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INEQUALITIES FOR THE ANISOTROPIC POISSON POLYTOPE

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Abstract

The typical cell of a stationary Poisson hyperplane tessellation in the d-dimensional Euclidean space is called the Poisson polytope, and the cell containing the origin is called the Poisson 0-polytope. The intention of the paper is to show that the cells of the anisotropic tessellations are in some sense larger than those of the isotropic tessellations. Under the condition of equal intensities, it is proved that the moments of order $n = 1, 2, \ldots$ for the volume of the Poisson 0-polytope in the anisotropic case are not smaller than the corresponding moments in the isotropic case. Similar results are derived for the Poisson polytope. Finally, generalizations are mentioned.

LINE PROCESS; HYPERPLANE PROCESS; RANDOM TESSELLATION; TYPICAL CELL; 0-CELL; POISSON POLYGON

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1. Introduction

Let Y be a stationary Poisson line tessellation in $\mathbb{R}^2$. Its distribution is uniquely determined by the Steiner compact S [1], [2]. Especially, the intensity j of Y, i.e. the mean total length of edges per unit area is given by

$$j = L(S)/2, \tag{1.1}$$

where $L(S)$ denotes the perimeter of S. If $s\colon [0, 2\pi) \to [0, \infty)$ is the support function of S, we obtain from (1.1),

$$j = \int_0^\pi s(a)\, da. \tag{1.2}$$

By D_0, we denote the 0-cell of the stationary Poisson line tessellation Y, i.e. the cell containing the origin. It is called the Poisson 0-polygon. According to [2], the first moment for the area $A(D_0)$ of D_0 is given by

$$\boldsymbol{E}A(D_0) = (1/2) \int_0^\pi s^{-2}(a)\, da. \tag{1.3}$$

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Jensen's inequality leads to

$$(1.4) \qquad (1/\pi)\int_0^\pi s^{-2}(a)\,da \geqq \left((1/\pi)\int_0^\pi s(a)\,da\right)^{-2}.$$

Combining (1.2), (1.3), (1.4), we obtain

$$(1.5) \qquad \boldsymbol{E}A(D_0) \geqq \pi^3/(2j^2).$$

The right-hand side of inequality (1.5) equals the mean area of the 0-cell in a stationary and *isotropic* Poisson line tessellation with intensity j. Thus, we have proved the following result.

Proposition 1.1. The mean area of the 0-cell in a stationary Poisson line tessellation in $\mathbb{R}^2$ with intensity j is not smaller than the mean area of the 0-cell in a stationary and *isotropic* Poisson line tessellation with the same intensity j.

This is the special case $d = 2$ of a result derived by Weil and Wieacker in [8] concerning the mean volume of the 0-cell in a d-dimensional stationary Poisson hyperplane tessellation ($d = 2, 3, \cdots$).

It is shown in Section 3 that a corresponding inequality is valid for *all moments* of the volume of the 0-cell in a d-dimensional stationary Poisson hyperplane tessellation ($d = 2, 3, \cdots$).

Consequences relating to moments of the volume for the *typical* cell in a stationary Poisson hyperplane tessellation are derived in Section 4.

2. Volume of the Poisson 0-polytope

Let Y be a stationary Poisson hyperplane process in $\mathbb{R}^d$ (d fixed), i.e. a 'point process' in the sense of [2], [5] on the space $\mathbb{E}_{d-1}$ of hyperplanes in $\mathbb{R}^d$, the distribution of which is invariant under translations [1], [2]. Its distribution is uniquely determined by the intensity $0 < j < \infty$ and the directional distribution P_0. The intensity equals the mean $(d-1)$-content of the hyperplane process per unit volume. The directional distribution is a probability measure on the set $\mathbb{U}_1$ of all 1-subspaces (lines through the origin considered as normals to hyperplanes of the process) in $\mathbb{R}^d$, cf. [1], [2].

The hyperplane process Y is supposed to divide the $\mathbb{R}^d$ into bounded convex polytopes. This means that the directional distribution P_0 must not be concentrated on a subset of $\mathbb{U}_1$ generating a lower-dimensional subspace. Then there is a so-called Poisson hyperplane tessellation corresponding to Y. The cell D_0 of the tessellation containing the origin $\boldsymbol{0}$ is said to be the Poisson 0-polytope.

We are interested in the moments $\boldsymbol{E}V^n(D_0)$ for the volume $V(D_0)$ of the Poisson 0-polytope; $n = 1, 2, \cdots$. These moments depend on the intensity j and the

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directional distribution P_0 only, hence we may write

$$\boldsymbol{E}V^n(D_0) = M_n(j, P_0).$$

Because

$$V(D_0) = \int I_{D_0}(\boldsymbol{x})\, d\boldsymbol{x},$$

we have

$$(2.1)\quad M_n(j, P_0) = \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \boldsymbol{E}(I_{D_0}(\boldsymbol{x}_1) \cdots I_{D_0}(\boldsymbol{x}_n))\, d\boldsymbol{x}_1 \cdots d\boldsymbol{x}_n; \qquad n = 1, 2, \cdots,$$

where I_A denotes the indicator of a set A.

All points $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n \in \mathbb{R}^d$ lie inside D_0 if and only if the convex hull $\operatorname{conv}(\boldsymbol{0}, \boldsymbol{x}_1, \cdots, \boldsymbol{x}_n)$ of the points $\boldsymbol{0}, \boldsymbol{x}_1, \cdots, \boldsymbol{x}_n$ is not intersected by the hyperplanes of Y. Since Y is a stationary *Poisson* hyperplane process of intensity j and directional distribution P_0, we obtain from (2.1),

$$(2.2)\quad \begin{aligned} M_n(j, P_0) = \int \cdots \int \exp\left(-j \int_{\mathbb{U}_1} b(h; \operatorname{conv}(\boldsymbol{0}, \boldsymbol{x}_1, \cdots, \boldsymbol{x}_n)) P_0(dh)\right) \\ \times d\boldsymbol{x}_1, \cdots, d\boldsymbol{x}_n; \qquad n = 1, 2, \cdots, \end{aligned}$$

where $b(h; K)$ denotes the length of the orthogonal projection of a bounded convex set K onto the line h.

In the isotropic case, the directional distribution is given by the uniform distribution G on $\mathbb{U}_1$ (distribution invariant under all rotations about the origin). The 0-cell D_0 is then called the isotropic Poisson 0-polytope.

Equation (2.2) implies

$$(2.3)\quad M_n(j, G) = \int \cdots \int \exp(-j\bar{b}(\operatorname{conv}(\boldsymbol{0}, \boldsymbol{x}_1, \cdots, \boldsymbol{x}_n)))\, d\boldsymbol{x}_1 \cdots d\boldsymbol{x}_n,$$

where $\bar{b}(K)$ denotes the mean breadth of a bounded convex set K.

Remark . For $n = 1$, Equation (2.2) simplifies to

$$(2.4)\quad M_1(j, P_0) = d!\, k_d \int \left(j \int |\cos(g, h)| P_0(dh)\right)^{-d} G(dg),$$

where k_d is the volume of the d-dimensional unit ball, $\cos(g, h)$ the cosine of the angle between the lines g, $h \in \mathbb{U}_1$, and (2.3) leads to

$$(2.5)\quad M_1(j, G) = 2(\pi/j)^d \Gamma(d) \Gamma^d((d+1)/2) / \Gamma^{d+1}(d/2),$$

[1], [3].

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3. Inequalities

Let SO_d be the group of rotations of $\mathbb{R}^d$ about the origin. Since the Lebesgue measure on $\mathbb{R}^d$ is invariant under SO_d, we obtain for every rotation $R \in SO_d$,

$$M_n(j, P_0) = \int \cdots \int \exp\left(-j \int b(h; \operatorname{conv}(\mathbf{0}, R^{-1}\boldsymbol{x}_1, \cdots, R^{-1}\boldsymbol{x}_n))P_0(dh)\right) d\boldsymbol{x}_1 \cdots d\boldsymbol{x}_n$$

or

$$M_n(j, P_0) = \int \cdots \int \exp\left(-j \int b(Rh; \operatorname{conv}(\mathbf{0}, \boldsymbol{x}_1, \cdots, \boldsymbol{x}_n))P_0(dh)\right) \times d\boldsymbol{x}_1 \cdots d\boldsymbol{x}_n; \qquad R \in SO_d. \tag{3.1}$$

Integration with respect to the invariant probability measure N on SO_d transforms (3.1) into

$$M_n(j, P_0) = \int d\boldsymbol{x}_1 \cdots \int d\boldsymbol{x}_n \int N(dR) \times \exp\left(-j \int b(Rh; \operatorname{conv}(\mathbf{0}, \boldsymbol{x}_1, \cdots, \boldsymbol{x}_n))P_0(dh)\right). \tag{3.2}$$

First, let $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n$ be fixed and write for brevity

$$C = \operatorname{conv}(\mathbf{0}, \boldsymbol{x}_1, \cdots, \boldsymbol{x}_n). \tag{3.3}$$

Jensen's inequality leads to

$$\int \exp\left(-j \int b(Rh; C)P_0(dh)\right)N(dR) \geqq \exp\left(-j \int\int b(Rh; C)P_0(dh)N(dR)\right). \tag{3.4}$$

We have

$$\int\int b(Rh; C)P_0(dh)N(dR) = \int\int_{\mathbb{U}_1} b(g; C)G(dg)P_0(dh) = \bar{b}(C) \int P_0(dh),$$

and hence

$$\int\int b(Rh; C)P_0(dh)N(dR) = \bar{b}(C). \tag{3.5}$$

Combining (3.2)–(3.5), we observe that

$$M_n(j, P_0) \geqq \int \cdots \int \exp\left(-j\bar{b}(\operatorname{conv}(\mathbf{0}, \boldsymbol{x}_1, \cdots, \boldsymbol{x}_n))\right) d\boldsymbol{x}_1 \cdots d\boldsymbol{x}_n. \tag{3.6}$$

By comparison of (2.3) and (3.6), we obtain the main result of this paper,

$$M_n(j, P_0) \geqq M_n(j, G); \qquad n = 1, 2, \cdots. \tag{3.7}$$

It may also be formulated as follows.

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Theorem 3.1. For given intensity j, and for every $n = 1, 2, \cdots$, the nth moment of the volume of the anisotropic Poisson 0-polytope is not smaller than the nth moment of the volume of the isotropic Poisson 0-polytope.

Remark. If $P_0 \neq G$, it is even true that

$$M_n(j, P_0) > M_n(j, G).$$

In the case $n = 1$, the result has already been presented by Weil and Wieacker [8].

4. Consequences for the Poisson polytope

The typical cell D (see [1], [2], [3], [5]) of a stationary Poisson hyperplane tessellation in $\mathbb{R}^d$ is called a Poisson polytope. The nth moment of its volume $\boldsymbol{E}V^n(D)$ is uniquely determined by the intensity j and the directional distribution P_0, hence the notation

$$\boldsymbol{E}V^n(D) = m_n(j, P_0)$$

is justified.

The moments m_n and M_n are connected by the equation

$$M_{n-1}(j, P_0) = j_0(j, P_0) m_n(j, P_0); \qquad n = 0, 1, 2, \cdots, \tag{4.1}$$

where $j_0(j, P_0)$ denotes the vertex intensity (mean number of vertices per unit volume) of the random hyperplane tessellation with characteristics j, P_0, cf. [1], [2], [5].

According to Thomas [6], cf. Schneider [4], the vertex intensity takes its maximal value for given j in the isotropic case,

$$j_0(j, P_0) \leqq j_0(j, G). \tag{4.2}$$

We conclude from (3.7), (4.1), (4.2),

$$m_n(j, P_0) \geqq m_n(j, G); \qquad n = 1, 2, \cdots. \tag{4.3}$$

Thus the following result is derived, which is analogous to Theorem 3.1.

Theorem 4.1. For given intensity j, and for every $n = 1, 2, \cdots$, the nth moment for the volume of the anisotropic Poisson polytope is not smaller than the nth moment for the volume of the isotropic Poisson polytope.

Remarks

- If $P_0 \neq G$, it is even true that

$$m_n(j, P_0) > m_n(j, G).$$

- Inequality (4.2) is equivalent to

$$M_{-1}(j, P_0) \leqq M_{-1}(j, G).$$

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5. Generalizations

Let H be an arbitrary probability measure on the group SO_d of rotations about the origin and $P_0 * H$ the directional distribution defined by

$$\int f(h)(P_0 * H)(dh) = \int\int f(Rh)P_0(dh)H(dR);$$

$f : \mathbb{U}_1 \to [0, \infty)$ measurable. We may also write

$$P_0 * H = \int H(dR)(P_0 \circ R^{-1}).$$

Hence, $P_0 * H$ is a mixing of directional distributions all equal to P_0 up to a rotation.

A slight modification of the proof for Theorem 3.1, i.e. replacing N by H in (3.2), (3.4), leads to

$$M_n(j, P_0) \geqq M_n(j, P_0 * H); \qquad n = 1, 2, \cdots. \tag{5.1}$$

The relation (4.2) is a special case of the following more general result.

Proposition 5.1. The vertex intensities $j_0(j, P_0)$ and $j_0(j, P_0 * H)$ fulfil the inequality

$$j_0(j, P_0) \leqq j_0(j, P_0 * H). \tag{5.2}$$

Proof. Denote by $S(Q)$ the Steiner compact in the sense of Matheron [1] related to a finite measure Q on $\mathbb{U}_1$. Let r be a random rotation about the origin distributed according to H. Then

$$S(jP_0 * H) = \boldsymbol{E}(rS(jP_0)), \tag{5.3}$$

where the right-hand side means the set-valued expectation in the sense of [7] for the random set $rS(jP_0)$.

The generalized Brunn–Minkowski inequality of Vitale [7] implies

$$\boldsymbol{E}V^{1/d}(rS(jP_0)) \leqq V^{1/d}(\boldsymbol{E}(rS(jP_0))), \tag{5.4}$$

where V denotes volume. But $V(rS(jP_0))$ is a degenerated random variable, i.e. a constant,

$$V(rS(jP_0)) = V(S(jP_0)),$$

and hence (5.4) leads to

$$V(S(jP_0)) \leqq V(\boldsymbol{E}(rS(jP_0))). \tag{5.5}$$

Combining (5.3) and (5.5), we obtain

$$V(S(jP_0)) \leqq V(S(jP_0 * H)).$$

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According to [1], [2], [4], [6], we find

$$V(S(jP_0)) = j_0(j, P_0), \qquad V(S(jP_0 * H)) = j_0(j, P_0 * H),$$

which completes the proof.

Now, by a similar argument to that in Section 4, we conclude

$$m_n(j, P_0) \geqq m_n(j, P_0 * H); \qquad n = 1, 2, \cdots. \tag{5.6}$$

Remember that N denotes the invariant probability measure on the group SO_d, and G the isotropic directional distribution. It is clear that

$$P_0 * N = G.$$

Hence, in the special case $H = N$, (5.1) coincides with (3.7), (5.2) with (4.2), and (5.6) with (4.3).

References

[1] MATHERON, G. (1975) *Random Sets and Integral Geometry.* Wiley, New York.

[2] MECKE, J., SCHNEIDER, R., STOYAN, D. AND WEIL, W, (1990) *Stochastische Geometrie.* Birkhäuser, Basel.

[3] MILES, R. E. (1974) A synopsis of Poisson flats in Euclidean spaces. In *Stochastic Geometry*, ed. E. F. Harding and D. G. Kendall, pp. 202–227, Wiley, New York.

[4] SCHNEIDER, R. (1982) Random hyperplanes meeting a convex body. *Z. Wahrscheinlichkeitsth.* **61,** 379–387.

[5] STOYAN, D., KENDALL, W. S. AND MECKE, J. (1987) *Stochastic Geometry and Its Applications.* Wiley, New York.

[6] THOMAS, C. (1984) Extremum properties of the intersection densities of stationary Poisson hyperplane processes. *Math. Operationsforsch. Statist., Ser. Statist.* **15,** 443–449.

[7] VITALE, R. A. (1990) The Brunn–Minkowski inequality for random sets. *J. Multivariate Anal.* **33,** 286–293.

[8] WEIL, W. AND WIEACKER, J. A. (1993) Chapter 5.2 in *The Handbook of Convex Geometry*, ed. P. M. Gruber and J. M. Wills. Elsevier, Amsterdam.

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