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Magnetohydrodynamic waves in non-uniform flows I: a variational approach

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Abstract. The interaction of magnetohydrodynamic (MHD) waves in a non-uniform, time-dependent background plasma flow is investigated using Lagrangian field theory methods. The analysis uses Lagrangian maps, in which the position of the fluid element $\mathbf{x}^*$ is expressed as a vector sum of the position vector $\mathbf{x}$ of the background plasma fluid element plus a Lagrangian displacement $\boldsymbol{\xi}(\mathbf{x}, t)$ due to the waves. Linear, non-Wentzel–Kramer–Brillouin (WKB) wave interaction equations are obtained by expansion of the Lagrangian out to second order in $\boldsymbol{\xi}$ and ΔS, where ΔS is the Lagrangian entropy perturbation. The characteristic manifolds of the waves are determined by consideration of the Cauchy problem for the wave interaction equations. The manifolds correspond to the usual MHD waves modes, namely the Alfvén waves, the fast and slow magnetoacoustic waves and the entropy wave. The relationships between the characteristic manifolds, and the ray equations of geometrical MHD optics are developed using the theory of Cauchy characteristics for first-order partial differential equations. The first-order differential equations describing the singular manifolds are the dispersion equations for the MHD eigenmodes, where the wave vector $\mathbf{k} = \nabla\phi$ and frequency $\omega = -\phi_t$ correspond to the characteristic manifolds $\phi(\mathbf{x}, t) =$ constant. The form of the characteristic manifolds for both time-dependent and steady MHD flows are developed. The bi-characteristics for steady MHD waves in a steady background flow are related to the group velocity surface and Mach cone for the waves, and determine when the flow is elliptic, hyperbolic, or of mixed hyperbolic–elliptic type. The wave interaction equations are decomposed into coupled equations for the compressible and incompressible perturbations.

1. Introduction

Heinemann and Olbert (1980) obtained bi-directional evolution equations describing the propagation of toroidal Alfvén waves in the Solar Wind, in which the backward Alfvén wave is coupled to the forward Alfvén wave via large-scale gradients in the background flow. This is known as wave mixing in the space physics community. Zhou and Matthaeus (1990) and Marsch and Tu (1989) and others, subsequently

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developed theories for Alfvénic turbulence in the Solar Wind which naturally incorporated the effects of wave mixing on the turbulent fluctuations (e.g. Tu and Marsch 1995). The ponderomotive force of the Alfvén waves on the background wind flow has been invoked as an important element in accelerating the Solar Wind in both Wentzel–Kramer–Brillouin (WKB) models (e.g. Hollweg 1978; Alazraki and Couturier 1971; Jacques 1977; McKenzie 1994), and non-WKB models of wave accelerated winds (e.g. Heinemann and Olbert 1980; Lou 1993; MacGregor and Charbonneau 1994). Webb et al. (1999) developed wave mixing equations for the seven magnetohydrodynamic (MHD) wave modes in one Cartesian space dimension with application to wave interactions and instabilities in cosmic ray modified shocks. The relation between this formalism, based on the MHD eigenvectors, and a variational approach, for the case of one-dimensional sound waves and entropy waves in compressible gas dynamics was investigated in Webb et al. (1998a).

There is an extensive literature on the application of Lagrangian and Hamiltonian methods in MHD and fluid mechanics. Herivel (1955) considered a Lagrangian version of Hamilton's principle for incompressible fluids, whereas similar developments for compressible, non-homentropic fluids were considered by Serrin (1959). Early, Eulerian versions of Hamilton's principle for ideal fluids were those of Lin (1963) and Seliger and Whitham (1968). Whitham (1974) discusses averaged Lagrangian methods for linear and nonlinear dispersive waves. Newcomb (1962) obtained Lagrangian variational principles for MHD and Chew–Goldberg–Low (CGL) plasmas. Broer and Kobussen (1974) showed that the conversion from Eulerian to material (i.e. Lagrangian) coordinates could be viewed as a canonical transformation. The role of wave action conservation for waves in non-uniform media were elucidated by Bretherton (1971). Dewar (1970) developed a variational approach to the propagation of WKB magnetohydrodynamic waves in inhomogeneous media. Dewar (1970, 1977) discussed canonical and physical stress energy tensors for waves, and the background medium through which the waves propagate. Generalized Lagrangian mean flows, and the interaction of waves with the mean flow were introduced by Andrews and McIntyre (1978) (see also Grimshaw (1984) and Holm (1999)). A good general review of variational methods in fluid mechanics is given by Salmon (1988). Non-canonical Poisson brackets for MHD were introduced by Morrison and Greene (1980, 1982).

The main aim of this paper is to provide a framework for linear, non-WKB, MHD wave propagation in non-uniform media such as the Solar Wind, based on the MHD variational principles of Dewar (1970, 1977) and Newcomb (1962).

Sections 2–4 consider non-WKB, linear wave propagation in a non-uniform background medium. The analysis is based on Dewar's variational principle in which the Lagrangian is expanded out to second order in the Lagrangian wave displacement $\boldsymbol{\xi}$, and in terms of a Lagrangian perturbation in the background entropy ΔS. Variations of the action with respect to $\boldsymbol{\xi}$ yields a system of linear wave evolution equations for $\boldsymbol{\xi}$, which is equivalent to the perturbed momentum equation for the system modified by the effects of entropy waves. The entropy wave perturbation ΔS is advected with the background flow (i.e. $(\partial/\partial t + \mathbf{u}\cdot\nabla)\Delta S = 0$ where $\mathbf{u}$ is the background flow velocity).

Section 3 shows that the linearized wave equations are related to the Frieman and Rotenberg (1960) equations used to investigate the stability of steady MHD flows.

Section 4 studies the characteristic manifolds of the linear wave interaction equations derived in Sec. 2. The concept of a characteristic manifold for a

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system of partial differential equations can be defined as a manifold $\phi(\mathbf{x}) =$ constant (here $\mathbf{x}$ denotes the independent variables) on which the Cauchy problem does not have a unique solution. The characteristic manifolds of the linear wave equations are shown to correspond to the dispersion equations for linear MHD waves in which $\omega' = -(\phi_t + \mathbf{u}\cdot\nabla\phi)$ is the Doppler shifted frequency in the fluid frame and $\mathbf{k} = \nabla\phi$ is the wave vector. These manifolds correspond to Alfvén waves, the fast and slow magnetoacoustic waves and the entropy wave. Exactly the same characteristic manifold equations can also be obtained from analysis of the fully nonlinear MHD equations (e.g. Webb et al. (1996, 1998b); see also Courant and Hilbert (1989, ch. 2 and 6)). The bi-characteristics or ray equations of geometrical MHD optics correspond to the Cauchy characteristics of the first-order partial differential equations for ϕ described by the wave dispersion equations of the different eigenmodes. The characteristics for standing waves in steady MHD flows, the magnetoacoustic wave eikonal and group velocity surface and Mach cone for standing waves, and methods to split off the compressible and incompressible perturbations and their mutual interaction are discussed.

Section 5 concludes with a summary and discussion.

2. Variational principles for linear waves

In this section we derive equations for the evolution of linear MHD waves in non-uniform background flows, allowing for the effects of an external gravitational field potential $\phi(\mathbf{x})$, ($\mathbf{g} = -\nabla\phi$ is the acceleration due to gravity), and the role of entropy perturbations ΔS, on the propagation of the waves.

The basis of our analysis is the variational formulation for the propagation of MHD waves in non-uniform flows, developed by Dewar (1970). Dewar's variational principle describes the propagation of the waves in terms of the Lagrangian fluid displacement, $\boldsymbol{\xi}$. Dewar only applied his variational principle to the propagation of WKB waves, and did not consider the role of entropy perturbations, ΔS or the effects of a gravitational field. However, it is clear that Dewar's variational principle can in fact be applied to non-WKB, MHD wave propagation problems in non-uniform flows, which is the subject of the present analysis.

Dewar's analysis is based, in part, on the variational formulations of MHD and CGL plasmas, developed by Newcomb (1962), using ideas from Lagrangian fluid mechanics and MHD. In Newcomb's approach, the mass continuity equation, Faraday's law for the evolution of the magnetic field and the entropy advection equation, are expressed in Lagrangian form in terms of the transformation between the Lagrangian fluid coordinate $\mathbf{x}_0$, and the corresponding Eulerian position vector, $\mathbf{x}$ of the fluid element $\mathbf{x} = \mathbf{X}(\mathbf{x}_0, t)$, in which $\mathbf{u} = d\mathbf{x}/dt \equiv \partial\mathbf{X}(\mathbf{x}_0, t)/\partial t$ is the fluid velocity. In this approach, it is not necessary to include constraints in the variational principle, in order to ensure that the mass continuity equation, Faraday's law and the entropy equation are satisfied (see, e.g., Lundgren 1963). For the case of an adiabatic ideal gas, the entropy advection equation implies that p/ρ^γ is advected with the flow, where γ is the adiabatic index of the gas.

In addition to including the effects of an external gravitational field and entropy perturbations in the analysis, we also allow for a more general equation of state for the gas than that used by Dewar, in which the internal energy per unit volume $\varepsilon = \varepsilon(\rho, S)$ is a given, but arbitrary function of the density ρ and entropy S. Dewar and Newcomb, both considered the case of an ideal gas, with adiabatic

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index γ, for which $\varepsilon = p/(\gamma-1)$, where p is the gas pressure. The thermodynamics of the gas is governed by the second law of thermodynamics:

$$dQ \equiv T\,dS = dU + p\,d\tau, \tag{2.1}$$

where $\tau = 1/\rho$ is the specific volume of the gas. The second law relates the change in the heat energy dQ, to the change dU in the internal energy per unit mass, U ($U = \varepsilon(\rho, S)/\rho$), and the mechanical work done by the pressure forces $p\,d\tau$ due to expansion or contraction of the plasma. From (2.1) one obtains the standard expressions:

$$p = \rho\frac{\partial \varepsilon}{\partial \rho} - \varepsilon, \quad T = \frac{1}{\rho}\frac{\partial \varepsilon}{\partial S}, \quad h = \frac{\varepsilon + p}{\rho} = \frac{\partial \varepsilon}{\partial \rho}, \tag{2.2}$$

for the gas pressure p, temperature T, and enthalpy h in terms of the internal energy density $\varepsilon(\rho, S)$.

The first step in the analysis is to write down the action for the combined system of waves and background plasma in the form:

$$A = \int d^3x^* \int dt \mathscr{L}^*, \tag{2.3}$$

where

$$\mathscr{L}^* = \frac{1}{2}\rho^* u^{*2} - \varepsilon(\rho^*, S^*) - \frac{B^{*2}}{2\mu} - \rho^*\phi(\mathbf{x}^*), \tag{2.4}$$

is the Lagrangian density for the system. In (2.4), the terms in the Lagrangian density $\mathscr{L}^*$ correspond respectively to the kinetic energy of the plasma flow ($u = |\mathbf{u}|$ is the magnitude on the fluid velocity $\mathbf{u}$); the internal energy density ε, the magnetic energy density ($\mathbf{B}$ is the magnetic field induction, and μ is the magnetic permeability), and the gravitational potential energy $\rho\phi$. The position coordinate $\mathbf{x}^* = \mathbf{x} + \boldsymbol{\xi}(\mathbf{x}, t)$ where $\mathbf{x}$ is the position of the background plasma element, and $\boldsymbol{\xi}$ is the Lagrangian displacement of the fluid element due to the waves. The entropy $S^* = S + \Delta S$ in (2.3), where ΔS is the Lagrangian entropy perturbation. The volume element

$$d^3x^* = J^* d^3x, \tag{2.5}$$

where

$$\begin{aligned} J^* &= \det\left(\frac{\partial x^{*i}}{\partial x^j}\right) = \det\left(\delta^i_j + \frac{\partial \xi^i}{\partial x^j}\right) \\ &= 1 + \nabla\cdot\boldsymbol{\xi} + \frac{1}{2}[(\nabla\cdot\boldsymbol{\xi})^2 - \nabla\boldsymbol{\xi}{:}\nabla\boldsymbol{\xi}] + \frac{1}{6}[(\nabla\cdot\boldsymbol{\xi})^3 \\ &\quad + 2(\nabla\boldsymbol{\xi}\cdot\nabla\boldsymbol{\xi}){:}\nabla\boldsymbol{\xi} - 3(\nabla\cdot\boldsymbol{\xi})\nabla\boldsymbol{\xi}{:}\nabla\boldsymbol{\xi}], \end{aligned} \tag{2.6}$$

is the Jacobian of the transformation between $\mathbf{x}^*$ and $\mathbf{x}$ (see, e.g., Kumar et al. 1994). The Lagrangian transformations

$$\rho^* = \frac{\rho}{J^*}, \quad B^{*i} = \frac{\partial x^{*i}}{\partial x^j}\frac{B^j}{J^*}, \tag{2.7}$$

correspond to mass continuity and Faraday's law (e.g. Newcomb 1962). Using (2.7) in (2.3) we obtain the action in the form

$$A = \int d^3x \int dt \mathscr{L} \quad \text{where } \mathscr{L} = J^*\mathscr{L}^*. \tag{2.8}$$

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The exact Lagrangian density $\mathscr{L}$ can be written more explicitly in the form

$$\mathscr{L} = \frac{1}{2}\rho(|\mathbf{u}|^2 + 2\mathbf{u}\cdot\dot{\boldsymbol{\xi}} + |\dot{\boldsymbol{\xi}}|^2) - J^*\varepsilon\left(\frac{\rho}{J^*}, S + \Delta S\right) - \frac{1}{2\mu J^*}(x^{*i}_j B^j x^{*i}_s B^s) - \rho\phi(\mathbf{x}+\boldsymbol{\xi}). \tag{2.9}$$

The transformation for $\mathbf{B}$ in (2.7) is the frozen in field theorem in MHD (see, e.g., Parker 1979, ch. 4, for a detailed exposition). Using the transformations (2.6) and (2.7), we obtain the expansion

$$A = \int d^3x \int dt[\mathscr{L}_0 + \mathscr{L}_1 + \mathscr{L}_2 + O(\xi^3)], \tag{2.10}$$

for the action of the system, where

$$\mathscr{L}_0 = \frac{1}{2}\rho u^2 - \varepsilon(\rho, S) - \frac{B^2}{2\mu} - \rho\phi, \tag{2.11}$$

$$\mathscr{L}_1 = \rho\mathbf{u}\cdot\dot{\boldsymbol{\xi}} - (\rho T\Delta S - p\nabla\cdot\boldsymbol{\xi}) + \frac{B^2}{2\mu}\nabla\cdot\boldsymbol{\xi} - \frac{\mathbf{B}\cdot\nabla\boldsymbol{\xi}\cdot\mathbf{B}}{\mu} - \rho\boldsymbol{\xi}\cdot\nabla\phi, \tag{2.12}$$

$$\mathscr{L}_2 = \frac{1}{2}\rho|\dot{\boldsymbol{\xi}}|^2 - \frac{1}{2}[(\rho a^2 - p)(\nabla\cdot\boldsymbol{\xi})^2 + p\nabla\boldsymbol{\xi}{:}\nabla\boldsymbol{\xi} - 2p_S\Delta S(\nabla\cdot\boldsymbol{\xi}) + \varepsilon_{SS}(\Delta S)^2]$$
$$+ \frac{\mathbf{B}\cdot\nabla\boldsymbol{\xi}\cdot\mathbf{B}}{\mu}\nabla\cdot\boldsymbol{\xi} - \frac{(\mathbf{B}\cdot\nabla\boldsymbol{\xi})^2}{2\mu} - \frac{B^2}{4\mu}((\nabla\cdot\boldsymbol{\xi})^2 + \nabla\boldsymbol{\xi}{:}\nabla\boldsymbol{\xi}) - \frac{1}{2}\rho\boldsymbol{\xi}\boldsymbol{\xi}{:}\nabla\nabla\phi. \tag{2.13}$$

In (2.13)

$$\dot{\boldsymbol{\xi}} = \frac{\partial\boldsymbol{\xi}}{\partial t} + \mathbf{u}\cdot\nabla\boldsymbol{\xi} \quad \text{and} \quad a = \left(\frac{\partial p}{\partial\rho}\right)^{1/2}, \tag{2.14}$$

denote the Lagrangian velocity perturbation, moving with the fluid (note $\mathbf{u}^* = \mathbf{u} + \dot{\boldsymbol{\xi}}$) and the adiabatic sound speed, respectively. Dewar considered the case of an adiabatic gas, with adiabatic index γ, in which case $\varepsilon = p/(\gamma - 1)$.

In the derivation of (2.10), it is assumed that the entropy S and the Lagrangian entropy perturbation ΔS are advected with the flow, i.e.

$$\frac{dS}{dt} = 0, \quad \frac{d\Delta S}{dt} = 0, \tag{2.15}$$

where $d/dt = \partial_t + \mathbf{u}\cdot\nabla$ is the Lagrangian time derivative moving with the flow.

In the absence of waves, the total Lagrangian $\mathscr{L} \equiv \mathscr{L}_0$ in (2.10) and (2.11), and the variational principle (2.10) obtained by varying the background plasma, taking into account the Lagrangian constraints (i.e. the mass continuity equation, Faraday's equation, and the entropy advection equation in Lagrangian form) yields the MHD momentum equation for the background plasma (Newcomb 1962). Newcomb obtained: (i) both the Lagrangian and Eulerian form of the MHD momentum equation by using both Lagrangian and Eulerian forms of the variational principle; (ii) the energy principle for static, MHD equilibria of Bernstein et al. (1958); and (iii) an energy principle for some steady, azimuthal MHD flows.

Dewar (1970) applied the variational principle (2.10) to derive equations for WKB, MHD waves in a non-uniform background flow. He used an averaged Lagrangian method, similar to that used by Whitham (1965), in which the Lagrangian density $\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_1 + \mathscr{L}_2 + O(\xi^3)$ is averaged over the periodic, fast variations of the wave phase ϕ. Variations of the wave amplitude, in the averaged action principle using the averaged Lagrangian density $\langle\mathscr{L}_2\rangle$, results in the MHD

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wave eigenvector equations and dispersion relation, whereas slow variations of the wave phase (i.e. of $\mathbf{k} = \nabla\phi$ and $\omega = -\phi_t$) results in the wave action equation.

2.1. Linear waves in non-uniform flows

We now consider equations for linear waves in non-uniform flows, in which the wave amplitudes are supposed to be sufficiently small, that the waves do not affect the background flow. The action principle (2.10) can be written as

$$A = \int d^3x \int dt[\mathscr{L}_{\rm b} + \mathscr{L}_{\rm w} + O(\xi^3)], \tag{2.16}$$

where

$$\mathscr{L}_{\rm b} = \mathscr{L}_0, \quad \mathscr{L}_{\rm w} = \mathscr{L}_1 + \mathscr{L}_2 + O(\xi^3), \tag{2.17}$$

represent the background Lagrangian density $\mathscr{L}_{\rm b}$ and the wave Lagrangian density $\mathscr{L}_{\rm w}$. We also use the notation

$$A_j = \int d^3x \int dt\ \mathscr{L}_j \quad (j = 0, 1, 2) \tag{2.18}$$

to denote the action components due to $\mathscr{L}_0$, $\mathscr{L}_1$ and $\mathscr{L}_2$, respectively.

Using (2.12) we find

$$\frac{\delta A_1}{\delta \xi} = -\left[\frac{\partial}{\partial t}(\rho\mathbf{u}) + \nabla\cdot\left(\rho\mathbf{u}\mathbf{u} + \left(p + \frac{B^2}{2\mu}\right)\mathbf{I} - \frac{\mathbf{B}\mathbf{B}}{\mu}\right) + \rho\nabla\phi\right] = 0. \tag{2.19}$$

The equation $\delta A_1/\delta\xi = 0$ is recognizable as the momentum equation for the undisturbed background flow. Equation (2.19) can also be obtained by varying the background variables in the action $A_0 = \int d^3x \int dt \mathscr{L}_0$ (see, e.g., Newcomb 1962).

Variations of the action A_2 with respect to $\boldsymbol{\xi}$, and setting $\mathbf{P}^{(\rm D)} = -\delta A_2/\delta\boldsymbol{\xi} = 0$, gives the linearized momentum equation:

$$\begin{aligned}\mathbf{P}^{(\rm D)} = \frac{\partial}{\partial t}(\rho\dot{\boldsymbol{\xi}}) + \nabla\cdot\Bigg\{&\rho\mathbf{u}\dot{\boldsymbol{\xi}} + ((p - \rho a^2)\nabla\cdot\boldsymbol{\xi} - p_{\rm S}\Delta S)\mathbf{I} - p(\nabla\boldsymbol{\xi})^t \\ &+ \left(\frac{\mathbf{B}\cdot\nabla\boldsymbol{\xi}\cdot\mathbf{B}}{\mu} - \frac{B^2}{2\mu}\nabla\cdot\boldsymbol{\xi}\right)\mathbf{I} - \frac{B^2}{2\mu}(\nabla\boldsymbol{\xi})^t + \frac{\mathbf{B}}{\mu}((\nabla\cdot\boldsymbol{\xi})\mathbf{B} - \mathbf{B}\cdot\nabla\boldsymbol{\xi})\Bigg\} \\ &+ \rho\boldsymbol{\xi}\cdot\nabla\nabla\phi = 0.\end{aligned} \tag{2.20}$$

In (2.20) we use the notation $\mathbf{P}^{(\rm D)}$ to denote the linearized momentum flux, where the superscript D, refers to Dewar's variational principle. Equation (2.20), coupled with the advection equation:

$$\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)\Delta S = 0, \tag{2.21}$$

for the Lagrangian entropy perturbation, are the fundamental equations governing the interaction of linear MHD waves and the entropy wave in non-uniform background flows, in the presence of an external gravitational potential $\phi(\mathbf{x})$. For the case of an ideal gas, with adiabatic index γ, the thermodynamics of the gas are governed by the equations:

$$\varepsilon = \frac{p}{\gamma - 1}, \quad p = p_0\left(\frac{\rho}{\rho_0}\right)^\gamma \exp\left(\frac{S - S_0}{C_v}\right), \quad S = C_v \ln\left(\frac{p}{\rho^\gamma}\right), \tag{2.22}$$

where $C_{\rm v}$ is the specific heat of the gas at constant volume, in which case $p_{\rm S} = p/C_{\rm v}$ in (2.20).

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3. The Frieman and Rotenberg equations

Frieman and Rotenberg (1960) generalized the energy principle of Bernstein et al. (1958) to study the stability of steady MHD flows including the effects of gravity. The energy principle of Bernstein et al. (1958) only applies to magnetostatic equilibria. Similar equations were also used by Ferraro and Plumpton (1958) in a study of MHD wave propagation in the gravitationally stratified, Solar atmosphere. Frieman and Rotenberg's perturbation equations for $\boldsymbol{\xi}$ can be obtained by perturbing the MHD momentum equation:

$$\rho\frac{d\mathbf{u}}{dt} = -\nabla\left(p + \frac{B^2}{2\mu}\right) + \frac{\mathbf{B}\cdot\nabla\mathbf{B}}{\mu} + \rho\mathbf{g}, \tag{3.1}$$

where $\mathbf{g} = -\nabla\phi$ is the acceleration due to gravity.

The Eulerian perturbations $\delta\psi$ and the Lagrangian perturbation $\Delta\psi$ of a physical quantity ψ are related by the equation

$$\delta\psi = \Delta\psi - \boldsymbol{\xi}\cdot\nabla\psi, \tag{3.2}$$

where $\boldsymbol{\xi}$ is the Lagrangian displacement of the fluid element (e.g. Newcomb 1962; Lundgren 1963). The Lagrangian perturbations Δp, $\Delta\rho$, $\Delta\mathbf{u}$, and $\Delta\mathbf{B}$ in linear perturbation theory are given by

$$\begin{gathered}\Delta p = p_{\mathrm{S}}\Delta S - a^2\rho\nabla\cdot\boldsymbol{\xi}, \quad \Delta\rho = -\rho\nabla\cdot\boldsymbol{\xi},\\ \Delta\mathbf{u} = \dot{\boldsymbol{\xi}} = \boldsymbol{\xi}_t + \mathbf{u}\cdot\nabla\boldsymbol{\xi}, \quad \Delta\mathbf{B} = \mathbf{B}\cdot\nabla\boldsymbol{\xi} - \mathbf{B}\nabla\cdot\boldsymbol{\xi}.\end{gathered} \tag{3.3}$$

The corresponding Eulerian perturbations using (3.2) are given by

$$\begin{gathered}\delta p = p_{\mathrm{S}}\Delta S - a^2\rho\nabla\cdot\boldsymbol{\xi} - \boldsymbol{\xi}\cdot\nabla p, \quad \delta\rho = -\nabla\cdot(\rho\boldsymbol{\xi}).\\ \delta\mathbf{u} = \boldsymbol{\xi}_t + \mathbf{u}\cdot\nabla\boldsymbol{\xi} - \boldsymbol{\xi}\cdot\nabla\mathbf{u}, \quad \delta\mathbf{B} = \nabla\times(\boldsymbol{\xi}\times\mathbf{B}).\end{gathered} \tag{3.4}$$

Linearizing the momentum equation (3.1) using Eulerian perturbations, gives the perturbed momentum equation:

$$\mathbf{P}^{(\mathrm{FR})} \equiv \rho\boldsymbol{\xi}_{tt} + 2\rho\mathbf{u}\cdot\nabla(\boldsymbol{\xi}_t) - \mathbf{F}(\boldsymbol{\xi}) = 0, \tag{3.5}$$

where the force-like term $\mathbf{F}(\boldsymbol{\xi})$ does not depend on $\boldsymbol{\xi}_t$, and has the form

$$\begin{aligned}\mathbf{F}(\boldsymbol{\xi}) = -\nabla\cdot\boldsymbol{\Pi} + \frac{\mathbf{B}\cdot\nabla\delta\mathbf{B} + \delta\mathbf{B}\cdot\nabla\mathbf{B}}{\mu} - \mathbf{g}\nabla\cdot(\rho\boldsymbol{\xi}) + \nabla\cdot\left(\rho\boldsymbol{\xi}\frac{d\mathbf{u}}{dt} - \rho\mathbf{u}\mathbf{u}\cdot\nabla\boldsymbol{\xi}\right)\\ - \frac{\partial}{\partial t}(\rho\mathbf{u})\cdot\nabla\boldsymbol{\xi},\end{aligned} \tag{3.6}$$

$$\boldsymbol{\Pi} = p_{\mathrm{S}}\Delta S - a^2\rho\nabla\cdot\boldsymbol{\xi} - \boldsymbol{\xi}\cdot\nabla p + \frac{\mathbf{B}\cdot\delta\mathbf{B}}{\mu}. \tag{3.7}$$

For the case of a steady background flow, $d\mathbf{u}/dt = \mathbf{u}\cdot\nabla\mathbf{u} = 0$ and $(\rho\mathbf{u})_t = 0$, and for the case of zero entropy perturbations, $\Delta S = 0$. In this case the perturbed momentum equation (3.5) reduces to that obtained by Frieman and Rotenberg (1960).

It is interesting to compare the perturbed momentum equation $\mathbf{P}^{(\mathrm{FR})} = 0$ in (3.5) (the superscript FR refers to Frieman and Rotenberg), with the perturbed momentum equation $\mathbf{P}^{(\mathrm{D})} = 0$, obtained in (2.20) from Dewar's variational principle. From (2.20) and (3.5) we find:

$$\mathbf{P}^{(\mathrm{D})} - \mathbf{P}^{(\mathrm{FR})} = \nabla\cdot\left\{\boldsymbol{\xi}\left[\rho\frac{d\mathbf{u}}{dt} + \nabla\left(p + \frac{B^2}{2\mu}\right) - \frac{\mathbf{B}\cdot\nabla\mathbf{B}}{\mu} + \rho\nabla\phi\right]\right\}. \tag{3.8}$$

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If the background momentum equation is unaffected by the waves, then the right-hand side of (3.8) vanishes by virtue of the background MHD momentum equation (2.19). Hence in this case, $\mathbf{P}^{(\mathrm{FR})} = 0$ is equivalent to $\mathbf{P}^{(\mathrm{D})} = 0$. In cases where $\Delta S \neq 0$, the perturbed momentum equation (3.5) is coupled with the advection equation (2.21), $d\Delta S/dt = 0$.

For the steady flows considered by Frieman and Rotenberg (1960) ($\partial/\partial t = 0$ and $\Delta S = 0$), (3.5) has solutions of the form $\boldsymbol{\xi} = \tilde{\boldsymbol{\xi}}(\mathbf{r}) \exp i\omega t$, where $\tilde{\boldsymbol{\xi}}(\mathbf{r})$ satisfies the equation

$$-\omega^2 \rho \tilde{\boldsymbol{\xi}} + 2i\omega\rho \mathbf{u} \cdot \nabla \tilde{\boldsymbol{\xi}} - \mathbf{F}(\tilde{\boldsymbol{\xi}}) = 0. \tag{3.9}$$

In (3.9), $i\rho\mathbf{u} \cdot \nabla$ is a Hermitian operator (i.e. it is a self-adjoint operator, with respect to the complex inner product $\langle f, g \rangle = \int f g^* d^3x$). The operator $\mathbf{F}$ is a self-adjoint operator (i.e. $\int_{-\infty}^{\infty} \tilde{\boldsymbol{\eta}} \mathbf{F}(\tilde{\boldsymbol{\xi}}) d^3x = \int_{-\infty}^{\infty} \tilde{\boldsymbol{\xi}} \mathbf{F}(\tilde{\boldsymbol{\eta}}) d^3x$). The proof that $\mathbf{F}$ is self-adjoint is facilitated by noting $\mathbf{P}^{(\mathrm{D})} \equiv \mathbf{P}^{(\mathrm{FR})}$, using integration by parts, and dropping surface terms. Frieman and Rotenberg discussed sufficient conditions for stability and variational principles to determine the eigenvalues ω. Van der Holst et al. (1999) considered the problem of the stability of shear flows in gravitating plane plasmas, and investigate both the continuous spectra and the discrete spectra for ω as well as cluster spectra. A non-standard approach, for studying waves in shear flows, may be traced back to the work of Kelvin (1887). The Kelvin modes are either periodic in, or independent of each space coordinate, but the wavenumber and amplitude associated with each mode are functions of time which depend on the shearing rate of the fluid. Examples of exact solutions for wave interactions in shear flows governed by the incompressible Navier–Stokes equations have been obtained, for example, by Craik and Criminale (1986). Related work on the interaction and transformation of MHD waves in shear flows, using this approach have been investigated by Chagelishvili et al. (1996), Poedts et al. (1998), Kaghashvili (1999) and Bodo et al. (2001).

4. Characteristic manifolds and equations

In this section, we consider the characteristics of the linear wave interaction equations (2.20) and (2.21). We consider the characteristics for time dependent flows (Sec. 4.1) and also for steady MHD flows (Sec. 4.2). The concept of a characteristic manifold of a system of hyperbolic or non-strictly hyperbolic system of equations has been developed in detail by a number of authors (e.g. Courant and Hilbert 1989; Chorin and Marsden 1979). For a hyperbolic system, the characteristic wave speeds are all real and distinct, whereas for a non-strictly hyperbolic system (such as MHD), the wave speeds $\{\lambda_j\}$, are all real, but in general are not all distinct. In MHD, two or more of the eigenvalues λ_j, coincide for propagation parallel and perpendicular to the mean magnetic field. For hyperbolic waves, the dispersion equations for the waves (in a uniform medium) are scale invariant (i.e. $\omega \propto k$, where ω and k are the wave frequency and wave number, respectively). This means that there is no dispersion of waves with different k. However, the waves in MHD are anisotropic, meaning that the wave speeds are dependent on the direction of propagation of the wave relative to the direction of the background magnetic field.

The concept of a characteristic manifold for a partial differential equation system can be defined as a manifold $\phi(\mathbf{x}) =$ constant (here $\mathbf{x}$ denotes the independent variables), on which the Cauchy problem does not have a unique solution. In less

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technical jargon, this means that if the initial data is specified on a characteristic manifold $\phi(\mathbf{x}) = \text{constant}$, then the problem does not have a unique solution.

The characteristic manifolds of the wave equations (2.20) and (2.21) describing linear wave propagation and interaction in a non-uniform background flow, turn out to be equivalent to the characteristic manifolds for the fully nonlinear MHD equations. The characteristic manifolds for (2.20) and (2.21), thus correspond to the Alfvén waves, the fast and slow magnetoacoustic waves, and the entropy wave. Alternatively, one can think of the characteristic manifolds as corresponding to the wave fronts of short wavelength (WKB) disturbances in the medium (e.g. Whitham 1974). Appendix A sketches how the characteristic manifolds may be related to WKB analysis by using the variational approach of Dewar (1970).

4.1. Time-dependent characteristics

Equations (2.20) and (2.21) describing linear MHD waves in a non-uniform flow, may be written in the form

$$\mathbf{L}(\boldsymbol{\xi}) + \mathbf{R}(\boldsymbol{\xi}, \Delta S) = 0, \tag{4.1}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)\Delta S = 0, \tag{4.2}$$

where

$$\begin{aligned}\mathbf{L}(\boldsymbol{\xi}) = {} & \boldsymbol{\xi}_{tt} + 2\mathbf{u}\cdot\nabla(\boldsymbol{\xi}_t) + \mathbf{uu}:\nabla\nabla\boldsymbol{\xi} - (a^2+b^2)\nabla(\nabla\cdot\boldsymbol{\xi}) \\ & + \mathbf{b}\cdot\nabla(\nabla\boldsymbol{\xi})\cdot\mathbf{b} - \mathbf{bb}:\nabla\nabla\boldsymbol{\xi} + [\mathbf{b}\cdot\nabla(\nabla\cdot\boldsymbol{\xi})]\mathbf{b},\end{aligned} \tag{4.3}$$

corresponds to the second derivatives of $\boldsymbol{\xi}$ in (2.20), and

$$\begin{aligned}\mathbf{R}(\boldsymbol{\xi}, \Delta S) = \frac{1}{\rho}\Bigg\{ & \nabla\cdot\boldsymbol{\xi}\left[\nabla(p - a^2\rho) + \frac{(\mathbf{B}\cdot\nabla)\mathbf{B}}{\mu} - \nabla\left(\frac{B^2}{2\mu}\right)\right] \\ & - [\nabla\boldsymbol{\xi} + (\nabla\boldsymbol{\xi})^t]\cdot\nabla\left(p + \frac{B^2}{2\mu}\right) \\ & + \nabla(p_{\mathrm{S}}\Delta S) + \nabla\left(\frac{\mathbf{BB}}{\mu}\right):\nabla\boldsymbol{\xi} + \rho\boldsymbol{\xi}\cdot\nabla\nabla\phi\Bigg\},\end{aligned} \tag{4.4}$$

corresponds to lower-order derivatives of $\boldsymbol{\xi}$, terms linear in $\boldsymbol{\xi}$ (the gravitational term) and terms independent of $\boldsymbol{\xi}$ (the entropy wave contribution). In (4.3),

$$\mathbf{b} = \frac{\mathbf{B}}{\sqrt{\mu\rho}}, \tag{4.5}$$

is the Alfvén velocity and a is the adiabatic sound speed (2.14).

It is clear from the form of (4.1) and (4.2), that the entropy wave perturbation ΔS can affect the propagation of the other MHD waves represented by the Lagrangian fluid displacement $\boldsymbol{\xi}$ since ΔS appears as a source term in (4.1) for $\boldsymbol{\xi}$, but the entropy wave perturbation ΔS is unaffected by changes in $\boldsymbol{\xi}$ in (4.2).

To consider the Cauchy problem for (4.1) and (4.2) we introduce new independent variables $(\phi^0, \phi^1, \phi^2, \phi^3)$ where $\phi^j = \phi^j(\mathbf{x})$, and $\mathbf{x} = (t, x, y, z) = (x^0, x^1, x^2, x^3)$ are the independent variables. We have in mind, the problem of specifying initial data on the manifold $\phi^0(\mathbf{x}) = \text{constant}$, and determining when it is possible (or not possible) to obtain a unique solution for $\boldsymbol{\xi}$ and ΔS. At least locally, what is required to obtain a unique solution is that the Taylor series for the solution can be determined, using the initial data, and by calculating the higher-order derivatives

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required from the differential equation system and its differential consequences. In the analysis below, we use the notation $\phi^0(\mathbf{x}) \equiv \phi(\mathbf{x})$, in order to emphasize that the initial data is specified on the manifold $\phi(\mathbf{x}) =$ constant.

In the new variables $\{\phi^j\}$, the entropy advection equation (4.2) for ΔS becomes:

$$\frac{\partial \Delta S}{\partial t} + \mathbf{u}\cdot\nabla \Delta S = \frac{\partial \Delta S}{\partial \phi}\left[\frac{\partial \phi}{\partial t} + \mathbf{u}\cdot\nabla\phi\right] + \sum_{j=1}^{3}\frac{\partial \Delta S}{\partial \phi^j}\left[\frac{\partial \phi^j}{\partial t} + \mathbf{u}\cdot\nabla\phi^j\right] = 0. \quad (4.6)$$

If we choose ϕ such that

$$\frac{\partial \phi}{\partial t} + \mathbf{u}\cdot\nabla\phi = 0, \quad (4.7)$$

and specify initial data for ΔS and $\boldsymbol{\xi}$ on the surface $\phi(\mathbf{x}) =$ constant then it will not be possible to solve (4.6) for $\partial\Delta S/\partial\phi$, and higher-order derivatives of ΔS with respect to ϕ, since the coefficient of $\partial\Delta S/\partial\phi$ is zero by virtue of the choice (4.7) for the evolution of ϕ. Thus, solutions of (4.7), correspond to the characteristic manifold for the entropy wave, and it is not possible to obtain a solution for ΔS off the characteristic surface $\phi =$ constant, if the initial data was specified on $\phi =$ constant. If in fact, we had specified initial data on a surface $\phi =$ constant not satisfying (4.7), then (4.6) could be solved uniquely for $\partial\Delta S/\partial\phi$.

To determine the characteristic manifolds of (4.1) we first rewrite (4.1) in the form

$$A_j^{i\alpha\beta}\xi^j_{\alpha\beta} + R^i(\boldsymbol{\xi}, \Delta S) = 0, \quad (4.8)$$

where

$$A_j^{i\alpha\beta} = \delta^i_j[\delta^\alpha_0\delta^\beta_0 + 2u^\beta\delta^\alpha_0 + u^\alpha u^\beta - b^\alpha b^\beta] - (a^2+b^2)\delta^\alpha_i\delta^\beta_j + b^\beta b^j\delta^\alpha_i + b^\alpha b^i\delta^\beta_j. \quad (4.9)$$

In (4.9) we have defined $b^0 = 0$ and $u^0 = 0$ (i.e. $\mathbf{b}$ and $\mathbf{u}$ are vectors in three-dimensional position space). In (4.8) and (4.9) the indices i, j take the values 1,2,3, but the indices α, β refer to the independent variables $(x^0, x^1, x^2, x^3) \equiv (t, x, y, z)$ and take the values 0,1,2,3. These conventions are appropriate for non-relativistic MHD, but a four-vector formalism would be appropriate for relativistic MHD. The term $R^i(\boldsymbol{\xi}, \Delta S)$ in (4.8) is the ith component of the vector $\mathbf{R}$ in (4.4), which can be written in the form

$$R^i = B^{i\alpha}_j\xi^j_\alpha + C^i_j\xi^j + D^i, \quad (4.10)$$

and consists of first-order derivatives of $\boldsymbol{\xi}$, linear terms in $\boldsymbol{\xi}$ and terms independent of $\boldsymbol{\xi}$. The detailed form of R^i does not play a role in the nature of the characteristic manifolds.

Using new independent variables $\{\phi^\alpha(\mathbf{x})\}$, the wave equation (4.8) for $\boldsymbol{\xi}$ takes the form

$$A_j^{i\alpha\beta}\frac{\partial\phi^\mu}{\partial x^\alpha}\frac{\partial\phi^\nu}{\partial x^\beta}\frac{\partial^2\xi^j}{\partial\phi^\mu\partial\phi^\nu} + \left(A_j^{i\alpha\beta}\frac{\partial^2\phi^\mu}{\partial x^\alpha\partial x^\beta} + B_j^{i\alpha}\frac{\partial\phi^\mu}{\partial x^\alpha}\right)\frac{\partial\xi^j}{\partial\phi^\mu} + C^i_j\xi^j + D^i = 0. \quad (4.11)$$

For the purposes of characteristic analysis, we write (4.11) as

$$A_j^{i\alpha\beta}\frac{\partial\phi}{\partial x^\alpha}\frac{\partial\phi}{\partial x^\beta}\frac{\partial^2\xi^j}{\partial\phi^2} + S^i = 0, \quad (4.12)$$

where we have isolated the second derivatives of ξ^j with respect to $\phi \equiv \phi^0$ and S^i represents the remaining terms in (4.11). As in our discussion of the characteristic

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manifold of the entropy advection equation in (4.6) et seq., we consider the initial value problem in which the data is specified on the manifold $\phi =$ constant. The initial data on the manifold $\phi =$ constant can be written in the form

$$\xi^j = \tilde{\xi}^j(\phi^1, \phi^2, \phi^3), \quad \xi^j_{,\phi} = \tilde{n}^j(\phi^1, \phi^2, \phi^3), \quad \Delta S = \tilde{s}(\phi^1, \phi^2, \phi^3), \tag{4.13}$$

where $\tilde{\xi}^j$, $\tilde{n}^j$ and $\tilde{s}$ specify the initial data in terms of ϕ^1,ϕ^2, and ϕ^3. The initial data (4.13) is sufficient to determine the source term S^i in (4.12). To obtain a unique solution for $\partial^2\xi^j/\partial\phi^2$ on the manifold $\phi =$ constant, requires the matrix

$$\tilde{A}^i_j = A^{i\alpha\beta}_j \phi_\alpha \phi_\beta \tag{4.14}$$

to be non-singular, i.e. $\det(\tilde{\mathbf{A}}) \neq 0$. If $\det(\tilde{\mathbf{A}}) = 0$, then (4.12) does not possess a unique solution for $\xi^j_{\phi\phi}$. Thus,

$$\det(\tilde{\mathbf{A}}) \equiv \det(A^{i\alpha\beta}_j \phi_\alpha \phi_\beta) = 0, \tag{4.15}$$

defines the characteristic manifolds $\phi =$ constant for the wave equation (4.1).

The matrix $\tilde{\mathbf{A}}$ in (4.14) can be expressed in the form

$$\tilde{A}^i_j = [\omega'^2 - (\mathbf{b}\cdot\mathbf{k})^2]\delta^i_j - (a^2+b^2)k^i k^j + (\mathbf{b}\cdot\mathbf{k})(b^i k^j + b^j k^i), \tag{4.16}$$

where

$$\mathbf{k} = \nabla\phi, \quad \omega = -\phi_t, \quad \omega' = \omega - \mathbf{k}\cdot\mathbf{u}, \tag{4.17}$$

are identified with the wave number $\mathbf{k}$ and frequency ω associated with the wave surface $\phi =$ constant, and $\omega' = \omega - \mathbf{k}\cdot\mathbf{u}$ is the Doppler shifted frequency in the fluid frame. Taking the determinant of (4.16) we obtain

$$\det(\tilde{\mathbf{A}}) = [\omega'^2 - (\mathbf{b}\cdot\mathbf{k})^2]\{\omega'^4 - (a^2+b^2)\omega'^2k^2 + a^2k^2(\mathbf{b}\cdot\mathbf{k})^2\}. \tag{4.18}$$

Thus, $\det(\tilde{\mathbf{A}}) = 0$, if

$$F_A \equiv \omega'^2 - (\mathbf{b}\cdot\mathbf{k})^2 = (\phi_t + \mathbf{u}\cdot\nabla\phi)^2 - (\mathbf{b}\cdot\nabla\phi)^2 = 0, \tag{4.19}$$

corresponding to the Alfvén wave characteristic manifolds, or alternatively, $\det(\tilde{\mathbf{A}}) = 0$ if

$$\begin{aligned} F_{\rm MS} &\equiv \omega'^4 - (a^2+b^2)\omega'^2k^2 + a^2k^2(\mathbf{b}\cdot\mathbf{k})^2 \\ &= (\phi_t + \mathbf{u}\cdot\nabla\phi)^4 - (a^2+b^2)(\phi_t + \mathbf{u}\cdot\nabla\phi)^2|\nabla\phi|^2 + a^2(\mathbf{b}\cdot\nabla\phi)^2|\nabla\phi|^2 \\ &= 0, \end{aligned} \tag{4.20}$$

which defines the characteristic surfaces for the magnetosonic modes.

The different dispersion equation branches (4.19) for the Alfvén waves, (4.20) for the magnetoacoustic waves, and (4.7) for the entropy wave can be expressed generically in the form

$$\omega = \Omega(\mathbf{k}, \mathbf{x}, t), \quad \text{where} \quad \Omega = kV_{\rm p}(\mathbf{n}, \mathbf{x}, t), \tag{4.21}$$

where $\mathbf{n} = \mathbf{k}/k$ is the wave normal, and $V_{\rm p} = \omega/k$ is the phase velocity of the wave. Using the identifications $\omega = -\phi_t$, and $\mathbf{k} = \nabla\phi$ the dispersion equations for the different MHD modes can be written in the generic form

$$F \equiv \phi_t + \Omega(\nabla\phi, \mathbf{x}, t) = 0. \tag{4.22}$$

Equation (4.22) is the Hamilton–Jacobi equation for the wave eikonal function

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$\phi(\mathbf{x},t)$ with Hamiltonian Ω. Writing $k^0=\phi_t$ and using the four vector notation $(t,x,y,z)=(x^0,x^1,x^2,x^3)$, (4.22) has Cauchy characteristics (e.g. Sneddon 1957):

$$\frac{dx^\alpha}{d\tau}=\frac{\partial F}{\partial k^\alpha},\quad \frac{d\phi}{d\tau}=k^\alpha\frac{\partial F}{\partial k^\alpha},\quad \frac{dk^\alpha}{d\tau}=-\left(\frac{\partial F}{\partial x^\alpha}+\frac{\partial F}{\partial \phi}k^\alpha\right), \tag{4.23}$$

where τ is a parameter along the trajectory ((4.23) are sometimes referred to as the bi-characteristics, since the characteristic manifolds (4.19) and (4.20) may also be thought of as characteristic equations). Alternatively, using ω and Ω, the characteristics, or ray equations can be written in the form:

$$\frac{dx^i}{d\tau}=\frac{\partial\Omega}{\partial k^i},\quad \frac{dk^i}{d\tau}=-\frac{\partial\Omega}{\partial x^i},\quad 1\leqslant i\leqslant 3, \tag{4.24}$$

$$\frac{dt}{d\tau}=1,\quad \frac{d\phi}{d\tau}=0,\quad \frac{d\omega}{d\tau}=\frac{\partial\Omega}{\partial t}. \tag{4.25}$$

The ray equations (4.24) are Hamilton's equations, with Hamiltonian $H=\Omega(\mathbf{k},\mathbf{x},t)$. If we use the usual Cartesian three-dimensional vectors $\mathbf{x}$ and $\mathbf{k}=(k^1,k^2,k^3)$ for the corresponding vectors in $\mathbf{k}$ space, then $d\mathbf{x}/d\tau=\partial\Omega/\partial\mathbf{k}\equiv\mathbf{V}_{\rm g}$, where $\mathbf{V}_{\rm g}$ is the group velocity of the waves. To prove $d\phi/d\tau=0$ in (4.25) we use the results:

$$\omega=\Omega=kV_{\rm p}(\mathbf{n},\mathbf{x},t),\quad \mathbf{n}=\mathbf{k}/k, \tag{4.26}$$

$$\mathbf{V}_{\rm g}=\frac{\partial\omega}{\partial\mathbf{k}}=V_{\rm p}\mathbf{n}+\frac{1}{k}(\mathbf{I}-\mathbf{nn}){:}\nabla_{\mathbf{n}}V_{\rm p}. \tag{4.27}$$

Using (4.27) it follows that $\mathbf{V}_{\rm g}{\cdot}\mathbf{k}=kV_{\rm p}\equiv\Omega$. Using this result in (4.23) we find $d\phi/d\tau=0$, by virtue of the dispersion equation (4.21).

4.2. Steady flow characteristics

The characteristics for steady MHD flows can be obtained from the characteristic equations $F_{\rm A}=0$ and $F_{\rm MS}=0$ in (4.19) and (4.20) in which we set $\omega=-\phi_t=0$ and $\omega'=-\mathbf{u}\cdot\mathbf{k}\equiv-\mathbf{u}\cdot\nabla\phi$. Thus, the characteristics for steady flows correspond to standing, MHD waves in a steady background flow (e.g. Crapper (1965); McKenzie (1991) and Woodward and McKenzie (1993)). Kogan (1960) and Cabannes (1970) consider, steady ($\omega=0$), linear MHD waves in two-dimensional flows, in Cartesian geometry, in the xy-plane, in which the z coordinate is ignorable, and the unperturbed background flow velocity $\mathbf{u}=(u^x,0,0)^{\rm T}$ is directed along the x-axis. Cabannes derived a fourth-order partial differential equation for the perturbed current, for the magnetoacoustic waves, which is distinct from the characteristic equation $F_{\rm MS}=0$ ((4.20) for steady flows with $\omega=-\phi_t=0$). The analysis of Kogan (1960) is of particular interest, since he considered the magnetoacoustic bi-characteristics for two-dimensional flow, which correspond to the solutions of a fourth-order differential equation for dy/dx, which gives the directions of the bi-characteristics in terms of the parameters defining the background flow. The bi-characteristics for Cabannes equation are the same as those obtained by Kogan (1960). Kogan's analysis describes when the flow is elliptic (complex characteristics), or hyperbolic (real characteristics). The characteristics for steady, field-aligned MHD flow, is described in the books by Jeffrey and Taniuti (1964) and Landau et al. (1984). These characteristics also arise naturally as a special case of Kogan's analysis, even although Kogan's analysis was strictly for the case of a uniform background flow.

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4.2.1. The magnetoacoustic wave eikonal. From (4.20), the magnetoacoustic wave eikonal equation for standing MHD waves, or equivalently, the characteristic manifold equation for $\phi(\mathbf{r}, t)$ is

$$F'(\omega', \mathbf{k}, \mathbf{x}) = \omega'^4 - (a^2 + b^2)\omega'^2 k^2 + a^2(\mathbf{b}\cdot\mathbf{k})^2 k^2 = 0, \tag{4.28}$$

where

$$\omega' = -\mathbf{u}\cdot\mathbf{k} \equiv -\mathbf{u}\cdot\nabla\phi, \tag{4.29}$$

is the frequency of the wave in the fluid frame. In the fixed frame, the frequency $\omega = -\phi_t = 0$. If we eliminate the reference to ω' in (4.28), (4.28) can be written in the form:

$$F(\mathbf{k}, \mathbf{x}) = (\mathbf{u}\cdot\mathbf{k})^4 - (a^2 + b^2)(\mathbf{u}\cdot\mathbf{k})^2 k^2 + a^2(\mathbf{b}\cdot\mathbf{k})^2 k^2 = 0. \tag{4.30}$$

The Cauchy characteristics (or bi-characteristics, or ray equations) for the first-order partial differential equation (4.30) for ϕ are given by (4.23), except that the index α is now restricted to the spatial components of k^α (i.e. $\alpha = 1, 2, 3$, $\alpha \neq 0$). Equation (4.28) can also be cast in the form:

$$c^4 - (a^2 + b^2)c^2 + a^2(\mathbf{b}\cdot\mathbf{n})^2 = 0, \tag{4.31}$$

for the wave phase speed $c = \omega'/k$ of the waves in the fluid frame, where $\mathbf{n} = \mathbf{k}/k$ is the wave normal.

Using (4.27), the group velocity, $\mathbf{V}'_{\mathrm{g}}$ in the fluid frame is given by

$$\mathbf{V}'_{\mathrm{g}} = \frac{\partial\omega'}{\partial\mathbf{k}} = c\mathbf{n} - \frac{a^2(\mathbf{b}\cdot\mathbf{n})(\mathbf{b} - \mathbf{b}\cdot\mathbf{nn})}{c[2c^2 - (a^2 + b^2)]}, \tag{4.32}$$

where the phase speed c is one of the roots of (4.31). Note from (4.32) that $\mathbf{V}'_{\mathrm{g}}\cdot\mathbf{n} = c$. An alternative, equivalent expression for the group velocity $\mathbf{V}'_{\mathrm{g}}$ can be obtained by noting that $\omega' = c(\vartheta)k$ where $\cos\vartheta = \mathbf{n}\cdot\mathbf{e}_1$, $\mathbf{e}_1 = \mathbf{B}/B$ is the unit vector along the magnetic field, and $c(\vartheta)$ satisfies the dispersion equation (4.31). In this approach (Whitham 1974), the group velocity has the form:

$$\mathbf{V}'_{\mathrm{g}} = \frac{\partial\omega'}{\partial\mathbf{k}} = c(\vartheta)\mathbf{n} + c'(\vartheta)\mathbf{e}_\vartheta \tag{4.33}$$

where $c'(\vartheta) = \partial c/\partial\vartheta$, and

$$\mathbf{n} = \cos\vartheta\mathbf{e}_1 + \sin\vartheta\mathbf{e}_2, \tag{4.34}$$

$$\mathbf{e}_\vartheta = -\sin\vartheta\mathbf{e}_1 + \cos\vartheta\mathbf{e}_2, \tag{4.35}$$

are the wave normal $\mathbf{n} = \mathbf{k}/k$ and $\mathbf{e}_\vartheta$ is the unit vector in the direction of increasing ϑ. In (4.33)–(4.35) cylindrical coordinates are used in which the polar axis ($\mathbf{e}_1$) is along the magnetic field, and $(x, y, z) = (x_1, x_2\cos\varphi, x_2\sin\varphi)$ is the position vector using cylindrical coordinates (x_2, φ, x_1). From (4.33)–(4.35) the group velocity surface $\mathbf{r}' = \mathbf{V}'_{\mathrm{g}}t$ can be written in the parametric form

$$\begin{aligned} x_1 &= V'_{\mathrm{g}1}t = [c(\vartheta)\cos\vartheta - c'(\vartheta)\sin\vartheta]t, \\ x_2 &= V'_{\mathrm{g}2}t = [c'(\vartheta)\cos\vartheta + c(\vartheta)\sin\vartheta]t. \end{aligned} \tag{4.36}$$

The group velocity surface (4.36) can also be described locally, by the envelope of the wave eikonal function

$$S = k(x_1\cos\vartheta + x_2\sin\vartheta - c(\vartheta)t), \tag{4.37}$$

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obtained by varying the wave normal angle ϑ. In other words, the envelope of the family of plane wave fronts $S = 0$, tangent to the group velocity surface (4.36) obtained by solving $S = 0$ and $S_\vartheta = 0$ simultaneously for x_1 and x_2 yields the group velocity surface (4.36). The group velocity of the waves in the stationary frame, $\mathbf{V}_{\rm g}$ is given by the usual Galilean transformation

$$\mathbf{V}_{\rm g} = \mathbf{V}'_{\rm g} + \mathbf{u}, \tag{4.38}$$

for the transformation of velocity between frames.

Below we discuss the connection between the characteristics of (4.30), the group velocity $\mathbf{V}_{\rm g}$ and the Mach cone for magnetoacoustic waves. The Cauchy characteristics for (4.30) are

$$\frac{d\mathbf{x}}{d\tau} = \frac{\partial F}{\partial \mathbf{k}}, \quad \frac{d\phi}{d\tau} = \mathbf{k}\cdot\frac{\partial F}{\partial \mathbf{k}}, \quad \frac{d\mathbf{k}}{d\tau} = -\frac{\partial F}{\partial \mathbf{x}}. \tag{4.39}$$

Because $F(\mathbf{k},\mathbf{x})$ is a homogeneous function of $\mathbf{k}$ of degree four, i.e. $F(\lambda\mathbf{k},\mathbf{x}) = \lambda^4 F(\mathbf{k},\mathbf{x})$, it follows that

$$\frac{d\phi}{d\tau} = \mathbf{k}\cdot\frac{\partial F}{\partial \mathbf{k}} = 4F(\mathbf{k},\mathbf{x}) = 0, \tag{4.40}$$

and hence ϕ is constant on the characteristics. Differentiation of the magnetosonic dispersion equation (4.28) with respect to $\mathbf{k}$ gives the expression

$$\mathbf{V}'_{\rm g} = -F'_{\mathbf{k}}/F'_{\omega'}, \tag{4.41}$$

for the group velocity in the fluid frame. Using (4.41), and noting $F(\mathbf{k},\mathbf{x}) \equiv F'(\omega',\mathbf{k},\mathbf{x})$ with $\omega' = -\mathbf{u}\cdot\mathbf{k}$ in the characteristic equations (4.39) we find

$$\frac{d\mathbf{x}}{d\tau} = \frac{\partial F'}{\partial \omega'}\left(\frac{\partial \omega'}{\partial \mathbf{k}} + F'_{\mathbf{k}}/F'_{\omega'}\right) = -\frac{\partial F'}{\partial \omega'}(\mathbf{u} + \mathbf{V}'_{\rm g}) = -\frac{\partial F'}{\partial \omega'}\mathbf{V}_{\rm g}. \tag{4.42}$$

In (4.42) $\partial\omega'/\partial\mathbf{k} = -\mathbf{u}$, since $\omega' = -\mathbf{k}\cdot\mathbf{u}$ for standing waves. The important point in (4.42) is that the direction of the characteristic vector field $d\mathbf{x}/d\tau$ is parallel to the group velocity $\mathbf{V}_{\rm g}$ in the fixed frame.

From (4.39) and (4.40) we find

$$\mathbf{k}\cdot\frac{d\mathbf{x}}{d\tau} = \mathbf{k}\cdot\frac{\partial F}{\partial \mathbf{k}} = \frac{d\phi}{d\tau} = 0. \tag{4.43}$$

Then from (4.43) it follows that

$$\mathbf{k}\cdot\mathbf{V}_{\rm g} = 0, \tag{4.44}$$

on the characteristics. Thus, for standing waves, the characteristic direction $d\mathbf{x}/d\tau$ (or equivalently, the group velocity direction in the fixed frame) is perpendicular to the wave vector $\mathbf{k} = \nabla\phi$, and hence the group velocity lies in the plane $\phi =$ constant. Equation (4.44) can also be derived by noting that

$$\mathbf{k}\cdot\mathbf{V}_{\rm g} = \mathbf{k}\cdot(\mathbf{u} + \mathbf{V}'_{\rm g}) = \mathbf{k}\cdot\mathbf{u} + ck = -\omega' + \omega' = 0, \tag{4.45}$$

on the characteristics.

4.2.2. Non-field aligned two-dimensional flow. As an example of the above ideas, consider the form of the characteristics for flow in two Cartesian space dimensions in the xy-plane, with ignorable coordinate z (see, e.g., Kogan 1960). Since ϕ is

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constant on the characteristics (4.40), then $\phi_x dx + \phi_y dy = 0$, and hence

$$\frac{dy}{dx} = -\frac{\phi_x}{\phi_y}, \tag{4.46}$$

on the characteristics. Using (4.46) in the characteristic manifold equation (4.30), implies that $y' = dy/dx$ satisfies the fourth-order differential equation

$$(u_y - u_x y')^4 - (a^2 + b^2)(u_y - u_x y')^2(1 + y'^2) + a^2(b_y - b_x y')^2(1 + y'^2) = 0. \tag{4.47}$$

Equation (4.47) can be written in the form

$$a_4 y'^4 + a_3 y'^3 + a_2 y'^2 + a_1 y' + a_0 = 0, \tag{4.48}$$

where

$$\begin{aligned} a_4 &= u_x^4 - (a^2 + b^2)u_x^2 + a^2 b_x^2, \\ a_3 &= -4u_x^3 u_y + 2(a^2 + b^2)u_x u_y - 2a^2 b_x b_y, \\ a_2 &= 6u_x^2 u_y^2 - (a^2 + b^2)u^2 + a^2 b^2, \\ a_1 &= -4u_x u_y^3 + 2(a^2 + b^2)u_x u_y - 2a^2 b_x b_y, \\ a_0 &= u_y^4 - (a^2 + b^2)u_y^2 + a^2 b_y^2. \end{aligned} \tag{4.49}$$

Equation (4.48) can also be obtained by re-arranging the characteristic equation $dy/dx = V_{gy}/V_{gx}$ which follows from (4.42). Equation (4.48) is related to the fourth-order differential equation for the characteristics obtained by Kogan (1960). Kogan (1960) considered the special case of linear, small amplitude waves in a uniform flow along the x-axis, in which $\mathbf{u} = u_x \mathbf{e}_x$, and the background magnetic field was uniform, and confined to the xy-plane. In this case, the fourth-order differential equation (4.48) for y' can be reduced to the form

$$\begin{aligned} y'^4[(M_s^2 - N_x^2)(1 - M_s^2) + N_y^2 M_s^2] + y'^3(2N_x N_y) \\ + y'^2[M_s^2 + (N_x^2 + N_y^2)(M_s^2 - 1)] + y'(2N_x N_y) - N_y^2 = 0, \end{aligned} \tag{4.50}$$

where the parameters

$$M_s = \frac{u_x}{a}, \quad N_x = \frac{b_x}{a}, \quad N_y = \frac{b_y}{a}, \tag{4.51}$$

specify the background flow. Here M_s is the sonic Mach number of the background flow, whereas $\mathbf{N} = \mathbf{b}/a$ is the ratio of the Alfvén velocity to the sound speed a. Equation (4.50) is the equation for the characteristics derived by Kogan (1960, (1.3)).

The characteristic equations (4.48)–(4.50) do not take into account the symmetry of the fluid frame group velocity surface about the magnetic field. If one uses for example, spatial coordinates x_1 and x_2 in the fixed frame, where x_1 is parallel to $\mathbf{B}$ and x_2 is perpendicular to $\mathbf{B}$, then since ϕ is constant on the characteristics $\phi_{x_1} dx_1 + \phi_{x_2} dx_2 = 0$ and $dx_2/dx_1 = -\phi_{x_1}/\phi_{x_2} \equiv -\cot\vartheta$ where ϑ is the wave normal angle between $\mathbf{k}$ and $\mathbf{B}$ (note that locally, $\phi = k(x_1 \cos\vartheta + x_2 \sin\vartheta)$). Using these relations in the characteristic equation (4.30) gives the fourth-order polynomial equation

$$\tilde{a}_4 w^4 + \tilde{a}_3 w^3 + \tilde{a}_2 w^2 + \tilde{a}_1 w + \tilde{a}_0 = 0, \tag{4.52}$$

for the bi-characteristics where

$$w = \frac{dx_2}{dx_1} = -\cot\vartheta, \tag{4.53}$$

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and

$$\tilde{a}_4 = (u_1^2 - a^2)(u_1^2 - b^2),$$
$$\tilde{a}_3 = 2u_1u_2(a^2 + b^2 - 2u_1^2),$$
$$\tilde{a}_2 = 6u_1^2u_2^2 - (a^2 + b^2)u^2 + a^2b^2, \tag{4.54}$$
$$\tilde{a}_1 = 2u_1u_2(a^2 + b^2 - 2u_2^2),$$
$$\tilde{a}_0 = u_2^2[u_2^2 - (a^2 + b^2)].$$

The fluid velocity components u_1 and u_2 are given by

$$u_1 = u_x \cos\alpha + u_y \sin\alpha, \quad u_2 = -u_x \sin\alpha + u_y \cos\alpha, \tag{4.55}$$

where α is the angle between $\mathbf{u}$ and $\mathbf{B}$. Equation (4.52) is in effect a fourth-order polynomial equation for $\cot\vartheta$ where ϑ is the wave normal angle. Real characteristic roots of this equation for ϑ correspond to hyperbolic characteristics.

Exactly the same equation for $w = -\cot\vartheta$ arises from noting that

$$\mathbf{n} \cdot \mathbf{V}_g = c(\vartheta) + \mathbf{u} \cdot \mathbf{n} = c + u_1 \cos\vartheta + u_2 \sin\vartheta = 0, \tag{4.56}$$

and that $c(\vartheta)$ satisfies the dispersion equation

$$c^4 - (a^2 + b^2)c^2 + a^2b^2 \cos^2\vartheta = 0. \tag{4.57}$$

Solving (4.56) for c in terms of $\mathbf{u}$ and ϑ and substituting the result in (4.57) yields (4.52).

4.2.3. The Mach cone. Kogan (1960) noted that the two-dimensional flow described above, in general has four roots of the characteristic equation (4.50). The nature of the roots of (4.50) for y' depends on the values of the parameters defining the background flow. The flow can have up to four possible real roots for y'. Four real, distinct roots for y' correspond to hyperbolic flow, whereas no real roots (i.e. the roots occur as complex conjugate pairs), correspond to elliptic flow. The existence of real roots for y' correspond to cases where the characteristics form a Mach cone about the original disturbance (see e.g. Landau and Lifshitz (1987, ch. 9, p. 313), for a discussion of the Mach cone formed by sound waves in a supersonic flow). The simplest way in which to understand the formation of Mach cones, corresponding to real roots for y' is obtained by constructing the Mach cone associated with the transformation of the group velocity (4.38) between frames. Real characteristics correspond to the case where the group velocity $\mathbf{V}_g$ in the fixed frame can be drawn as a tangent to either the fast or slow mode group velocity surface.

Equation (4.56) can be written in the form:

$$c = -\mathbf{u} \cdot \mathbf{n} = -|u| \cos(\alpha - \vartheta) = |u| \sin A, \tag{4.58}$$

or

$$\sin A = \frac{1}{M}, \tag{4.59}$$

where

$$M = \frac{|u|}{c} \quad \text{and} \quad A = \vartheta - \alpha - \frac{\pi}{2}. \tag{4.60}$$

In (4.59) and (4.60), M is the Mach number of the flow on the Mach cone, and A is the Mach cone angle. For real characteristics $M > 1$ and $|\sin A| < 1$. Equations (4.59)–(4.60) are analogous to similar results for the Mach cone angle for sound waves

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(e.g. Landau and Lifshitz (1987, ch. 9, p. 313)), except that in the present case, the magnetosonic speed $c(\vartheta)$ can correspond to either the fast or slow magnetosonic wave speed.

An alternative proof of (4.59) follows from noting that

$$\sin^2 A = \frac{|\mathbf{x}_\tau \times \mathbf{u}|^2}{|\mathbf{x}_\tau|^2 u^2}. \tag{4.61}$$

Since ϕ is constant on the characteristics, $\phi_x x_\tau + \phi_y y_\tau = 0$ in (4.61). This implies $y_\tau = -\phi_x/\phi_y x_\tau$, and hence

$$\mathbf{x}_\tau \times \mathbf{u} = (x_\tau u_y - y_\tau u_x)\mathbf{e}_z = \frac{x_\tau}{\phi_y}(\phi_x u_x + \phi_y u_y)\mathbf{e}_z = \frac{x_\tau(\mathbf{k}\cdot\mathbf{u})}{k_y}\mathbf{e}_z. \tag{4.62}$$

Using (4.62) in (4.61) we find

$$\sin^2 A = \frac{(\mathbf{k}\cdot\mathbf{u})^2}{k^2u^2} = \frac{(\mathbf{n}\cdot\mathbf{u})^2}{u^2} = \frac{c^2}{u^2} = \frac{1}{M^2}, \tag{4.63}$$

which is equivalent to (4.59).

Illustrative examples of the group velocity surface for standing magnetoacoustic waves in a background flow with velocity $\mathbf{u}$ are given for example by Sears (1960), and Woodward and McKenzie (1993). In general, there are four roots of the characteristic equation (4.52), but depending on the magnitude and direction of the background flow relative to the background magnetic field, not all roots of (4.52) will necessarily be real. Real solutions of (4.52) correspond to cases where there are real characteristics which can be drawn tangent to the group velocity surface. These conditions determine whether the flow is elliptic or hyperbolic, or of a mixed hyperbolic-elliptic type. The Alfvén waves also have a Mach cone for super-Alfvénic flow which are commonly referred to as Alfvén wings (e.g. Woodward and McKenzie (1993)).

4.3. Compressible ($\delta\rho \neq 0$) and incompressible ($\delta\rho = 0$) perturbations

The characteristic analysis of the coupled wave equations (4.1) and (4.2) for $\boldsymbol{\xi}$ and ΔS in Secs 4.1 and 4.2 does not explicitly attempt to isolate off the compressible wave perturbations ($\delta\rho \neq 0$) from the incompressible perturbations ($\delta\rho = 0$). The Eulerian density perturbation $\delta\rho$, from (3.4) is given by

$$\delta\rho = -\nabla\cdot(\rho\boldsymbol{\xi}). \tag{4.64}$$

The Eulerian MHD eigenvectors (e.g. Webb et al. (1999)) show that the linear, fast and slow magnetoacoustic waves and the entropy wave are compressible ($\delta\rho \neq 0$), whereas the Alfvén wave is incompressible ($\delta\rho = 0$).

From electromagnetic theory (e.g. Panofsky and Phillips (1964, ch. 1)), a three-dimensional vector field $\mathbf{V}$ with finite divergence ($\nabla\cdot\mathbf{V} = \sigma$) and curl ($\nabla\times\mathbf{V} = \mathbf{c}$) in which $|\sigma|$ and $|\mathbf{c}|$ vanish sufficiently fast as $\mathbf{r}\to\infty$ can be represented in the form $\mathbf{V} = \nabla\psi + \nabla\times\mathbf{w}$. The scalar and vector potentials ψ and $\mathbf{w}$ are solutions of the Poisson equations $\nabla^2\psi = \sigma$ and $\nabla^2\mathbf{w} = -\mathbf{c}$, where the guage potential defining $\mathbf{w}$ has been chosen so that $\nabla\cdot\mathbf{w} = 0$. Using this idea we represent the vector field $\mathbf{q} = \rho\boldsymbol{\xi}$ in the form

$$\mathbf{q} = \rho\boldsymbol{\xi} = \nabla\psi + \nabla\times\mathbf{w}. \tag{4.65}$$

Using (4.65) it follows that

$$\delta\rho = -\nabla\cdot\mathbf{q} = -\nabla^2\psi \tag{4.66}$$

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and

$$\nabla \times \mathbf{q} = \nabla(\nabla \cdot \mathbf{w}) - \nabla^2 \mathbf{w} \equiv -\nabla^2 \mathbf{w}, \tag{4.67}$$

where we have chosen the guage such that $\nabla \cdot \mathbf{w} = 0$. Equations (4.65)–(4.67) show that the compressible perturbations ($\delta\rho \neq 0$) are described in terms of the scalar potential ψ whereas the incompressible perturbations ($\delta\rho = 0$) are described by $\mathbf{w}$. However, it should be noted that the formulas $\Delta\rho = -\rho\nabla\cdot\xi$ and $\delta\rho = -\nabla\cdot(\rho\xi)$ only apply for small amplitude, linear waves, whereas $\Delta\rho = \rho^* - \rho = (\rho/J - \rho)$ in the general nonlinear case where $J = \det(\partial x^{*i}/\partial x^j)$ is the Jacobian determinant (2.6).

The action A and the wave action functionals A_j ($j = 1, 2$) in (2.18) depend on the Lagrangian wave displacement vector $\boldsymbol{\xi}$. Using (4.65) we obtain the variational derivative transformations:

$$\frac{\delta \tilde{F}}{\delta \psi} = -\nabla \cdot \left(\frac{1}{\rho} \frac{\delta F}{\delta \boldsymbol{\xi}} \right) \quad \text{and} \quad \frac{\delta F}{\delta \mathbf{w}} = \nabla \times \left(\frac{1}{\rho} \frac{\delta F}{\delta \boldsymbol{\xi}} \right). \tag{4.68}$$

for the variational functional $\tilde{F}[\psi, \mathbf{w}] = F[\boldsymbol{\xi}]$. Using the transformations (4.68) we obtain

$$\frac{\delta A_2}{\delta \psi} = -\nabla \cdot \left(\frac{1}{\rho} \frac{\delta A_2}{\delta \boldsymbol{\xi}} \right) \equiv \nabla \cdot \left(\frac{1}{\rho} \mathbf{P}^{\mathrm{D}} \right) = 0, \tag{4.69}$$

$$\frac{\delta A_2}{\delta \mathbf{w}} = \nabla \times \left(\frac{1}{\rho} \frac{\delta A_2}{\delta \boldsymbol{\xi}} \right) \equiv -\nabla \times \left(\frac{1}{\rho} \mathbf{P}^{\mathrm{D}} \right) = 0, \tag{4.70}$$

where $\mathbf{P}^{\mathrm{D}} = 0$ is the perturbed momentum equation (2.20). From (2.20), (4.69) and (4.70) we obtain the equations

$$\mathbf{P}^{\mathrm{D}} = 0, \quad \nabla \cdot \mathbf{P}^{\mathrm{D}} = 0, \quad \nabla \times \mathbf{P}^{\mathrm{D}} = 0, \tag{4.71}$$

describing the wave perturbations.

4.3.1. Wave equations for ψ and $\mathbf{w}$. Below, we provide two propositions which characterize the interaction of the compressible ($\delta\rho \neq 0$) and incompressible ($\delta\rho = 0$) perturbations.

PROPOSITION 1. *The equations $\nabla\cdot\mathbf{P}^{\mathrm{D}} = 0$ and $\nabla \times \mathbf{P}^{\mathrm{D}} = 0$ in (4.71) may be rewritten, respectively, as wave equations for ψ and $\mathbf{w}$ of the form*

$$\left(\frac{\partial^2}{\partial t^2} + 2\mathbf{u} \cdot \nabla \frac{\partial}{\partial t} + \mathbf{uu}{:}\nabla\nabla + \mathbf{bb} : \nabla\nabla - (a^2 + b^2)\nabla^2 \right) \nabla^2 \psi + \mathbf{bb} : \nabla\nabla \times (\nabla^2 \mathbf{w}) + S_1 = 0, \tag{4.72}$$

$$\left(\frac{\partial^2}{\partial t^2} + 2\mathbf{u} \cdot \nabla \frac{\partial}{\partial t} + \mathbf{uu}{:}\nabla\nabla - \mathbf{bb} : \nabla\nabla \right) \nabla^2 \mathbf{w} + \mathbf{b} \times [(\mathbf{b} \cdot \nabla)\nabla^2 \psi] + \mathbf{S}_2 = 0, \tag{4.73}$$

where $a = (\gamma p/\rho)^{1/2}$ and $\mathbf{b} = \mathbf{B}/(\mu\rho)^{1/2}$ denote the gas sound speed and Alfvén velocity, respectively. In (4.72) and (4.73) the source terms consist of lower-order derivatives of ψ and $\mathbf{w}$, which describe the interaction of finite wavelength waves, whereas the higher-order, explicit derivative terms (of order 4) represent the dominant effects at short wavelengths.

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The source terms S_1 and $\mathbf{S}_2$ in (4.72) and (4.73) can be written in terms of $\mathbf{q}$ (or ψ and $\mathbf{w}$). The detailed forms of S_1 and $\mathbf{S}_2$ are given in Appendix B. Note, that if the Lagrangian entropy perturbation $\Delta S \neq 0$, then ΔS is advected with the background flow (i.e. ΔS satisfies (2.21)).

The proof of Proposition 1 follows in a straightforward fashion by first writing $\mathbf{P}^{\mathrm{D}}$ of (2.20) in terms of $\mathbf{q} = \rho\boldsymbol{\xi}$ in the form

$$\begin{aligned}\mathbf{P}^{\mathrm{D}} = {} & \frac{\partial}{\partial t}(\mathbf{q}_t + \nabla\cdot(\mathbf{uq})) \\ & + \nabla\cdot\left\{\mathbf{u}[\mathbf{q}_t + \nabla\cdot(\mathbf{uq})] + \left[\left(\frac{p}{\rho} - a^2\right)(\nabla\cdot\mathbf{q} - \nabla(\ln\rho)\cdot\mathbf{q}) - p_S\Delta S\right]\mathbf{I}\right. \\ & - \frac{p}{\rho}[(\nabla\mathbf{q})^t - \mathbf{q}\nabla(\ln\rho)] \\ & + \left(\mathbf{bb}:(\nabla\mathbf{q} - \nabla(\ln\rho)\mathbf{q}) - \frac{b^2}{2}[\nabla\cdot\mathbf{q} - \mathbf{q}\cdot\nabla(\ln\rho)]\right)\mathbf{I} \\ & \left. - \frac{b^2}{2}((\nabla\mathbf{q})^t - \mathbf{q}\nabla(\ln\rho)) + \mathbf{bb}[\nabla\cdot\mathbf{q} - \mathbf{q}\cdot\nabla(\ln\rho)] - \mathbf{bb}\cdot(\nabla\mathbf{q} - \nabla(\ln\rho)\mathbf{q})\right\} \\ & + \mathbf{q}\cdot\nabla\nabla\phi. \end{aligned} \tag{4.74}$$

To obtain (4.72) and (4.73) the highest-order derivatives in $\nabla\cdot\mathbf{P}^{\mathrm{D}} = 0$ and $\nabla\times\mathbf{P}^{\mathrm{D}} = 0$ are separated off from the lower-order derivatives which are represented by the source terms S_1 and $\mathbf{S}_2$ respectively.

PROPOSITION 2. *The density perturbation $\delta\rho = -\nabla^2\psi$ satisfies the magnetoacoustic type wave equation*

$$\hat{L}_{\mathrm{MS}}\delta\rho + \hat{S}_1 = 0, \tag{4.75}$$

where $\hat{L}_{\mathrm{MS}}$ are the highest-order derivative terms in the magnetoacoustic operator

$$L_{\mathrm{MS}} = \partial_{t'}^4 - (a^2 + b^2)\nabla^2\partial_{t'}^2 + a^2\mathbf{bb}:\nabla\nabla, \tag{4.76}$$

and

$$\partial_{t'} = \partial_t + \mathbf{u}\cdot\nabla, \tag{4.77}$$

is the Lagrangian time derivative following the flow. The source term $\hat{S}_1$ in (4.75) involves lower-order derivatives of ψ and $\mathbf{w}$. In other words, we use the convention that the differential operator $\hat{L}$ is obtained from the operator L by expanding L in terms of ∂_t and ∇ and regarding the background flow quantities $\mathbf{u}$, a and $\mathbf{b}$ as constants in the process.

To prove (4.75) we note that the wave equations (4.72) and (4.73) can be written in the form

$$\hat{L}_1\nabla^2\psi + \mathbf{bb}:\nabla(\nabla\times\nabla^2\mathbf{w}) + S_1 = 0, \tag{4.78}$$

$$\hat{L}_{\mathrm{A}}\nabla^2\mathbf{w} + \mathbf{b}\times[(\mathbf{b}\cdot\nabla)\nabla(\nabla^2\psi)] + \mathbf{S}_2 = 0, \tag{4.79}$$

where

$$L_1 = \partial_{t'}^2 + \mathbf{bb}:\nabla\nabla - (a^2 + b^2)\nabla\nabla. \tag{4.80}$$

and

$$\hat{L}_{\mathrm{A}} = \left(\frac{\partial^2}{\partial t^2} + 2\mathbf{u}\cdot\nabla\frac{\partial}{\partial t} + \mathbf{uu}:\nabla\nabla - \mathbf{bb}:\nabla\nabla\right), \tag{4.81}$$

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is the second-order wave operator associated with Alfvénic perturbations. From (4.78) and (4.79):

$$\{\hat{L}_{\rm A}\hat{L}_1 - \mathbf{bb} : \nabla\nabla \times [\mathbf{b} \times (\mathbf{b}\cdot\nabla)\nabla]\}\nabla^2\psi + S = 0, \tag{4.82}$$

where

$$S = -\mathbf{bb} : \nabla\nabla \times \mathbf{S}_2 + \hat{L}_{\rm A}S_1 + \hat{L}_{\rm A}(\mathbf{bb}) : \nabla\nabla \times \nabla^2\mathbf{w}. \tag{4.83}$$

Expanding the operator in curly brackets in (4.82) and collecting the highest-order derivative terms, and noting that $\delta\rho = -\nabla^2\psi$ we obtain the magnetoacoustic type wave equation (4.75) where $\hat{S}_1$ consists of lower-order derivative terms.

5. Summary and discussion

The main aim of this paper was to provide a framework for MHD wave propagation in non-uniform media, such as the Solar Wind, based on the MHD variational principles of Newcomb (1962) and Dewar (1970, 1977).

The paper has concentrated on linear, non-WKB wave propagation in a non-uniform background medium, based on Dewar's (1970) variational principle in which the Lagrangian is expanded out to second order in the Lagrangian wave displacement $\boldsymbol{\xi}(\mathbf{x}, t)$, and also in terms of the Lagrangian entropy perturbation ΔS, which is advected with the background flow. These equations are generalizations of the Frieman and Rotenberg (1960) equations used to study the stability of steady MHD flows (Sec. 3). The characteristic manifolds $\phi(\mathbf{x}, t)$ = constant of the wave equations correspond to the usual dispersion equations for the MHD eigenmodes (i.e. the Alfvén, fast and slow magnetoacoustic waves and the entropy wave), in which $\omega' = -(\phi_t + \mathbf{u}\cdot\nabla\phi)$ and $\mathbf{k} = \nabla\phi$ define the local Doppler shifted frequency of the waves in the fluid frame and $\mathbf{k}$ is the wavenumber. The dispersion equations can be regarded as first-order partial differential equations for the singular manifold function $\phi(\mathbf{x}, t)$, with Cauchy characteristics that define the ray equations of MHD geometrical optics. For standing waves in a steady flow, the fixed frame frequency $\omega = -\phi_t = 0$, and in this case the ray equations define the group velocity surface and Mach cone for the magnetoacoustic waves and Alfvén wings, described in earlier analyses of Kogan (1960), Sears (1960), Bazer and Hurley (1963), Crapper (1965) and Woodward and McKenzie (1993). However, these earlier analyses did not emphasize the connection between the theory of Cauchy characteristics of first-order partial differential equations and the ray equations used in the present analysis. Methods to split the wave equations into compressible and incompressible components were also discussed (Sec. 4).

It is interesting to note that an inadvertent loss of two of the MHD characteristics occurs in many discussions of MHD flows based on the generalized Grad–Shafranov equation used to describe steady MHD winds and jets with one ignorable spatial coordinate. The loss of two of the characteristics can be traced back to the assumption that the electric field in the ignorable coordinate direction is zero (see e.g. Contopoulos 1996; Webb et al. 2001). Similarly, the equations used to describe steady, two-dimensional field-aligned flow (e.g. Jeffrey and Taniuti 1964) do not contain the slow mode characteristics. For field-aligned flow, the slow mode phase speed in the fluid frame for wave propagation perpendicular to the magnetic field is zero, but the group velocity is directed parallel to the magnetic field and equals the MHD cusp speed $V_{\rm c} = ab/(a^2 + b^2)^{1/2}$, where a is the sound speed and b is

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the Alfvén speed. This result turns out to be important in the discussion of the possibility of a slow mode, hydrodynamical, 3–4 type bow shock in MHD models of the interaction of the Solar Wind with an incoming, field aligned, supersonic, but sub-Alfvénic interstellar flow (Ratkiewicz and Webb (2004), Pogorelov and Matsuda (2004)).

Webb et al. (2003) and Webb (2004) have discussed the use of Noether's theorem in the derivation of stress-energy tensors for the waves and background flow, for fully nonlinear, non-WKB waves. This analysis is similar to Dewar's (1970) derivation of stress-energy tensors for the background and WKB waves in a non-uniform background flow, except that the WKB assumption is discarded, and the analysis was carried out using the exact MHD Lagrangian consisting of wave and background components. A detailed derivation of these results, and other uses of Noether's theorem for MHD waves in non-uniform, time-dependent background flows will be given in a forthcoming paper.

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Appendix A

In this appendix we point out that the characteristic manifolds for MHD waves discussed in Sec. 4.1, also arise naturally from an analysis of WKB waves in non-uniform flows in the averaged Lagrangian variational principle of Dewar (1970). The Hermitian matrix $\tilde{\mathbf{A}}$ in (4.16) and the Alfvén and magnetoacoustic dispersion equations (4.19) and (4.20) also occur in the analysis of Dewar (1970). By using the identifications $\omega = -\phi_t$ and $\mathbf{k} = \nabla\phi$, the dispersion equations (4.19) and (4.20) can be identified as the wave eikonal equations of WKB theory for the wave phase ϕ. Dewar (1970) considered WKB, waves in a non-uniform background flow for which $\boldsymbol{\xi} = \mathbf{a}\exp(i\phi) + \mathbf{a}^*\exp(-i\phi)$, in which ϕ is the fast varying wave phase and $\mathbf{a}$ is the complex wave amplitude. By averaging over the fast wave phase variable ϕ, Dewar obtained

$$\langle \mathscr{L}_2 \rangle = \rho \mathbf{a}^t \tilde{\mathbf{A}} \mathbf{a}^*, \tag{A 1}$$

for the averaged wave Lagrangian, where $\tilde{\mathbf{A}}$ is the matrix (4.16). Variation of the action $\langle A_2 \rangle$ corresponding to $\langle \mathscr{L}_2 \rangle$ in (2.16) with respect to the wave amplitude $\mathbf{a}$ gave the eigenvector equation:

$$\frac{\delta \langle A_2 \rangle}{\delta \mathbf{a}} = \rho \tilde{\mathbf{A}} \mathbf{a}^* = 0. \tag{A 2}$$

The eigen-equation (A 2) has a non-trivial solution for $\mathbf{a}^*$ if $\det(\tilde{\mathbf{A}}) = 0$, which is the determinantal equation (4.15) for the characteristic MHD manifolds (note $\det \tilde{\mathbf{A}}$ is given by (4.18)). By varying the slow wave phase (i.e. by varying $\mathbf{k} = \nabla\phi$ and $\omega = -\phi_t$), Dewar obtained the wave action equation for WKB, MHD waves.

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Appendix B

In this appendix we list the lower-order derivative source terms that appear in the wave equations (4.72) and (4.73) for the potentials ψ and $\mathbf{w}$ describing compressible and incompressible perturbations. The source terms S_1 and $\mathbf{S}_2$ are given by

$$S_1 = 2\nabla\mathbf{q} : \nabla\mathbf{u} + \nabla\nabla(\mathbf{q}\cdot\nabla)\colon \mathbf{uu} - \nabla(a^2+b^2)\cdot\nabla(\nabla^2\psi)$$
$$+ [\nabla(\mathbf{bb})\cdot\nabla]\colon \nabla\mathbf{q} + \nabla(\mathbf{bb})\cdot\nabla(\nabla^2\psi) - \nabla\nabla(\mathbf{q}\cdot\nabla)\colon \mathbf{bb} + \nabla\cdot\mathbf{R}, \tag{B 1}$$

$$\mathbf{S}_2 = -\{2\nabla u^s \times \nabla_s \mathbf{q}_t + \nabla(u^s u^\alpha)\times\nabla_s\nabla_\alpha(\mathbf{q})$$
$$- \nabla(a^2+b^2)\times\nabla(\nabla\cdot\mathbf{q}) + \nabla(b^s b^p)\times\nabla(\nabla_s q^p)$$
$$+ \nabla\times(\mathbf{b}b^s)\nabla_s(\nabla\cdot\mathbf{q}) - \nabla(b^\alpha b^s)\times\nabla_\alpha\nabla_s\mathbf{q} + \nabla\times\mathbf{R}\}, \tag{B 2}$$

where

$$\mathbf{R} = \mathbf{u}_t\cdot\nabla\mathbf{q} + \frac{\partial}{\partial t}(\mathbf{q}\nabla\cdot\mathbf{u}) + (\nabla\cdot\mathbf{u})\mathbf{q}_t + \nabla\cdot(\mathbf{uu})\cdot\nabla\mathbf{q} + \nabla\cdot[\mathbf{uq}(\nabla\cdot\mathbf{u})]$$
$$+ \nabla\left(\frac{p}{\rho} - a^2\right)\nabla\cdot\mathbf{q} - \nabla\left[\left(\frac{p}{\rho}-a^2\right)\mathbf{q}\cdot\nabla\ln(\rho) - p_S\Delta S\right]$$
$$+ \nabla\mathbf{q}\cdot\nabla\left(\frac{p}{\rho}\right) + \nabla\cdot\left(p\mathbf{q}\frac{\nabla\rho}{\rho^2}\right)$$
$$+ \mathbf{q}\cdot\nabla\nabla\phi + \nabla(\mathbf{bb})\colon\nabla\mathbf{q} - \nabla[\mathbf{bb}:\nabla(\ln\rho)\mathbf{q}] - \nabla\left(\frac{b^2}{2}\right)\nabla\cdot\mathbf{q}$$
$$+ \nabla\left(\frac{b^2}{2}\nabla(\ln\rho)\cdot\mathbf{q}\right) - \nabla\mathbf{q}\cdot\nabla\left(\frac{b^2}{2}\right) + \nabla\cdot\left(\mathbf{q}\nabla(\ln\rho)\frac{b^2}{2}\right)$$
$$+ \nabla\cdot(\mathbf{bb})\nabla\cdot\mathbf{q} - \nabla\cdot[\mathbf{bb}(\mathbf{q}\cdot\nabla(\ln\rho))] - \nabla\cdot(\mathbf{bb})\cdot\nabla\mathbf{q} + \nabla\cdot[\mathbf{bq}(\mathbf{b}\cdot\nabla(\ln\rho))]. \tag{B 3}$$

In (B 2) we have used the Einstein summation convention for repeated indices, where the indices range from 1 to 3.

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