}h. \tag{2.4}$$

Hence, $\mathbb{1}_\Omega \mathcal{L}h = \mathcal{L}(\mathbb{1}_\Omega h)$ almost surely. The Lasota–Yorke inequality is proven in Lemma 3.1. It follows that $\mathcal{L}$ is a bounded operator from $L^1(\Omega, m)$, and from $BV(\Omega)$, to itself.

Using standard arguments (see [**1**] for a general discussion, here I use Hennion's theorem in [**21**]), Lemma 3.1 implies that $\mathcal{L}$ is a quasi-compact operator with spectral radius one and essential spectral radius bounded by λ^{-1}. In addition, one is an eigenvalue since $\int h = \int \mathcal{L}h$, for each $h \in L^1$, implying that one belongs to the spectrum of the dual operator $\mathcal{L}'$. Thus, there exists a spectral gap between the eigenvalues of modulus one and the rest of the spectrum. Accordingly, we can write $\mathcal{L} = \Pi + R$, where Π is a finite-rank operator with spectrum contained in $\{z \in \mathbb{C} : |z| = 1\} \cup \{0\}$, while the spectral radius of R is strictly smaller than one and $\Pi R = R\Pi = 0$. In addition, the second of the inequalities (3.1) below shows that Π must be power bounded, hence it cannot contain Jordan blocks. Thus, we can further decompose $\Pi = \sum_j e^{i\theta_j} \Pi_{\theta_j}$, $\Pi_{\theta_j} \Pi_{\theta_k} = \delta_{jk} \Pi_{\theta_j}$‡. By the spectral gap,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{-i\theta k} \mathcal{L}^k = \begin{cases} \Pi_{\theta_j} & \text{if } \theta = \theta_j, \\ 0 & \text{otherwise,} \end{cases} \tag{2.5}$$

where the limit is in the strong operator topology in $L(BV, BV)$. Again, the second of the inequalities (3.1) below implies that the Π_{θ_j} are bounded operators from L^1 to BV. In particular, $\Pi_0 1 \in BV$ is the density of an invariant measure. On the other hand, if μ is an absolutely continuous invariant measure with density h, (2.5) implies that $h = \Pi_0 h$, hence $h \in BV$. That is, all the invariant densities belong to the finite-dimensional range of Π_0.

† In fact, (2.4) is often taken as the definition of $\mathcal{L}$. Formula (2.3) follows then by changing variables, remembering hypothesis (0) and using the Lebesgue monotone convergence theorem to prove almost sure convergence of the right-hand side.

‡ Note that one should complexify BV. I do not give details, as they are standard.

https://doi.org/10.1017/S0143385711000939 Published online by Cambridge University Press

Next, observe that $\Pi_{\theta_j}(g) = \sum_\kappa \ell_{j,\kappa}(g) h_{j,\kappa}$, where $h_{j,\kappa} \in BV$ and $\ell_{j,\kappa} \in (L^1)'$ and $\ell_{j,\kappa}(h_{r,s}) = \delta_{j,r}\delta_{\kappa,s}$. That is, there exists $\rho_{j,\kappa} \in L^\infty$ such that $\ell_{j,\kappa}(g) = \int_\Omega \rho_{j,\kappa} \cdot g$ for all $g \in L^1$. Moreover, $\Pi_{\theta_0} := \Pi_0$ is a positive projector and $\int \Pi_0 h = \int h$, hence $\sum_\kappa m(h_{0,\kappa})\rho_{0,\kappa} = 1$, which implies that $\rho_{0,\kappa} = \mathbb{1}_{A_\kappa} m(h_{0,\kappa})^{-1}$, where the $\{A_\kappa\}$ are disjoint indecomposable invariant sets† and must therefore correspond to the ergodic decomposition of Lebesgue. In addition, for all $g \in L^1$,

$$\int \rho_{j,\kappa}\rho_{j',\kappa'}\mathcal{L}g = \int (\rho_{j,\kappa}\rho_{j',\kappa'}) \circ T \cdot g = e^{i(\theta_j+\theta'_j)} \int \rho_{j,\kappa}\rho_{j',\kappa'} g,$$
$$\int \bar{\rho}_{j,\kappa}\mathcal{L}g = e^{-i\theta_j} \int \bar{\rho}_{j,\kappa} g.$$

Hence, $e^{-i\theta_j} \in \sigma(\mathcal{L}') = \sigma(\mathcal{L})$ and either $\rho_{j,\kappa}\rho_{j',\kappa'} = 0$ or $e^{i(\theta_j+\theta_{j'})} \in \sigma(\mathcal{L})$. That is, the peripheral eigenvalues form finitely many cyclic groups, each related to a different ergodic component (see [**1**] for details).

If the peripheral spectrum consists of one as a simple eigenvalue, then Π is a rank-one operator and, setting $h_* = \Pi 1$ for each $h \in BV$ and $f \in L^d$ holds (remembering (A.1)),

$$\int_\Omega f \circ T^n h = \int_\Omega f\mathcal{L}^n h = \int_\Omega f h_* \cdot \int_\Omega h + \mathcal{O}(\|f\|_{L^d}\|R\|^n_{BV}\|h\|_{BV}),$$

which gives the announced exponential decay of correlations. Similar arguments work for $(T^N, \widetilde{\Omega}_j, \mu)$ in the general case.

The only thing left to prove is the openness of the ergodic components. If $d = 1$, then it follows from the fact that, in one dimension only, BV functions are strictly positive over some ball (i.e. interval) and by the standard argument in the last paragraph of the proof of Lemma 4.1. If $d > 1$, then the result follows from hypothesis (6) and is proven in Lemma 4.1. □

Remark 2.5. In the following, we will use $C_\#$ to designate a generic constant, depending only on the map T and the constants appearing in the hypotheses (0)–(6), which values can change from line to line. On the contrary, we will use $C_{a,b,c,\ldots}$ for a generic constant depending on the parameters $a, b, c, \ldots$.

3. *Lasota–Yorke inequality*

LEMMA 3.1. *For each map T satisfying hypotheses (0)–(5) and for each $\sigma \in (\lambda^{-1}, 1)$, there exists $B > 0$ such that, for all $n \in \mathbb{N}$ and $h \in BV$, we have*

$$\begin{aligned} \|\mathcal{L}^n h\|_{L^1(\Omega,m)} &\le \|h\|_{L^1(\Omega,m)}, \\ \|\mathcal{L}^n h\|_{BV(\Omega)} &\le \sigma^n \|h\|_{BV(\Omega)} + B\|h\|_{L^1(\Omega,m)}. \end{aligned} \tag{3.1}$$

† Remember that, by construction, $\mathcal{L}\Pi_{\theta_j} = \Pi_{\theta_j}\mathcal{L} = e^{i\theta_j}\Pi_{\theta_j}$, which implies that $\mathcal{L}'\ell_{0,\kappa} = \ell_{0,\kappa}$ and thus $\rho_{0,\kappa} \circ T = \rho_{0,\kappa}$ almost surely. In turn, this means that $\rho_{0,\kappa}$ can take only one value (besides zero), otherwise we could construct new invariant sets, and hence invariant measures, which we know do not exist. For the same reasons, we must have $\rho_{0,\kappa}\rho_{0,j} = \delta_{\kappa,j}\rho^2_{0,\kappa}$.

https://doi.org/10.1017/S0143385711000939 Published online by Cambridge University Press

Proof. For each $h \in L^1(\Omega, m)$ and $\varphi \in L^\infty(\mathbb{R}^d, m)$, we have

$$\int_\Omega \mathcal{L}h \cdot \varphi = \int_\Omega h \cdot \varphi \circ T \le \|h\|_{L^1}\|\varphi\|_{L^\infty}.$$

Thus, $\|\mathcal{L}h\|_{L^1} \le \|h\|_{L^1}$†.

To prove the second of the inequalities (3.1), let $h \in BV$ and $\varphi \in \mathcal{C}^1_0(\mathbb{R}^d, \mathbb{R}^d)$ with $\|\varphi\|_{L^\infty} \le 1$,

$$\begin{aligned}\int_\Omega \mathcal{L}h \cdot \sum_{i=1}^d \partial_{x_i}\varphi_i &= \int_\Omega h \sum_{i=1}^d (\partial_{x_i}\varphi_i)\circ T = \sum_{i,k}\int_{\Omega_k} h(\partial_{x_i}\varphi_i)\circ T \\ &= \sum_{i,k,j}\int_{\Omega_k} h\partial_{x_j}\{[(DT)^{-1}]_{ji}[\varphi_i \circ T]\} \\ &\quad - \sum_{i,k,j}\int_{\Omega_k} h[\varphi_i \circ T]\partial_{x_j}[(DT)^{-1}]_{ji}. \end{aligned} \tag{3.2}$$

To proceed, we need to recreate proper test functions. To this end, remember from hypothesis (3) that for each $j \in \{1, \dots, d\}$ and $x_{\ne j} \in \mathbb{R}^{d-1}$, we have defined intervals $I_{k,\ell}(x_{\ne j}) =: (a_{k,\ell}(x_{\ne j}), b_{k,\ell}(x_{\ne j}))$. Let us define $J_k(x_{\ne j}) := \bigcup_\ell \overline{I_{k,\ell}(x_{\ne j})}$ and, for each $x \in \mathbb{R}^d$‡,

$$\begin{aligned}\Psi_{j,k}(x) &= \sum_{i=1}^d [(D_x T)^{-1}]_{ji} \cdot \varphi_i \circ T(x) \cdot \mathbb{1}_{J_k(x_{\ne j})}(x_j), \\ \Psi^-_{j,k,\ell}(x_{\ne j}) &= \Psi_{j,k}(x_1, \dots, x_{j-1}, a_{k,\ell}, x_{j+1}, \dots, x_d), \\ \Psi^+_{j,k,\ell}(x_{\ne j}) &= \Psi_{j,k}(x_1, \dots, x_{j-1}, b_{k,\ell}, x_{j+1}, \dots, x_d).\end{aligned}$$

Next, define functions $\eta_{j,k,\ell,x_{\ne j}} \in L^\infty(\mathbb{R}, m)$ by

$$\eta_{j,k,\ell,x_{\ne j}}(y) = \begin{cases} 0 & \text{for all } y \in (-\infty, a_{k,\ell} - \delta], \\ \Psi^-_{j,k,\ell}(x_{\ne j}) \cdot (y - a_{k,\ell} + \delta)\delta^{-1} & \text{for all } y \in (a_{k,\ell} - \delta, a_{k,\ell}), \\ 0 & \text{for all } y \in [a_{k,\ell}, b_{k,\ell}], \\ \Psi^+_{j,k,\ell}(x_{\ne j}) \cdot (b_{k,\ell} + \delta - y)\delta^{-1} & \text{for all } y \in (b_{k,\ell}, b_{k,\ell} + \delta), \\ 0 & \text{for all } y \in [b_{k,\ell} + \delta, +\infty), \end{cases}$$

where $\delta > 0$ satisfies hypothesis (3), and

$$\begin{aligned}\overline{\Psi}_{j,k}(x) &= \Psi_{j,k}(x) + \sum_\ell \eta_{j,k,\ell,x_{\ne j}}(x_j) = \sum_\ell \theta_{k,\ell}, \\ \theta_{k,\ell} &= [\Psi_{j,k}(x)\mathbb{1}_{\overline{I_{k,\ell}(x_{\ne j})}}(x_j) + \eta_{j,k,\ell,x_{\ne j}}(x_j)]; \quad \Theta_j := \sum_k \overline{\Psi}_{j,k}.\end{aligned}$$

The point of introducing such functions is that, by construction, the $\theta_{k,\ell}$ are continuous functions in the x_j variable, for each $x_{\ne j}$, and the same holds for the functions Θ_j,

† Note that equality holds if $h \ge 0$.

‡ The numbers $\Psi^\pm_{j,k,\ell}(x_{\ne j})$ are well defined by hypothesis (2). In addition, if $x \in \{x_{\ne j}\} \times \overline{J_k(x_{\ne j})}$ but not in $\{x_{\ne j}\} \times J_k(x_{\ne j})$, then x is a point of accumulation of intervals and, hence, by hypothesis (3), it is natural to define $\Psi_{j,k}(x) = 0$.

https://doi.org/10.1017/S0143385711000939 Published online by Cambridge University Press

which are, by definition and hypothesis (3), a uniformly convergent series of continuous functions. Also, for $x \notin \partial\Omega$, (3) implies that

$$\sup_j |\Theta_j(x)| \le \sup_j \sum_{k,\ell} \left[\sup_{x_j \in I_{k,\ell}(x_{\neq j})} |\Psi_{j,k}(x)| \right] \mathbb{1}_{[a_{k,\ell}-\delta, b_{k,\ell}+\delta]}(x_j) \le \lambda^{-1}. \tag{3.3}$$

Moreover†,

$$\Theta_j(x) = \sum_{k,\ell} \int_0^{x_j} dy \partial_y \theta_{k,\ell}(x_{\neq j}, y) = \int_0^{x_j} dy \sum_{k,\ell} \partial_y \theta_{k,\ell}(x_{\neq j}, y)$$

implies that the function Θ_j is Lebesgue almost surely differentiable with respect to the variable x_j and $\partial_{x_j}\Theta_j(x_{\neq j}, \cdot) \in L^1(\mathbb{R})$. Thus, almost surely‡,

$$\begin{aligned} \operatorname{div} \Theta(x) = &\sum_{i,k,j} \partial_{x_j} \{ [(D_x T)^{-1}]_{ji} \varphi_i \circ T \} \cdot \mathbb{1}_{J_k(x_{\neq j})}(x_j) \\ &+ \delta^{-1} \sum_{j,k,\ell} \{ \Psi^-_{j,k,\ell}(x_{\neq j}) \mathbb{1}_{[a_{k,\ell}-\delta, a_{k,\ell}]}(x_j) - \Psi^+_{j,k,\ell}(x_{\neq j}) \mathbb{1}_{[b_{k,\ell}, b_{k,\ell}+\delta]}(x_j) \}. \end{aligned}$$

This implies, by (A.2), that Θ has enough properties to be used as a test function.

Having set up the above machinery, we rewrite (3.2) as

$$\begin{aligned} \left| \int_\Omega \mathcal{L}h \cdot \operatorname{div} \varphi \right| &\le \left| \int_{\mathbb{R}^d} h \operatorname{div} \Theta \right| + \frac{d\|h\|_{L^1}}{\lambda\delta} + \left| \sum_{i,k,j} \int_{\Omega_k} h[\varphi_i \circ T] \partial_{x_j} [(DT)^{-1}]_{ji} \right| \\ &\le \frac{\|h\|_{BV}}{\lambda} + \frac{d\|h\|_{L^1}}{\lambda\delta} + \left| \sum_{i,k,j} \int_{\mathbb{R}^d} h[\varphi_i \circ T] \partial_{x_j} [(DT)^{-1}]_{ji} \mathbb{1}_{\Omega_k} \right|. \end{aligned} \tag{3.4}$$

To conclude, note that one can split the last term of (3.4) as

$$\begin{aligned} &\sum_{i,k,j} \int_{\mathbb{R}^d} h[\varphi_i \circ T] \partial_{x_j} [(DT)^{-1}]_{ji} \mathbb{1}_{\Omega_k^\varepsilon} \\ &\quad + \sum_j \int_{\mathbb{R}^d} h \partial_{x_j} \int_0^{x_j} \sum_{i,k} [\varphi_i \circ T] \partial_{x_j} [(DT)^{-1}]_{ji} \mathbb{1}_{\partial^\varepsilon \Omega_k}. \end{aligned}$$

Thanks to hypothesis (4), for each $\sigma \in (\lambda^{-1}, 1)$, one can then choose ε so that

$$\left\| \left(\int_0^{x_j} dy \sum_{i,k,j} \varphi_i \circ T(x_{\neq j}, y) \cdot \partial_y [(D_{(x_{\neq j}, y)} T)^{-1}]_{ji} \cdot \mathbb{1}_{\partial^\varepsilon \Omega_k}(x_{\neq j}, y) \right) \right\|_{L^\infty} \le \sigma - \lambda^{-1}.$$

Finally, note that if $\Omega_k^\varepsilon \neq \emptyset$, then Ω_k must contain a ball of radius ε, but only finitely many such balls can fit in Ω, thus only finitely many Ω_k^ε are non-empty. Accordingly, by hypothesis (1) and (A.2), there exists $C_\sigma > 0$ such that, taking the sup over φ,

$$\|\mathcal{L}h\|_{BV} \le \lambda^{-1}\|h\|_{BV} + d\delta^{-1}\lambda^{-1}\|h\|_{L^1} + (\sigma - \lambda^{-1})\|h\|_{BV} + C_\sigma \|h\|_{L^1},$$

which (iterating) proves the lemma with $B = (1-\sigma)^{-1}(C_\sigma + d\delta^{-1}\lambda^{-1})$. □

† The first equality holds by hypotheses (1) and (2), the second follows from hypotheses (3), (4), (5) and the Lebesgue dominated convergence theorem.

‡ Note that the last term of the next equation is bounded by $d\delta^{-1}\lambda^{-1}$ due to hypothesis (3).

https://doi.org/10.1017/S0143385711000939 Published online by Cambridge University Press

4. *Openness of the ergodic components*

Finally, I deal with the openness of the ergodic components. As far as I know, the only general result of this type (but limited to finite partitions and bounded derivatives) is in [**35**], where it can be easily obtained thanks to the particular Banach space used there. Yet, I do not see how to use such a Banach space in the unbounded derivative case. In addition, the argument in [**35**] does not seem to be constructive. On the contrary, the radius ε_* in the following theorem is constructive and could, with some extra work, be estimated explicitly in concrete examples.

LEMMA 4.1. *For each map T satisfying hypotheses (0)–(6), there exists $\varepsilon_* > 0$ such that each ergodic component of the Lebesgue measure is open and contains a ball of radius ε_* on which the density of the associated absolutely continuous invariant measure (a.c.i.m.) is uniformly positive.*

Proof. We argue along known lines (see [**23**] for similar arguments in a closely related case). The basic idea is to compare the true dynamics with one in which only trajectories that are always far away from the singularities are considered. Such trajectories experience only the smooth part of the dynamics. To do so, we need first to define a transfer operator restricted to such trajectories and then to obtain an *a priori* estimate on the difference between the real system and the *restricted* one.

Consider the sets $\Gamma_\varepsilon = \bigcup_k \partial^\varepsilon \Omega_k$, and the operators $\mathcal{L}_{\varepsilon,n} f = \mathcal{L}((1 - \mathbb{1}_{\Gamma_{\varepsilon\nu^n}}) f)$†, where

$$\nu \in (\nu_0^{1/a}, 1), \qquad \nu_0 = \|DT^{-1}\|_{L^\infty}$$

(the interval is not empty by assumption (3)). Setting $\widetilde{\mathcal{L}}_{\varepsilon,n} = \mathcal{L}_{\varepsilon,0}\mathcal{L}_{\varepsilon,1}\cdots\mathcal{L}_{\varepsilon,n-1}$, $\widetilde{\mathcal{L}}_{\varepsilon,0} = \mathbb{1}$, for each $f \in BV$, $f \geq 0$, we have‡

$$\begin{aligned}
\|\mathcal{L}^n f - \widetilde{\mathcal{L}}_{\varepsilon,n} f\|_{L^1} &\leq \sum_{k=0}^{n-1} \|\mathcal{L}^{k+1}\mathcal{L}_{\varepsilon,k+1}\cdots\mathcal{L}_{\varepsilon,n-1} f - \mathcal{L}^k \mathcal{L}_{\varepsilon,k}\cdots\mathcal{L}_{\varepsilon,n-1} f\|_{L^1} \\
&= \sum_{k=0}^{n-1} \|\mathbb{1}_{\Gamma_{\varepsilon\nu^k}} \cdot \mathcal{L}_{\varepsilon,k+1}\cdots\mathcal{L}_{\varepsilon,n-1} f\|_{L^1} \leq \sum_{k=0}^{n-1} \|\mathbb{1}_{\Gamma_{\varepsilon\nu^k}} \cdot \mathcal{L}^{n-k-1} f\|_{L^1} \\
&\leq \sum_{k=0}^{n-1} m(\Gamma_{\varepsilon\nu^k})^{1/d} \|\mathcal{L}^{n-k-1} f\|_{L^{d/(d-1)}} \\
&\leq \sum_{k=0}^{n-1} C_\# \varepsilon^{\alpha/d} \nu^{\alpha k/d} \|\mathcal{L}^{n-k-1} f\|_{BV} \\
&\leq C_\# \varepsilon^{\alpha/d} / 1 - \nu^{\alpha/d} \|f\|_{BV}. \qquad (4.1)
\end{aligned}$$

Next, we choose some mollifier η and, for each $\delta \in (0, 1)$, we introduce the regularization h_δ of the invariant measure as defined in Appendix B; we will be interested in estimates uniform in ε for δ arbitrarily small.

† This corresponds to an open dynamics where the trajectory disappears when it gets closer than $\nu^n \varepsilon$ to the boundary of an Ω_k.

‡ In the second line, we have equality since $\mathcal{L}$ is an L^1 isometry on positive functions, in the third we have used the Hölder inequality (A.1) and hypothesis (6). The fourth line follows from Lemma 3.1.

https://doi.org/10.1017/S0143385711000939 Published online by Cambridge University Press

Suppose that, for some $x_0 \in \Omega\backslash\Gamma_\varepsilon$, $\widetilde{\mathcal{L}}_{\varepsilon,n}1(x_0) > 0$. This means that, for each $y \in T^{-n}\{x_0\}$ that contributes a non-zero term to the sum defining $\widetilde{\mathcal{L}}_{\varepsilon,n}$, we have $T^k y \notin \Gamma_{\varepsilon\nu^{n-k-1}}$ for each $k \in \{0, \dots, n-1\}$. Hence, if $\|x_0 - z\| \le \varepsilon/2$, it follows that there exists $w \in T^{-n}\{z\}$ such that, for each $k \in \{0, \dots, n-1\}$, $\|T^k w - T^k y\| \le \nu_0^{n-k-1}\varepsilon/2$, hence $T^k w \notin \Gamma_{\varepsilon\nu^{n-k-1}/2}$. A standard distortion argument together with (B.1) yields

$$\begin{aligned}|\det(D_w T^n)^{-1} h_\delta(w)| &= e^{-\sum_{k=0}^{n-1} \ln|\det(D_{T^k w}T)|} h_\delta(w)\\ &\ge e^{-C_\eta\{\sum_{k=0}^{n-1} \nu_0^{(n-k)}\nu^{-(n-k)a}\varepsilon^{1-a} + \nu_0^n \varepsilon\delta^{-d-1}\}} |\det(D_y T^n)^{-1}| h_\delta(y),\end{aligned}$$

where we have used assumption (6). The above implies that, for each $\varepsilon > 0$, there exists $\gamma_\varepsilon > 0$ such that, for each $n \in \mathbb{N}$ and $\delta \ge \delta_n = \nu_0^{n/(d+1)}$, we have

$$\mathcal{L}^n h_\delta(z) \ge \widetilde{\mathcal{L}}_{\varepsilon/2,n} h_\delta(z) \ge \gamma_\varepsilon \widetilde{\mathcal{L}}_{\varepsilon,n} h_\delta(x_0). \tag{4.2}$$

It remains to find good points x_0. Let $A = \{x \in \Omega : h(x) \ge \frac{1}{2}\}$; then we have $m(A) > C_\#$†. Consider $B_{\delta,n} = \{x \in A : \widetilde{\mathcal{L}}_{\varepsilon,n} h_\delta(x) \le \frac{1}{4}\}$; then, by (4.1) and Lemma B.1,

$$\tfrac{1}{4} m(B_{\delta,n}) \ge \|\mathbb{1}_{B_{\delta,n}} h\|_{L^1} - \|h - \widetilde{\mathcal{L}}_{\varepsilon,n} h_\delta\|_{L^1} \ge \tfrac{1}{2} m(B_{\delta,n}) - C_\eta(\delta^{\alpha/d} + \varepsilon^{\alpha/d}).$$

Thus, provided δ, ε are both small enough, we have $m(A\backslash(B_{\delta,n} \cup \Gamma_\varepsilon)) \ge \frac{1}{2} m(A)$ for all $n \in \mathbb{N}$.

To conclude, set $\tilde{B}_{n,\varepsilon} = B_{\delta_n,n} \cup \Gamma_\varepsilon$. The above implies that there exist a positive measure set $D \subset A$ such that for each $x_0 \in D$ there exists an infinite sequence $\{n_j\}$ such that $x_0 \in A\backslash\tilde{B}_{n_j,\varepsilon}$ for all $j \in \mathbb{N}$‡. It follows that for all $x_0 \in D$ and all z in an $\varepsilon/2$ neighborhood of x_0, by (4.2), we have $\mathcal{L}^{n_j} h_{\delta_{n_j}}(z) \ge \frac{1}{4}\gamma_\varepsilon$§. On the other hand, $f_j := \mathcal{L}^{n_j} h_{\delta_{n_j}}$ converges to h in L^1¶, hence there exists a subsequence f_{j_k} which converges to h almost surely. Finally, this implies that $h \ge \frac{1}{4}\gamma_\varepsilon$ almost surely in the ball $\{z \in \Omega : \|z - x_0\| \le \varepsilon/2\}$, which shows that the support of any invariant measure must contain an open ball.

We are left with the task of proving the openness of the ergodic components. Given an ergodic component $\Delta \subset \Omega$, let h be the density of the a.c.i.m. μ supported on it. By the previous arguments, we know that there exists an open ball $B \subset \Delta$, such that $\mathbb{1}_B \le C_\# h$. Then $\mathcal{L}^n \mathbb{1}_B \le C_\# \mathcal{L}^n h = C_\# h$ implies that $\Lambda_0 := \bigcup_{n=0}^\infty T^n B$ is contained in the support of h. Next, notice that for all connected open sets $U \subset \Omega$, $U_1 = U\backslash T^{-1}\partial\Omega$ is open‖. Accordingly, $TU_1 \subset \Omega$ is open, which, by hypothesis (0), shows that TU differs

† Indeed, $1 = m(\mathbb{1}_A h) + m(\mathbb{1}_{A^c} h) \le C_\# \|h\|_{BV} m(A)^{1/d} + \frac{1}{2} m(A^c)$. Thus, $1 \le 2C_\# \|h\|_{BV} m(A)^{1/d} - m(A)$, which implies that $m(A) \ge (2C_\# \|h\|_{BV})^{-d}$. But, the second of the inequalities (3.1) implies that $\|h\|_{BV} = \|\mathcal{L}h\|_{BV} \le \sigma\|h\|_{BV} + B$. Hence, $m(A) \ge (2C_\# B)^{-d}(1-\sigma)^d$.

‡ By the monotone convergence theorem,

$$\int_\Omega \sum_{n=0}^\infty \mathbb{1}_{A\backslash\tilde{B}_{n,\varepsilon}} = \sum_{n=0}^\infty m(A\backslash\tilde{B}_{n,\varepsilon}) = \infty;$$

thus, the sum must diverge on a positive measure set.

§ Since $x_0 \in A\backslash\tilde{B}_{n_j,\varepsilon}$, it must be $\widetilde{\mathcal{L}}_{\varepsilon,n_j} 1(x_0) > 0$.

¶ In fact, $\|h - f_j\|_{L^1} = \|\mathcal{L}^{n_j}(h - h_{\delta_{n_j}})\|_{L^1} \le \|h - h_{\delta_{n_j}}\|_{L^1} \le C_\eta \delta_{n_j}^{\alpha/d} \|h\|_{BV}$, by Lemma B.1.

‖ Indeed, since U is connected, it must be contained in some Ω_k; then $T_k = T|_{\Omega_k}$ is continuous by hypothesis, hence $T^{-1}(\partial\Omega) \cap \Omega_k = T_k^{-1}(\partial\Omega)$ is closed.

https://doi.org/10.1017/S0143385711000939 Published online by Cambridge University Press

from an open set by a zero measure set. Iterating this argument† shows that there exists an open set $V \subset \Lambda_0$ such that $\mu(\Lambda_0 \backslash V) = 0$. On the other hand, $T^{-1}\Lambda_0 \supset \Lambda_0$, hence $\mu(T^{-1}\Lambda_0 \backslash \Lambda_0) = 0$. This implies that there exists $\Lambda_1 \supset \Lambda_0$, $\mu(\Lambda_1 \backslash \Lambda_0) = 0$, such that $T^{-1}\Lambda_1 = \Lambda_1$, hence $0 = \mu(\Delta \backslash \Lambda_1) = m(\Delta \backslash V)$, which implies the lemma. □

A. *Appendix. BV functions*

Here we collect some basic facts about BV functions‡.

LEMMA A.1. *There exists $C_d > 0$ such that, for each $h \in BV(\Omega)$, $A \subset \mathbb{R}^d$,*

$$\|\mathbb{1}_A h\|_{L^1} \le m(A)^{1/d} \|h\|_{L^{d/(d-1)}} \le C_d m(A)^{1/d} \|h\|_{BV}. \tag{A.1}$$

The set $B_1 := \{h \in BV : \|h\|_{BV} \le 1\}$ is relatively compact in the L^1 topology.

For all $j \in \{1, \dots, d\}$, $h \in BV(\mathbb{R}^d)$ and almost all $x_{\neq j}$, $h(x_{\neq j}, \cdot) \in BV(\mathbb{R}) \subset L^\infty(\mathbb{R}, m)$.

For each $h \in BV(\mathbb{R}^d)$ and all $\varphi \in L^\infty(\mathbb{R}^d, m)$ of compact support and such that $x_{\neq j}$ almost surely, $\varphi_j(x_{\neq j}, \cdot) \in \mathcal{C}^0(\mathbb{R}, \mathbb{R})$, $(\partial_{x_j}\varphi_j)(x_{\neq j}, \cdot) \in L^1(\mathbb{R}, m)$, we have

$$\left| \int_{\mathbb{R}^d} h \operatorname{div} \varphi \right| \le \|h\|_{BV(\mathbb{R}^d)} \|\varphi\|_{L^\infty}. \tag{A.2}$$

Proof. The first inequality of (A.1) is a Hölder inequality:

$$\int_A |h| \le m(A)^{1/d} \|h\|_{L^{d/(d-1)}},$$

while the second is a Sobolev-type inequality; see [**16**, §5.6.1, Theorem 1(i)].

The second statement follows directly from [**16**, §5.2.3, Theorem 4], since $B_1 \subset \{h \in BV((-1, 2)^d) : \|h\|_{BV((-1,2)^d)} \le 1\}$.

To prove the third statement, let

$$V_j h(x_{\neq j}) = \sup_{\substack{\varphi \in \mathcal{C}^1_0(\mathbb{R}, \mathbb{R}) \\ |\varphi|_{L^\infty} \le 1}} \int_{\mathbb{R}} h(x_{\neq j}, t) \varphi'(t)\, dt.$$

Then [**16**, §5.10.2, Theorem 2] states that $V_j h \in L^1(\mathbb{R}^{d-1}, m)$. Hence, $V_j h$ must be almost surely finite, from which the statement follows (taking into account (A.1) for $d = 1$).

To prove (A.2), we have to deal with the fact that φ is not a legal test function since it may be discontinuous. Nevertheless, this is a superficial problem: given a positive function $\eta \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ with integral one and supported on $[-1, 1]$, define $\eta_\varepsilon(y) = \varepsilon^{-1}\eta(\varepsilon^{-1}y)$, $y \in \mathbb{R}$, and $\bar\eta_{j,\varepsilon}(x) = \prod_{\substack{k=1 \\ k \neq j}}^d \eta_\varepsilon(x_k)$ and let $*$ stand for the usual convolution. We then define $\varphi_{\varepsilon_1,\varepsilon_2} \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R}^d)$ by§ $(\varphi_{\varepsilon_1,\varepsilon_2})_j := \bar\eta_{j,\varepsilon_1} * (\eta_{\varepsilon_2} * \varphi_j)$. Note that $\|\varphi_{\varepsilon_1,\varepsilon_2}\|_{L^\infty} \le \|\varphi\|_{L^\infty}$.

† By considering a connected component at a time.

‡ For the general theory of BV functions, see [**16**], or see [**24**] for a quick introduction to the properties relevant to the present context.

§ The second convolution is with respect to the variable x_j.

https://doi.org/10.1017/S0143385711000939 Published online by Cambridge University Press

For each $h \in BV(\mathbb{R}^d)$, the second statement of the lemma implies, $x_{\neq j}$ almost surely†,

$$\begin{aligned}
&\int dx_j h(x_{\neq j}, x_j)\partial_{x_j}\varphi_j(x_{\neq j}, x_j) \\
&\quad = \lim_{\varepsilon_2\to 0}\int dx_j h(x_{\neq j}, x_j)\int_{\mathbb{R}} dy\,\partial_y\varphi_j(x_{\neq j}, y)\eta_{\varepsilon_2}(x_j - y) \\
&\quad = \lim_{\varepsilon_2\to 0}\int dx_j h(x_{\neq j}, x_j)\partial_{x_j}\int_{\mathbb{R}} dy\,\varphi_j(x_{\neq j}, y)\eta_{\varepsilon_2}(x_j - y).
\end{aligned}$$

In addition, for each $\varepsilon_2 > 0$, the right-hand side is bounded by $V_j h\|\varphi_j\|_{L^\infty}$, which, by the above results, belongs to $L^1(\mathbb{R}^{d-1}, m)$. Hence, by Lebesgue's dominated convergence theorem,

$$\begin{aligned}
\int_{\mathbb{R}^d} h\partial_{x_j}\varphi_j &= \lim_{\varepsilon_2\to 0}\int_{\mathbb{R}^d} h\partial_{x_j}(\eta_{\varepsilon_2} * \varphi_j) \\
&= \lim_{\varepsilon_2\to 0}\lim_{\varepsilon_1\to 0}\int_{\mathbb{R}^d}\bar{\eta}_{j,\varepsilon_1} * h\cdot(\partial_{x_j}\eta_{\varepsilon_2} * \varphi_j) \\
&= \lim_{\varepsilon_2\to 0}\lim_{\varepsilon_1\to 0}\int_{\mathbb{R}^d} h\cdot\partial_{x_j}[\bar{\eta}_{j,\varepsilon_1} * (\eta_{\varepsilon_2} * \varphi_j)].
\end{aligned}$$

Thus,

$$\begin{aligned}
\left|\int_{\mathbb{R}^d} h\,\mathrm{div}\,\varphi\right| = \lim_{\varepsilon_2\to 0}\lim_{\varepsilon_1\to 0}\left|\int_{\mathbb{R}^d} h\,\mathrm{div}\,\varphi_{\varepsilon_1,\varepsilon_2}\right| &\le \lim_{\varepsilon_2,\varepsilon_1\to 0}\|h\|_{BV(\mathbb{R}^d)}\|\varphi_{\varepsilon_1,\varepsilon_2}\|_{L^\infty} \\
&\le \|h\|_{BV(\mathbb{R}^d)}\|\varphi\|_{L^\infty}. \qquad \square
\end{aligned}$$

B. *Appendix. A regularization scheme*

Here I describe a way to regularize a $BV(\Omega)$ density and some relevant related properties. This is fairly standard; I add it here just for the reader's convenience.

Let us fix $\eta \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}_+)$, $\mathrm{supp}\,\eta \subset [-1, 1]$, $\int \eta = 1$ and set, for $\delta > 0$,

$$\bar{\eta}_\delta(x) = \delta^{-d}\prod_{i=1}^{d}\eta(\delta^{-1}x_i).$$

Let $h \in BV(\Omega)$ be such that, $h \ge 0$, $\int h = 1$. For each $\delta > 0$, define

$$h_\delta(x) = \int_\Omega \bar{\eta}_\delta(x - y)h(y)\,dy + \delta.$$

LEMMA B.1. *If Ω satisfies the first inequality of hypothesis (6), then the following properties hold true:*

$$\begin{aligned}
h_\delta &\ge \delta, \\
\|\nabla h_\delta\|_{L^1} &\le \|h\|_{BV}, \\
\|\nabla h_\delta\|_{L^\infty} &\le C_\eta\delta^{-d}\|h\|_{BV}, \\
\|h - h_\delta\|_{L^1} &\le C_\eta\delta^{\alpha/d}\|h\|_{BV}.
\end{aligned}$$

† The integration by parts in the last line is allowed, since φ_j is, $x_{\neq j}$ almost surely, of bounded variation in the x_j variable.

https://doi.org/10.1017/S0143385711000939 Published online by Cambridge University Press

Proof. The first inequality is trivial. The second follows from

$$\|\nabla h_\delta\|_{L^1} = \sup_{\substack{\varphi\in\mathcal{C}^1_0\\ \|\varphi\|_{L^\infty}\le 1}} \sum_{k=1}^d \int_{\mathbb{R}^d} dx\varphi_k(x) \int_\Omega dy \partial_{x_k}\bar\eta_\delta(x-y)h(y)$$

$$= \sup_{\substack{\varphi\in\mathcal{C}^1_0\\ \|\varphi\|_{L^\infty}\le 1}} \sum_{k=1}^d \int_\Omega h(x)\partial_{x_k}(\bar\eta_\delta * \varphi_k)(x).$$

The next inequality follows from $\|\bar\eta_\delta\|_{L^\infty} \le C_\eta \delta^{-d}$. To prove the last inequality, let $\chi_\delta \in \mathcal{C}^1(\mathbb{R}^d, [0,1])$ be such that $\operatorname{supp}\chi_\delta \subset \Omega^{\delta/2}$ and $\operatorname{supp}(1-\chi_\delta) \subset (\Omega^\delta)^c$. Then, by (A.1),

$$\|h - h_\delta\|_{L^1} \le \|\chi_\delta(h - \bar\eta_\delta * h)\|_{L^1} + \delta\|h\|_{L^1} + C\delta^{\alpha/d}\|h\|_{BV}.$$

Next, for each $g \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R})$, and $\varphi \in \mathcal{C}^0(\Omega, \mathbb{R})$, let $\varphi_\delta = \chi_\delta\varphi$ and

$$\begin{aligned}
&\int_\Omega dx \int_{\mathbb{R}^d} dy\, \varphi_\delta(x)\bar\eta_\delta(x-y)(g(y)-g(x))\\
&= \int_{\Omega\times\mathbb{R}^d} \int_0^1 dt\, \varphi_\delta(x)\bar\eta_\delta(x-y)\langle\nabla g(x+(y-x)t),\, y-x\rangle\\
&= \int_{\Omega\times\mathbb{R}^d} \int_0^1 dt\, t^{-d-1}\varphi_\delta(x)\bar\eta_\delta(t^{-1}(x-\xi))\langle\nabla g(\xi),\, \xi-x\rangle\\
&= \delta\int_0^1 dt \int_{\mathbb{R}^d} \operatorname{div}\,(\hat\eta_{t\delta} * \varphi_\delta)\cdot g,
\end{aligned}$$

where $\hat\eta_\delta(x) = \bar\eta_\delta(x)x\delta^{-1}$. Hence,

$$\|\chi_\delta(g - \bar\eta_\delta * g)\|_{L^1(\Omega)} \le C_\eta\delta\|g\|_{BV(\mathbb{R}^d)}$$

and the same then holds for $g \in BV$†. Accordingly, for each $h \in BV$,

$$\|h - h_\delta\|_{L^1} \le C_\eta\delta^{\alpha/d}\|h\|_{BV(\Omega)}. \qquad \square$$

Here is a useful consequence of the above lemma: for each two points $x,\, y \in \mathbb{R}^d$,

$$h_\delta(x) \le e^{C_\eta\delta^{-d-1}\|x-y\|}h_\delta(y). \tag{B.1}$$

This follows since, setting $g(t) = h_\delta(tx + (1-t)y)$, we have

$$\left|\ln\frac{h_\delta(x)}{h_\delta(y)}\right| = |\ln g(1) - \ln g(0)| \le C_\#\|x-y\|\frac{\|\nabla h_\delta\|_{L^\infty}}{\inf h_\delta} \le C_\eta\|x-y\|\delta^{-d-1}.$$

Acknowledgements. It is a pleasure to thank Viviane Baladi and Gabriella Tarantello for helpful discussions. Also, I would like to thank the Fields Institute, Toronto, where the paper was completed. The work was supported by the European Advanced Grant MALADY (ERC AdG 246953). Lastly, I would like to thank the anonymous referee for pointing out a really silly mistake and saving me from public embarrassment.

† Indeed, if $g \in BV$, then there exists $\{g_n\} \subset \mathcal{C}^1$, $\sup_n \|g_n\|_{BV} < \infty$ such that g_n converges to g in L^1 (see [**16**, ¶5.2.2, Theorem 2]) and the claim follows since convolutions are L^1 contractions.

https://doi.org/10.1017/S0143385711000939 Published online by Cambridge University Press

References

[1] V. Baladi. *Positive Transfer Operators and Decay of Correlations (Advanced Series in Nonlinear Dynamics, 16)*. World Scientific, River Edge, NJ, 2000.

[2] V. Baladi. Anisotropic Sobolev spaces and dynamical transfer operators: $\mathcal{C}^\infty$ foliations. *Algebraic and Topological Dynamics (Contemporary Mathematics, 385)*. Eds. S. Kolyada, Y. Manin and T. Ward. American Mathematical Society, Providence, RI, 2005, pp. 123–135.

[3] V. Baladi and S. Gouëzel. Good Banach spaces for piecewise hyperbolic maps via interpolation. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **26** (2009), 1453–1481.

[4] V. Baladi and S. Gouëzel. Banach spaces for piecewise cone hyperbolic maps. *J. Mod. Dyn.* **4** (2010), 91–137.

[5] V. Baladi and C. Liverani. Exponential decay of correlations for piecewise cone hyperbolic contact flows. *Preprint*, arXiv:1105.0567v1.

[6] V. Baladi and M. Tsujii. Anisotropic Hölder and Sobolev spaces for hyperbolic diffeomorphisms. *Ann. Inst. Fourier* **57** (2007), 127–154.

[7] V. Baladi and M. Tsujii. Dynamical determinants and spectrum for hyperbolic diffeomorphisms. *Geometric and Probabilistic Structures in Dynamics (Contemporary Mathematics, 469)*. Eds. K. Burns, D. Dolgopyat and Ya. Pesin. American Mathematical Society, Providence, RI, 2008, pp. 29–68.

[8] M. Blank, G. Keller and C. Liverani. Ruelle–Perron–Frobenius spectrum for Anosov maps. *Nonlinearity* **15**(6) (2001), 1905–1973.

[9] O. Butterley and C. Liverani. Smooth Anosov flows: correlation spectra and stability. *J. Mod. Dyn.* **2** (2007), 301–322.

[10] J. Buzzi. Absolutely continuous invariant probability measures for arbitrary expanding piecewise $\mathbb{R}$-analytic mappings of the plane. *Ergod. Th. & Dynam. Sys.* **20**(3) (2000), 697–708.

[11] J. Buzzi. No or infinitely many a.c.i.p. for piecewise expanding C^r maps in higher dimensions. *Comm. Math. Phys.* **222**(3) (2001), 495–501.

[12] J. Buzzi. Absolutely continuous invariant measures for generic multi-dimensional piecewise affine expanding maps. Discrete dynamical systems. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **9**(9) (1999), 1743–1750.

[13] W. J. Cowieson. Stochastic stability for piecewise expanding maps in R^d. *Nonlinearity* **13**(5) (2000), 1745–1760.

[14] W. J. Cowieson. Absolutely continuous invariant measures for most piecewise smooth expanding maps. *Ergod. Th. & Dynam. Sys.* **22**(4) (2002), 1061–1078.

[15] M. Demers and C. Liverani. Stability of statistical properties in two-dimensional piecewise hyperbolic maps. *Trans. Amer. Math. Soc.* **360** (2008), 4777–4814.

[16] L. C. Evans and R. F. Gariepy. *Measure Theory and Fine Properties of Functions (Studies in Advanced Mathematics)*. CRC Press, Boca Raton, FL, 1992, viii+268 pp.

[17] P. Góra and A. Boyarsky. Absolutely continuous invariant measures for piecewise expanding C^2 transformation in R^N. *Israel J. Math.* **67**(3) (1989), 272–286.

[18] S. Gouëzel. Almost sure invariance principle for dynamical systems by spectral methods. *Ann. Probab.* **38**(4) (2010), 1639–1671.

[19] S. Gouëzel and C. Liverani. Banach spaces adapted to Anosov systems. *Ergod. Th. & Dynam. Sys.* **26**(1) (2006), 189–217.

[20] S. Gouëzel and C. Liverani. Compact locally maximal hyperbolic sets for smooth maps: fine statistical properties. *J. Differential Geom.* **79** (2008), 433–477.

[21] H. Hennion. Sur un théorème spectral et son application aux noyaux lipchitziens. *Proc. Amer. Math. Soc.* **118**(2) (1993), 627–634.

[22] G. Keller. Ergodicité et mesures invariantes pour les transformations dilatantes par morceaux d'une région bornée du plan. *C. R. Acad. Sci. Paris Sér. A–B* **289**(12) (1979), A625–A627.

[23] G. Keller. Mixing for finite systems of coupled tent maps. *Tr. Mat. Inst. Steklova* **216** (1997), 320–326 Din. Sist. i Smezhnye Vopr., translation in *Proc. Steklov Inst. Math.* **216**(1) (1997), 315–332.

[24] G. Keller and C. Liverani. A spectral gap for a one-dimensional lattice of coupled piecewise expanding interval maps. *Dynamics of Coupled Map Lattices and of Related Spatially Extended Systems (Lecture Notes in Physics, 671)*. Eds. J. Chazottes and B. Fernandez. Springer, Berlin, 2005, pp. 115–151.

[25] G. Keller and C. Liverani. Stability of the spectrum for transfer operators. *Ann. Sc. Norm. Super. Pisa Cl. Sci.*(4) **28**(1) (1999), 141–152.

[26] G. Keller and C. Liverani. Rare events, escape rates and quasistationarity: some exact formulae. *J. Stat. Phys.* **135**(3) (2009), 519–534.

https://doi.org/10.1017/S0143385711000939 Published online by Cambridge University Press

[27] C. Liverani. Decay of correlations. *Ann. of Math.* (2) **142** (1995), 239–301.
[28] C. Liverani. Central limit theorem for deterministic systems. *International Conference on Dynamical Systems (Montevideo, 1995) (Pitman Research Notes in Mathematics Series, 362)*. Longman, Harlow, 1996, pp. 56–75.
[29] C. Liverani. On contact Anosov flows. *Ann. of Math.* (2) **159**(3) (2004), 1275–1312.
[30] C. Liverani. Fredholm determinants, Anosov maps and Ruelle resonances. *Discrete Contin. Dyn. Syst.* **13**(5) (2005), 1203–1215.
[31] C. Liverani and M. Tsujii. Zeta functions and dynamical systems. *Nonlinearity* **19**(10) (2006), 2467–2473.
[32] R. Murray. Existence, mixing and approximation of invariant densities for expanding maps on $\mathbb{R}^r$. *Nonlinear Anal.* **45**(1) (2001), 37–72 Ser. A: Theory Methods.
[33] H. Proppe, P. Góra and A. Boyarsky. Inadequacy of the bounded variation technique in the ergodic theory of higher-dimensional transformations. *Nonlinearity* **3**(4) (1990), 1081–1087.
[34] M. Rychlik. Bounded variation and invariant measures. *Studia Math.* **76**(1) (1983), 69–80.
[35] B. Saussol. Absolutely continuous invariant measures for multidimensional expanding maps. *Israel J. Math.* **116** (2000), 223–248.
[36] D. Thomine. A spectral gap for transfer operators of piecewise expanding maps. *Discrete Contin. Dyn. Syst.* **30**(3) (2011), 917–944.
[37] M. Tsujii. Piecewise expanding maps on the plane with singular ergodic properties. *Ergod. Th. & Dynam. Sys.* **20**(6) (2000), 1851–1857.
[38] M. Tsujii. Absolutely continuous invariant measures for expanding piecewise linear maps. *Invent. Math.* **143**(2) (2001), 349–373.

https://doi.org/10.1017/S0143385711000939 Published online by Cambridge University Press