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Expanders Are Universal for the Class of All Spanning Trees

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A graph is called *universal* for a given graph class $\mathcal{F}$ (or, equivalently, *$\mathcal{F}$-universal*) if it contains a copy of every graph in $\mathcal{F}$ as a subgraph. The construction of sparse universal graphs for various classes $\mathcal{F}$ has received a considerable amount of attention. There is particular interest in tight $\mathcal{F}$-universal graphs, that is, graphs whose number of vertices is equal to the largest number of vertices in a graph from $\mathcal{F}$. Arguably, the most studied case is that when $\mathcal{F}$ is some class of trees. In this work, we are interested in $\mathcal{T}(n,\Delta)$, the class of all n-vertex trees with maximum degree at most Δ. We show that every n-vertex graph satisfying certain natural expansion properties is $\mathcal{T}(n,\Delta)$-universal. Our methods also apply to the case when Δ is some function of n. Since random graphs are known to be good expanders, our result implies, in particular, that there exists a positive constant c such that the random graph $G(n, cn^{-1/3}\log^2 n)$ is asymptotically almost surely (a.a.s.) universal for $\mathcal{T}(n, O(1))$. Moreover, a corresponding result holds for the random regular graph of degree $cn^{2/3}\log^2 n$. Another interesting consequence is the existence of locally sparse n-vertex $\mathcal{T}(n,\Delta)$-universal graphs. For example, we show that one can (randomly) construct n-vertex $\mathcal{T}(n, O(1))$-universal graphs with clique number at most five. This complements the construction of Bhatt, Chung, Leighton and Rosenberg (1989), whose $\mathcal{T}(n,\Delta)$-universal graphs with merely $O(n)$ edges contain large cliques of size $\Omega(\Delta)$. Finally, we show that random graphs are robustly $\mathcal{T}(n,\Delta)$-universal in the context of the Maker–Breaker tree-universality game.

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1. Introduction

A graph G is *universal* for a class of graphs $\mathcal{F}$ (equivalently, we say that G is *$\mathcal{F}$-universal*) if a copy of every member of $\mathcal{F}$ is contained in G. Since $\mathcal{F}$-universality implies that the

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maximum degree of G is at least as large as the maximum degrees of all graphs in $\mathcal{F}$, it is natural to consider only classes with bounded maximum degree. There exists a rich literature [2, 3, 4, 6, 12, 15, 16, 17, 18, 19, 20, 27, 30, 43] on explicit and randomized constructions of universal graphs.

One of the classes for which universality has been studied extensively is the class of bounded-degree trees. For a positive integer n and a positive real number Δ, let $\mathcal{T}(n,\Delta)$ be the class of all n-vertex trees with maximum degree at most Δ. Bhatt, Chung, Leighton and Rosenberg [12] gave an explicit construction of $\mathcal{T}(n,\Delta)$-universal graphs on n vertices that are very sparse and of maximum degree bounded by a function in Δ. For $\Delta \in O(1)$, their universal graphs have only $O(n)$ edges.

In this work, instead of constructing specific universal graphs, we are rather interested in determining for which edge densities almost all n-vertex graphs become universal for the class $\mathcal{T}(n,\Delta)$. In particular, we want to know for which edge probabilities p the binomial random graph $G(n,p)$ asymptotically almost surely (a.a.s.) becomes universal for the class of all bounded-degree spanning trees. Moreover, we want to identify particular pseudo-random properties that guarantee this universality and that are a.a.s. satisfied by $G(n,p)$.

To this end, we first revise known results in the remainder of this Introduction. Afterwards, we present our new results in Section 2.

Since every connected component of a graph contains a spanning tree, we know that, for large values of c, the random graph $G(n,c/n)$ a.a.s. contains a copy of some tree that covers a significant proportion of the vertices of G. We may now ask whether this is true for every *specific* tree with size linear in n. For paths, that is, trees with maximum degree two, Ajtai, Komlós and Szemerédi [1] showed that if $c > 1$, the random graph $G(n,c/n)$ indeed a.a.s. contains a path of length linear in n. On the other hand, for a fixed c, the random graph $G(n,c/n)$ a.a.s. has maximum degree $(1+o(1))\log n/\log\log n$ and therefore we cannot expect to embed trees of larger maximum degree. Thus, a more reasonable question is to ask whether, for every bounded-degree tree T with size linear in n, the random graph $G(n,c/n)$ a.a.s. contains a copy of T.

This question was first addressed by Fernandez de la Vega [25], who showed that, for fixed $\Delta \geqslant 2$ and $7/8 \leqslant \alpha < 1$, there exists a constant $c := c(\Delta)$ with $\Delta - 1 < c \leqslant 8(\Delta-1)$ such that, for every specific tree $T \in \mathcal{T}((1-\alpha)n,\Delta)$, the random graph $G(n,c/n)$ a.a.s. contains a copy of T. Alon, Krivelevich and Sudakov [5] showed for all $\varepsilon \in (0,1)$ the existence of a constant $c = c(\Delta,\varepsilon)$ such that $G(n,c/n)$ a.a.s. contains a copy of every tree in $\mathcal{T}((1-\varepsilon)n,\Delta)$, thus extending the result of Fernandez de la Vega to *almost-spanning trees*, that is, to arbitrarily small values of ε. A better bound for $c(\Delta,\varepsilon)$ and a resilience version of this result were obtained by Balogh, Csaba, Pei and Samotij [7] and by Balogh, Csaba and Samotij [8], respectively.

Besides being valid for small values of ε, the results of Alon, Krivelevich and Sudakov [5] as well as those of Balogh, Csaba, Pei and Samotij [7] exhibit a substantial difference to that of Fernandez de la Vega. Instead of showing that (i) a *fixed* tree in $\mathcal{T}((1-\varepsilon)n,\Delta)$ is a.a.s. contained in $G(n,p)$, they show instead that (ii) $G(n,p)$ is a.a.s. *universal* for the whole class $\mathcal{T}((1-\varepsilon)n,\Delta)$, that is, it contains a copy of every tree in $\mathcal{T}((1-\varepsilon)n,\Delta)$ simultaneously. Note that, for $\Delta \geqslant 3$, the size of $\mathcal{T}((1-\varepsilon)n,\Delta)$ is exponential in n and therefore the union bound is not sufficient to derive (ii) from (i).

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In order to show that $G(n, c/n)$ is a.a.s. universal for $\mathcal{T}((1-\varepsilon)n, \Delta)$, both Alon, Krivelevich and Sudakov [5] and Balogh, Csaba, Pei and Samotij [7] showed that $G(n, c/n)$ exhibits certain pseudo-random properties that imply large expansion of small sets of vertices (after we delete a few vertices with very small degrees). This allowed them to apply the classical tree-embedding result of Friedman and Pippenger [27] (as was done in [5]) or its somewhat stronger version due to Haxell [30] (as was done in [7]) to embed every bounded-degree tree that covers all but an ε-fraction of the vertices of $G(n, c/n)$. Recently, Sudakov and Vondrák [44] gave a randomized algorithm to efficiently embed bounded-degree almost-spanning trees in graphs with certain expansion properties. We discuss the result of Haxell in Section 3.

For $pn > \log n$, the situation changes drastically. In this regime, the random graph $G(n, p)$ is connected and we may ask for the existence of *spanning* trees. The very specific case of embedding a Hamilton path was resolved by Komlós and Szemerédi [35] and, independently, Bollobás [13], who proved that if $pn \geqslant \log n + \log\log n + \omega(1)$ (where $\omega(1)$ is any function which tends to infinity as $n \to \infty$), then $G(n, p)$ a.a.s. contains a Hamilton cycle. Frieze and Krivelevich [28] and Krivelevich and Sudakov [38] investigated pseudo-random conditions expressed in terms of the spectral gap of the host graph which guarantee the existence of Hamilton paths. Hefetz, Krivelevich and Szabó [31] showed *Hamilton-connectedness* (that is, the existence of a Hamilton path between any two vertices) of graphs with expansion properties similar to those we introduce in the next section. We discuss their result in Section 3.

Addressing the question of embedding a *fixed* tree $T \in \mathcal{T}(n, \Delta)$, Krivelevich [37] showed that if $np \geqslant \frac{40}{\varepsilon}\Delta \log n + n^{\varepsilon}$ for some $\varepsilon > 0$, then the random graph $G(n, p)$ a.a.s. contains a copy of T. Moreover, it is shown in [37] that this bound on p is asymptotically tight in the order of magnitude if $n^{\varepsilon} \leqslant \Delta \leqslant n/\log n$. Extending and improving a result of Alon, Krivelevich and Sudakov [5], Hefetz, Krivelevich and Szabó [32] showed that if, in addition, T has a linear number of leaves or contains a bare path (that is, a path in which all vertices have degree two in T) of length linear in n, then $G(n, p)$ a.a.s. contains a copy of T already when $pn = (1 + o(1))\log n$.

To the best of our knowledge, up to now there have been no results directly addressing the question of $\mathcal{T}(n, \Delta)$-universality of $G(n, p)$ for $p \in o(1)$. For $\mathcal{G}(n, \Delta)$, the class of all graphs on n vertices with maximum degree at most Δ, Dellamonica, Kohayakawa, Rödl and Ruciński [22] showed that there exists a constant $c := c(\Delta)$ such that, for $pn \geqslant cn^{1-1/(2\Delta)}\log^{1/\Delta} n$, the random graph $G(n, p)$ is a.a.s. $\mathcal{G}(n, \Delta)$-universal, improving an earlier result in [4]. As a special case, this result also applies to the subclass $\mathcal{T}(n, \Delta)$ and implies that the random graph $G(n, p)$ is a.a.s. $\mathcal{T}(n, \Delta)$-universal for such values of p. Recently, Dellamonica, Kohayakawa, Rödl and Ruciński [21] improved their result to $pn \geqslant cn^{1-1/\Delta}\log n$. However, in all these bounds $pn = n^{1-o(1)}$ for $\Delta \to \infty$.

1.1. Outline

This paper is organized as follows. In the next section, we present all results of this work and put them into context with existing research. The subsequent sections are devoted to the proofs of these results. We give respective references to these proofs whenever we state a result in the next section.

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1.2. Notation

Let $\mathbb{N}$ and $\mathbb{R}^+$ be the sets of positive integers and positive real numbers, respectively. For two functions $f, g : \mathbb{N} \to \mathbb{R}^+$, we write $f \in o(g)$ or, equivalently, $f \ll g$, to denote the fact that $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$ and $f \in O(g)$ or, equivalently, $g \in \Omega(f)$, to denote the fact that there exists an absolute constant $c \in \mathbb{R}^+$ such that $f(n) \leqslant cg(n)$ for all $n \in \mathbb{N}$.

Given a graph G, we denote its vertex set by $V(G)$ and its edge set by $E(G)$. All graphs considered in this work are finite (where typically $|V(G)|$ will be denoted by n), simple, and undirected. For $X \subseteq V(G)$, let $G[X]$ be the subgraph of G induced by X and let $N_G(X)$ be the *external* neighbourhood of X, that is,

$$N_G(X) := |\{y \in V(G) \setminus X \mid \exists\, x \in X : \{x, y\} \in E(G)\}|.$$

For $v \in V(G)$, let $N_G(v) := N_G(\{v\})$ and let $\deg_G(v) := |N_G(v)|$. Given two sets $X, Y \subseteq V(G)$, $e_G(X, Y)$ is the number of ordered pairs (x, y) with $x \in X$ and $y \in Y$ such that $\{x, y\} \in E(G)$. Note that if X and Y intersect, then all edges in the intersection are counted twice.

Finally, for two graphs H and G, an *embedding* φ of H in G is an injective graph homomorphism, that is, an injective function $\varphi : V(H) \to V(G)$ such that $\{v, w\} \in E(H)$ implies $\{\varphi(v), \varphi(w)\} \in E(G)$. We say that φ embeds H *onto* G if φ is bijective and we say that G *contains a copy of* H if there exists an embedding of H in G.

2. Our results

2.1. Tree-universality of (n, d)-expanders

The main contribution of this work is establishing tree-universality for the members of a certain class of graphs with good expansion properties, which we term (n, d)-*expanders*. The following notion is an adaptation of the expansion properties investigated in [7] and [31].

Definition. Let $n \in \mathbb{N}$ and let $d \in \mathbb{R}^+$. A graph G is an (n, d)-*expander* if $|V(G)| = n$ and G satisfies the following two conditions:

(E1) $|N_G(X)| \geqslant d|X|$ for all $X \subseteq V(G)$ with $1 \leqslant |X| < \left\lceil \frac{n}{2d} \right\rceil$,

(E2) $e_G(X, Y) > 0$ for all disjoint $X, Y \subseteq V(G)$ with $|X| = |Y| = \left\lceil \frac{n}{2d} \right\rceil$.

A simple calculation (Lemma 3.1) shows that these properties are monotone, that is, if $3 \leqslant d_0 \leqslant d \leqslant n/6$, then every (n, d)-expander is also an (n, d_0)-expander.

The main result of this work is the following theorem, which states that (n, d)-expanders are tree-universal.

Theorem 2.1 (Tree-Universality). *There exists an absolute constant $c \in \mathbb{R}^+$ such that the following statement holds. Let $n \in \mathbb{N}$ and $\Delta \in \mathbb{R}^+$ satisfy $\log n \leqslant \Delta \leqslant cn^{1/3}$. Then every $(n, 7\Delta n^{2/3})$-expander is universal for $\mathcal{T}(n, \Delta)$.*

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Note that the class $\mathcal{T}(n, \log n)$ also includes all n-vertex trees of maximum degree Δ smaller than $\log n$ (for example, for $\Delta \in O(1)$ and n large). Thus, Theorem 2.1 also applies to the situation where $\Delta < \log n$ by setting Δ to $\log n$. The proof of Theorem 2.1 is given in Section 4.

There are many known constructions of expanders (see, *e.g.*, [39]). Thus, by verifying the two conditions (E1) and (E2) defined in Section 2.1, one obtains explicit constructions of relatively sparse graphs that are universal for $\mathcal{T}(n, \Delta)$.

2.2. Random graphs

Random graphs are well known to typically exhibit strong expansion properties. For example, the random graph $G(n, p)$ with $pn \geqslant 7d \log n$ is a.a.s. an (n, d)-expander (Lemma 5.2) and thus, as a direct consequence of Theorem 2.1, tree-universal.

Theorem 2.2. *There exists an absolute constant $c \in \mathbb{R}^+$ such that the following statement holds. Let $\Delta : \mathbb{N} \to \mathbb{R}^+$ satisfy $\Delta \geqslant \log n$. Then the random graph $G(n, p)$ is a.a.s. universal for $\mathcal{T}(n, \Delta)$, provided that $pn \geqslant c\Delta n^{2/3} \log n$.*

Theorem 2.2 implies that $G(n, cn^{-1/3} \log^2 n)$ is a.a.s. universal for $\mathcal{T}(n, \log n)$ and thus also for $\mathcal{T}(n, O(1))$ if c is a large enough constant.

Likewise, the random r-regular n-vertex graph with $\max\{7d \log n, \sqrt{n} \log n\} \leqslant r \ll n$ (where rn is even) is a.a.s. an (n, d)-expander (Lemma 5.3) and thus, as a direct consequence of Theorem 2.1, tree-universal.

Theorem 2.3. *There exists an absolute constant $c \in \mathbb{R}^+$ such that the following statement holds. Let $\Delta : \mathbb{N} \to \mathbb{R}^+$ satisfy $\Delta \geqslant \log n$. Then the random r-regular graph on n vertices is a.a.s. universal for $\mathcal{T}(n, \Delta)$, provided that $c\Delta n^{2/3} \log n \leqslant r \ll n$, and rn is even.*

Note that the restriction $r \ll n$ in Theorem 2.3 is likely to be an artifact resulting from the use of a switching argument in the proof of Lemma 5.3, and we believe the result should extend to linear values of $r = r(n)$. As before, Theorem 2.3 implies that the random r-regular graph on n vertices with $r \geqslant cn^{2/3} \log^2 n$ (where rn is even) is a.a.s. universal for $\mathcal{T}(n, O(1))$ if c is a large enough constant.

2.3. Locally sparse expanders

Bhatt, Chung, Leighton and Rosenberg [12] gave an explicit construction of n-vertex $\mathcal{T}(n, \Delta)$-universal graphs whose maximum degree is bounded by a function of Δ. Thus, for constant Δ, the number of edges in this graph is in $O(n)$. In comparison, the random graph we consider in Theorem 2.2 a.a.s. has $\Theta(n^{5/3} \log^2 n)$ edges. However, the graph constructed in [12] is *locally dense*, that is, it contains a large number of cliques of size $\Omega(\Delta)$ (*cf.* [12, Lemma 8]). In this section, we show how to construct *locally sparse* graphs that are universal for all bounded-degree trees.

Consider the random graph G drawn according to $G(n, p)$. The expected number of cliques of size k in G is $\binom{n}{k} p^{\binom{k}{2}}$. Therefore, by Markov's inequality, if $n^k p^{(k(k-1))/2} \ll 1$, then G a.a.s. does not contain any clique of size k. Consequently, for $p(n) = n^{-1/3} \log^2 n$, the

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random graph $G(n,p)$ is a.a.s. both K_8-free and $\mathcal{T}(n,\log n)$-universal. Thus, for sufficiently large n, a $\mathcal{T}(n,\log n)$-universal graph with clique number at most seven exists. We can strengthen this observation by showing that, for an appropriate choice of d, r, and p, the random graph $G(n,p)$ is still a.a.s. an (n,d)-expander even if we remove from it all copies of K_r by deleting a carefully chosen set of edges (Lemma 6.1). Together with Theorem 2.1, this implies the existence of locally sparse tree-universal graphs.

Theorem 2.4. *There exists an absolute constant $c \in \mathbb{R}^+$ such that the following statement holds. Let $n \in \mathbb{N}$ and let $r \in \mathbb{N}$ with $r \geqslant 5$. Then there exists a graph with clique number at most r that is universal for $\mathcal{T}(n, cn^{1/3-2/(r+2)}/\log n)$.*

In particular, Theorem 2.4 implies that there exists a $\mathcal{T}(n, cn^{1/21}/\log n)$-universal graph with clique number at most five for all $n \in \mathbb{N}$ if c is a small enough constant.

2.4. Lower bound constructions

In Theorem 2.1 we gave an *upper* bound of $7\Delta n^{2/3}\log n$ on the minimum value d^* such that, for all $d \geqslant d^*$, every (n,d)-expander is $\mathcal{T}(n,d)$-universal. We now discuss *lower* bounds on d^*, that is, constructions of (n,d)-expanders with (relatively) large values of d which are not universal for $\mathcal{T}(n,\Delta)$.

For random graphs, Krivelevich [37] showed that, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $n^\varepsilon \leqslant \Delta \leqslant n/\log n$, then there exists a tree $T \in \mathcal{T}(n,\Delta)$ such that the random graph $G(n,p)$ with $pn = \delta\Delta\log n$ a.a.s. does not contain a copy of T. In contrast, Theorem 2.2 shows that there exist an absolute constant c such that $pn = c\Delta n^{2/3}\log n$ is sufficient for $G(n,p)$ to become universal for $\mathcal{T}(n,\Delta)$. This huge gap of order $n^{2/3}$ seems to be mainly an artifact of the proof of Theorem 2.1.

In contrast to the random graph setting, we can show that, for $\Delta \in n^{\Omega(1)}$, the smallest value of d necessary for every (n,d)-expander to be $\mathcal{T}(n,\Delta)$-universal grows faster than $\Omega(\Delta\log n)$. To this end, recall that the radius of a connected graph G is defined as

$$r(G) := \min_{u\in V}\max_{v\in V}\operatorname{dist}(u,v),$$

where the *distance* $\operatorname{dist}(u,v)$ between two vertices u and v is the length of a shortest path connecting u and v in G. For example, the radius of the star-graph $K_{1,n}$ is 1 and the radius of a path of even length $2k$ is k. A crucial observation is that we cannot embed a spanning graph onto a host graph with a strictly larger radius: the embedding itself would be proof that the host graph has small radius too. We will show that $\mathcal{T}(n,\Delta)$ contains trees with a relatively small radius, whereas there are (n,d)-expanders (with quite large d) with a fairly large radius.

Now, consider the complete $(\lfloor\Delta\rfloor-1)$-ary tree on n vertices with $\Delta \geqslant 3$, that is, the rooted $(\lfloor\Delta\rfloor-1)$-ary tree in which every level, except possibly the last, is completely filled. This tree is in $\mathcal{T}(n,\Delta)$ and its radius is strictly smaller than $1+\log n/\log(\lfloor\Delta\rfloor-1)$. On the other hand, we show below that there exist very strong expanders with radius at least $1+\log n/\log(\lfloor\Delta\rfloor-1)$.

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Definition. Let $n \in \mathbb{N}$ and let H be a graph on n vertices. We define $\mathcal{G}_H$ to be the class of $2n$-vertex graphs obtained from H by replacing each vertex $v \in V(H)$ by two vertices u_v and u'_v and each edge $\{v,w\} \in E(H)$ by either the two edges $\{u_v,u_w\}$ and $\{u'_v,u'_w\}$ or the two edges $\{u_v,u'_w\}$ and $\{u'_v,u_w\}$.

For the class $\mathcal{G}_H$, we will show the following result in Section 6.

Lemma 2.5. *There exists an absolute constant $c \in \mathbb{R}^+$ such that the following statement holds. Let $n,r \in \mathbb{N}$ satisfy $cn^{1/r} \geqslant 3\log n$. Then there exists an n-vertex graph H such that all graphs in $\mathcal{G}_H$ have radius at least $r+2$ and a graph chosen uniformly at random from $\mathcal{G}_H$ is a.a.s. a $(2n, cn^{1/r}\log^{-1} n)$-expander.*

Combining Lemma 2.5 with our discussion of the radius of the complete $(\lfloor\Delta\rfloor - 1)$-ary tree, we get the following theorem.

Theorem 2.6. *There exists an absolute constant $c \in \mathbb{R}^+$ such that the following statement holds. Let $n,r \in \mathbb{N}$ and let $\Delta := n^{1/(r+1)} + 2$ satisfy $\Delta \geqslant c^{-1}\log n$. Then there exists an $(n, c\Delta^{1+1/r}\log^{-1} n)$-expander which is not $\mathcal{T}(n,\Delta)$-universal.*

Note that although all graphs in $\mathcal{G}_H$ have an even number of vertices, we do not require this restriction in Theorem 2.6. If we duplicate a vertex of an (n,d)-expander with radius r and connect the duplicate to all of the vertex's neighbours, we obtain an $(n+1, d/2)$-expander which still has radius r.

Theorem 2.6 implies that there even exist $(n, cn/\log n)$-expanders which are not universal for $\mathcal{T}(n,(1+o(1))\sqrt{n})$. In comparison, we do not expect the same to be true a.a.s. for $G(n,7c)$, which (a.a.s.) is a canonical example of an $(n,cn/\log n)$-expander. We remark that our construction complements a result of Komlós, Sárközy and Szemerédi [34]. They showed that, for every given δ, there exists a constant $c := c(\delta)$ such that every graph of minimum degree $(1+\delta)n/2$ is $\mathcal{T}(n,cn/\log n)$-universal. It is clear that this result is best possible, since if we allow the minimum degree to be at most $\lfloor n/2\rfloor - 1$, then the host graph may be disconnected. However, our $(n/2-1)$-regular construction for even values of n ($\mathcal{G}_H$ with $H := K_{n/2}$) shows that even host graphs that have an edge between all disjoint pairs of vertex sets of relatively small size $O(\log n)$ may not be $\mathcal{T}(n,(1+o(1))\sqrt{n})$-universal. Finally, we remark that Böttcher, Taraz and Würfl [14] observed a similar effect in the context of (ε,δ)-regular graphs and independently proposed a construction with similar properties to $\mathcal{G}_H$.

2.5. Universality for almost all spanning trees

For $n \in \mathbb{N}$, let $\mathcal{T}_n$ be the family of all labelled trees on the vertex set $\{1,\dots,n\}$. Bender and Wormald [11] showed that for every fixed constant $p \in (0,1)$ there is a subfamily $\mathcal{T}_n^* \subseteq \mathcal{T}_n$ with $|\mathcal{T}_n^*| = (1-o(1))|\mathcal{T}_n|$ such that the random graph $G(n,p)$ is a.a.s. universal for $\mathcal{T}_n^*$. Note that this notion differs substantially from the (weaker) notion of being *almost-universal* for $\mathcal{T}_n$ (see, *e.g.*, [15, 29]), which means that if T is drawn uniformly at random from all trees of $\mathcal{T}_n$, then G a.a.s. contains a copy of T.

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It is well known (see, *e.g.*, [42]) that a tree chosen uniformly at random from $\mathcal{T}_n$ a.a.s. has maximum degree at most $(1+o(1))\log n/\log\log n$. Therefore, the subfamily $\mathcal{T}_n^*$ of all trees in $\mathcal{T}_n$ with maximum degree at most $2\log n/\log\log n$ satisfies $|\mathcal{T}_n^*| = (1-o(1))|\mathcal{T}_n|$. Thus, Theorem 2.2 strengthens the result of Bender and Wormald as it allows us to replace the constant $p \in (0,1)$ with a function $p \in o(1)$.

Theorem 2.7. *There exists an absolute constant $c \in \mathbb{R}^+$ and a subfamily $\mathcal{T}_n^* \subseteq \mathcal{T}_n$ with $|\mathcal{T}_n^*| = (1-o(1))|\mathcal{T}_n|$ for every $n \in \mathbb{N}$ such that the random graph $G(n,p)$ with $p \geqslant cn^{-1/3}\log^2 n$ is a.a.s. universal for $\mathcal{T}_n^*$.*

2.6. The Maker–Breaker game

In recent studies of extremal properties of random graphs (such as tree-universality), a central concept is that of *robustness*. This means that we require some property to be not only *typically* present in the respective graph class (that is, to appear a.a.s. in a random graph drawn from this class), but also to persist after a modification of the random instance. There are numerous ways to model robustness, for example by the notion of resilience or via positional games. Here, we study the robustness of tree-universality in expanders in the setting of a Maker–Breaker game.

An $(a:b)$ Maker–Breaker game is played on a finite hypergraph $(X,\mathcal{F})$ between two players, *Maker* and *Breaker*. The vertex set of the hypergraph is the *board* and the hyperedges are the *winning sets* in the game. The game is played in turns, starting with Maker's turn. In each of their turns, Maker claims a and Breaker claims b previously unclaimed vertices. The numbers a and b are called the *biases* of Maker and Breaker, respectively. Maker's objective is to claim all elements of a winning set by the end of the game. In this case, Maker wins the game. Breaker's objective is to claim at least one element in each winning set by the end of the game. In this case, Breaker wins the game. The game ends when all vertices have been claimed, by which time either Maker or Breaker has won.

We say that an $(a:b)$ Maker–Breaker game on $(X,\mathcal{F})$ is *Maker's win* if Maker has a strategy that allows him to win the game regardless of Breaker's strategy, otherwise the game is *Breaker's win*. Clearly, every $(a:b)$ Maker–Breaker game $(X,\mathcal{F})$ is either Maker's or Breaker's win and the decision which of the two holds depends only on the parameters a, b, X, and $\mathcal{F}$. For a more detailed discussion, we refer the reader to [10].

We now formulate a Maker–Breaker game for preserving tree-universality of a graph. Given a graph, Maker tries to claim a set of edges, which induces a $\mathcal{T}(n,\Delta)$-universal subgraph.

Definition. For $n \in \mathbb{N}$ and $\Delta \in \mathbb{R}^+$, the *Maker–Breaker $\mathcal{T}(n,\Delta)$-universality game* on a graph G is the Maker–Breaker game on the hypergraph $(E(G),\mathcal{F})$, where $\mathcal{F}$ consists of all edge sets $F \subseteq E(G)$ such that the subgraph $(V(G),F)$ is $\mathcal{T}(n,\Delta)$-universal.

Our main finding in this section is a condition for Maker's win in the $(1:b)$ Maker–Breaker tree-universality game on an (n,d)-expander.

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Theorem 2.8. *There exists an absolute constant $c \in \mathbb{R}^+$ such that the following statement holds. Let $n, b \in \mathbb{N}$ and let $\Delta \in \mathbb{R}^+$ satisfy $\Delta \geqslant \log n$. Then the $(1:b)$ Maker–Breaker $\mathcal{T}(n,\Delta)$-universality game is Maker's win on every (n,d)-expander with $d \geqslant cb\Delta n^{2/3} \log n$.*

Theorem 2.8 implies that, for $\Delta \leqslant \log n$ and in particular for $\Delta \in O(1)$ and sufficiently large n, the $(1:b)$ Maker–Breaker $\mathcal{T}(n,\Delta)$-universality game is Maker's win on every (n,d)-expander with $d \geqslant cbn^{2/3} \log^2 n$.

The proof of Theorem 2.8 is given in Section 7. Together with Lemma 5.2, it implies the following condition for Maker's win in the tree-universality game on binomial random graphs.

Corollary 2.9. *There exists an absolute constant $c \in \mathbb{R}^+$ such that the following statement holds. Let $b : \mathbb{N} \to \mathbb{N}$ and let $\Delta : \mathbb{N} \to \mathbb{R}^+$ satisfy $\Delta \geqslant \log n$. Then the $(1:b)$ Maker–Breaker $\mathcal{T}(n,\Delta)$-universality game is a.a.s. Maker's win on the random graph $G(n,p)$, provided that $pn \geqslant cb\Delta n^{2/3} \log^2 n$.*

Correspondingly, Theorem 2.8 and Lemma 5.3 imply the following conditions for Maker's win for the tree-universality game on random regular graphs.

Corollary 2.10. *There exists an absolute constant $c \in \mathbb{R}^+$ such that the following statement holds. Let $b : \mathbb{N} \to \mathbb{N}$ and let $\Delta : \mathbb{N} \to \mathbb{R}^+$ satisfy $\Delta \geqslant \log n$. Then the $(1:b)$ Maker–Breaker $\mathcal{T}(n,\Delta)$-universality game is a.a.s. Maker's win on the random r-regular graph, provided that $r \geqslant cb\Delta n^{2/3} \log^2 n$, $r \in o(n)$, and rn is even.*

3. Properties of (n,d)-expanders

We now present all properties of (n,d)-expanders that are needed to prove the universality results in the remainder of this work. First, we observe that together the expansion properties (E1) and (E2) defined in Section 2.1 are monotone in d.

Lemma 3.1. *Let $n \in \mathbb{N}$ and let $d, d_0 \in \mathbb{R}^+$ satisfy $3 \leqslant d_0 \leqslant d \leqslant n/6$. Then every (n,d)-expander is also an (n,d_0)-expander.*

Proof. Let $m := \lceil \frac{n}{2d} \rceil$, let $m_0 := \lceil \frac{n}{2d_0} \rceil$, and let G be an (n,d)-expander. Since $d_0 \leqslant d$, condition (E2) holds immediately for the parameters n and d_0. Moreover, we have $|N_G(X)| \geqslant d_0|X|$ for all $X \subseteq V(G)$ with $1 \leqslant |X| < m$. Hence, in order to show condition (E1) for the parameters n and d_0, it is sufficient to verify that $|N_G(X)| \geqslant d_0|X|$ holds for all $X \subseteq V(G)$ with $m \leqslant |X| < m_0$. For such a set X, we have by condition (E2) that

$$|N_G(X)| \geqslant n - |X| - m \geqslant 2d_0(m_0 - 1) - 2m_0 \geqslant d_0 m_0 \geqslant d_0|X|.$$

The lemma follows. □

Next, we give a lower bound on the number of edges between two large disjoint sets that satisfy condition (E2) defined in Section 2.1.

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Lemma 3.2. *Let $m \in \mathbb{N}$. Let G be a graph such that $e_G(X, Y) > 0$ holds for all disjoint $X, Y \subseteq V(G)$ with $|X| = |Y| = m$. Then*

$$e_G(X, Y) \geqslant \frac{|X||Y|}{4m}$$

holds for all disjoint $X, Y \subseteq V(G)$ with $|X| \geqslant m$ and $|Y| \geqslant 2m$.

Proof. Partition X into

$$k := \left\lfloor \frac{|X|}{m} \right\rfloor \geqslant \frac{|X|}{2m}$$

disjoint parts $X_1, \ldots, X_k$, each of size at least m. By the prerequisite of the lemma, we have $|N_G(X_i) \cap Y| \geqslant |Y| - m \geqslant |Y|/2$ for all $i \in \{1, \ldots, k\}$. Thus, $e_G(X, Y) \geqslant k|N_G(X_i) \cap Y|$ and Lemma 3.2 follows. □

The following fact is an important insight into the structure of sparse expanders and is frequently used in the proof of Theorem 2.1. It allows us to bound the number of vertices with small neighbourhood in any sufficiently large vertex set of an expander.

Lemma 3.3 (Small Exceptional Sets). *Let G be a graph, let $m \in \mathbb{N}$, and let the set $W \subseteq V(G)$ satisfy $|W| \geqslant m^2$. We call a vertex in $V(G) \setminus W$* exceptional *with respect to W and m if it has at most $m - 1$ neighbours in W. Suppose that $e_G(X, Y) > 0$ for all $X \subseteq V(G) \setminus W$ and all $Y \subseteq W$ that satisfy $|X| = |Y| = m$. Then there are at most $m - 1$ exceptional vertices with respect to W and m.*

Proof. Let $V := V(G)$. Assume for contradiction that there exists a set $X \subseteq V \setminus W$ with $|X| = m$ such that $|N_G(v) \cap W| < m$ for all $v \in X$.

On the one hand, since $|X| = m$ and therefore $|V \setminus (X \cup N_G(X))| \leqslant m - 1$, we have

$$e_G(X, W) \geqslant |W \cap N_G(X)| \geqslant |W| - (m - 1) \geqslant m^2 - m + 1.$$

On the other hand,

$$e_G(X, W) = \sum_{x \in X} |N_G(x) \cap W| \leqslant m(m - 1) = m^2 - m,$$

which is clearly a contradiction. Thus, no such set X exists and there are at most $m - 1$ vertices in $V \setminus W$ with fewer than m neighbours in W. □

3.1. Partitioning expanders

We now show that we can partition the vertex set of an (n, d)-expander in such a way that the neighbourhoods of small expanding sets distribute between the parts according to the sizes of the parts. In Section 4, this technique plays a major role in the proof of our main result, the tree-universality of sparse expanders (Theorem 2.1).

Lemma 3.4 (Partition Lemma). *There exists an absolute constant $n_0 \in \mathbb{N}$ such that the following statement holds. Let $k, n \in \mathbb{N}$ and $d \in \mathbb{R}^+$ satisfy $n \geqslant n_0$ and $k \leqslant \log n$.*

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Furthermore, let $n_1, \dots, n_k \in \mathbb{N}$ *satisfy* $n = n_1 + \cdots + n_k$ *and, for all* $i \in \{1, \dots, k\}$, *let* $d_i := \frac{n_i}{5n} d$ *satisfy* $d_i \geqslant 2 \log n$.

Then, for every (n, d)*-expander* G, *the vertex set* $V(G)$ *can be partitioned into* k *disjoint sets* $U_1, \dots, U_k$ *of sizes* $n_1, \dots, n_k$, *respectively, such that*

$$|N_G(X) \cap U_i| \geqslant d_i |X| \tag{3.1}$$

holds for all sets $X \subseteq V$ *with* $1 \leqslant |X| < \lceil \frac{n}{2d} \rceil$ *and all* $i \in \{1, \dots, k\}$. *Moreover, the induced subgraph* $G[W_i]$ *is a* $(|W_i|, d_i)$*-expander for all* $i \in \{1, \dots, k\}$ *and all* $W_i \subseteq V(G)$ *with* $U_i \subseteq W_i$.

Before we give the proof of Lemma 3.4, we first state a well-known result (see, *e.g.*, [33, Theorem 2.10]) for bounding the tail probabilities of the hypergeometric distribution $\textsc{Hyp}(n, m, \ell)$. A random variable X distributed according to $\textsc{Hyp}(n, m, \ell)$ models the number of white balls found among ℓ balls drawn without replacement from an urn containing n balls, m of which are white. Recall that $\mathbb{P}(X = k) = \binom{m}{k}\binom{n-m}{\ell-k} / \binom{n}{\ell}$ for all $0 \leqslant k \leqslant n$ and that $\mathbb{E}[X] = m\ell/n$.

Theorem 3.5. *Let* ε *be a positive constant satisfying* $\varepsilon \leqslant 3/2$ *and let* $X \sim \textsc{Hyp}(n, m, \ell)$. *Then*

$$\mathbb{P}\big[|X - \mathbb{E}[X]| > \varepsilon \mathbb{E}[X]\big] \leqslant \mathrm{e}^{-\frac{\varepsilon^2}{3}\mathbb{E}[X]}.$$

Proof of Lemma 3.4. Choose n_0 such that $\log n \leqslant n^{2/15}$ for all $n \geqslant n_0$. Let $V = V(G)$ and $m := \lceil \frac{n}{2d} \rceil$. We choose a partition $U_1, \dots, U_k$ of V into disjoint sets of respective sizes $n_1, \dots, n_k$ uniformly at random. We show that with positive probability, the relation (3.1) holds for all sets $X \subseteq V$ with $1 \leqslant |X| < m$ and all $i \in \{1, \dots, k\}$.

Let $X \subseteq V$ with $1 \leqslant |X| < m$ and let $i \in \{1, \dots, k\}$. Then the random variable $|N_G(X) \cap U_i|$ is distributed according to $\textsc{Hyp}(n, n_i, |N_G(X)|)$ with

$$\mathbb{E}[|N_G(X) \cap U_i|] = \frac{n_i}{n} |N_G(X)| \geqslant \frac{n_i}{n} d\,|X| = 5d_i|X|.$$

We apply Theorem 3.5 with $\varepsilon = 4/5$ and obtain

$$\mathbb{P}[|N_G(X) \cap U_i| \leqslant d_i|X|] \leqslant \mathrm{e}^{-\frac{16}{15} d_i |X|} \leqslant \mathrm{e}^{-\frac{32}{15}|X| \log n} = n^{-\frac{32}{15}|X|}.$$

Let q be the probability that there exists a set $X \subseteq V$ with $1 \leqslant |X| < m$ and an $i \in \{1, \dots, k\}$ which violates property (3.1). Then, by the union bound,

$$q \leqslant \sum_{i=1}^{k} \sum_{j=1}^{m} \binom{n}{j} n^{-\frac{32}{15} j} < \sum_{i=1}^{k} \sum_{j=1}^{n} n^j n^{-\frac{32}{15} j} \leqslant k n^{-\frac{2}{15}} \leqslant 1$$

for sufficiently large n.

We have shown that with positive probability the randomly chosen partition $U_1, \dots, U_k$ satisfies property (3.1); therefore, such a partition exists and the first statement of Lemma 3.4 holds.

Finally, let $U_1, \dots, U_k$ be a partition that satisfies property (3.1). Let $i \in \{1, \dots, k\}$ and consider a set $W \subset V$ with $U_i \subset W$ and the induced graph $H := G[W]$. Then, by the

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choice of d_i, we have

$$\left\lceil \frac{|W|}{2d_i} \right\rceil \geqslant \left\lceil \frac{n_i}{2d_i} \right\rceil \geqslant \left\lceil \frac{n}{2d} \right\rceil.$$

Thus, condition (E2) defined in Section 2.1 with $n := |W|$ and $d := d_i$ holds for H since G is an (n,d)-expander. By (3.1), $|N_H(X)| \geqslant d_i|X|$ holds for all $X \subseteq V(H)$ with $1 \leqslant |X| < \lceil \frac{n}{2d} \rceil$. Hence, similar to the proof of Lemma 3.1, it is sufficient to verify that $|N_H(X)| \geqslant d_i|X|$ holds also for all $X \subseteq V(H)$ with

$$\left\lceil \frac{n}{2d} \right\rceil \leqslant |X| < \left\lceil \frac{|W|}{2d_i} \right\rceil.$$

Since G is an (n,d)-expander, we have for such a set X that

$$|N_H(X)| \geqslant |W| - |X| - \left\lceil \frac{n}{2d} \right\rceil,$$

and therefore

$$|N_H(X)| \geqslant 2d_i\left(\left\lceil \frac{n_i}{2d_i} \right\rceil - 1\right) - 2\left\lceil \frac{n_i}{2d_i} \right\rceil \geqslant d_i\left\lceil \frac{n_i}{2d_i} \right\rceil \geqslant d_i|X|,$$

and the second statement of Lemma 3.4 holds. □

3.2. Almost-spanning trees, Hamilton paths and star matchings

We now summarize three known results on embedding almost-spanning trees, Hamilton paths, and star matchings in graphs with large expansion. These results are crucial for the proof of Theorem 2.1 (Tree-Universality).

In [30], Haxell extended a result of Friedman and Pippenger [27] and showed that one can embed every almost-spanning tree with bounded maximum degree in a graph with sufficiently large expansion. Here, we present a formulation of this result in the flavour of Theorem 3 of [7].

Theorem 3.6. *Let $d,m,k \in \mathbb{N}$ and let H be a non-empty graph satisfying the following two conditions:*

(i) $|N_H(X)| \geqslant d|X| + 1$ *for all* $X \subseteq V(H)$ *with* $1 \leqslant |X| \leqslant m$,
(ii) $|N_H(X)| \geqslant d|X| + k$ *for all* $X \subseteq V(H)$ *with* $m < |X| \leqslant 2m$.

Then H contains a copy of every tree T with $|V(T)| \leqslant k+1$ and maximum degree at most d.

In terms of (n,d)-expanders, we may reformulate the previous theorem as follows.

Corollary 3.7 (Almost-Spanning Tree Embedding). *Let $n, \Delta \in \mathbb{N}$ and let $d \in \mathbb{R}^+$ with $d \geqslant 2\Delta$. Then every (n,d)-expander is $\mathcal{T}(n - 4\Delta\lceil \frac{n}{2d} \rceil, \Delta)$-universal.*

Proof. Without loss of generality we may suppose that $\Delta \geqslant 2$. Let $m := \lceil \frac{n}{2d} \rceil$ and let H be an (n,d)-expander. Let $k = n - 4\Delta m$ and let $T \in \mathcal{T}(k,\Delta)$.

Then, for all $X \subseteq V(H)$ with $1 \leqslant |X| \leqslant m$, we have by (E1) that

$$|N_H(X)| \geqslant 2\Delta|X| \geqslant \Delta|X| + 1.$$

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For all $X \subseteq V(H)$ with $m \leqslant |X| \leqslant 2m$, we have

$$|N_H(X)| \geqslant n - |X| - m \geqslant k + 4\Delta m - 3m \geqslant k + 2\Delta m \geqslant \Delta|X| + k.$$

The corollary then follows from Theorem 3.6. □

Next, we state a result of Hefetz, Krivelevich and Szabó [31] on the existence of Hamilton paths in expanders with edge-connectivity between large sets. For this, let us briefly revisit the notion of *Hamilton-connectedness*. An *x–y-Hamilton path* in a graph is a path with end-vertices x and y that visits each vertex of the graph exactly once. A graph is *Hamilton-connected* if there exists an x–y-Hamilton path for every pair of vertices x and y in the graph. The following theorem is a simplified version of the results in [31].

Theorem 3.8. *Let $n, h \in \mathbb{N}$ satisfy that n is sufficiently large and $12 \leqslant h \leqslant \sqrt{n}$. Let H be a graph on n vertices satisfying the following two conditions:*

(i) $|N_H(X)| \geqslant h|X|$ *for all* $X \subseteq V(H)$ *with* $0 < |X| \leqslant \frac{n \log h}{h \log n}$,

(ii) $e_H(X, Y) > 0$ *for all disjoint* $X, Y \subseteq V(H)$ *with* $|X| = |Y| \geqslant \frac{n \log h}{1035 \log n}$.

Then H is Hamilton-connected.

As before, we give a reformulation of this result in terms of (n, d)-expanders.

Corollary 3.9 (Hamilton Connectivity). *There exists an absolute constant $n_0 \in \mathbb{N}$ such that the following statement holds. Let $n \in \mathbb{N}$ with $n \geqslant n_0$ and let $d \in \mathbb{R}^+$ with $d \geqslant \log n$. Then every (n, d)-expander is Hamilton-connected.*

Proof. Let G be an (n, d)-expander. For $h = \mathrm{e}^{1035}$, consider the two conditions of Theorem 3.8. Condition (i) holds by Lemma 3.1 for sufficiently large n. Condition (ii) holds since

$$\left\lceil \frac{n}{2d} \right\rceil \leqslant \frac{n}{d} \leqslant \frac{n}{\log n} = \frac{n \log h}{1035 \log n}.$$

Thus, G is Hamilton-connected by Theorem 3.8. □

Finally, we state a version of Hall's marriage theorem for expanders, which shows that we can embed a star matching in a bipartite graph with large expansion in one direction and large minimum degree in the other direction.

Lemma 3.10 (Star Matching). *Let $d, m \in \mathbb{N}$ and let G be a graph. Suppose that two disjoint sets $U, W \subseteq V(G)$ satisfy the following three conditions:*

(i) $|N_G(X) \cap W| \geqslant d|X|$ *for all* $X \subseteq U$ *with* $1 \leqslant |X| \leqslant m$,

(ii) $e_G(X, Y) > 0$ *for all* $X \subseteq U$ *and* $Y \subseteq W$ *with* $|X| = |Y| \geqslant m$,

(iii) $|N_G(w) \cap U| \geqslant m$ *for all* $w \in W$.

Then, for every function $k : U \to \{1, \ldots, d\}$ that satisfies $\sum_{u \in U} k(u) = |W|$, the set W can be partitioned into $|U|$ disjoint subsets $\{W_u\}_{u \in U}$ such that, for every $u \in U$, we have $|W_u| = k(u)$

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and $W_u \subseteq N_G(u) \cap W$. We call the set of edges between the vertices of U and their respective parts in W a star matching.

Proof. To prove this lemma, we show that the generalized Hall's condition holds, that is,

$$|N_G(X) \cap W| \geqslant \sum_{x \in X} k(x), \tag{3.2}$$

for all $X \subseteq U$. We distinguish three cases.

(1) If $|X| < m$, then $k(x) \leqslant d$ for all $x \in X$ and (i) implies (3.2).
(2) If $m \leqslant |X| \leqslant |U| - m$, then $k(u) \geqslant 1$ for all $u \in U \setminus X$ and (ii) implies (3.2).
(3) If $|U| - m < |X|$, then (iii) directly implies (3.2).

Thus, (3.2) holds for all $X \subseteq U$ and the lemma is a direct consequence of the Max-Flow Min-Cut Theorem [23, 26]. □

4. Tree-universality of (n, d)-expanders

This section is devoted to the proof of our main result, Theorem 2.1 (Tree-Universality). The proof is based on a case distinction on whether the embedded tree contains a long bare path or many leaves. This extends the ideas in [37].

Definition. Let T be a tree. A *leaf* of T is a vertex of degree one in T. A *bare path* is a path in T whose vertices have all degree two in T. If we remove all leaves from T, we call the leaves and bare paths in the remaining tree *second-level leaves* and *second-level bare paths*, respectively. For distinction, we call the leaves and bare paths of the original tree T *first-level leaves* and *first-level bare paths.*

The following observation was made in [37] and states that a tree with bounded maximum degree contains a long bare path or many leaves.

Lemma 4.1. *Let T be a tree, let P be a bare path of maximum length in T, and let L be the set of leaves in T. Then*

$$2\big(|V(P)| + 1\big)\big(|L| - 1\big) \geqslant |V(T)|.$$

Proof. Let $V := V(T)$ and let $B \subseteq V$ the set of vertices of degree at least three in T. Then we have $|B| \leqslant |L| - 2$, since

$$-2 = 2|E(T)| - 2|V(T)| = \sum_{v \in V} (\deg_T(v) - 2) \geqslant |B| - |L|.$$

Next, we root T at an arbitrary leaf. This allows us to injectively map the bare paths of T to the set $L \cup B$ by assigning every bare path to the leaf or branching vertex adjacent to it farther away from the root. Therefore, the number of bare paths is at most $|L| + |B|$. Since every vertex in V is either in L, in B, or in a bare path of T, and since P is of maximum length, this implies that $|V| \leqslant |L| + |B| + |V(P)|(|L| + |B|)$ and therefore Lemma 4.1 holds. □

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Consider the setting of Theorem 2.1. Let $c \in \mathbb{R}^+$ be sufficiently small and assume that n is sufficiently large. Let $T \in \mathcal{T}(n, \Delta)$ with $\log n \leqslant \Delta \leqslant cn^{1/3}$, and let G be an (n, d)-expander with $d := 7\Delta n^{2/3}$.

Let $m := \lceil \frac{n}{2d} \rceil$. Lemma 4.1 tells us that T contains a bare path on $50\Delta m$ vertices or has at least $25\Delta m^2$ leaves, since $2(50\Delta m + 1)(25\Delta m^2 - 1) < n$ for sufficiently large n and small c. In fact, $2(50\Delta m + 1)(25\Delta m^2 - 1) < n/\Delta$ for sufficiently large n. Therefore, if L is the set of leaves in T, then $T - L$ still contains a bare path on $50\Delta m$ vertices or has at least $25\Delta m^2$ leaves, since $|T - L| \geqslant n/\Delta$. Based on this observation, we consider three cases.

Case 1. T contains a first-level bare path on at least $50\Delta m$ vertices.

In this case, we use Corollary 3.7 (Almost-Spanning Tree Embedding) to first embed all of T in G except for the bare path (whose removal splits T into two rooted trees). Then we apply Corollary 3.9 (Hamilton Connectivity) to also embed the bare path by connecting the two roots by a path covering all the unused vertices in G. The details of this argument are given in Proposition 4.2.

Case 2. T has at least $25\Delta m^2$ first-level leaves and contains a second-level bare path on at least $50\Delta m$ vertices.

In this case, T has many leaves. We use Corollary 3.7 (Almost-Spanning Tree Embedding) to first embed all of T in G except for the second-level bare path and the leaves. Then we use Corollary 3.9 (Hamilton Connectivity) to embed the bare path. Finally, we use Lemma 3.10 (Star Matching) to embed the leaves of T. Note that once T without the leaves is embedded, we know which vertices in G are the images of the parents of the leaves of T. We call these vertices in G the *portals* of the leaves. In order to embed the leaves, we need to find a star matching in G between the set of portals and the set of vertices which remain free after the embedding of T without the leaves. However, if we are not careful, then after embedding T without the leaves (and thus fixing the set of portals), some of the remaining vertices of G may not be connected to any of the portals. As these vertices would prevent us from finding a star matching, we call them *exceptional* vertices. We solve this problem by forcing the second-level bare path to cover all exceptional vertices. The details of this argument are given in Proposition 4.3.

Case 3. T has at least $25\Delta m^2$ first-level leaves and at least $25\Delta m^2$ second-level leaves.

In this case, T has many (first-level) leaves that are attached to second-level leaves. We again use Corollary 3.7 (Almost-Spanning Tree Embedding) to embed T without these two levels of leaves and then embed the leaves of each level separately using Lemma 3.10 (Star Matching). Here, there may again exist a set of exceptional vertices which can spoil the embedding of the second-level leaves. We apply an argument similar to that in Case 2, only this time the original leaves of T take the role that the second-level bare path played

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before, that is, they cover the set of exceptional vertices. The details for this argument are given in Proposition 4.4.

In the remainder of this section we show three results, Propositions 4.2, 4.3, and 4.4, which cover the three cases discussed above. Theorem 2.1 is a direct consequence of these three propositions.

Proposition 4.2 (Case 1). *The statement of Theorem 2.1 holds with $\mathcal{T}(n,\Delta)$ restricted to trees that contain a first-level bare path on at least $50\Delta m$ vertices.*

Proof. We construct an embedding φ of T onto G as follows. We split T into two parts and embed them consecutively. These parts are a (first-level) bare path P on exactly $50\Delta m$ vertices (chosen as a subpath of a longest bare path in T) and the remaining forest $F := T[V(T) \setminus V(P)]$ on $n - 50\Delta m$ vertices which consists of two trees. Note that $|V(F)| \geqslant |V(P)|$ since $50\Delta m \leqslant n/3$ for sufficiently large n. Let s_P and t_P be the two end-vertices of P and let s_F and t_F, respectively, be their two neighbours in F.

We construct an embedding φ of T in G in two steps. The first step is to find an embedding φ_F of the forest F in G and the second step is to find an embedding φ_P of the path P in $G[V \setminus \varphi_F(F)]$. In so doing, we make sure that these embeddings satisfy $\{\varphi_P(s_P), \varphi_F(s_F)\} \in E(G)$ and $\{\varphi_P(t_P), \varphi_F(t_F)\} \in E(G)$.

We start by partitioning V into U_F and U_P which (partially) host the embeddings of F and P. For this, we apply Lemma 3.4 (Partition Lemma) to partition V into two sets U_F and U_P with $|U_F| = |V(F)| + 4\Delta m$ and $|U_P| = |V(P)| - 4\Delta m$. Note that U_F and U_P are each of size at least $20\Delta m$. Since

$$\frac{|U_P|}{5n}\, d \geqslant \frac{|U_P|}{10m} \geqslant 2\Delta \ (\geqslant 2\log n),$$

the prerequisites of Lemma 3.4 are satisfied. Thus, $G[U_F]$ is a $(|U_F|, 2\Delta)$-expander and also $G[W_P]$ is a $(|W_P|, 2\Delta)$-expander for every set W_P with $U_P \subseteq W_P \subseteq V$.

Now, we turn to the actual constructions of φ_F and φ_P. First, we determine φ_F. By Corollary 3.7 (Almost-Spanning Tree Embedding), there exists an embedding φ_F of F in $G[U_F]$. Note that since this result only allows us to embed almost-spanning trees, U_F was chosen to be somewhat larger than $|V(F)|$.

Next, we move the unused $4\Delta m$ vertices of U_F to U_P and embed P by applying Corollary 3.9 (Hamilton Connectivity). Let $W_P := V \setminus \varphi_F(F)$. Since $U_P \subseteq W_P$, we already know that $G[W_P]$ is a $(50\Delta m, 2\Delta)$-expander. Moreover, by the properties given in Lemma 3.4 (Partition Lemma), we can find two (distinct) vertices $v \in N_G(\varphi_F(s_F)) \cap U_P$ and $w \in N_G(\varphi_F(t_F)) \cap U_P$ into which we embed s_P and t_P, respectively. Since $2\Delta \geqslant \log n$, the subgraph $G[W_P]$ contains a Hamilton path connecting v and w by Corollary 3.9 (Hamilton Connectivity). Let φ_P be the embedding of the bare path P into this Hamilton path. This concludes the construction of φ and the proof of the proposition. □

Proposition 4.3 (Case 2). *The statement of Theorem 2.1 holds with $\mathcal{T}(n,\Delta)$ restricted to trees that have at least $25\Delta m^2$ first-level leaves and contain a second-level bare path on at least $50\Delta m$ vertices.*

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Proof. Let $L \subseteq V(T)$ be the set of (first-level) leaves of T. By the assumptions of the proposition, we have $|L| \geqslant 25\Delta m^2$. Let K be the set of neighbours of the leaves in T, that is, $K = N_T(L)$.

Since T contains a second-level bare path on at least $50\Delta m$ vertices, we can find two vertex-disjoint second-level bare paths on exactly $25\Delta m$ vertices in T (*e.g.*, two subpaths of a longest bare path). Of these two second-level bare paths, let P be the one that contains *at most* $|K|/2$ vertices of K.

Let F be the forest $T[V(T) \setminus (L \cup V(P))]$; note that F consists of two trees. As in Proposition 4.2, let s_P and t_P be the end-vertices of P and let s_F and t_F be their respective neighbours in F.

We partition K into $K_F := K \cap V(F)$ and $K_P := K \cap V(P)$. Since each vertex in K is adjacent to at most Δ vertices in L, we have $|K| \geqslant 25m^2$. Moreover, $|K_F| \geqslant |K_P|$ by the choice of P, and therefore

$$|K_F| \geqslant m^2. \tag{4.1}$$

As in Proposition 4.2, we construct an embedding φ of T in G in several steps. First, we construct an embedding φ_F of the forest F, then an embedding φ_P of the path P, and finally an embedding φ_L of the leaves L. In so doing, we make sure that the images of s_P and t_P are adjacent to those of s_F and t_F, respectively, and that the image of each leaf in L is adjacent to the image of its respective neighbour in K.

Again, we partition G into sets which partially host the embeddings of F, P, and L. For this, we apply Lemma 3.4 (Partition Lemma) to partition V into the three parts U_F, U_P, and U_L satisfying $|U_F| = |V(F)| + 4\Delta m$, $|U_L| = |L| + \Delta m$, and $|U_P| = |V(P)| - 5\Delta m$. Then U_F and U_P are each of size at least $20\Delta m$ and U_L is of size at least $20\Delta m^2$. Hence, the subgraph $G[U_F]$ is a $(|U_F|, 2\Delta)$-expander and also $G[W_P]$ is a $(|W_P|, 2\Delta)$-expander for every set W_P with $U_P \subseteq W_P \subseteq V$. Moreover, for every set $X \subseteq V$ with $1 \leqslant |X| < m$, it holds that

$$|N_G(X) \cap U_L| \geqslant 2\Delta m|X|,$$

and therefore, for every set W_L of size $|L|$ with $W_L \subseteq U_L$, we have

$$|N_G(X) \cap W_L| \geqslant |N_G(X) \cap U_L| - \Delta m \geqslant \Delta|X| \tag{4.2}$$

for all sets $X \subseteq V \setminus W_L$ with $1 \leqslant |X| < m$.

In order to construct φ, we first apply Corollary 3.7 (Almost-Spanning Tree Embedding) to find an embedding φ_F of F in U_F. Let $W_F := \varphi(F)$. Later, we move the remaining $4\Delta m$ vertices in $U_F \setminus W_F$ to U_P.

Next, we embed the second-level bare path P. However, before doing so, we identify the exceptional set of vertices $Z \subseteq U_L$ which might later spoil the application of Lemma 3.10 (Star Matching) to the embedding of L. This set, denoted by Z, contains all vertices in U_L that have fewer than m neighbours in $\varphi(K)$, the set of portals. At this point in the construction of φ, we only know $\varphi(K_F)$, which is equal to $\varphi_F(K_F)$. However, since $|K_F| \geqslant |K_P|$ and therefore $|\varphi_F(K_F)| \geqslant |\varphi(K)|/2$, we may already define Z.

Let $Z := \{u \in U_L \mid |N_G(u) \cap \varphi_F(K_F)| < m\}$. Then we have already seen that $|\varphi(K_F)| = |K_F| \geqslant m^2$. Thus, since G is an (n, d)-expander, we have by Lemma 3.3 (Small Exceptional

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Sets) that $|Z| \leqslant m \leqslant \Delta m$. Let $W_L \subseteq U_L$ be an arbitrary set of size $|L|$ that contains no vertex in Z. In the third step of the embedding, W_L will be the image of L under φ_L. Note that however we embed K_P, this choice of W_L ensures that $|N_G(u) \cap \varphi(K)| \geqslant m$ holds for every vertex $u \in W_L$. In fact, the only reason why we separated the embedding of the second-level bare path P from the embedding of F is to take care of the exceptional set Z.

Now, we return to the embedding φ_P of the second-level bare path P. So far, we have constructed the embedding φ_F of F into the set $W_F \subseteq U_F$ of size $|V(F)|$ and reserved the set $W_L \subseteq U_L$ of size $|L|$ for the embedding φ_L of L. Let $W_P := V \setminus (W_F \cup W_L)$. Then $|W_P| = |P|$, and we have already seen that $G[W_P]$ is a $(|W_P|, 2\Delta)$-expander. Moreover, as in the proof of the previous proposition, we can choose two distinct vertices $v \in N_G(\varphi_F(s_F)) \cap U_P$ and $w \in N_G(\varphi_F(t_F)) \cap U_P$ as the images of s_P and t_P in φ_P, respectively. Afterwards, we define φ_P by embedding P into a Hamilton path between v and w in $G[W_P]$ given by Corollary 3.9 (Hamilton Connectivity).

Finally, we construct an embedding φ_L of L by applying Lemma 3.10 (Star Matching). At this point, the embedding of K is already given by the embeddings φ_F of K_F and φ_P of K_P. Thus, it suffices to verify that the conditions of Lemma 3.10 are satisfied. Condition (i) holds by (4.2), condition (ii) holds since G is an (n, d)-expander, and condition (iii) holds since we excluded Z from U_L when choosing W_L. Thus, we find an embedding φ_L of L into G that respects the edges between K and L in T. This concludes the construction of φ and the proof of the proposition. □

Proposition 4.4 (Case 3). *The statement of Theorem 2.1 holds with $\mathcal{T}(n, \Delta)$ restricted to trees that have at least $25\Delta m^2$ (first-level) leaves and at least $25\Delta m^2$ second-level leaves.*

Proof. Let L' be the set of first-level leaves of T and let M' be the set of second-level leaves of T, that is, the set of leaves of $T[V(T) \setminus L]$. By the assumptions of the proposition, $|L'| \geqslant 25\Delta m^2$ and $|M'| \geqslant 25\Delta m^2$. Note that every second-level leaf $v \in M'$ has at least one first-level leaf attached to it since otherwise v would have been in L' to begin with.

As in the proof of Proposition 4.3, we split T into three parts. Let M be an arbitrary subset of M' of size exactly $25\Delta m^2$, let $L := N_T(M) \cap L'$ be the set of first-level leaves with neighbours in M, and let F be the induced subtree $F := T[V(T) \setminus (L \cup M)]$. Let $K := N_{T-L}(M)$ be the neighbours of the second-level leaves M in T without the first-level leaves. Then $|K| \geqslant 25m^2$. Note that by definition M is the set of portals of L, that is, $M = N_T(L)$. Also note that $|M \cup L| \leqslant 25\Delta^2 m^2$ and thus $|V(F)| \geqslant 16\Delta m$ for sufficiently large n.

We construct an embedding φ of T in G by defining three partial embeddings φ_F, φ_M, and φ_L of F, M, and L, respectively. As before, we make sure that these embeddings respect the edges linking K to M and M to L in T. A crucial step in this process will be again to handle the exceptional set Z that might spoil the embedding of M. In the proof of Proposition 4.3, we forced Z to be covered by the image of the second-level path; now we will force Z to be covered by the images of L.

We again apply Lemma 3.4 (Partition Lemma) and partition V into three parts U_F, U_M, and U_L satisfying $|U_F| = |V(F)| + 4\Delta m$, $|U_M| = |M| + \Delta m$, and $|U_L| = |L| - 5\Delta m$.

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This implies that U_F is of size at least $20\Delta m$ and that U_M and U_L are each of size at least $20\Delta m^2$. Hence, $G[U_F]$ is a $(|U_F|, 2\Delta)$-expander. Moreover, for every set W_L of size $|L|$ with $U_L \subseteq W_L$ and for all sets $X \subseteq V$ with $1 \leqslant |X| < m$, we have that

$$|N_G(X) \cap W_L| \geqslant |N_G(X) \cap U_L| \geqslant 2\Delta m|X|. \tag{4.3}$$

Furthermore, for every set W_M of size $|M|$ with $W_M \subseteq U_M$ and for all sets $X \subseteq V$ with $1 \leqslant |X| < m$, we have that

$$|N_G(X) \cap W_M| \geqslant |N_G(X) \cap U_M| - \Delta m \geqslant \Delta m|X|. \tag{4.4}$$

The construction of the embedding φ closely follows that of the previous proposition. We first apply Corollary 3.7 (Almost-Spanning Tree Embedding) to find an embedding φ_F of F in U_F. Let $W_F := \varphi_F(F)$. We later move $U_F \setminus W_F$ to U_L.

Next, we give an embedding φ_M of M. Let Z be the exceptional vertices in M, that is, let $Z := \{u \in U_M \mid |N_G(u) \cap \varphi_F(K)| < m\}$. Then $|Z| < m$ by Lemma 3.3 (Small Exceptional Sets). Let W_M be an arbitrary subset of $U_M \setminus Z$ of size $|M|$ and let W_L be the set given by $V \setminus (W_F \cup W_M)$. Then, by the choice of W_M and by (4.3) and (4.4), we can apply Lemma 3.10 (Star Matching) to find an embedding φ_M of M in U_M which respects the edges between K and M in T.

Finally, we give an embedding φ_L of L. Because of (4.3) and (4.4), the preconditions of Lemma 3.10 (Star Matching) also hold for L. Note that this time there is no exceptional set, since (4.4) guarantees the minimum degree constraint (iii) in Lemma 3.10. Hence, we can embed L such that the edges between M and L in T are respected. This concludes the proof of the proposition and hence of Theorem 2.1. □

5. Random graphs

The typical random graph is one of the most prominent examples of a strong expander. Random graphs a.a.s. have the property that between all pairs of sufficiently large vertex sets the edge density is concentrated around its expectation. The following result links this pseudo-random property to (n, d)-expansion and allows us to show that binomial random graphs and random regular graphs with sufficiently large (expected) degree are (n, d)-expanders.

Lemma 5.1. *Let $C \in \mathbb{R}^+$, let $(\mathcal{G}_n)_{n\in\mathbb{N}}$ be a sequence of probability distributions over all graphs on n vertices, and let $d\colon \mathbb{N} \to \mathbb{R}^+$ satisfy $d \geqslant 3$. Suppose there exists an absolute constant $n_0 \in \mathbb{N}$ such that, for all $n \geqslant n_0$, for G drawn according to $\mathcal{G}_n$, and for all disjoint sets $X, Y \subseteq V(G)$ that satisfy $1 \leqslant |X| < \lceil \frac{n}{2d} \rceil$ and $|Y| = n - \lceil (d+1)|X| \rceil + 1$ or satisfy $|X| = |Y| = \lceil \frac{n}{2d} \rceil$, it holds that*

$$\mathbb{P}[e_G(X, Y) = 0] \leqslant Cn^{-\frac{6d|X||Y|}{n}}. \tag{5.1}$$

Then a random graph drawn according to $\mathcal{G}_n$ is a.a.s. an (n, d)-expander.

Proof. Let G be drawn according to $\mathcal{G}_n$, let $V := V(G)$, and let $m := \lceil \frac{n}{2d} \rceil$. Furthermore, let ($\overline{\text{E1}}$) be the event that there exist two disjoint sets $X, Y \subseteq V$ with $1 \leqslant |X| < m$ and

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$|Y| = n - \lfloor (d+1)|X| \rfloor$ for which $e_G(X,Y) = 0$ holds, and let $(\overline{\mathrm{E2}})$ be the event that there exist two disjoint sets $X, Y \subseteq V$ with $|X| = |Y| = m$ for which $e_G(X,Y) = 0$ holds.

Clearly, G is an (n,d)-expander if neither the event $(\overline{\mathrm{E1}})$ nor the event $(\overline{\mathrm{E2}})$ occurs. Since $6dm^2/n \geqslant 3m$, we have by (5.1) and the union bound that

$$\mathbb{P}[(\overline{\mathrm{E2}})] \leqslant C\binom{n}{m}\binom{n-m}{m}n^{-3m} \leqslant Cn^{-1}.$$

For $1 \leqslant k < m$, we have

$$\frac{6dk(n - \lfloor (d+1)k \rfloor)}{n} \geqslant 6dk - 3\lfloor (d+1)k \rfloor \geqslant 2dk.$$

Thus,

$$\mathbb{P}[(\overline{\mathrm{E1}})] \leqslant C\sum_{k=1}^{m}\binom{n}{k}\binom{n}{\lfloor dk \rfloor}n^{-2dk} \leqslant C \cdot n \cdot n^k \cdot n^{dk} \cdot n^{-2dk} \leqslant Cn^{-1}.$$

Hence, by the union bound, $\mathbb{P}[(\overline{\mathrm{E1}}) \vee (\overline{\mathrm{E2}})] = o(1)$, and therefore G a.a.s. satisfies (E1) and (E2), that is, G is a.a.s. an (n,d)-expander. □

If the average degree pn (actually, $p(n-1)$) of the random graph $G(n,p)$ is of order $d \log n$, then $G(n,p)$ has expansion d.

Lemma 5.2. *Let $d : \mathbb{N} \to \mathbb{R}^+$ satisfy $d \geqslant 3$. Then the random graph $G(n, 7dn^{-1}\log n)$ is a.a.s. an (n,d)-expander.*

Together with Theorem 2.1, this lemma immediately implies Theorem 2.2.

Proof. This lemma is a direct consequence of Lemma 5.1, since

$$\mathbb{P}[e_G(X,Y) = 0] = (1-p)^{|X||Y|} \leqslant \mathrm{e}^{-p|X||Y|}$$

holds for all disjoint vertex sets X and Y in a graph G drawn from $G(n,p)$. □

A second example of strong expanders is random regular graphs. Similar to the random graph $G(n,p)$, a random regular n-vertex graph is a.a.s. an (n,d)-expander if its degree is of order $d \log n$. Note that our proof is restricted to random regular graphs with a relatively large degree (at last of order $\sqrt{n}\log n$). However, in our case this is not a problem since in the main result, Theorem 2.1, we require an even higher expansion (at least of order $n^{2/3}\log n$).

Lemma 5.3. *Let $d : \mathbb{N} \to \mathbb{R}^+$ and $r : \mathbb{N} \to \mathbb{N}$ satisfy $d \geqslant \sqrt{n}\log n$, $7d \log n \leqslant r \ll n$, and that rn is even. Then the random r-regular graph is a.a.s. an (n,d)-expander.*

Together with Theorem 2.1, Lemma 5.3 implies Theorem 2.3. To prove Lemma 5.3, we follow the proof of [40, Theorem 2.2], which uses a switching argument originally introduced in [41].

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Proof. Let n be a sufficiently large integer. Let V be a vertex set of size n and let X and Y be two sets such that $|X| = |Y| = \lceil \frac{n}{2d} \rceil$ or $1 \leqslant |X| < \lceil \frac{n}{2d} \rceil$ and $|Y| = n - (d+1)|X|$. By Lemma 5.1, it is sufficient to show that

$$\mathbb{P}[e_G(X, Y) = 0] \leqslant n^{-\frac{6d|X||Y|}{n}}$$

in order to prove the theorem.

Let

$$k := \left\lfloor \frac{2r|X||Y|}{n} \right\rfloor$$

and, for $j \in \{0, \ldots, k\}$, let $\mathcal{C}_j$ be the class of n-vertex r-regular graphs on V that have exactly j edges with end-vertices in X and Y.

First, let $j \in \{1, \ldots, k\}$ and let $G \in \mathcal{C}_j$. We consider the following procedure. We choose an ordered vertex pair $(x, y) \in X \times Y$ and two other ordered vertex pairs $(v, w) \in V \times V$ and $(v', w') \in V \times V$ such that $\{x, y\}$, $\{v, w\}$, and $\{v', w'\}$ are edges of G. We delete these three edges from $E(G)$ and then add each of the three pairs $\{x, v\}$, $\{y, v'\}$, and $\{w, w'\}$ to $E(G)$, provided each of them forms a new edge in G. If the resulting graph is in $\mathcal{C}_{j-1}$, we call this procedure a *forward switching*. The total number of forward switchings in G is at most

$$jr^2n^2$$

since there are exactly j choices for (x, y) and at most $2|E(G)| = rn$ choices each for (v, w) and (v', w').

Next, also for $j \in \{1, \ldots, k\}$, let $G \in \mathcal{C}_{j-1}$ and consider the following procedure. We choose three ordered vertex pairs $(x, v) \in X \times V$, $(y, v') \in Y \times V$, and $(w, w') \in V \times V$, such that $\{x, v\}$, $\{y, v'\}$, and $\{w, w'\}$ are edges of G. We delete all three edges from $E(G)$ and add each of the three pairs $\{x, y\}$, $\{v, w\}$, and $\{v', w'\}$ to $E(G)$, provided each of them forms a new edge in G. If the new graph is in $\mathcal{C}_j$, we call this procedure a *reverse switching*.

There are $|X||Y| - j$ ways to choose vertices $(x, y) \in X \times Y$ such that $\{x, y\} \notin E(G)$. Then there are at least $rn - j - 2r$ ways to choose the pair (w, w'), since we can choose (and count twice) any edge which does not run between X and Y and which is not incident to x or y. Finally, there are at least $(r-3)^2 - 2j = (1 - o(1))r^2$ ways to choose the vertices v and v', since we can choose any combination of a neighbour of x and a neighbour of y except for w, w', and v or v', respectively, and except for the j edges between X and Y. Therefore, the total number of reverse switchings in G is at least

$$(|X||Y| - j)(rn - j - 2r)((r-3)^2 - 2j).$$

Now, we have that

$$j \leqslant k \leqslant \frac{2r|X||Y|}{n}.$$

Since $r = o(n)$, we get $j = o(|X||Y|)$. Furthermore, since

$$|X||Y| \leqslant \frac{n}{d} \cdot \frac{n}{2} \leqslant \frac{rn}{14 \log^3 n},$$

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we also get $j = o(r^2)$ which implies $j = o(rn)$. Thus, the total number of reverse switchings in G is at least

$$(1 - o(1))|X||Y|r^3 n.$$

Every pair of a forward switching and a graph in $\mathcal{C}_j$ can be identified with the corresponding pair of a backward switching and a graph in $\mathcal{C}_{j-1}$. Thus,

$$\frac{jr^2n^2}{4}|\mathcal{C}_j| \geqslant (1 - o(1))\frac{|X||Y|r^3 n}{2}|\mathcal{C}_{j-1}|.$$

Hence

$$|\mathcal{C}_j| \geqslant (1 - o(1))\frac{k}{j}|\mathcal{C}_{j-1}|,$$

and therefore

$$|\mathcal{C}_k| \geqslant (1 - o(1))^k \frac{k^k}{k!}|\mathcal{C}_0| \geqslant \mathrm{e}^{(1-o(1))k}|\mathcal{C}_0|.$$

Thus, for sufficiently large n,

$$\mathbb{P}[e_G(X, Y) = 0] = \mathbb{P}[G \in \mathcal{C}_0] \leqslant \mathrm{e}^{-(1-o(1))k} \leqslant n^{-\frac{6d|X||Y|}{n}},$$

which concludes the proof of Theorem 5.3. □

6. Locally sparse expanders

So far, the only examples of (n, d)-expanders we have considered are the binomial random graph and random regular graphs. Note that even if $d \in O(1)$, then $G(n, p)$ becomes an (n, d)-expander only when $p \gg \log n/n$. But when the edge probability is so high, then a.a.s. $G(n, p)$ also contains a triangle. Nevertheless, in this subsection we will see that there exist (n, d)-expanders with almost logarithmic girth. Moreover, we will show that even for d slightly smaller than $\sqrt{n}$, there still exist (n, d)-expanders which are triangle-free, while a random graph with the same expansion already contains a copy of K_5.

Definition. We call a graph (k, ℓ)-*locally sparse* if all of its induced k-vertex subgraphs have at most $\ell - 1$ edges.

For example, a $(r + 1, \binom{r+1}{2})$-locally sparse graph is K_{r+1}-free, that is, has clique number at most r, and a connected $(r + 1, r + 1)$-locally sparse graph does not contain a cycle of length at most $r + 1$, that is, has girth at least $r + 2$. Note that, for $r = 2$, the two previous observations coincide.

Lemma 6.1. *There exists an absolute constant $c \in \mathbb{R}^+$ such that the following statement holds. Let $n, k, \ell \in \mathbb{N}$ with $\ell \geqslant 2$ and let $d := cn^{1-(k-2)/(\ell-1)} \log^{-1} n$ satisfy $d \geqslant 3$. Then there exists an (n, d)-expander H that is (k, ℓ)-sparse. Moreover, H satisfies*

$$e_H(X, Y) \geqslant \frac{48d|X||Y| \log n}{n} \tag{6.1}$$

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for all (not necessarily disjoint) sets $X, Y \subseteq V(H)$ *which satisfy* $1 \leqslant |X| < \lceil \frac{n}{2d} \rceil$ *and* $|Y| = n - \lceil (d+1)|X| \rceil + 1$ *or satisfy* $|X| = |Y| = \lceil \frac{n}{2d} \rceil$.

If we apply Lemma 6.1 with $k = r+1$ and $\ell = \binom{r+1}{2}$, then the resulting graph has clique number at most r. This, together with Theorem 2.1, implies Theorem 2.4. Note that if we were able to improve the bound on d in Theorem 2.1 to $cn^{1/r}\log^{-1} n$, then Lemma 6.1 with $k = r+1$ and $\ell = r+1$ would immediately give the existence of tree-universal graphs with girth at least $r+2$.

Before proving Lemma 6.1, we first state the well-known Chernoff inequality (see, *e.g.*, [33, Theorem 2.3]) which we use to bound the tail probabilities of the binomial distribution $\textsc{Bin}(n,p)$. Recall that, for a random variable X that is distributed according to $\textsc{Bin}(n,p)$, we have $\mathbb{P}[X = k] = \binom{n}{k} p^k (1-p)^{n-k}$ for all $0 \leqslant k \leqslant n$ and $\mathbb{E}[X] = pn$.

Theorem 6.2 (Chernoff inequality). *Let* ε *be a positive constant satisfying* $\varepsilon \leqslant 3/2$ *and let* $X \sim \textsc{Bin}(n,p)$. *Then*

$$\mathbb{P}[|X - \mathbb{E}[X]| > \varepsilon \mathbb{E}[X]] \leqslant \mathrm{e}^{-\frac{\varepsilon^2}{3}\mathbb{E}[X]}.$$

We are now ready to prove Lemma 6.1. The main idea of this proof originated in [36].

Proof of Lemma 6.1. In order to prove the lemma, we will show that the random graph $G(n,p)$ with $p := 144dn^{-1}\log n$ is a.a.s. an (n,d)-expander, even if we remove all edges of some maximal family of edge-disjoint subgraphs on k vertices and ℓ edges.

Let $c \in \mathbb{R}^+$ be sufficiently small that $p \leqslant \mathrm{e}^{-7} n^{-(k-2)/(\ell-1)}$ and assume that n is sufficiently large. Furthermore, let $\mathcal{F}$ be the class of all graphs with k vertices and ℓ edges and, for every n-vertex graph G, let $\mathcal{F}(G)$ be an arbitrary (but fixed) maximal family of edge-disjoint copies of graphs in $\mathcal{F}$ in G.

Now, let G be a random graph drawn according to $G(n,p)$ and let H be the random graph resulting from G by removing all edges covered by $\mathcal{F}(G)$. Because the family $\mathcal{F}(G)$ is maximal, H is (k,ℓ)-sparse.

For all (not necessarily disjoint) vertex sets X and Y in H which satisfy $1 \leqslant |X| < \lceil \frac{n}{2d} \rceil$ and $|Y| = n - \lceil (d+1)|X| \rceil + 1$ or which satisfy $|X| = |Y| = \lceil \frac{n}{2d} \rceil$, we are going to show that a.a.s. (6.1) in Lemma 6.1 holds. Clearly, this implies that H is an (n,d)-expander and that there indeed exists a graph which satisfies the properties stated in Lemma 6.1.

Let $X, Y \subseteq V(H)$ be two sets which satisfy $1 \leqslant |X| < \lceil \frac{n}{2d} \rceil$ and $|Y| = n - \lceil (d+1)|X| \rceil + 1$ or which satisfy $|X| = |Y| = \lceil \frac{n}{2d} \rceil$. Then, by Chernoff's inequality (Theorem 6.2), we have

$$\mathbb{P}\left[e_G(X,Y) \leqslant \frac{p|X|(|Y|-1)}{2} \right] \leqslant \mathrm{e}^{-\frac{p|X|(|Y|-1)}{12}}.$$

Note that the -1 in the product $|X|(|Y|-1)$ is due to the fact that X and Y may intersect and G is simple, that is, has no loops. Since $|Y| \geqslant 5$ and thus $5(|Y|-1) \geqslant 4|Y|$ for sufficiently large n, this implies

$$\mathbb{P}\left[e_G(X,Y) \leqslant \frac{2p|X||Y|}{5} \right] \leqslant \mathrm{e}^{-\frac{p|X||Y|}{15}}. \tag{6.2}$$

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Next, let $\mathcal{F}(X,Y)$ be the family of sets in $\mathcal{F}(G)$ which contain at least one edge between X and Y. For every $t \in \mathbb{N}$, there are at most $\binom{|X||Y|}{t} n^{(k-2)t}$ candidates for a collection of t edge-disjoint copies in G of graphs in $\mathcal{F}(G)$ which cover at least one edge between X and Y: there are $\binom{|X||Y|}{t}$ choices for the t different (potential) edges between X and Y and at most $n^{(k-2)t}$ choices for the remaining $k-2$ vertices of each of the copies. Since the t copies are edge-disjoint and all graphs in $\mathcal{F}(G)$ have ℓ edges, each candidate collection occurs with probability

$$p^{\ell t} = p^t p^{(\ell-1)t} \leqslant \left(\frac{p}{\mathrm{e}^{7(\ell-1)}}\right)^t n^{-(k-2)t}$$

since $p \leqslant \mathrm{e}^{-7} n^{-(k-2)/(\ell-1)}$. Now, for $\ell \geqslant 2$ we have

$$\mathrm{e}^{7(\ell-1)} = \mathrm{e}^{6\ell-7}\mathrm{e}^{\ell} \geqslant 15\mathrm{e}\ell\mathrm{e}^{\ell}$$

and therefore

$$p^{\ell t} \leqslant \left(\frac{p}{15\mathrm{e}\ell}\right)^t \mathrm{e}^{-\ell t} n^{-(k-2)t}.$$

Thus, since

$$\binom{|X||Y|}{t} \leqslant \left(\frac{\mathrm{e}|X||Y|}{t}\right)^t,$$

we get by the union bound that

$$\mathbb{P}[|\mathcal{F}(X,Y)| \geqslant t] \leqslant \binom{|X||Y|}{t} n^{(k-2)t} \left(\frac{p}{15\mathrm{e}\ell}\right)^t \mathrm{e}^{-\ell t} n^{-(k-2)t} \leqslant \left(\frac{p|X||Y|}{15\ell t}\right)^t \mathrm{e}^{-\ell t}$$

and, for $t = \frac{p|X||Y|}{15\ell}$, this implies that

$$\mathbb{P}\left[|\mathcal{F}(X,Y)| \geqslant \frac{p|X||Y|}{15\ell}\right] \leqslant \mathrm{e}^{-\frac{p|X||Y|}{15}}. \tag{6.3}$$

Observe that by deleting the edges contained in a graph from $\mathcal{F}(X,Y)$ we can reduce $e_G(X,Y)$ by at most ℓ. Thus, by again applying the union bound to (6.2) and (6.3), we get

$$\mathbb{P}\left[e_H(X,Y) \leqslant \frac{p|X||Y|}{3}\right] \leqslant \mathbb{P}\left[e_G(X,Y) \leqslant \frac{2p|X||Y|}{5} \ \vee\ |\mathcal{F}(X,Y)| \geqslant \frac{p|X||Y|}{15\ell}\right],$$

and therefore

$$\mathbb{P}\left[e_H(X,Y) \leqslant p|X||Y|/3\right] \leqslant 2\mathrm{e}^{-\frac{p|X||Y|}{15}}.$$

By substituting $p = 144dn^{-1}\log n$, we obtain

$$\mathbb{P}\left[e_H(X,Y) < \frac{48d|X||Y|\log n}{n}\right] \leqslant 2n^{-\frac{48d|X||Y|}{5n}}.$$

Now, since $48/5 \geqslant 6$, Lemma 6.1 follows from the same union bound argument over all sets X and Y as in Lemma 5.1. □

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Proof of Lemma 2.5. Let c be any positive constant that is smaller than that in Lemma 6.1 and which satisfies $c \leqslant 1/60$. Let n be sufficiently large. Let $d := cn^{1/r} \log^{-1} n$ and note that $d \geqslant 3$ by the choice of r.

For $r = 1$, let H be the complete graph K_n, and for $r \geqslant 2$, let H be the graph from Lemma 6.1 with $k = \ell = r + 1$. In both cases H has girth $r + 2$ (note that the graphs constructed in Lemma 6.1 are expanders and thus they are always connected). Let G be a random graph drawn uniformly at random from $\mathcal{G}_H$. Thus, independently for every $\{v, w\} \in E(H)$, the graph G contains either the two edges $\{u_v, u_w\}$ and $\{u'_v, u'_w\}$ or the two edges $\{u_v, u'_w\}$ and $\{u_v, u'_w\}$, where each choice has probability $1/2$.

Our first observation is that the radius of G is at least as large as the girth of H, that is, at least $r + 2$, since each path between two vertices u_v and u'_v in G corresponds to a non-trivial closed walk in H.

Next, we show that G is a.a.s. a $(2n, d)$-expander. Let $X \subseteq V(G)$ and $Y \subseteq V(G)$ be two disjoint sets which satisfy $1 \leqslant |X| < \lceil \frac{2n}{2d} \rceil$ and $|Y| = n - \lceil (d+1)|X| \rceil + 1$ or which satisfy $|X| = |Y| = \lceil \frac{2n}{2d} \rceil$. We need to show that

$$\mathbb{P}[e_G(X, Y) = 0] \leqslant n^{-\frac{6d|X||Y|}{n}}.$$

Let $A := \{v \in V(H) \mid u_v \in X \vee u'_v \in X\}$ and $B := \{v \in V(H) \mid u_v \in Y \vee u'_v \in Y\}$ be the projections of X and Y to $V(H)$. Note that A and B may intersect. Clearly, we have $|A| \geqslant |X|/2$ and $|B| \geqslant |Y|/2$. Consider all ordered pairs $(v, w) \in A \times B$. For each of them mark an arbitrary pair from $\{u_v, u'_v\} \times \{u_w, u'_w\}$ in $X \times Y$. Then there exist exactly $e_H(A, B)$ marked pairs, each of which forms an edge in G between X and Y independently with probability $1/2$. Thus,

$$\mathbb{P}[e_G(X, Y) = 0] \leqslant 2^{-e_H(A,B)} \leqslant \mathrm{e}^{-\frac{e_H(A,B)}{2}}. \tag{6.4}$$

For the case $r = 1$, we have $H = K_n$. Then, since $e_H(A, B)$ counts edges between ordered pairs and since A and B may intersect, we have for sufficiently large Y that

$$e_H(A, B) = |A||B| - |A| = \frac{|X||Y| - 2|X|}{4} \geqslant \frac{|X||Y|}{5}$$

and therefore, since $d \leqslant \frac{1}{60} n \log^{-1} n$, we have

$$\mathbb{P}[e_G(X, Y) = 0] \leqslant \mathrm{e}^{-\frac{|X||Y|}{10}} \leqslant n^{-\frac{6d|X||Y|}{n}}.$$

For the case $r \geqslant 2$, H is the graph provided by Lemma 6.1. Since H satisfies (6.1), that is,

$$e_H(A, B) \geqslant \frac{48d|A||B| \log n}{n},$$

we get, by inequality (6.4) and by $|A| \geqslant |X|/2$ and $|B| \geqslant |Y|/2$, that

$$\mathbb{P}[e_G(X, Y) = 0] \leqslant n^{-\frac{6d|X||Y|}{n}}.$$

In both cases, we have

$$\mathbb{P}[e_G(X, Y) = 0] \leqslant n^{-\frac{6d|X||Y|}{n}} \leqslant (2n)^{-\frac{6d|X||Y|}{2n}}$$

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and Lemma 2.5 follows from Lemma 5.1. □

7. The Maker–Breaker expander game

This section is devoted to the proof of Theorem 2.8. To this end, we first formulate the *Maker–Breaker Expander Game*, in which Maker tries to claim a subset of the edges that induces an (n,d)-expander.

Definition. Let $n \in \mathbb{N}$, let $d \in \mathbb{R}^+$, and let G be a graph. Then the *Maker–Breaker* (n,d)*-expander game on* G is the Maker–Breaker game on the hypergraph $(E(G),\mathcal{F})$, where $\mathcal{F}$ consists of all edge sets $F \subseteq E(G)$ such that the subgraph $(V(G),F)$ is an (n,d)-expander.

If the $(1:b)$ Maker–Breaker expander game is played on an $(n, 15bd\log n)$-expander, then Maker can always secure the edges of an (n,d)-expander.

Theorem 7.1. *There exists an absolute constant $n_0 \in \mathbb{N}$ such that the following statement holds. Let $n,b \in \mathbb{N}$ and $d \in \mathbb{R}^+$ satisfy $n \geqslant n_0$ and $d \geqslant 3$. Then the $(1:b)$ Maker–Breaker (n,d)-expander game is Maker's win on every $(n, 15bd\log n)$-expander.*

Theorem 7.1, together with Theorem 2.1, implies Theorem 2.8. The key to the proof of this theorem is Beck's generalization of the Erdős–Selfridge criterion for Breaker's win [9, 24].

Theorem 7.2. *Let $a,b \in \mathbb{N}$. Let $(X,\mathcal{F})$ be a finite hypergraph. Suppose that*

$$\sum_{F\in\mathcal{F}} (1+b)^{-|F|/a} < \frac{1}{1+b}.$$

Then the $(a:b)$ Maker–Breaker game on $(X,\mathcal{F})$ is Breaker's win.

Proof of Theorem 7.1. Let $n, d_0, d, b \in \mathbb{N}$ satisfy $d \geqslant 3$ and $d_0 \geqslant 15bd\log n$. Furthermore, let $m := \lceil \frac{n}{2d} \rceil$, let $m_0 := \lceil \frac{n}{2d_0} \rceil$, and let G be an (n,d_0)-expander. Then we have $d_0 \geqslant 10bd\log_2 n$ and also $m_0 \leqslant n/(20bd\log_2 n)$ for sufficiently large n.

Since Theorem 7.2 only allows us to bound the bias for Breaker's win, we reformulate the $(1:b)$ Maker–Breaker (n,d)-expander game on G so that the roles of Maker and Breaker are reversed. For this, consider the hypergraph $(E(G),\mathcal{F})$ where $F \in \mathcal{F}$ if F consists of all edges of G between two disjoint sets $X \subseteq V(G)$ and $Y \subseteq V(G)$ such that either $1 \leqslant |X| < m$ and $|Y| = n - \lceil (d+1)|X| \rceil + 1$ hold, or such that $|X| = |Y| = m$ holds. Furthermore, consider the outcome of one game of the $(b:1)$ Maker–Breaker game on $(E(G),\mathcal{F})$ and let H be the subgraph of G defined by the edges claimed by Breaker. If Breaker wins, then H satisfies the conditions (E1) and (E2) defined in Section 2.1 and H is an (n,d)-expander. Thus, if we show that the generalized Erdős–Selfridge criterion,

$$\sum_{F\in\mathcal{F}} 2^{-|F|/b} < \frac{1}{2},$$

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holds for $\mathcal{F}$, then the $(b:1)$ Maker–Breaker game on $(E(G),\mathcal{F})$ is *Breaker's win* and therefore the $(1:b)$ Maker–Breaker (n,d)-expander game on G is *Maker's win*. Also note that because of the exchanged roles Breaker starts the game. This, however, only strengthens the result.

For $k,\ell \in \{1,\dots,n\}$, let $\mathcal{F}(k,\ell)$ be the family of edge sets $F \in \mathcal{F}$ such that F contains all edges in G between a set X of size k and a set Y of size ℓ. Then, for every $F \in \mathcal{F}$, either $F \in \mathcal{F}\big(k, n-(d+1)k+1\big)$ holds for some $k \in \{1,\dots,m\}$ or it holds that $F \in \mathcal{F}(m,m)$. Therefore,

$$\sum_{F\in\mathcal{F}} 2^{-|F|/b} = \sum_{k=1}^{m} \sum_{F\in\mathcal{F}(k,n-(d+1)k+1)} 2^{-|F|/b} + \sum_{F\in\mathcal{F}(m,m)} 2^{-|F|/b}.$$

Let $X \subseteq V(G)$ and $Y \subseteq V(G)$ be two disjoint sets that satisfy $1 \leqslant |X| < m$ and $|Y| = n-(d+1)|X|+1$. First, suppose that $|X| < m_0$. Since G is an (n,d_0)-expander, we have $|N_G(X)| \geqslant d_0|X|$. Thus, since $d \geqslant 3$,

$$e_G(X,Y) \geqslant |N_G(X) \cap Y| \geqslant d_0|X| - d|X| \geqslant b(d+2)|X|\log_2 n.$$

Next, suppose that $m_0 \leqslant |X| < m$. Since $d \geqslant 3$ and thus $|Y| \geqslant \frac{n}{3}$, we have by Lemma 3.2 that

$$e_G(X,Y) \geqslant \frac{|X||Y|}{4m_0} \geqslant \frac{n}{12m_0}|X| \geqslant b(d+2)|X|\log_2 n.$$

Thus, we have for $d \geqslant 3$

$$\sum_{k=1}^{m} \sum_{F\in\mathcal{F}(k,n-(d+1)k+1)} 2^{-|F|/b} \leqslant \sum_{k=1}^{m} \binom{n}{k}\binom{n}{dk} n^{-(d+2)k} \leqslant 2n^{-1}.$$

Finally, suppose that $|X| = |Y| = m$. Then, again by Lemma 3.2, we have

$$e_G(X,Y) \geqslant \frac{|X||Y|}{4m_0} \geqslant \frac{mn}{8dm_0} \geqslant \frac{5}{2}bm\log_2 n.$$

Therefore,

$$\sum_{F\in\mathcal{F}(m,m)} 2^{-|F|/b} \leqslant \binom{n}{m}\binom{n}{m} n^{-5m/2} \leqslant n^{-1/2}.$$

Hence,

$$\sum_{F\in\mathcal{F}} 2^{-|F|/b} \leqslant \frac{3}{n^{1/2}} < \frac{1}{2},$$

and Beck's generalization of the Erdős–Selfridge criterion is indeed satisfied. □

8. Conclusion

We have shown that, for sufficiently large n, every $(n, 8n^{2/3}\max\{\Delta, \log n\})$-expander is universal for the class of all n-vertex trees with maximum degree at most Δ. This implies that binomial random graphs and random regular graphs with sufficiently large (average)

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degree are a.a.s. $\mathcal{T}(n,\Delta)$-universal. Our result also leads to constructions of locally sparse $\mathcal{T}(n,\Delta)$-universal graphs. We have also discussed $\mathcal{T}(n,\Delta)$-universality in the setting of the Maker–Breaker game.

One major open problem is to establish the smallest value of p for which $G(n,p)$ becomes a.a.s. $\mathcal{T}(n,\Delta)$-universal. Here, our work leaves a substantial gap of $n^{2/3}$ compared to the lower bound in [37]. Also, it would be interesting to see why the corresponding lower bound for (n,d)-expanders in Theorem 2.6 differs so drastically from that in [37] and to possibly find pseudo-random sufficient conditions which do not yield this discrepancy. In the spirit of Theorem 2.4, it would be interesting to see constructions of tree-universal graphs which are triangle-free or even have large girth. Finally, although our embedding results are (for the most part) constructive, they do not give an efficient algorithm to find the embeddings. Here, an algorithmic version would also be desirable.

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