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Embeddings of dynamical systems into cellular automata

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Abstract. We prove that general topological dynamical systems on metric Cantor spaces can be embedded into cellular automata defined on Cayley graphs. Furthermore, we prove that it is possible to construct this embedding in such a way that topological entropy is preserved.

1. *Introduction*

Symbolic dynamics has been a useful tool for studying hyperbolic (smooth) dynamical systems (see [**1**]). Specifically, an invariant subsystem of the original dynamical system is searched which is topologically conjugate to a subshift over some finite alphabet. So, symbolic embeddings and extensions (or conversely, factors) of dynamical systems $(\mathfrak{X}, f)$ are investigated (here, $\mathfrak{X}$ is assumed to be a compact metric space and $f : \mathfrak{X} \to \mathfrak{X}$ is continuous). If f is a homeomorphism, there are several results in the literature. First of all, there is a classical result of Hedlund stating that a dynamical system $(\mathfrak{X}, f)$ is conjugate to a subshift if and only if f is expansive and $\mathfrak{X}$ is a Cantor space. If symbolic extensions (Σ, σ) of $(\mathfrak{X}, f)$ are considered – that is, if there is a continuous surjection of $\varphi : \Sigma \to \mathfrak{X}$ onto $\mathfrak{X}$ with $f \circ \varphi = \varphi \circ \sigma$ – there is a result due to Reddy [**11**]. Namely, $(\mathfrak{X}, f)$ has a symbolic extension if and only if f is expansive. More recently, Boyle *et al* [**2**] have given a specialization of this result when the topological entropy is considered: if f is asymptotically h-expansive, then there is a symbolic extension preserving topological entropy.

These results are far from being complete and of general nature in some sense. So, it is tempting not to look for (one-dimensional) shifts as supersystems but rather for higher-dimensional symbolic supersystems. The simplest higher-dimensional symbolic systems

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which form a proper superset of the shifts are the cellular automata since they have a finite description [**5**]. To be more general, the grid of the cellular automaton is not restricted to $\mathbb{Z}^d$ for some dimension d. Instead, the grid is assumed to be a Cayley graph of a finitely generated group. Then a general embedding theorem can be proved: any dynamical system on a metric Cantor space can be embedded into a cellular automaton. Furthermore, the embedding can be chosen such that topological entropy is preserved.

In the next section, basic definitions and a first general theorem are stated. In §3, the case of a topological dynamical system on a metric Cantor space $\mathfrak{X}$ with bijective f is considered. Here, $(\mathfrak{X}, f)$ can be embedded into a cellular automaton where the global function is a shift. However, if we demand preservation of the topological entropy, the global function is a little more complex. Finally, in §4, the general case is considered where f need not be bijective. In this general case, a cellular automaton can be found for which $(\mathfrak{X}, f)$ is embedded and the topological entropy is preserved.

2. *Setting*

Let $\mathfrak{X}$ denote a metric Cantor space, and $f \in C^0(\mathfrak{X}, \mathfrak{X})$ a continuous function. We know that the state space of a cellular automaton is a metric Cantor space, and that the global function of any cellular automaton is a continuous function on this Cantor space.

Conversely, given such a function f, is it possible to relate f somehow to a cellular automaton? Or, can we understand which subset of this function can be related, and which subset cannot? We use a generalization of cellular automata from the lattice $\mathbb{Z}^2$ to Cayley graphs which is introduced in [**10**] or [**13**]; see also [**4, 12**].

Definition 2.1. (1) Let $G = \langle \tau_1, \ldots, \tau_m \rangle$ denote the group generated by the elements $\tau_1, \ldots, \tau_m$. The Cayley graph associated with these generators has G as nodes. An edge connects $g, h \in G$ if there is a $\tau \in \{\tau_1, \ldots, \tau_m\}$ such that either $g = h\tau$ or $h = g\tau$. $d_\Gamma : G \times G \to \mathbb{N}_0$ denotes the path metric (length of the shortest path connecting two nodes).

(2) A cellular automaton consist of a four-tuple (Γ, U_0, E, f_0) where:

(i) Γ is a Cayley graph;
(ii) $U_0 \subset \Gamma$ is a finite subgraph, called a neighbourhood of the unit $e \in G$;
(iii) E is a finite set of elementary states;
(iv) $f_0 : E^{U_0} \to E$ denotes the local function.

(3) For $g \in G$, the operators $\sigma_g : E^\Gamma \to E^\Gamma$, $\sigma_g(u)\,(h) = u(gh)$ are called shift operators.

(4) The global function of a cellular automaton (Γ, U_0, E, f_0) is defined via

$$f : E^\Gamma \to E^\Gamma, \quad f(u)\,(g) = f_0((u \circ \sigma_g)|_{U_0}).$$

(5) The set E^Γ is equipped with the Cantor metric. For $u, v \in E^\Gamma$ and $u = v$, let $d(u, v) = 0$. If $u \neq v$, find some $g \in G$ with

(i) $u(g) \neq v(g)$ such that
(ii) $k = d_\Gamma(e, g)$ is minimal.

Then $d(u, v) = 1/(1 + k)$.

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Note that the neighbourhood U_0 of $e \in \Gamma$ does not necessarily agree with the set of nodes that are neighbours of e in the Cayley graph, that is, that have an edge that connects them with e. Note, furthermore, that the metric space (E^Γ, d) is a metric Cantor space.

As a first hint as to how a function on a metric Cantor space can be related to a cellular automaton, we introduce some more notation and state some theorems.

Definition 2.2. Let $(\mathfrak{X}, d)$ be a metric Cantor space.

(1) Let $\sigma_1, \ldots, \sigma_m$ be bijective functions on $\mathfrak{X}$, and $G = \langle \sigma_1, \ldots, \sigma_m \rangle$ the group generated by these functions. If there is a real constant $\delta > 0$ such that for any $x, y \in \mathfrak{X}$, $x \neq y$, there exists $g \in G$ with

$$d(g(x), g(y)) > \delta,$$

then $\sigma_1, \ldots, \sigma_m$ are called jointly expansive and we say that the group G acts expansively on $\mathfrak{X}$.

(2) Let $f \in C^0(\mathfrak{X}, \mathfrak{X})$ be a continuous function. The group $A(f)$ is defined by

$$A(f) = \{g : \mathfrak{X} \to \mathfrak{X} \mid g \text{ continuous, bijective and } f \circ g = g \circ f\}.$$

(3) We say that a topological dynamical system $(\mathfrak{X}, f)$ can be embedded into a topological dynamical system $(\mathfrak{Y}, g)$, if there is a continuous, injective map $\varphi : \mathfrak{X} \to \mathfrak{Y}$ such that $\varphi \circ f = g \circ \varphi$.

(4) We say that a topological dynamical system $(\mathfrak{X}, f)$ can be (weakly) embedded into a cellular automaton (Γ, U_0, E, g_0) with global function g, if $(\mathfrak{X}, f)$ can be embedded into the topological dynamical system (E^Γ, g), that is, if there is a continuous, injective function $\varphi : \mathfrak{X} \to E^\Gamma$ such that $\varphi \circ f = g \circ \varphi$. If $\varphi(\mathfrak{X})$ is invariant with respect to the shift operators, we call this embedding strong.

Suppose that a topological dynamical system $(\mathfrak{X}, f)$ can be embedded into a topological dynamical system $(\mathfrak{Y}, g)$. Then $(\mathfrak{X}, f)$ is a subsystem of $(\mathfrak{Y}, g)$. The question is now whether there is an embedding of $(\mathfrak{Y}, g)$ with no reasonable loss of the complexity of orbit structure. Topological entropy measures the complexity of the orbit structure of a dynamical system; see [**3, 8, 14**].

Definition 2.3. Let $(\mathfrak{X}, f)$ be a topological dynamical system and let d be the corresponding metric on $\mathfrak{X}$. Let $n \in \mathbb{N}$ and $\epsilon > 0$; then a subset $A \subseteq \mathfrak{X}$ is called (n, ϵ)-*spanning* if for every $x \in \mathfrak{X}$ there is some $y \in A$ such that $d_n(x, y) < \epsilon$. Here, the metric $d_n : \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}$ is given by $d_n(x, y) = \max_{0 \leq k < n} d(f^k(x), f^k(y))$. Let $\text{span}(n, \epsilon, f)$ be be the minimum cardinality of an (n, ϵ)-spanning set. Then the topological entropy $h(f)$ of f is defined by

$$h(f) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{\log(\text{span}(n, \epsilon, f))}{n}.$$

Note that the limit exists and is finite (see [**3, 14**]). If the topological dynamical system is a cellular automaton, the formula for the topological entropy assumes a simple form. Let $f : E^\Gamma \to E^\Gamma$ be the global function of the cellular automaton. Then define a function $\varphi : E^\Gamma \to E^{\Gamma \times \mathbb{N}}$ by setting $\varphi(u)(g, n) = f^n(u)(g)$. Hence, the subset $\tilde{U} \subseteq E^{\Gamma \times \mathbb{N}}$ defined by $\tilde{U} = \varphi(E^\Gamma)$ can be seen as a kind of space–time diagram. Let $R(l, t)$

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denote the number of distinct configurations in the space–time diagram with spacial extent l and temporal extent t. More formally, $R(l, t) = |\{\tilde{u}|_{\Gamma_l \times [0,t-1]} \mid \tilde{u} \in \tilde{U}\}|$ where $\Gamma_l = \{g \in G \mid d_\Gamma(e, g) \leq l\}$ is the (finite) subgraph of Γ. Now, the topological entropy of the cellular automaton depends only on $R(l, t)$.

PROPOSITION 2.1. *Let (Γ, U_0, E, f_0) be a cellular automaton with global function f. Then the topological entropy of the cellular automaton, $h(E^\Gamma, f)$, is given by*

$$h(E^\Gamma, f) = \lim_{l\to\infty} \lim_{t\to\infty} \frac{\log R(l, t)}{t}$$

where $R(l, t)$ is the number of distinct space–time configurations defined as above.

Proof. Let (Γ, U_0, E, f_0) be a cellular automaton with global function f. Then (E^Γ, f) is a topological dynamical system. Now consider the space $E^{\Gamma\times\mathbb{N}}$ with the usual topology. This space is compact and metrizable, and the metric is given by $d_e(\tilde{u}, \tilde{v}) = 1/(1 + k)$ with $k = \min\{\max\{d_\Gamma(g, e), n\} \mid \tilde{u}(g, n) \neq \tilde{v}(g, n)\}$. Now consider the function $\varphi : E^\Gamma \to E^{\Gamma\times\mathbb{N}}$, defined by $\varphi(u)\,(g, n) = f^n(u)\,(g)$ for all $g \in \Gamma$, $n \in \mathbb{N}$. Then

$$\varphi \circ f(u)\,(g, n) = f^n(f(u))\,(g) = f^{n+1}(u)\,(g) = \varphi(u)\,(g, n + 1) = \sigma_t \circ \varphi(u)\,(g, n)$$

holds for all $u \in E^\Gamma$, where $\sigma_t : E^{\Gamma\times\mathbb{N}} \to E^{\Gamma\times\mathbb{N}}$ is defined by $\sigma_t(\tilde{u})\,(g, n) = \tilde{u}(g, n + 1)$. Hence, $\varphi \circ f = \sigma_t \circ \varphi$ holds. Furthermore, φ is injective as can be easily seen: if $u, v \in E^\Gamma$ and $u \neq v$, there is some $g \in \Gamma$ with $u(g) \neq v(g)$. Hence, $\varphi(u)\,(g, 0) = u(g) \neq v(g) = \varphi(v)\,(g, 0)$ holds, so $\varphi(u) \neq \varphi(v)$. Next, by the definition of φ, $d_e(\varphi(u), \varphi(v)) \leq d(u, v)$ holds for all $u, v \in E^\Gamma$. So, φ is continuous. Finally, let $U \subseteq E^{\Gamma\times\mathbb{N}}$ be given by $U = \varphi(E^\Gamma)$. Then U is closed and, since $\sigma_t(U) = \sigma_t(\varphi(E^\Gamma)) = \varphi(f(E^\Gamma)) \subseteq U$, U is σ_t-invariant.

So, the restriction $\tilde{\varphi} : E^\Gamma \to U$ of φ is a homeomorphism and, since $\tilde{\varphi} \circ f = \sigma_t \circ \tilde{\varphi}$, is also a conjugacy between the dynamical systems (E^Γ, f) and (U, σ_t). Hence, the topological entropies are equal, $h(E^\Gamma, f) = h(U, \sigma_t)$. The latter, however, is easily computable.

Let $n \geq 1$ and $B_n(u, m) = \{v \in U \mid d_{n,e}(u, v) < 1/(1 + m)\}$ be a ball in the $d_{n,e}$-metric with centre $u \in U$ and radius $1/(1 + m)$. Here, the metric $d_{n,e} : E^{\Gamma\times\mathbb{N}} \times E^{\Gamma\times\mathbb{N}} \to \mathbb{R}$ is given by $d_{n,e}(u, v) = \max_{0\leq i<n}\{d_e(\sigma_t^i(u), \sigma_t^i(v))\}$. Then $B_n(u, m) = \{v \in U \mid u|_{\Gamma_m\times[0,m+n-1]} = v|_{\Gamma_m\times[0,m+n-1]}\}$ holds. Let $r(n, 1/(1 + m))$ denote the minimum cardinality of an $(n, 1/(1 + m))$-spanning set for U. Then $r(n, 1/(1 + m)) = R(m, n + m)$ holds with $R(l, t) = |\{u|_{\Gamma_l\times[0,t-1]} \mid u \in U\}|$. Finally, for the topological entropy,

$$h(U, \sigma_t) = \lim_{m\to\infty} \limsup_{n\to\infty} \frac{\log R(m, n + m)}{n} = \lim_{m\to\infty} \lim_{n\to\infty} \frac{\log R(m, n)}{n}$$

follows. □

The following theorem is essentially a reformulation of the ideas of Hedlund [7] that serve our purposes.

THEOREM 2.1. *Let $\mathfrak{X}$ be a metric Cantor space and $f : \mathfrak{X} \to \mathfrak{X}$ a continuous function. The topological dynamical system $(\mathfrak{X}, f)$ can be strongly embedded into a cellular automaton (Γ, U_0, E, g_0), if and only if the group $A(f)$ contains a finitely generated subgroup that acts expansively on $\mathfrak{X}$.*

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Proof. Assume that there is a strong embedding. Let the Cayley graph Γ be generated by $\tau_1, \ldots, \tau_m$. If g denotes the global function of the cellular automaton, then $\sigma_{\tau_i} \in A(g)$ for all $i = 1, \ldots, m$. Furthermore, $G = \langle \sigma_{\tau_1}, \ldots, \sigma_{\tau_m} \rangle$ acts in an expansive way on E^Γ. Since $\varphi : \mathfrak{X} \to E^\Gamma$ is injective and since $\varphi(\mathfrak{X})$ is invariant under all shift operators, the inverse of φ is well defined on $\sigma_{\tau_i}(\varphi(\mathfrak{X}))$ – that is, $\varphi^{-1}(\sigma_{\tau_i}(\varphi(x)))$ is also well defined. Define the function $\zeta_i : \mathfrak{X} \to \mathfrak{X}$ by

$$\zeta_i : \mathfrak{X} \to \mathfrak{X}, \quad x \mapsto \varphi^{-1} \circ \sigma_{\tau_i} \circ \varphi(x).$$

Then ζ_i are continuous, bijective and commute with f, that is, $\zeta_i \in A(f)$. We now show that the group $H = \langle \zeta_1, \ldots, \zeta_m \rangle$ acts in an expansive way on $\mathfrak{X}$. If we assume that this is not the case, then there are two sequences $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}} \subset \mathfrak{X}$, such that $x_n \neq y_n$ for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \sup_{\eta\in H} d(\eta(x_n), \eta(y_n)) = 0.$$

Thus, $\lim_{n\to\infty} \sup_{\gamma\in G} d(\varphi^{-1} \circ \gamma \circ \varphi(x_n), \varphi^{-1} \circ \gamma \circ \varphi(y_n)) = 0$. That is, we find sequences $u_n = \varphi(x_n)$, $v_n = \varphi(y_n) \in E^\Gamma$, such that the distance from the origin vanishes for $n \to \infty$, no matter which elements of $\gamma \in G$ we apply to u_n, v_n. Thus there are $\gamma_n \in G$ such that $d(\gamma_n(u_n), \gamma_n(v_n)) \geq \delta$. Let $\hat{u}_n = \gamma_n(u_n)$, $\hat{v}_n = \gamma_n(v_n) \in E^\Gamma$. These two sequences satisfy $d(\hat{u}_n, \hat{v}_n) \geq \delta$ and

$$\lim_{n\to\infty} d(\varphi^{-1}(\hat{u}_n), \varphi^{-1}(\hat{v}_n)) = 0.$$

Since E^Γ is compact, the sequence $\{(\hat{u}_n, \hat{v}_n)\}_{n\in\mathbb{N}} \subset E^\Gamma \times E^\Gamma$ has a converging subsequence that converges to $(\hat{u}, \hat{v})$. Since φ is a bijective, continuous function between the compact set $\mathfrak{X}$ and the compact set $\varphi(\mathfrak{X}) \subset E^\Gamma$, the inverse function of φ on $\varphi(\mathfrak{X})$ is also continuous. Thus, $d(\hat{u}, \hat{v}) \geq \delta$ and $d(\varphi^{-1}(\hat{u}), \varphi^{-1}(\hat{v})) = 0$. That is, $\varphi^{-1}(\hat{u}) = \varphi^{-1}(\hat{v})$ but $\hat{u} \neq \hat{v}$. This contradicts the definition of φ^{-1} as the inverse of a function.

Now assume that $A(f)$ contains a finitely generated subgroup $H = \langle \zeta_1, \ldots, \zeta_m \rangle$ that acts in an expansive way on $\mathfrak{X}$ with constant $\delta > 0$. Then there is a finite number of clopen, pairwise disjoint and non-empty sets $U_1, \ldots, U_n \subset \mathfrak{X}$ with diameter less than δ that cover $\mathfrak{X}$,

$$\mathfrak{X} = \bigcup_{i=1}^{n} U_i.$$

Let Γ be the Cayley graph corresponding to H. We identify $\mathfrak{X}$ in the usual way with a subset of E^Γ, $E = \{1, \ldots, n\}$: a point $x \in \mathfrak{X}$ corresponds to $u \in E^\Gamma$ if and only if

$$\eta(x) \in U_{u(\eta)} \quad \text{for all } \eta \in H.$$

Standard arguments show that

$$\varphi : \mathfrak{X} \to E^\Gamma, \quad x \mapsto u$$

is a continuous and injective map. Then the Hedlund–Lyndon–Curtis theorem [**7**] can be applied and shows that the function

$$\tilde{g} : \varphi(\mathfrak{X}) \to \varphi(\mathfrak{X}), \quad u \mapsto \varphi \circ f \circ \varphi^{-1}(u)$$

can be represented as the global function of a cellular automaton (Γ, U_0, E, g_0) on E^Γ. In general, this automaton is only uniquely defined on $\varphi(\mathfrak{X})$, but can be extended (in a non-unique way) to E^Γ. Thus, $(\mathfrak{X}, f)$ is strongly embedded into (Γ, U_0, E, g_0). □

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The extension in the above theorem can be chosen in a way such that the topological entropy is preserved.

PROPOSITION 2.2. *Let $\mathfrak{X}$ be a metric Cantor space and $f : \mathfrak{X} \to \mathfrak{X}$ a continuous function. If the topological dynamical system $(\mathfrak{X}, f)$ can be strongly embedded into a cellular automaton (Γ, U_0, E, g_0), there also exists an embedding (Γ, U_0, F, g_0') such that the topological entropy is preserved, that is, $h(\mathfrak{X}, f) = h(F^\Gamma, g')$.*

Proof. Consider the construction of the embedding from Theorem 2.1. The local function $\tilde{g}_0 : E^{U_0} \cap \varphi(\mathfrak{X})|_{U_0} \to E$, corresponding to the global function $\tilde{g} : \varphi(\mathfrak{X}) \to \varphi(\mathfrak{X})$, has to be extended in an appropriate way. Assume that 0 is not element of E. Then extend the alphabet E by the additional symbol 0, $F = E \cup \{0\}$. Then define $g_0'(x) = 0$ for all $x \in F^{U_0} \setminus (E^{U_0} \cap \varphi(\mathfrak{X})|_{U_0})$. The global function $g' : F^\Gamma \to F^\Gamma$ is then defined in a standard way.

Now observe that by construction, for all $u \in F^\Gamma \setminus \varphi(\mathfrak{X})$ there is some $\eta \in H$ such that $g'(u)(\eta) = 0$ holds. As a consequence, for any $m \in \mathbb{N}$ there is some $N \in \mathbb{N}$ such that for all $u \in F^\Gamma \setminus \varphi(\mathfrak{X})$, $g'^n(u)(\eta) = 0$ holds for all $\eta \in \Gamma_m$, $n \geq N$. Consequently, if $R(l, t)$ denotes the number of space–time configurations of (Γ, U_0, E, g_0) and $R'(l, t)$ the number of space–time configurations of (Γ, U_0, F, g_0'), then for all $l \in \mathbb{N}$ there are constants $C, t \in \mathbb{N}$ such that $R'(l, t) = R(l, t) + C$ holds for all $t \geq T$. So

$$h(\mathfrak{X}, f) = h(\varphi(\mathfrak{X}), \tilde{g}) = \lim_{l\to\infty} \lim_{t\to\infty} \frac{R(l,t)}{t} = \lim_{l\to\infty} \lim_{t\to\infty} \frac{R'(l,t)}{t} = h(F^\Gamma, g')$$

follows. Note that $h(\mathfrak{X}, f) = h(\varphi(\mathfrak{X}), \tilde{g})$ holds since the restriction $\tilde{\varphi} : \mathfrak{X} \to \varphi(\mathfrak{X})$ of φ is a conjugacy between $(\mathfrak{X}, f)$ and $(\varphi(\mathfrak{X}), \tilde{g})$. □

If $(\mathfrak{X}, f)$ can be embedded weakly but not strongly into a cellular automaton, then $A(f)$ does not contain a finitely generated group that acts in an expansive way on $\mathfrak{X}$. However, the cellular automaton itself can be understood as an extension of f for which such an expansive group exists.

COROLLARY 2.1. *If $(\mathfrak{X}, f)$ can be weakly embedded into a cellular automaton, then there is a continuous extension g of f in the sense that there exist a metric Cantor space $\tilde{\mathfrak{X}}$, a continuous function $g \in C^0(\tilde{\mathfrak{X}}, \tilde{\mathfrak{X}})$ and a continuous, injective function $\varphi : \mathfrak{X} \to \tilde{\mathfrak{X}}$ with*

$$\varphi \circ f = g \circ \varphi$$

such that $(\tilde{\mathfrak{X}}, g)$ can be strongly embedded into a cellular automaton. That is, $A(g)$ contains a finitely generated expansive group.

3. *Bijective functions*

We first consider a topological dynamical system $(\mathfrak{X}, f)$ where f is a continuous and bijective function. In this case, it is possible to construct directly a cellular automaton that allows an embedding of the dynamical system. In this section and the next we will use the following definitions.

Definition 3.1. (1) Let $E_n = \{0, \ldots, n-1\}$ denote the alphabet with n letters, equipped with the discrete topology.

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(2) Let Σ_n be the set of all bisequences over E_n,

$$\Sigma_n = \{x = ((x_i))_{i\in\mathbb{Z}} \mid x_i \in E_n\}.$$

The metric on Σ_n is the standard Cantor metric.

(3) Let $\mathfrak{Y}_n$ be the one-sided shift over the alphabet Σ_n, where we identify $(\Sigma_n)^{\mathbb{N}_0}$ with $\mathfrak{Y}_n = E_n^{\mathbb{N}_0\times\mathbb{Z}}$, and equip this set with the usual Cantor metric: for $u, v \in E^{\mathbb{N}_0\times\mathbb{Z}}$ define $d_{\mathfrak{Y}}(u, v) = 0$ if $u = v$; if $u \neq v$, determine $k = \inf\{i \in \mathbb{N}_0 \mid \exists j \in \mathbb{Z} : |j| \leq i$ and $u(i, j) \neq v(i, j)\}$ and define $d_{\mathfrak{Y}}(u, v) = 1/(k + 1)$.

(4) Let $\sigma_{\mathfrak{Y}}$ be the left-shift on $\mathfrak{Y}_n$, that is, for $u \in \mathfrak{Y}_n$ define $\sigma_{\mathfrak{Y}}(u)\,(i, j) = u(i + 1, j)$ for $i \in \mathbb{N}_0$, $j \in \mathbb{Z}$.

PROPOSITION 3.1. *Let $(\mathfrak{X}, f)$ be a topological system, $\mathfrak{X}$ a metric Cantor space and f a bijective, continuous function. Furthermore, let (Γ, U_0, E, g_0) be a cellular automaton with $\Gamma = \mathbb{Z}^2$, $E = \{0, 1\}$ and U_0 the von Neumann neighbourhood $U_0 = \{(0, 0), (0, 1), (1, 0), (0, -1), (-1, 0)\}$. Let the global function g of this cellular automaton be the shift operator in the x_1-direction of the grid. Then $(\mathfrak{X}, f)$ can be embedded into the cellular automaton.*

Proof. Since $\mathfrak{X}$ is a metric Cantor space, it is homeomorphic to Σ_2. Thus, there is a bijective function $\varphi_1 \in C^0(\mathfrak{X}, \Sigma_2)$. Let $f_1 = \varphi_1 \circ f \circ \varphi_1^{-1} : \Sigma_2 \to \Sigma_2$. Thus, we find $\varphi_1 \circ f = f_1 \circ \varphi_1$, that is, $(\mathfrak{X}, f)$ and (Σ_2, f_1) are conjugated.

Next, we identify the function f_1 with its forward–backward trajectory: let $\varphi_2 : \Sigma_2 \to E_2^{\mathbb{Z}^2}$ defined by

$$\varphi_2(x)\,(i, j) = f_1^{(i)}(x)\,(j), \quad i, j \in \mathbb{Z}^2.$$

Furthermore, let

$$\sigma_1 : E_2^{\mathbb{Z}^2} \to E_2^{\mathbb{Z}^2}, \quad u \mapsto v \quad \text{with } v(i, j) = u(i + 1, j).$$

Then

$$\varphi_2(f_1(x)) = f_1^{(i+1)}(x)\,(j) = \sigma_1(\varphi_2(x)).$$

The function φ_2 is continuous and injective. Thus, $(\mathfrak{X}, f)$ is embedded into the cellular automaton that represents the shift operator on $E_2^{\mathbb{Z}^2}$. □

Remark 3.1. From this proposition we can conclude that for each bijective $f \in C^0(\mathfrak{X})$ there exists a bijective, continuous function on $\mathfrak{X}$ that commutes with f and, together with f, is jointly expansive; $A(f)$ contains $\mathbb{Z}^2$ as an expansive subgroup.

The above proposition allows us to characterize all bijective functions on a metric Cantor spaces.

LEMMA 3.1. *All bijective, continuous functions on a metric Cantor space are conjugated to the shift operator acting on a (topologically and with respect to all shift operators) closed subset of $\{0, 1\}^{\mathbb{Z}^2}$. We can identify subsets of $\{0, 1\}^{\mathbb{Z}^2}$ that are topologically closed and closed under the shift operator with the bijective, continuous functions on metric Cantor spaces.*

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The embedding constructed in the proof of Proposition 3.1 obviously does not preserve the topological entropy, since then $h(\mathfrak{X}, f)$ would be constant, independent of the dynamical system $(\mathfrak{X}, f)$. By adding one more state to E, the embedding can be made entropy-preserving (cf. Proposition 2.2).

PROPOSITION 3.2. *Let $(\mathfrak{X}, f)$ be a topological system as in Proposition 3.1. Furthermore, let (Γ, U_0, E, g_0) be a cellular automaton with $\Gamma = \mathbb{Z}^2$, $E = \{0, 1, 2\}$ and U_0 the von Neumann neighbourhood. Let the global function g of this cellular automaton be the shift operator in the x_1-direction of the grid. Then $(\mathfrak{X}, f)$ can be embedded into the cellular automaton and $h(\mathfrak{X}, f) = h(E^\Gamma, g)$ holds.*

Proof. Construct $\varphi_2 : \Sigma_2 \to E_2^{\mathbb{Z}^2}$ as in the proof of Proposition 3.1. The function φ_2 is continuous and injective but generally not bijective. Now apply the construction in the proof of Theorem 2.2. First expand the set of states from E_2 to E_3. Next define a bijective extension of $\tilde{\sigma}_1 : \varphi_2(\Sigma_2) \to \varphi_2(\Sigma_2)$ to some $f_2 : E_3^{\mathbb{Z}^2} \to E_3^{\mathbb{Z}^2}$ as is described in Theorem 2.2. □

Remark 3.2. The function $f_2 : E_3^{\mathbb{Z}^2} \to E_3^{\mathbb{Z}^2}$ in the proof of the above theorem is explicitly given by

$$f_2(x)\ (i, j) = \begin{cases} \sigma_1(x)\ (i, j) & \text{if } x(i, j) \neq 2 \text{ and } \sigma_1(x)\ (i, j) = f(x)\ (i, j), \\ 2 & \text{otherwise.} \end{cases}$$

If we drop the assumption that f is bijective and consider a general continuous function on a Cantor space, we can repeat the construction above, using the positive orbit $(f^{(i)}(x))_{i \in \mathbb{N}}$ instead of the complete orbit. If we extend this sequence for $i < 0$ by zero, say, we again derive a cellular automaton (the shift on $\{0, 1\}^{\mathbb{Z}}$). However, the dynamical system $(\mathfrak{X}, f)$ is not exactly embedded into the cellular automaton, since we are forced to use the restriction operator to non-negative indices in order to work out the connection. Thus, we will use another idea for non-bijective functions in the next section.

4. *The general case: construction of the cellular automaton*

In this section, given a continuous function on a metric Cantor space $(\mathfrak{X}, f)$, we construct a cellular automaton into which $(\mathfrak{X}, f)$ can be weakly embedded. The first step is a concrete representation of $(\mathfrak{X}, f)$. This is done in the next theorem.

THEOREM 4.1. *Let $\mathfrak{X}$ be a metric Cantor space with metric $d_{\mathfrak{X}}$ and let $f : \mathfrak{X} \to \mathfrak{X}$ be a continuous function. Then $(\mathfrak{X}, f)$ can be embedded into the full shift over the alphabet E_n, that is, in $(\mathfrak{Y}_n, \sigma_{\mathfrak{Y}})$, for all $n \geq 2$.*

Proof. Step 1: Representation of $\mathfrak{X}$. Since any metric Cantor space is homeomorphic to any other metric Cantor space, we find a topological homeomorphism $\varphi_1 : \mathfrak{X} \to \Sigma_n$ for any $n \geq 2$. Define $h : \Sigma_n \to \Sigma_n$ by $h(x) = \varphi_1 \circ f \circ \varphi_1^{-1}(x)$. Then $(\mathfrak{X}, f)$ and (Σ_n, h) are topologically conjugated.

Step 2: Representation of h. We represent h as a one-sided shift over the the alphabet Σ_n in defining $\varphi_2 : \Sigma_n \to \mathfrak{Y}_n$ by

$$\varphi_2(x)\ (i, j) = h^{(i)}(x)\ (j), \quad i \in \mathbb{N}_0,\ j \in \mathbb{Z}.$$

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We now state and prove various properties of φ_2, h and $\sigma_{\mathfrak{Y}}$.

(a) $\varphi_2 \circ h = \sigma_{\mathfrak{Y}} \circ \varphi_2$. This property is a direct consequence of the construction of φ_2.

(b) $\varphi_2 \in C^0(\Sigma_n, \mathfrak{Y})$. Consider $x \in \Sigma_n$ and chose $\varepsilon > 0$ arbitrary, fixed. We show that there is a constant $\delta > 0$, such that $d_n(x, y) < \delta$ implies $d_{\mathfrak{Y}}(\varphi_2(x), \varphi_2(y)) < \varepsilon$. Let $m = \lceil 1/\varepsilon \rceil + 1$. Since h is continuous, there is a $\delta > 0$ such that $h^{(i)}(x)\,(j) = h^{(i)}(y)\,(j)$ for $|j| \leq m$ and $i = 0, \ldots, m$ if $d_n(x, y) < \delta$. Then $d_{\mathfrak{Y}}(\varphi_2(x), \varphi_2(y)) < \varepsilon$ is also a consequence of $d_n(x, y) < \delta$.

(c) φ_2 is injective. If $x \neq y$, then $\varphi(x)\,(0, \cdot) = x \neq y = \varphi(y)\,(0, \cdot)$. Thus, this function is injective.

Properties (a)–(c) imply that $(\mathfrak{X}, f)$ is embedded into the shift $(\mathfrak{Y}_n, \sigma_{\mathfrak{Y}})$. □

Remark 4.1. (1) If $\hat{\mathfrak{Y}} \subset \mathfrak{Y}_3$ denotes the subshift created by $(\mathfrak{X}, f)$, then set of the first components of all elements in $\hat{\mathfrak{Y}}$ is the complete set Σ_3,

$$\{x \mid \exists u \in \varphi_2 \circ \varphi_1(\mathfrak{X}) : u(0, \cdot) = x(\cdot)\} = \Sigma_3.$$

(2) Since $\mathfrak{X}$ is a metric Cantor space and $\varphi_2 \circ \varphi_1$ is continuous and injective, $\hat{\mathfrak{Y}} = \varphi_2 \circ \varphi_1(\mathfrak{X})$ is also a metric Cantor space.

We now aim to construct elements of $A(\sigma_{\mathfrak{Y}})$. If we are able to find a finite number of elements that generate an expansive group, we are done.

Definition 4.1. (1) Define $\tau : \mathfrak{Y}_3 \to \mathfrak{Y}_3$ by

$$\tau(u)\,(i, j) = u(i, j + 1) \quad \text{for } i \in \mathbb{N}_0,\ j \in \mathbb{Z}.$$

(2) Let $\zeta : E_3 \times E_3 \to E_3$ be defined by

$$\begin{array}{lll} \zeta(0, 0) = 0, & \zeta(0, 1) = 2, & \zeta(0, 2) = 0, \\ \zeta(1, 0) = 1, & \zeta(1, 1) = 1, & \zeta(1, 2) = 1, \\ \zeta(2, 0) = 2, & \zeta(2, 1) = 0, & \zeta(2, 2) = 2 \end{array}$$

and define $\gamma : \mathfrak{Y}_3 \to \mathfrak{Y}_3$ by

$$\gamma(u)\,(i, j) = \zeta(u(i, j), u(i + 1, j)) \quad \text{for } i \in \mathbb{N}_0,\ j \in \mathbb{Z}.$$

(3) Let $\hat{\omega}$ be the cyclic permutation of E_3, that is, $\hat{\omega}(0) = 1$, $\hat{\omega}(1) = 2$, $\hat{\omega}(2) = 0$. The lift ω of $\hat{\omega}$ to $\mathfrak{Y}_3$ is defined by

$$\omega : \mathfrak{Y}_3 \to \mathfrak{Y}_3, \quad u \mapsto \omega(u) \quad \text{with } \omega(u)\,(i, j) = \hat{\omega}(u(i, j)).$$

Then for $i \in \{0, 1, 2\}$ define $\gamma_i : \mathfrak{Y}_3 \to \mathfrak{Y}_3$, $u \mapsto v$ by $\gamma_i = \omega^{-i} \circ \gamma \circ \omega^i$.

PROPOSITION 4.1. *The function $\tau : \mathfrak{Y}_3 \to \mathfrak{Y}_3$ is continuous, bijective and commutes with the map $\sigma_{\mathfrak{Y}} : \mathfrak{Y}_3 \to \mathfrak{Y}_3$.*

Proof. Continuity is ensured by the local definition of the shift τ. The inverse of τ is given by

$$u \mapsto v, \quad v(i, j) = v(i, j - 1) \quad \text{for } i \in \mathbb{N}_0,\ j \in \mathbb{Z}$$

so that τ is bijective. Finally, the shifts τ and $\sigma_{\mathfrak{Y}}$ act on different indices, and thus commute. □

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PROPOSITION 4.2. *The function* $\gamma : \mathfrak{Y}_3 \to \mathfrak{Y}_3$ *is (i) continuous, (ii) bijective and (iii) commutes with the maps* $\sigma_{\mathfrak{Y}} : \mathfrak{Y}_3 \to \mathfrak{Y}_3$ *and* $\tau : \mathfrak{Y}_3 \to \mathfrak{Y}_3$.

Proof. This proof follows closely [**7**, p. 335]; for the sake of completeness (and since the functions are defined in a slightly different way) we repeat the argument given there.

(i) The continuity follows as usual from the locality of the map σ_3.

(ii) The bijectivity of γ is the property that requires more subtle arguments. We first show injectivity.

Assume that $u, v \in \mathfrak{Y}_3$, $u \neq v$, and $\gamma(u) = \gamma(v) =: w$. Then there is an index $(i_0, j_0) \in \mathbb{N}_0 \times \mathbb{Z}$ such that $u(i_0, j_0) \neq v(i_0, j_0)$ and $\gamma(u)\ (i, j_0) = \gamma(v)\ (i, j_0)$ for all $i \in \mathbb{N}_0$. Since $\zeta(a, b) = 1$ if and only if $a = 1$, $w(i_0, j_0) \neq 1$. Hence, $w(i_0, j_0) \in \{0, 2\}$ and therefore also $u(i_0, j_0),\ v(i_0, j_0) \in \{0, 2\}$. Without loss of generality we may assume $u(i_0, j_0) = 0$ and $v(i_0, j_0) = 2$.

Assume $w(i_0, j_0) = 0$. Then necessarily $v(i_0 + 1, j_0) = 1$, which in turn implies that $w(i_0 + 1, j_0) = 1$. If $w(i_0 + 1, j_0) = 1$ we can conclude that $u(i_0 + 1, j_0) = 1$. However, in this case we find $\zeta(u(i_0, j_0), u(i_0 + 1, j_0)) = \zeta(0, 1) = 2 \neq 0 = w(i_0, j_0)$ such that this case cannot occur.

Now let us consider the second case, $w(i_0, j_0) = 2$. The argument parallels the previous one. If $w(i_0, j_0) = 2$, necessarily $u(i_0 + 1, j_0) = 1$, which in turn implies that $w(i_0 + 1, j_0) = 1$ and thus also $v(i_0 + 1, j_0) = 1$. However, $\zeta(v(i_0, j_0), v(i_0 + 1, j_0)) = \zeta(2, 1) = 0 \neq 2 = w(i_0, j_0)$ such that this case also cannot occur.

All that said, we conclude that γ is injective.

We show that γ is also surjective. First note that the map $\zeta(\cdot, \cdot)$ is permutative in the first entry (i.e. for given $b \in E_3$, the map $a \mapsto \zeta(a, b)$ is a permutation of E_3). Let us consider $v \in \mathfrak{Y}_3$. We construct a sequence of states $\{u_n\}_{n\in\mathbb{N}} \subset \mathfrak{Y}_3$, that approximate the origin of v under γ: let $u_n(i, j) = 0$ for $i > n$. Due to the permutativity, there is a unique state $u_n(n, j) \in E_3$ such that $\zeta(u_3(n, j), u_3(n + 1, j)) = \zeta(u_3(n, j), 0) = v(n, j)$. Recursively, we are able to obtain values $u_n(n - 2, j), \ldots, u_n(0, j)$. Thus, $\gamma(u_n)\ (i, j) = v(i, j)$ for $i = 0, \ldots, n$ and $j \in \mathbb{Z}$. As $\mathfrak{Y}_3$ is compact, there is a converging subsequence, $u_n \to u$. Since γ is continuous, $\gamma(u) = v$.

(iii) Finally, we show that γ commutes with τ and $\sigma_{\mathfrak{Y}}$. We find that

$$\sigma_Y \circ \gamma(u)\ (i, j) = \zeta(u(i + 1, j), u(i + 2, j)) = \gamma \circ \sigma_{\mathfrak{Y}}(u)\ (i, j),$$
$$\tau \circ \gamma(u)\ (i, j) = \zeta(u(i, j + 1), u(i + 1, j + 1)) = \gamma \circ \tau(u)\ (i, j). \qquad \square$$

Since the functions γ_i are equal to γ up to a local permutation of the states of a cell, the properties of γ carry over to γ_i, that is, we have the following corollary.

COROLLARY 4.1. *The functions* $\gamma_i : \mathfrak{Y}_3 \to \mathfrak{Y}_3$, $i = 0, 1, 2$, *are continuous, bijective and commute with the maps* $\sigma_{\mathfrak{Y}} : \mathfrak{Y}_3 \to \mathfrak{Y}_3$ *and* $\tau : \mathfrak{Y}_3 \to \mathfrak{Y}_3$.

Thus τ as well as γ_i and ω are in $A(\sigma_{\mathfrak{Y}})$. Is the group generated by τ, γ_0, γ_1, γ_2 expansive? What does 'expansiveness' mean? Now, if we have two configurations u, v and they differ in some coordinate $(i, j) \in \mathbb{N}_0 \times \mathbb{Z}$, the information about this disagreement should be transported by the group to the coordinate $(0, 0)$, say. If this is the case, the images of u and v under a suitable sequence of transformation carried out by group

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TABLE 1. All cases for the proof of Proposition 4.3. $A_1 = \zeta(x, y)$, $B_1 = \delta(x, z)$, $A_2 = \zeta(\hat{\omega}(x), \hat{\omega}(y))$, $B_2 = \zeta(\hat{\omega}(x), \hat{\omega}(z))$, $A_3 = \zeta(\hat{\omega}^2(x), \hat{\omega}^2(y))$, $B_3 = \zeta(\hat{\omega}^2(x), \hat{\omega}^2(z))$. Numbers A_i, B_i are bold and the corresponding cells are shaded if $A_i \neq B_i$.

x	y	z	A_1	B_1	$\hat{\omega}(x)$	$\hat{\omega}(y)$	$\hat{\omega}(z)$	A_2	B_2	$\hat{\omega}^2(x)$	$\hat{\omega}^2(y)$	$\hat{\omega}^2(z)$	A_3	B_3
0	0	1	**0**	**2**	1	1	2	1	1	2	2	0	2	2
0	0	2	0	0	1	1	0	1	1	2	2	1	**2**	**0**
0	1	0	**2**	**0**	1	2	1	1	1	2	0	2	2	2
0	1	2	**2**	**0**	1	2	0	1	1	2	0	1	**2**	**0**
0	2	0	0	0	1	0	1	1	1	2	1	2	**0**	**2**
0	2	1	**0**	**2**	1	0	2	1	1	2	1	0	**0**	**2**
1	0	1	1	1	2	1	2	**0**	**2**	0	2	0	0	0
1	0	2	1	1	2	1	0	**0**	**2**	0	2	1	**0**	**2**
1	1	0	1	1	2	2	1	**2**	**0**	0	0	2	0	0
1	1	2	1	1	2	2	0	2	2	0	0	1	**0**	**2**
1	2	0	1	1	2	0	1	**2**	**0**	0	1	2	**2**	**0**
1	2	1	1	1	2	0	2	2	2	0	1	0	**2**	**0**
2	0	1	**2**	**0**	0	1	2	**2**	**0**	1	2	0	1	1
2	0	2	2	2	0	1	0	**2**	**0**	1	2	1	1	1
2	1	0	**0**	**2**	0	2	1	**0**	**2**	1	0	2	1	1
2	1	2	**0**	**2**	0	2	0	0	0	1	0	1	1	1
2	2	0	2	2	0	0	1	**0**	**2**	1	1	2	1	1
2	2	1	**2**	**0**	0	0	2	0	0	1	1	0	1	1

elements have a fixed distance $\delta = 1$. The essential part is the ability of the group action to move information about local disagreement of two configurations to a given coordinate.

The function τ transports information in the direction of the second coordinate by a simple shift. Transport of information in the direction of the first coordinate is less simple, since the canonical shift in this direction, $\sigma_{\mathfrak{Y}}$, is not bijective. We thus need a more refined mechanism. The following proposition shows that the functions γ_i jointly are able to perform the required transport of information.

PROPOSITION 4.3. *Let $u, v \in \mathfrak{Y}$, $u(i, j) = v(i, j)$ for $i = 0, \ldots, n-1$ and $j \in \mathbb{Z}$, and $u(n, j_0) \neq v(n, j_0)$ for $n \in \mathbb{N}_0$, $j_0 \in \mathbb{Z}$. Then there exists $\gamma \in \{\gamma_0, \gamma_1, \gamma_2\}$ such that $\gamma(u)\,(n-1, j_0) \neq \gamma(v)\,(n-1, j_0)$.*

Proof. Let $x = u(n-1, j_0) = v(n-1, j_0)$, $y = u(n, j_0) \neq v(n, j_0) = z$. Then the number of combinations for (x, y) is nine, and – given (x, y) – the number of possibilities for z is two. Altogether there are 18 possibilities for (x, y, z). For these 18 cases we check whether there is an $i \in \{0, 1, 2\}$ such that

$$\hat{\omega}^{-i} \circ \zeta(\hat{\omega}^i(x)\hat{\omega}^i(y)) \neq \hat{\omega}^{-i} \circ \zeta(\hat{\omega}^i(x)\hat{\omega}^i(z)) \Leftrightarrow \zeta(\hat{\omega}^i(x)\hat{\omega}^i(y)) \neq \zeta(\hat{\omega}^i(x)\hat{\omega}^i(z)).$$

The 18 cases are listed in Table 1. If $\gamma_i(x, y) \neq \gamma_i(x, z)$, the results of the functions are shown in bold. As there are bold numbers in each line of the table, this concludes the proof of the proposition. □

PROPOSITION 4.4. *The group $\mathcal{G} = \langle \tau, \gamma_0, \gamma_1, \gamma_2 \rangle$ acts in an expansive way on $\mathfrak{Y}_3$.*

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Proof. Let $u \neq v$. Then there is (at least) one cell $(i_0, j_0) \in \mathbb{N}_0 \times \mathbb{Z}$, such that $u(i_0, j_0) \neq u(i_0, j_0)$. Without loss of generality, let i_0 be minimal, that is, $u(i, j) = v(i, j)$ for $i = 0, \ldots, i_0 - 1$, $j \in \mathbb{Z}$. According to Proposition 4.3, we find an element $\eta \in \langle \gamma_i | i = 0, 1, 2 \rangle$, $\eta = \gamma_{l^{(0)}} \circ \cdots \circ \gamma_{l^{(i_0-1)}}$, such that $\eta(u)\ (0, j_0) \neq \eta(v)\ (0, j_0)$. Applying τ a large enough number of times ($-j_0$ times suffice), we obtain

$$\tau^{(-j_0)} \circ \eta(u)\ (0, 0) \neq \tau^{(-j_0)} \circ \eta(v)\ (0, 0).$$

Thus,

$$d(\tau^{(-j_0)} \circ \eta(u), \tau^{(-j_0)} \circ \eta(v)) = 1$$

and the group $\mathcal{G}$ acts expansively on $\mathfrak{Y}_3$. □

We now formulate our central theorem.

THEOREM 4.2. *Consider a continuous map on a Cantor space* $(\mathfrak{X}, f)$. *Then there is a cellular automaton* (Γ, U_0, E, g_0) *with grid*

$$\Gamma = \langle \tau, \omega, \gamma | \tau \circ \omega = \omega \circ \tau,\ \tau \circ \gamma = \gamma \circ \tau,\ \omega^3 = e \} \rangle$$

and alphabet $E = \{0, 1, 2\}$, *such that* $(\mathfrak{X}, f)$ *can be weakly embedded into the cellular automaton. Furthermore, there is a cellular automaton* (Γ, U_0, F, g_0') *with alphabet* $F = \{0, 1, 2, 3\}$ *such that* $(\mathfrak{X}, f)$ *can be weakly embedded into the cellular automaton such that both dynamical systems have the same topological entropy.*

Proof. First of all, we can associate with $(\mathfrak{X}, f)$ a subshift $\mathfrak{Y}$ of the full shift $(\mathfrak{Y}_3, \sigma_{\mathfrak{Y}})$. We apply Theorem 2.1. $\mathfrak{Y}$ is a metric Cantor space. τ, ω and γ defined above are bijective, continuous, and commute with $\sigma_{\mathfrak{Y}}$. Since $\gamma_i = \omega^{(-i)} \circ \gamma \circ \omega^{(i)}$, the group generated by these functions acts expansively on $\mathfrak{Y}$. Furthermore, $\sigma_{\mathfrak{Y}}$ is a continuous function. Thus, we know that we can embed $(\mathfrak{Y}, \sigma_{\mathfrak{Y}})$ strongly into a cellular automaton.

The rest is a direct consequence of Proposition 2.2. □

Remark 4.2. (1) Note that in general $\mathfrak{Y}$ is not invariant under the translation operators of the grid Γ.

(2) The function g_0 depends on f. All possible U_0 and g_0 appear if f runs through all possible functions f.

(3) The structure of this graph realizes several no-go theorems: the graph $\mathbb{Z}^2$ is subgraph of Γ, such that in general injectivity and surjectivity of the global function are not decidable [**6, 9**]. Furthermore, the grid has an exponential growth function so that, in general, the Garden of Eden theorems need not hold [**10**].

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