triangle side lengths in pairs and all three make up the distance from a vertex to the point of tangency with the opposite e-circle, which is therefore constant. The interior angle bisectors are shown by dashed lines. Two similar right triangles are picked out in bold and reproduced in Figure 4.

We have

$$r = \frac{kR_A}{k + l + m} \tag{2}$$

and similar expressions involving R_B and R_C.

The area, Δ, of the triangle is the product of the semi-perimeter and r, whence

$$\Delta = (k + l + m)r = kR_A = lR_B = mR_C. \tag{3}$$

In the case of a right angle at A, $r = k$, so we have

$$\Delta = rR_A, \tag{4}$$

a result students can also derive using only the tangents-from-a-point theorem and the Pythagorean condition.

Acknowledgement

I would like to thank the referee for detailed suggestions which have, I hope, improved the clarity of my presentation.

References

1. Students may consult the Wikipedia entry ‘Dandelin spheres’.
2. For the photograph of an experimental set-up, go to www.magicmathworks.org, then Geomlab, then 7.

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101.41 On conics

The goal of this Note is to demonstrate the equivalence of three definitions of conic sections by means of a geometric construction. This also gives a construction of the directrices of any conic section by means of a pair of compasses and an unmarked straightedge only. Given the elementary nature of the arguments presented here, it is very likely that all the constructions presented below have long been known. This note has been submitted for publication only after the author's failure to find them in the literature on the subject available to him.

The ellipse

Points will be denoted by capital letters. AB is the straight segment between A and B. $|AB|$ is the length of AB. $\overline{AB}$ is the straight line through A

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and B. $A \mid B$ is the perpendicular bisector to AB. ABC is the triangle with vertices A, B and C. $\angle ABC$ is the interior angle of ABC at B.

Given F and C, the set of all P such that $|PF| + |PC|$ is constant is an ellipse. Alternatively, let Δ be a straight line, Q the foot of the perpendicular to Δ from P, and e a real number in $(0, 1)$. The set of all P such that $|PF| = e|PQ|$ is also an ellipse and Δ is called a directrix of the ellipse. A third definition of ellipse is: the set of all points equidistant to a circle and a point interior to it.

In this section we prove the equivalence of these three definitions. Taking the third definition of ellipse as starting point, let C be the centre of a circle and F a point interior to it. For each A on the circle, let $P = A \mid F \cap \overline{AC}$. Then $|AP| = |PF|$ and therefore all points P so constructed define an ellipse.

Note that the radius of the circle is $|AP| + |PC| = |PF| + |PC|$ which is constant for all points A and therefore for all P. Therefore we have deduced the first definition of ellipse from the third one and see that F and C are the foci.

For any given A (and so P) let Q be the point on the parallel to $\overline{FC}$ through P which is cocircular to A, P and F. This produces a system of similar triangles whose correspondingly equal angles are indicated in the Figure below, in which $S = \overline{FC} \cap \overline{AQ}$. We now show that S is the same for every A.

Note that

$$\frac{|AC|}{|SC|} = \frac{|FC|}{|AC|}, \tag{1}$$

which is a constant smaller than one that we call e. Then $|SC| = AC/e$ is also a constant, and so S is the same for every A. Note that $Q \in S \mid F$. We now show that $S \mid F$ is a directrix of the ellipse. Because $|PF| = |PA|$ we have

$$\frac{|PF|}{|PQ|} = \frac{|PA|}{|PQ|} = e, \tag{2}$$

or equivalently $|PF| = e|PQ|$. We have thus deduced the second definition of ellipse, in which $S \mid F$ is the directrix.

If for each P the segment FP is produced from P by a length $|PC|$ the resulting endpoints describe a circle centred at F with radius $|CA|$. Such an endpoint is denoted by A' in Figure 1, where the new circle is indicated with a dotted line. Then every point on the ellipse is equidistant to this new circle and to C. This establishes the invariance of the figure under exchange of C and F and consequently the existence of a second directrix constructed as above but with the roles of C and F reversed.

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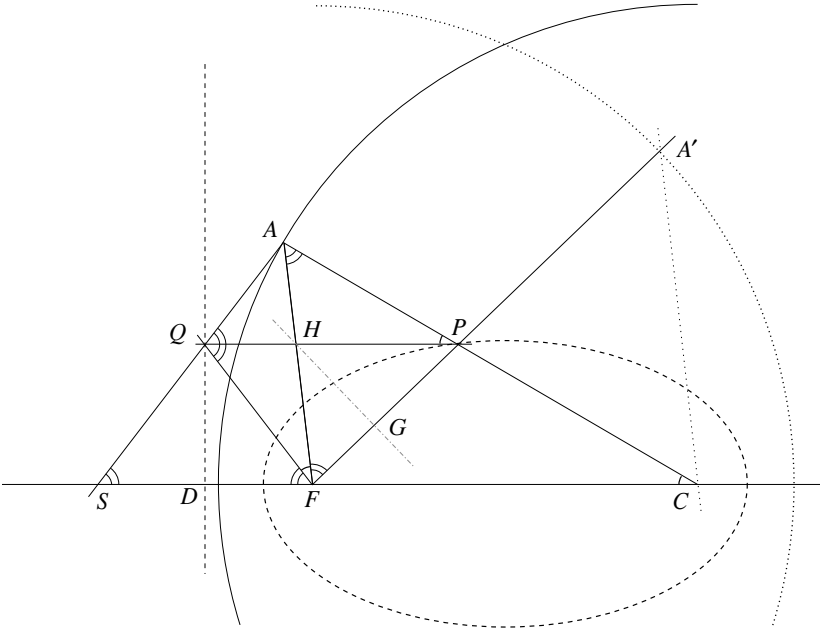


FIGURE 1

The hyperbola and the parabola

Let C be the centre of a circle and F a point exterior to it. The set of all points equidistant to the circle and F is a hyperbola. We shall show how the other two usual definitions of hyperbola follow from this.

Take A to be a point on the circle such that $\angle CAF$ is greater than a right angle. Then $P = A \mid F \cap \overline{AC}$ is a point on the hyperbola. Note that

$$|PC| = |PA| + |AC| = |PF| + |AC| \tag{3}$$

and therefore $|PC| - |PF|$ is a positive constant. This is the usual definitions of a hyperbola having foci C and F.

As we did for the ellipse, let Q be the point which is on the parallel to $\overline{FC}$ through P and concyclic with A, P and F. The ensuing system of similar triangles is shown in Figure 2.

It follows that

$$\frac{|CA|}{|CS|} = \frac{|CF|}{|CA|}, \tag{4}$$

which is a constant e now greater than 1. Therefore $|CS| = |CA|/e$ is a constant and so S is the same point for all A. Moreover, $Q \in S \mid F$ and we now show that $S \mid F$ is a directrix of the hyperbola. Because $|PF| = |PA|$,

$$\frac{|PF|}{|PQ|} = \frac{|PA|}{|PQ|} = e, \tag{5}$$

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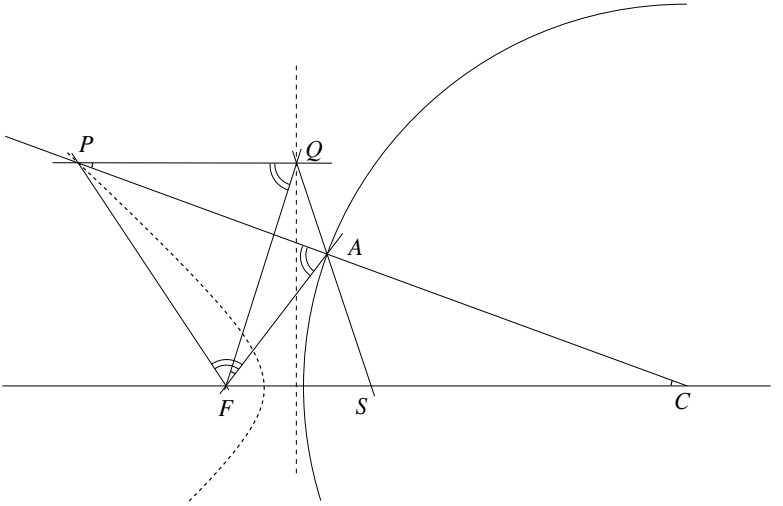


FIGURE 2

or equivalently $|PF| = e|PQ|$. This is another common definition of hyperbola, in which $S \mid F$ is a directrix.

Let A now be such that $\angle CAF$ is less than a right angle. P and A are obtained as before and the situation is summarised in Figure 3.

Note that the new point P lies on a separate branch of the hyperbola. Also, $|PF| - |PC| = |PA| - |PC| = |AC|$, which is the radius of the circle. Therefore the usual definition of the hyperbola applies to this second branch of the curve as well.

Observe that we still have

$$\frac{|CA|}{|CS|} = \frac{|CF|}{|CA|} = e, \tag{6}$$

therefore $|CS|$ is the same as before, and S is the same point as in the

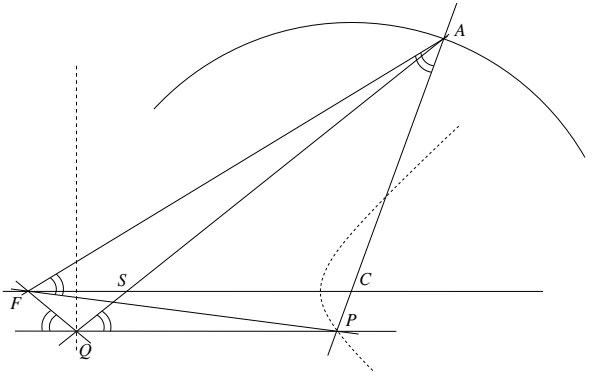


FIGURE 3

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construction of the first branch of the hyperbola. We deduce again that $|PF| = e|PQ|$ with same directrix $S \mid F$. The existence of a second directrix may be established by the same means as for the ellipse.

The parabola may be obtained as the limiting case of either the ellipse or the hyperbola in which the radius of the circle increases without bound while keeping the circle at a finite distance from F. The definition of parabola as the locus of points equidistant to a fixed point and a straight line, and the corresponding geometric construction, follows from this.

The equation of a conic

So far we have employed exclusively the methods of classical Euclidean geometry. No use has been made of coordinate systems or equations other than the equality of certain lengths or ratios of lengths. This was a deliberate choice, as the subject matter predates the application of Algebra to Geometry. That notwithstanding, and in acknowledgement of a suggestion by a referee, I shall indicate how the construction presented above leads directly to the equation of the ellipse. Identical arguments apply to the other conics.

Write $|AC| = 2a$ and $|FP| = r$. Let $\theta = \angle PFC$. Because DFQ is similar to FGH and $|FH| = e|QF|$ we have $|FG| = e|DF|$. Because $|SC| = 2a/e$ and $|DF| = 2ae$ we have $|DF| = a(1/e - e)$. Therefore $|FG| = a(1 - e^2)$. From $r = |FG| + |HP| \cos\theta$ and $|HP| = e|AP| = er$ we find $r(1 - e\cos\theta) = |FG|$. In conclusion, we have deduced that $r(1 - e\cos\theta) = a(1 - e^2)$.

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101.42 The Steiner point revisited

The circumcircle and the outer Steiner ellipse of a triangle meet in four points, namely, the vertices A, B, C and the Steiner point S.

In this Note we will use a linear transformation T on (the areal coordinates of) four concyclic points of the triangle to locate another circle passing through S.

The coordinates

Using $\Delta ABC = \Delta$ to denote the area of the triangle of reference, a point $P[x, y, z]$ has actual areal coordinates $x = \dfrac{\Delta PBC}{\Delta}$, $y = \dfrac{\Delta PCA}{\Delta}$, $z = \dfrac{\Delta PAB}{\Delta}$, where $x + y + z = 1$. We also use the cotangent notation $\alpha = \cot A, \beta = \cot B, \gamma = \cot C$, where $\beta\gamma + \gamma\alpha + \alpha\beta = 1$.

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