NON-MEASURABLE INTERPOLATION SETS

II. FUNCTIONS REGULAR IN AN ANGLE

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1. *Introduction.* In an earlier paper† the problem of the interpolation of an integral function $f(z)$ in terms of the values taken on a sequence z_n was studied by means of generalized σ-functions with zeros at z_n. The purpose of the present paper is to show that the methods used in I can be modified to deal with functions of finite order and type in the angle S_α, $|\arg z| \leqslant \alpha$. There is even a considerable saving since, for an S_α with $\alpha < \pi$, we can escape from the awkward convergence condition D_ρ (ii) of I. Most of our results, and to a great extent also their proofs, are direct generalizations of well-known results for the lattice points $m+in$ (cf. the references at the end of I, particularly (2), (5), (6), (14), (18)).

2. We shall say that a complex sequence $\{z_n\}$ satisfies conditions $D_\rho(\alpha)$ if in the notation of I

(i) for any two sectors $S_1(\theta_1, \phi_1)$, $S_2(\theta_2, \phi_2)$ such that $-\alpha \leqslant \theta_1 \leqslant \phi_1 \leqslant \alpha$, $-\alpha \leqslant \theta_2 \leqslant \phi_2 \leqslant \alpha$, as $r \to \infty$

$$\frac{n(r, S_1)}{n(r, S_2)} \to \frac{\phi_1 - \theta_1}{\phi_2 - \theta_2};$$

(ii) $$\lim_{\xi \to 0} \overline{\lim_{r \to \infty}} \frac{n\{(1+\xi)r, \alpha\} - n\{r, \alpha\}}{\{(1+\xi)^\rho - 1\} r^\rho} = \frac{D\alpha}{\pi} < \infty;$$

(iii) there exist positive Θ, d such that, for all sufficiently small positive ξ, ϵ, if $R > R_0(\xi, \epsilon)$ and $E(R)$ is the set of $r \leqslant R$ for which

$$\frac{n\{(1+\xi)r, \alpha\} - n\{r, \alpha\}}{\{(1+\xi)^\rho - 1\} r^\rho} \geqslant \frac{d\alpha}{\pi}(1-\epsilon),$$

$$\text{meas}\, E(R) \geqslant \Theta(1-\epsilon) R;$$

(iv) for some $h > 0$ if $m \neq n$

$$|z_m - z_n| \geqslant h \min\left(|z_n|^{1-\frac{1}{2}\rho}, |z_m|^{1-\frac{1}{2}\rho}\right).$$

The main results of the paper are

THEOREM A. *Suppose that for some $\alpha > 0$ a complex sequence $\{z_n\}$ satisfies $D_\rho(\alpha)$ and that $f(z)$ is a function regular in S_α for which for some γ_1, $\gamma > 0$ as $r \to \infty$*

$$\overline{\lim}\, r^{-\rho} \log |f(r \exp \pm i\alpha)| \leqslant \frac{d}{\rho}\{1 - \tau(1-\Theta)\} - \gamma, \tag{2·1}$$

$$\underline{\lim}\, r^{-\rho} \log M(r, \alpha) = \frac{d}{\rho}\{1 - \tau(1-\Theta)\} - \gamma_1, \tag{2·2}$$

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and as $|z_n| \to \infty$ $$\overline{\lim}\, |z_n|^{-\rho} \log |f(z_n)| \leqslant \frac{d}{\rho}\{1-\tau(1-\Theta)\}-\gamma. \tag{2·3}$$

Then $$\overline{\lim}\, r^{-\rho} \log M(r,\alpha) \leqslant \frac{d}{\rho} - \{1-\tau(1-\Theta)\}\gamma. \tag{2·4}$$

THEOREM B. *Let* $\{z_n\}$ *be a complex sequence satisfying* $D_\rho(\alpha)$ *for some* $\alpha > 0$ *and* $f(z)$ *regular and of order* ρ *in* S_α. *Suppose, moreover, that as* $r \to \infty$

$$\overline{\lim}\, r^{-\rho} \log M(r,\alpha) < \frac{d}{\rho}\{1-\tau(1-\Theta)\}, \tag{2·5}$$

and that for all r, z_n $$|f(r\exp \pm i\alpha)| \leqslant K_1, \tag{2·6}$$

$$|f(z_n)| \leqslant K_1. \tag{2·7}$$

Then for all z *in* S_α $$|f(z)| \leqslant K_1. \tag{2·8}$$

THEOREM C. *Let* $\{z_n\}$ *be a sequence satisfying* $D_\rho(\alpha)$ *and* $f(z)$ *a function regular in* S_α *and such that for all* z_n $$f(z_n) = 0. \tag{2·9}$$

Suppose also that either

(i) $\alpha = \pi/(2\rho)$, *and for some* A, B *and* z *in* S_α

$$|f(z)| \leqslant A \exp Br^\rho, \tag{2·10}$$

$$\overline{\lim}\, r^{-\rho} \log |f(r\exp \pm i\alpha)| < \frac{d}{\rho}\{1-\tau(1-\Theta)\}; \tag{2·11}$$

or

(ii) $\alpha > \pi/(2\rho)$ (2·10) *holds and also*

$$\underline{\lim}\, r^{-\rho} \log M(r,\alpha) < \frac{d}{\rho}\{1-\tau(1-\Theta)\}, \tag{2·12}$$

$$\overline{\lim}\, r^{-\rho} \log |f(r\exp \pm i\alpha)| \leqslant \frac{d}{\rho}\{1-\tau(1-\Theta)\}-\gamma, \tag{2·13}$$

where $$\gamma > \frac{\tau(1-\Theta)}{1-\tau(1-\Theta)}\frac{d}{\rho}.$$

Then $$f(z) \equiv 0. \tag{2·14}$$

There is also the following corollary of Theorem A which is analogous to certain results of M. L. Cartwright* and V. Ganapathy Iyer†:

COROLLARY. *If in Theorem A we have also* $\alpha \geqslant \pi/(2\rho)$ *and* $\gamma > \dfrac{d}{\rho\{1-\tau(1-\Theta)\}}$, *then*

$$f(z) \equiv 0. \tag{2·15}$$

In the various special cases when $\Theta = 1$ the enunciations can be considerably simplified and some restrictions relaxed; for example, in Theorem C we need only $\gamma > 0$ and all the τ terms disappear from the results. In the proofs we shall use the notations of I wherever relevant and also some of the results, particularly the construction of a $\sigma_g(z)$ as given in §§ 6, 7 and 9.

* Cartwright, M. L., *Quart. J. Math.* (Oxford), 7 (1936), 44–55.

† Ganapathy Iyer, V., *Quart. J. Math.* (Oxford), 9 (1938), 206.

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3. For sectors S_α there is no essential difference between the cases of non-integral and of integral ρ. Our proofs, however, involve the application of generalized σ functions that are in fact integral functions of order ρ. We can, nevertheless, avoid the expected trouble from the convergence condition by making a preliminary change of variable and translating theorems involving integral ρ into theorems about non-integral ρ. For if ρ is an integer, say p, we can write $re^{i\theta} = z = \zeta^{1+\eta} = Re^{i\psi}$, where $\eta > 0$ and is such that $p(1+\eta) = \rho'$ is not an integer, and in the definition of ζ the branch of $z^{1/(1+\eta)}$ is chosen which is positive when $\theta = 0$. It is easy to see that the hypotheses and conclusions of our theorems transform into those of theorems involving a $g(\zeta)$ regular in $|\arg\zeta| \leqslant \alpha/(1+\eta)$ and of order ρ'. It remains to observe that a set $\{z_n\}$ satisfying $\mathrm{D}_\rho(\alpha)$ is transformed into a set $\{\zeta_n\}$ satisfying $\mathrm{D}_{\rho'}\{\alpha/(1+\eta)\}$ with Θ, D, d transformed into Θ', $D(1+\eta)$, $d(1+\eta)$, where $\Theta' \geqslant \Theta/(1+\eta)$. The last statement is a consequence of the following lemma, the proof of which is an immediate corollary of the fact that any bounded measurable set can be approximated arbitrarily closely by a finite sum of intervals.

LEMMA 1. *Suppose that $\mathscr{E}$ is a set of $x \geqslant 1$ such that for all $X \geqslant 0$ $\mathscr{E} \wedge [O, X]$ is measurable and*

$$\lim_{X\to\infty} \frac{\operatorname{meas} \mathscr{E} \wedge [O, X]}{X} = d > 0,$$

and that $\mathscr{E}$ is transformed by $x = y^{1+\eta}$ with $\eta > 0$ into a set E of y. Then

$$\lim_{Y\to\infty} \frac{\operatorname{meas} E \wedge [O, Y]}{Y} \geqslant \frac{d}{1+\eta}.$$

The remaining statements are fairly obvious except perhaps for $\mathrm{D}_{\rho'}(\alpha/1+\eta)$ (iv). To verify this last we observe that if ζ_n, z_n and ζ_m, z_m are corresponding pairs

$$|z_n - z_m| = |\zeta_n^{1+\eta} - \zeta_m^{1+\eta}| = \left|\int_{C_{m,n}} (1+\eta)\, u^\eta\, du\right| \leqslant (1+\eta)\,|\zeta_m - \zeta_n|\max(|\zeta_m|^\eta, |\zeta_n|^\eta),$$

where $C_{m,n}$ is the linear segment ζ_m, ζ_n. It follows that by condition $\mathrm{D}_\rho(\alpha)$ (iv) for z_m, z_n

$$\begin{aligned} |\zeta_m - \zeta_n| &\geqslant \frac{1}{1+\eta} \min\left(\frac{1}{|\zeta_m|^\eta}, \frac{1}{|\zeta_n|^\eta}\right) h \min(|z_n|^{1-\frac{1}{2}p}, |z_m|^{1-\frac{1}{2}p}) \\ &\geqslant hK(\eta) \min(|\zeta_n|^{1-\frac{1}{2}\rho'}, |\zeta_m|^{1-\frac{1}{2}\rho'}), \end{aligned}$$

which is the required condition.

We can thus obtain by transformation the results for integral p from those for non-integral ρ' except that Θ is replaced by $\Theta/(1+\eta)$. The full result for integral p appears on letting η tend to 0.

Throughout the remainder of the proofs we shall assume that ρ is not an integer.

4. *Generalized σ-functions.* We begin by constructing a generalized σ-function $\sigma_g(z, \eta)$ which will be employed to produce an analogue of Whittaker's two-dimensional cardinal series. It is easy to see that a sequence $\{z_n\}$ satisfying $\mathrm{D}_\rho(\alpha)$ can be completed by adjoining z lying outside S_α in such a way that the completed sequence $\{Z_\nu\}$ fulfils the conditions D_ρ of I with constants D, d, Θ. By the argument of I, §§ 6, 7 and 9, it then follows that for every sufficiently small positive η we can construct a subsequence

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$Z_n(\eta)$ and choose an $R_0(\eta)$ such that if $\Sigma_g(z,\eta)$ is the canonical product of genus p formed from the $Z_n(\eta)$, $R > R_0$ and $|z| \leqslant R$

$$\log|\Sigma_g(z,\eta)| \leqslant \left(\frac{d}{\rho}+\eta\right) r^\rho, \tag{4·1}$$

$$\log|\Sigma_g(z,\eta)| \geqslant \left(\frac{d}{\rho}\{1-\tau(1-\Theta)\}-\eta\right) r^\rho, \tag{4·2}$$

except for a set of z of radial measure not exceeding ηR; and if $|Z_n(\eta)| \geqslant R_0$

$$\log|\Sigma_g'(Z_n(\eta),\eta)| \geqslant \left(\frac{d}{\rho}\{1-\tau(1-\Theta)\}-\eta\right)|z_n|^\rho. \tag{4·3}$$

Next we modify $\Sigma_g(z,\eta)$ in such a way that the modified function is large with $|z|$ if $|\arg z| = \alpha$. Let $\{z_n(\eta)\}$ be the subsequence of $\{Z_n(\eta)\}$ obtained by omitting the Z_n for which $||\arg Z_n| - \alpha| \leqslant \eta^{\frac{1}{4}}$ and let $\sigma_g(z,\eta)$ be the corresponding canonical product of genus p, $\omega(z,\eta)$ that of the Z_n for which $||\arg Z_n| - \alpha| \leqslant \eta^{\frac{1}{4}}$; so that

$$\sigma_g(z,\eta) = \Sigma_g(z,\eta)/\omega(z,\eta). \tag{4·4}$$

We note first that for some $r_0(\eta)$, if $r > r_0$

$$n(\omega,r) \leqslant K_\rho D\eta^{\frac{1}{4}} r^\rho, \tag{4·5}$$

from which it follows* that if $r > r_0$

$$\log M(\omega,r) \leqslant K_\rho D\eta^{\frac{1}{4}} r^\rho. \tag{4·6}$$

From (4·6) it immediately follows by a lemma of V. Bernstein† that if $R \geqslant r_0(\eta)$ and $|z| \leqslant R$

$$\log|\omega(z,\eta)| \geqslant -K_\rho D\eta^{\frac{1}{4}} \log\frac{1}{\eta} r^\rho, \tag{4·7}$$

except for z of radial measure not exceeding ηR.

On combining (4·1), (4·2), (4·6) and (4·7) it now follows that (4·1) and (4·2) hold also for σ_g, provided that on the right we replace η by $K_\rho D\eta^{\frac{1}{4}} \log 1/\eta$ and permit an exceptional set of radial measure not exceeding $2\eta R$.

It remains to examine the behaviour of $\sigma_g(z,\eta)$ when $|\arg z| = \alpha$. By the modified (4·2) we can choose an increasing sequence R_n with $R_n \leqslant (1+2\eta) R_n$ such that if $|z| = R_n$ the modified (4·2) holds. Moreover, if necessary, by rechoosing R_1 we can assume that those exceptional circles containing a z for which $R_n \leqslant |z| \leqslant R_{n+1}$ and $\arg z$ lies in one of the intervals I_j $(j = 1, 2, \ldots, 4)$,

$$[\alpha+\tfrac{3}{4}\eta^{\frac{1}{4}}, \alpha+\eta^{\frac{1}{4}}], \quad [\alpha-\eta^{\frac{1}{4}}, \alpha-\tfrac{3}{4}\eta^{\frac{1}{4}}], \quad [-\alpha+\tfrac{3}{4}\eta^{\frac{1}{4}}, -\alpha+\eta^{\frac{1}{4}}], \quad [-\alpha-\eta^{\frac{1}{4}}, -\alpha-\tfrac{3}{4}\eta^{\frac{1}{4}}],$$

have total radius not exceeding $2\eta \,.\, 2R_n$. If η is sufficiently small $4\eta R_n \leqslant \frac{1}{8}\eta^{\frac{1}{4}} R_n$, and it follows that there are θ_j $(j = 1, \ldots, 4)$ lying one in each of the intervals I_j such that if $R_n \leqslant |z| \leqslant R_{n+1}$ and $\arg z = \theta_j$,

$$\begin{aligned}\log|\sigma_g(z,\eta)| &\geqslant \left(\frac{d}{\rho}\{1-\tau(1-\Theta)\}-K_\rho D\eta^{\frac{1}{4}}\log\frac{1}{\eta}\right) r^\rho \\ &\geqslant \Re\left\{\left(\frac{d}{\rho}\{1-\tau(1-\Theta)\}-K_\rho D\eta^{\frac{1}{4}}\log\frac{1}{\eta}\right)(\exp \mp i\rho\alpha)\, z^\rho\right\},\end{aligned} \tag{4·8}$$

* Cf., for example, Valiron, G., *Integral functions* (Toulouse, 1923), p. 53.

† Bernstein, V., *Ann. Mat.* (4), 12 (1934), 173–96, Lemma II.

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the negative sign occurring if $j = 1, 2$ the positive if $j = 3, 4$. Applying the minimum modulus theorem to the two regions $R_n \leqslant |z| \leqslant R_{n+1}$, $\theta_2 \leqslant \arg z \leqslant \theta_1$, $\theta_4 \leqslant \arg z \leqslant \theta_3$ and the functions $\sigma_g(z) \exp\left\{-\left(\frac{d}{\rho}\{1-\tau(1-\Theta)\}\right) z^\rho \exp \mp i\rho\alpha\right\}$ respectively, which are regular and non-vanishing in the region concerned, and combining the results for all n, we have in particular

$$\log|\sigma_g(z,\eta)| \geqslant \left\{\frac{d}{\rho}\{1-\tau(1-\Theta)\} - K_\rho D\eta^{\frac{1}{2}} \log\frac{1}{\eta}\right\} r^\rho, \tag{4·9}$$

if $|z| = r > R_1$ and $|\arg z| = \alpha$.

Since (4·3) is always a consequence of (4·2) (cf. I, § 8), we have, combining our results and rechoosing η, the following:

LEMMA 2. *To any sufficiently small $\eta > 0$ corresponds $\eta_0 > 0$, an extension $\{Z_n(\eta)\}$ of $\{z_n\}$ satisfying D_ρ, a subsequence $\{z_n(\eta)\}$ of $Z_n(\eta)$ having no members for which*

$$||\arg z| - \alpha| < \eta_0$$

and such that the $z_n(\eta)$ for which $|\arg z_n(\eta)| \leqslant \alpha$ belong to $\{z_n\}$, and on $R_0(\eta)$; such that if $\sigma_g(z,\eta)$ is the canonical product of genus p formed from the $z_n(\eta)$, $R > R_0(\eta)$ and $|z| \leqslant R$

$$\log|\sigma_g(z,\eta)| \geqslant \left\{\frac{d}{\rho}\{1-\tau(1-\Theta)\} - \eta\right\} r^\rho = \mathfrak{d}(\eta)\, r^\rho, \tag{4·10}$$

$$\log|\sigma_g(z,\eta)| \leqslant \left(\frac{d}{\rho} + \eta\right) r^\rho = \mathscr{D}(\eta)\, r^\rho, \tag{4·11}$$

except for a set $\mathscr{E}_\eta(R)$ of z for which $m_r(\mathscr{E}) \leqslant \eta R$. Moreover, if $r > R_0$

$$\log|\sigma_g(r\exp \pm i\alpha, \eta)| \geqslant \mathfrak{d}(\eta)\, r^\rho, \tag{4·12}$$

and if $|z_n(\eta)| \geqslant R_0$

$$\log|\sigma_g'\{z_n(\eta), \eta\}| \geqslant \mathfrak{d}(\eta)\, |z_n|^\rho, \tag{4·13}$$

5. *Proof of Theorem A.* The rest of the proof follows closely a method of V. Ganapathy Iyer*. Suppose that $0 < \mu < \gamma$. Then in Lemma 2 we can choose $\eta < \frac{1}{16}\mu, \frac{1}{16}\gamma_1$ and then $\delta > 0$ such that $\mathscr{D}(\eta)\,\delta^\rho = \gamma - \mu$; by the lemma there exists a sequence $\{z_n(\eta)\}$ such that (4·10), (4·11), (4·12) and (4·13) holds for $\sigma_g(z,\eta)$, and also if $R > R_1(\eta) > R_0(\eta)$

$$\mathfrak{d}(\eta)\,(\delta r)^\rho \leqslant \log|\sigma_g(\delta z, \eta)| \leqslant \mathscr{D}(\eta)\,(\delta r)^\rho \tag{5·1}$$

if $|z| \leqslant R$ except for a set of z of radial measure not exceeding ηR. It follows that (4·10), (4·11), (4·12), (4·13) and (5·1) all hold if $R > R_1(\eta)$, except for a set of z of radial measure not exceeding $2\eta R$. We can therefore find a sequence R_n such that

$$R_{n+1} \leqslant R_n(1+3\eta), \tag{5·2}$$

and if $|z| = R_n$, (4·10), (4·11), (4·12), (4·13) and (5·1) hold, and

$$|z - z_\nu(\eta)| \geqslant |z_\nu(\eta)|^{-k}$$

for all $z_\nu(\eta)$, where k is such that $\sum_\nu |z_\nu(\eta)|^{-k} < \infty$. The sequence R_n contains a subsequence R_{n_ν} such that if $|z| = R_{n_\nu}$ we also have

$$\begin{aligned}\log M(R_{n_\nu}, \alpha) &\leqslant \left[\frac{d}{\rho}\{1-\tau(1-\Theta)\} - \gamma_1\right](1+\eta)(1+3\eta)^\rho R_{n_\nu}^\rho \\ &\leqslant \left[\frac{d}{\rho}\{1-\tau(1-\Theta)\} - 2\eta\right] R_{n_\nu}^\rho,\end{aligned} \tag{5·3}$$

* Ganapathy Iyer, V., *Quart. J. Math.* (Oxford), 9 (1938), 206–15.

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provided that in addition $\eta < \eta_1(d, \rho, \gamma_1)$. Suppose now that z belongs to S_α and that $|\zeta - z| \geqslant \epsilon$ for all ζ with $|\arg \zeta| = \alpha$. If $R_{n_\nu} > |z|$ consider

$$I_\nu(z) = \frac{1}{2\pi i}\int_{C_{n_\nu}} \frac{f(\zeta)\,\zeta^m}{\sigma_g(\zeta, \eta)\,(\zeta - z)}\,d\zeta,$$

where C_ν is the contour consisting of the portions of $|\arg \zeta| = \alpha$ for which $|\zeta| \leqslant R_n$ and the portion of $|z| = R_{n_\nu}$ included in S_α. For large ν by (4·11) and (5·3)

$$\frac{1}{2\pi}\left|\int_{|\zeta|=R_{n_\nu}} \frac{f(\zeta)\,\zeta^m}{\sigma_g(\zeta, \eta)\,(\zeta - z)}\,d\zeta\right| \leqslant R_{n_\nu}^m \exp(-\eta R_{n_\nu}^m) = o(1).$$

It follows from the residue theorem on letting ν tend to ∞ that

$$\frac{f(z)\,z^m}{\sigma_g(z, \eta)} = \frac{1}{2\pi i}\int_{\arg u = -\alpha} \frac{f(u)\,u^m}{\sigma_g(u, \eta)\,(u - z)}\,du - \frac{1}{2\pi i}\int_{\arg u = \alpha} \frac{f(u)\,u^m}{\sigma_g(u, \eta)\,(u - z)}\,du + \sum_{Z_n(\eta) \subset S_\alpha} \frac{f\{z_n(\eta)\}\,z_n(\eta)^m}{\sigma_g'\{Z_n(\eta), \eta\}\,\{z - z_n(\eta)\}}, \tag{5·4}$$

the series being absolutely convergent by (4·13) and (2·3). If now

$$\sigma_g(\delta z, \eta) = \sum_{m=0}^{\infty} a_m z^m,$$

then it is well known that by (2·1) and (2·3) and the choice of δ

$$\overline{\lim}\,(\log |f(r \exp \pm i\alpha)| \sum_m |a_m|\, r^m)\, r^{-\rho} \leqslant \frac{d}{\rho}\{1 - \tau(1 - \Theta)\} - \mu \leqslant \mathfrak{d}(\eta) - \eta,$$

$$\overline{\lim_{Z_n(\eta)}}\,(\log |f\{z_n(\eta)\}| \sum_m |a_m|\,|z_n(\eta)|^m)\,|z_n|^{-\rho} \leqslant \frac{d}{\rho}\{1 - \tau(1 - \Theta)\} - \mu \leqslant \mathfrak{d}(\eta) - \eta.$$

Forming the sum $\Sigma a_m z_m$ we can therefore invert the various double limits occurring in the second member of (5·4) obtaining

$$\frac{f(z)\,\sigma_g(\delta z, \eta)}{\sigma_g(z, \eta)} = \frac{1}{2\pi i}\int_{\arg u = -\alpha} \frac{f(u)\,\sigma_g(\delta u, \eta)}{\sigma_g(u, \eta)\,(u - z)}\,du - \frac{1}{2\pi i}\int_{\arg u = \alpha} \frac{f(u)\,\sigma_g(\delta u, \eta)}{\sigma_g(u, \eta)\,(u - z)}\,du + \sum_{Z_n(\eta) \subset S_\alpha} \frac{f\{z_n(\eta)\}\,\sigma_g\{z_n(\eta), \eta\}}{\sigma_g'\{z_n(\eta)\}\,\{z - z_n(\eta)\}}. \tag{5·5}$$

It follows that if $|\zeta - z| \geqslant \epsilon$ for all ζ with $|\arg \zeta| = \alpha$, $|z - z_n| \geqslant |z_n|^{-k}$ for all z_n in $\{z_n(\eta)\}$

$$\left|\frac{f(z)\,\sigma_g(\delta z, \eta)}{\sigma_g(z, \eta)}\right| \leqslant K(\eta, \delta, \epsilon, k). \tag{5·6}$$

In particular, if $\eta_1 < \eta_0$, where η_0 is chosen as in Lemma 2, (5·6) holds if $|\arg z| = \alpha - \eta_1$ and $|z| \geqslant r_0(\eta_1)$ or if $|z| = R_n$ and $|\arg z| \leqslant \alpha - \eta_1$ for sufficiently large n; that is, by choice of δ, on the boundary of the region $R_n \leqslant |z| \leqslant R_{n+1}$, $|\arg z| \leqslant \alpha - \eta_1$

$$|f(z)| \leqslant K(\eta, \delta, \epsilon, k)\left|\frac{\sigma_g(z, \eta)}{\sigma_g(\delta z, \eta)}\right| \leqslant K \exp\left(\mathscr{D}(\eta) - \frac{\mathfrak{d}(\eta)}{\mathscr{D}(\eta)}(\gamma - \tau)\right) r^\rho.$$

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The maximum modulus principle in conjunction with (5·2) then yields that throughout this region

$$|f(z)| \leqslant K \exp\left(\mathscr{D}(\eta) - \frac{\mathfrak{d}(\eta)}{\mathscr{D}(\eta)}(\gamma-\mu) + K(D)\eta\right) r^{\rho},$$

and therefore if $|\theta| \leqslant \alpha - \eta_1$

$$h(\theta) \leqslant \mathscr{D}(\eta) - \frac{\mathfrak{d}(\eta)}{\mathscr{D}(\eta)}(\gamma-\mu) + K(D)\eta.$$

Letting first η_1 then μ and finally η tend to 0 we conclude that if $|\theta| \leqslant \alpha$

$$h(\theta) \leqslant \frac{d}{\rho} - \gamma\{1-\tau(1-\Theta)\}.$$

The result of Theorem A then follows by a familiar Phragmén Lindelöf argument.

6. *Proof of Theorem B.* Arguing as in the previous section we can again obtain (5·4), but in this case since $f(z_n)$, $f(r\exp\pm i\alpha)$ are bounded we can carry through the argument leading to (5·5) with any function $g(z)$ for which

$$\overline{\lim}\, r^{-\rho} \log M(r,\alpha) < \mathfrak{d}(\eta),$$

in particular with $f(z)z^m$ for any integer m. We thus obtain if $z \neq z_n$ and $|\arg z| \neq \alpha$

$$\frac{f(z)^2 z^m}{\sigma_g(z,\eta)} = \sum_{z_n(\eta)\,c\,S_\alpha} \frac{f(z_n)^2 z_n(\eta)^m}{\sigma_g'\{z_n(\eta),\eta\}\{z-z_n(\eta)\}} + \frac{1}{2\pi i}\int_{\arg u=-\alpha} \frac{f(u)^2 u^m}{(u-z)\,\sigma_g(u,\eta)}\,du - \frac{1}{2\pi i}\int_{\arg u=\alpha} \frac{f(u)^2 u^m}{(u-z)\,\sigma_g(u,\eta)}\,du. \quad (6\cdot1)$$

Repeating the argument $(N-1)$ times with $m=0$ in the last step we obtain, if $N \geqslant 1$,

$$\frac{f(z)^N}{\sigma_g(z,\eta)} = \sum_{z_n(\eta)\,c\,S_\alpha} \frac{f(z_n)^N}{\sigma_g'\{z_n(\eta),\eta\}\{z-z_n(\eta)\}} + \frac{1}{2\pi i}\int_{\arg u=-\alpha} \frac{f(u)^N}{(u-z)\,\sigma_g(u,\eta)}\,du - \frac{1}{2\pi i}\int_{\arg u=\alpha} \frac{f(u)^N}{(u-z)\,\sigma_g(u,\eta)}\,du. \quad (6\cdot2)$$

If $|z-z_n| \geqslant \epsilon_1 > 0$, $|\zeta - z| \geqslant \epsilon_1$ for all ζ with $|\arg\zeta| = \alpha$ we therefore have

$$|f(z)| \leqslant K_1\left(|\sigma_g(z,\eta)|\,\frac{1}{\epsilon_1}K(\eta)\right)^{1/N}, \quad (6\cdot3)$$

and therefore letting $N \to \infty$ it follows that if $|\arg z| \neq \alpha$ and $z \neq z_n$

$$|f(z)| \leqslant K_1, \quad (6\cdot4)$$

from which the theorem follows immediately.

7. *Proof of Theorem C and Corollary.* We observe that by Theorem A the condition (ii) implies the condition (i); it is therefore sufficient to prove the theorem under conditions (i).

Choose η in Lemma 2 so that

$$\mathfrak{d}(\eta) > \overline{\lim}\, r^{-\rho} \log|f(r\exp\pm i\alpha)|, \quad (7\cdot1)$$

and write $g(z) = f(z)/\sigma_g(z,\eta)$. Then obviously $g(z)$ is regular in S_α and

$$h_g(\pm\alpha) = \overline{\lim}\, r^{-\rho}\log|g(r\exp\pm i\alpha)| < 0; \quad (7\cdot2)$$

and it is easy to see that* $\lim r^{-\rho}\log M_g(r,\alpha) = A_1 < \infty.$ (7·3)

* Cf. Pflüger, A., *Proc. London Math. Soc.* (2), 42 (1937), 367–72.

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By a well-known property of $h(\theta)$, (7·2) is impossible unless $g(z) \equiv 0$, that is, unless $f(z) \equiv 0$.

Moreover, if in Theorem A, $\gamma > \dfrac{d}{\rho\{1-\tau(1-\Theta)\}}$, the theorem yields $h(\theta) < 0$ for all $|\theta| \leqslant \alpha$. This is impossible if $\alpha \geqslant \pi/(2\rho)$ unless $f(z) \equiv 0$ which is the conclusion of the corollary.

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