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SPACES OF COMPACT OPEN SUBGROUPS AND OPEN MORPHISMS

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Abstract

For a morphism whose domain is a totally disconnected locally compact group, the openness test of Cowling, Hofmann and Morris ['Open mappings of locally compact groups', *J. Group Theory* **27**(6) (2024), 1143–1149] can be stated in terms of compact open subgroups. We use this formulation to study spaces of compact open subgroups. A surjective open morphism with compact kernel induces a quotient map between such spaces, and the inverse-image map identifies the target with a closed retract of the source. Conversely, for a surjective open morphism whose target has a compact open subgroup, compactness of the kernel is forced by the requirement that inverse images of compact open subgroups remain compact.

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1. Introduction

A morphism of locally compact groups is a continuous homomorphism between locally compact Hausdorff groups. Cowling, Hofmann and Morris gave several tests for the openness of such a morphism [2, Theorem 2 and Remark 3]. In particular, $\phi : G \to H$ is open as soon as $\phi(U)$ is open in H for every open subgroup U of G. They also gave quotient and measure versions of this fact.

The maps induced by homomorphisms on spaces of subgroups have been studied from several directions. For compact groups, Fisher and Gartside studied the Chabauty space of closed subgroups [4]. For locally compact abelian groups, Cornulier analysed natural maps attached to open subgroups and to quotients by compact subgroups; in particular, the natural projection from the Chabauty space of closed subgroups of G to that of G/K is continuous when K is compact [1]. Hamrouni and Kadri proved image-map, inverse-image-map and retraction statements for Chabauty spaces of closed subgroups under proper or open quotient hypotheses [5, Propositions 2.1, 2.5, and 2.8]. Jlali studied the induced map $L \mapsto \phi(L)$ on the subspace of compact

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subgroups in the Vietoris space of closed subgroups, relating the openness of ϕ to that of the induced map [8].

We work with the subspace of compact *open* subgroups, equipped with the relative Vietoris topology inherited from compact subsets, for a general locally compact group that need not be abelian or compact. These quotient and retraction statements have an elementary form on spaces of compact open subgroups once the kernel is compact, because inverse images of compact open subgroups then remain compact open. We prove the corresponding quotient and closed-retract statements, and isolate one precise role played by compactness of the kernel: as soon as the target has one compact open subgroup, the inverse-image assignment takes values in compact open subgroups only when the kernel is compact. The continuity of the image map used below is verified directly. The induced map is closely related to the subgroup-space maps studied in [8]. That work treats the subspace of all compact subgroups and relates the openness of ϕ to that of the induced map, whereas here, we use the relative Vietoris topology on the family of compact open subgroups and prove quotient-map and closed-retract statements, together with a kernel-compactness criterion.

In a totally disconnected locally compact group, compact open subgroups form a base at the identity. Proposition 3.1 recasts the openness criteria of [2, Theorem 2 and Remark 3] in terms of compact open subgroups. Its positive-measure clause is the compact open subgroup case of the measure criterion of [2, Corollary 8] and its category clause follows from the Baire category theorem.

The final examples separate the remaining hypotheses. The induced map need not be injective before it is restricted to $C_{o,\ker\phi}(G)$, openness cannot be replaced by continuity and the compact open subgroup test says nothing when the domain has no compact open subgroups.

2. Preliminaries

Throughout, all locally compact groups are assumed to be Hausdorff. Let G be a locally compact group. Put

$$C_o(G) = \{U \leq G : U \text{ is compact and open in } G\}.$$

If N is a compact normal subgroup of G, put

$$C_{o,N}(G) = \{U \in C_o(G) : N \subseteq U\}.$$

The Vietoris topology [9] on the family $\mathcal{K}(G)$ of nonempty compact subsets of G has basic sets of the form

$$\langle O \mid O_1, \ldots, O_n\rangle = \{K \in \mathcal{K}(G) : K \subseteq O \text{ and } K \cap O_i \neq \emptyset \text{ for } 1 \leq i \leq n\},$$

where $O, O_1, \ldots, O_n$ are open subsets of G. The case $n = 0$ is allowed, so $\langle O\rangle$ denotes $\{K \in \mathcal{K}(G) : K \subseteq O\}$. The spaces $C_o(G)$ and $C_{o,N}(G)$ are given the relative topology inherited from $\mathcal{K}(G)$. When G is compact, this agrees with the usual Chabauty topology on closed subgroups [9]. For a general locally compact group, we use the

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relative Vietoris topology just defined; the continuity statements we need are checked directly below.

We shall use the following standard facts. By van Dantzig's theorem [10], compact open subgroups form a base at the identity in every totally disconnected locally compact group. In addition, if N is a compact normal subgroup of a locally compact group G, then the quotient map $G \to G/N$ is open, closed and proper [6, 7].

3. Main results

We first state the open-mapping tests in terms of compact open subgroups.

PROPOSITION 3.1. *Let G be a totally disconnected locally compact group, let H be a locally compact group, let μ be a left Haar measure on H and let $\phi : G \to H$ be a morphism. The following statements are equivalent:*

(1) *ϕ is open;*
(2) *$\phi(U)$ is open in H for every $U \in \mathcal{C}_o(G)$;*
(3) *there is $U \in \mathcal{C}_o(G)$ such that $\phi(U)$ is open in H;*
(4) *there is a base $\mathcal{B} \subseteq \mathcal{C}_o(G)$ at the identity of G such that $\phi(U)$ is open in H for every $U \in \mathcal{B}$;*
(5) *$\mu(\phi(U)) > 0$ for some $U \in \mathcal{C}_o(G)$;*
(6) *$\phi(U)$ is nonmeagre in H for some $U \in \mathcal{C}_o(G)$.*

PROOF. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are immediate and (2) $\Rightarrow$ (4) follows from van Dantzig's theorem.

Assume statement (3) and choose $U \in \mathcal{C}_o(G)$ such that $\phi(U)$ is open in H. The restriction

$$\phi|_U : U \to \phi(U)$$

is a continuous surjective homomorphism from the compact group U onto the compact Hausdorff group $\phi(U)$, and hence, the quotient map by its kernel. In particular, it is open. Let O be open in G and take $x \in O$. Put

$$W = U \cap x^{-1}O.$$

Then, W is open in U with $xW \subseteq O$ and $\phi(W)$ is open in $\phi(U)$, and hence, in H because $\phi(U)$ is open there. Therefore, $\phi(xW) = \phi(x)\phi(W)$ is open in H and

$$\phi(O) = \bigcup_{x \in O} \phi(x(U \cap x^{-1}O))$$

is open. Hence, ϕ is open, giving statement (1).

For (4) $\Rightarrow$ (1), let O be open in G and take $x \in O$. Choose $U \in \mathcal{B}$ with $U \subseteq x^{-1}O$. Then, $xU \subseteq O$ and $\phi(xU) = \phi(x)\phi(U)$ is open in H. Hence, $\phi(O)$ is open.

If ϕ is open and $U \in \mathcal{C}_o(G)$, then $\phi(U)$ is a nonempty compact open subgroup of H, so it has positive Haar measure and is nonmeagre. This gives (1) $\Rightarrow$ (5) and (1) $\Rightarrow$ (6).

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For the two converses, recall that a subgroup with nonempty interior is open, so in each case, it suffices to find an interior point of $\phi(U)$. Under statement (5), $\phi(U)$ is a compact, and hence measurable, subgroup of positive Haar measure. By Steinhaus' theorem, $\phi(U)\phi(U)^{-1}$ is a neighbourhood of the identity and it equals $\phi(U)$ because $\phi(U)$ is a subgroup. Hence, $\phi(U)$ has nonempty interior. (This is the compact open subgroup case of the measure criterion of [2, Corollary 8].) Under statement (6), $\phi(U)$ is a compact, and hence closed, subgroup of H. Since locally compact groups are Baire and $\phi(U)$ is nonmeagre, it has nonempty interior. Either way, $\phi(U)$ is open in H and statement (3) holds. □

We shall use the following quotient calculation.

PROPOSITION 3.2. *Let G be a locally compact group, let N be a compact normal subgroup of G and let $q : G \to G/N$ be the quotient morphism. Then,*

$$q_{\#} : C_{o,N}(G) \to C_o(G/N), \quad q_{\#}(U) = q(U),$$

is a homeomorphism.

PROOF. Analogous quotient statements are known for Chabauty spaces of closed subgroups under proper quotient maps [5, Corollaries 2.4 and 2.7]. We give the direct verification needed here, in the relative Vietoris topology.

If $U \in C_{o,N}(G)$, then $q(U)$ is a compact open subgroup of G/N. Conversely, if $V \in C_o(G/N)$, then $q^{-1}(V)$ is an open subgroup of G containing $N = \ker q$ and it is compact because q is proper. Thus, $q^{-1}(V) \in C_{o,N}(G)$. Since

$$q^{-1}(q(U)) = UN = U$$

for $U \in C_{o,N}(G)$ and since $q(q^{-1}(V)) = V$, the map $q_{\#}$ is bijective with inverse $V \mapsto q^{-1}(V)$.

Continuity of $q_{\#}$ follows immediately from the Vietoris topology: for a basic open set $\langle W \mid W_1, \dots, W_n\rangle$ in $\mathcal{K}(G/N)$, its inverse image under $q_{\#}$ is

$$\langle q^{-1}(W) \mid q^{-1}(W_1), \dots, q^{-1}(W_n)\rangle \cap C_{o,N}(G).$$

It remains to check continuity of the inverse. Let $V \in C_o(G/N)$ and suppose that

$$q^{-1}(V) \in \langle O \mid O_1, \dots, O_n\rangle \cap C_{o,N}(G)$$

for open subsets $O, O_1, \dots, O_n$ of G. Since $q^{-1}(V) \subseteq O$,

$$V \subseteq (G/N) \setminus q(G \setminus O).$$

The set $(G/N) \setminus q(G \setminus O)$ is open because q is closed. Moreover, each $q(O_i)$ is open and $V \cap q(O_i) \neq \emptyset$ since $q^{-1}(V) \cap O_i \neq \emptyset$. Hence,

$$\langle (G/N) \setminus q(G \setminus O) \mid q(O_1), \dots, q(O_n)\rangle \cap C_o(G/N)$$

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is a neighbourhood of V and every member L of this neighbourhood satisfies

$$q^{-1}(L) \in \langle O \mid O_1, \dots, O_n \rangle \cap C_{o,N}(G).$$

Thus, $V \mapsto q^{-1}(V)$ is continuous. For related quotient maps in Chabauty spaces of locally compact abelian groups, see [1]. □

The quotient case gives the corresponding statement for surjective open morphisms.

THEOREM 3.3. *Let $\phi : G \to H$ be a surjective open morphism of locally compact groups with compact kernel. Then,*

$$\phi_\# : C_o(G) \to C_o(H), \quad \phi_\#(U) = \phi(U),$$

is a quotient map. Moreover, the map

$$s : C_o(H) \to C_o(G), \quad s(V) = \phi^{-1}(V),$$

is a continuous section of $\phi_\#$ whose image is the closed subspace $C_{o,\ker\phi}(G)$. In particular, $C_{o,\ker\phi}(G)$ is a closed retract of $C_o(G)$ and the restriction

$$\phi_\#\big|_{C_{o,\ker\phi}(G)} : C_{o,\ker\phi}(G) \to C_o(H)$$

is a homeomorphism.

PROOF. Since ϕ is open, $\phi(U) \in C_o(H)$ whenever $U \in C_o(G)$. Being a restriction of the image map on compact subsets, $\phi_\#$ is continuous: for a basic Vietoris set $\langle O \mid O_1, \dots, O_n \rangle$ in $\mathcal{K}(H)$, its inverse image under $\phi_\#$ is

$$\langle \phi^{-1}(O) \mid \phi^{-1}(O_1), \dots, \phi^{-1}(O_n) \rangle \cap C_o(G).$$

This direct verification is the standard continuity of the induced hyperspace map in the Vietoris topology, as in [3, 9].

Put $N = \ker\phi$. Since ϕ is open and surjective, the induced bijective morphism $\bar{\phi} : G/N \to H$, with $\phi = \bar{\phi} \circ q$ and $q : G \to G/N$ the quotient map, is an isomorphism of topological groups and such an isomorphism carries spaces of compact open subgroups homeomorphically onto one another, since the image map of a homeomorphism is a homeomorphism for the Vietoris topology. Write $\bar{\phi}_\# : C_o(G/N) \to C_o(H)$, $\bar{\phi}_\#(W) = \bar{\phi}(W)$, for the map it induces. Since N is compact, Proposition 3.2 gives the homeomorphism $q_\# : C_{o,N}(G) \to C_o(G/N)$ and composing inverses gives a continuous map

$$s := q_\#^{-1} \circ \bar{\phi}_\#^{-1} : C_o(H) \to C_o(G), \quad s(V) = q^{-1}(\bar{\phi}^{-1}(V)) = \phi^{-1}(V),$$

with image $C_{o,N}(G)$. Here, $\phi^{-1}(V)$ is compact because $\phi = \bar{\phi} \circ q$ is proper. Since $\phi_\#(s(V)) = V$ for every $V \in C_o(H)$, the map s is a continuous section and $\phi_\#$ is surjective. Its image $C_{o,N}(G)$ is closed in $C_o(G)$. Indeed, if $L \in C_o(G)$ does not contain N, choose $n \in N \setminus L$. Then, $O = G \setminus \{n\}$ is an open set containing L and $\langle O \rangle \cap C_o(G)$ is a neighbourhood of L that misses $C_{o,N}(G)$.

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Since $\phi_\# \circ s$ is the identity on $C_o(H)$, the restriction of $\phi_\#$ to $C_{o,N}(G)$ and the map s are inverse homeomorphisms. In addition, the map

$$r = s \circ \phi_\# : C_o(G) \to C_o(G)$$

is continuous, maps $C_o(G)$ into $C_{o,N}(G)$ and is the identity on that subspace: if $L = s(V)$, then $r(L) = s(\phi_\#(s(V))) = s(V) = L$. Hence, r is a retraction onto $C_{o,N}(G)$. This parallels the closed-subgroup retraction for proper quotient morphisms in [5, Proposition 2.8].

Finally, a continuous surjection that admits a continuous section is a quotient map: if $A \subseteq C_o(H)$ has $\phi_\#^{-1}(A)$ open in $C_o(G)$, then

$$A = s^{-1}(\phi_\#^{-1}(A))$$

is open in $C_o(H)$. □

The compactness of the kernel can also be read off from the inverse-image assignment itself.

PROPOSITION 3.4. *Let $\phi : G \to H$ be a surjective open morphism of locally compact groups and suppose that $C_o(H) \neq \emptyset$. The following statements are equivalent:*

(1) $\ker \phi$ *is compact;*
(2) $\phi^{-1}(V) \in C_o(G)$ *for every* $V \in C_o(H)$*;*
(3) $\phi^{-1}(V)$ *is compact for some* $V \in C_o(H)$*.*

When these conditions hold, the map $V \mapsto \phi^{-1}(V)$ is the continuous section in Theorem 3.3.

PROOF. The implication (1) ⇒ (2) is contained in Theorem 3.3, and (2) ⇒ (3) follows from $C_o(H) \neq \emptyset$. If statement (3) holds for some $V \in C_o(H)$, then $e_H \in V$, so $\ker \phi \subseteq \phi^{-1}(V)$. The kernel is closed in G and, hence, it is a closed subset of the compact set $\phi^{-1}(V)$. Thus, $\ker \phi$ is compact. □

Theorem 3.3 cannot be strengthened by making $\phi_\#$ injective on all of $C_o(G)$. The restriction to $C_{o,\ker\phi}(G)$ is necessary.

EXAMPLE 3.5. Let $G = C_2$ be the two-element discrete group, let H be the trivial group and let $\phi : G \to H$ be the quotient morphism. Then, ϕ is a surjective open morphism with compact kernel. Both $\{e\}$ and G belong to $C_o(G)$, and

$$\phi_\#(\{e\}) = \phi_\#(G) = H.$$

Hence, $\phi_\# : C_o(G) \to C_o(H)$ is not injective. It becomes a homeomorphism only after restriction to the compact open subgroups that contain $\ker \phi$.

The openness assumption in Theorem 3.3 cannot be replaced by continuity alone.

EXAMPLE 3.6. Let G be the additive group of $\mathbb{R}$ with the discrete topology, let H be the additive group of $\mathbb{R}$ with its usual topology and let $\iota : G \to H$ be the identity homomorphism. Then, ι is a continuous bijective morphism with compact kernel, but

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it is not open. The compact open subgroups of G consist only of $\{0\}$, while $\{0\}$ is not open in H. Hence, the compact open subgroup correspondence in Theorem 3.3 fails without the openness assumption.

The test of Proposition 3.1 also depends on the domain: with no compact open subgroups available, it says nothing.

EXAMPLE 3.7. Let $G = \mathbb{R}$ and $H = \mathbb{R}$, both with their usual locally compact group topologies, and let $\phi : G \to H$ be the zero morphism. Since $\mathbb{R}$ has no compact open subgroup, the requirement that $\phi(U)$ be open for every $U \in C_o(G)$ holds vacuously. Yet ϕ is not open, as it sends every nonempty open interval to $\{0\}$.

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