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Normal numbers from infinite convolution measures

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Abstract. Let r, s be natural numbers, $r, s \geq 2$, such that $\log r/\log s$ is irrational. Let $\mu_j = \sum_{k=0}^{s-1} p_k^{(j)} \delta_{k/s^j}$, where $p_k^{(j)} \in [0, 1)$, $\sum_{k=0}^{s-1} p_k^{(j)} = 1$, $j = 1, 2, \ldots$ and δ_x denotes the probability atom at x. We prove that if there exists $k_0 \in \{0, \ldots, s-2\}$, such that $\inf\{p_{k_0}^{(j)}, p_{k_0+1}^{(j)} : j \in \mathbb{N}\} > 0$ and μ is the infinite convolution measure $\mu = *_{j=1}^{\infty} \mu_j$, where the convergence is in the weak* sense, then μ-almost all numbers are normal to base r. This generalizes some earlier results of Feldman, Smorodinsky and Pushkin.

1. *Introduction*

Let r, s be natural numbers, $r, s \geq 2$, such that $\log r/\log s$ is irrational ($r \not\sim s$). Let $\mu_j = \sum_{k=0}^{s-1} p_k^{(j)} \delta_{k/s^j}$, where $p_k^{(j)} \in [0, 1)$, $\sum_{k=0}^{s-1} p_k^{(j)} = 1$, $j = 1, 2, \ldots$ and δ_x denotes the probability atom at x. Let μ be the infinite convolution measure

$$\mu = *_{j=1}^{\infty} \mu_j,$$

where the convergence is in the weak* sense. Then μ is a probability measure on $[0, 1]$; see [**1**] for more details. In this paper we prove the following.

THEOREM 1. *Let μ be a measure as above.*

(i) *If there exists $k_0 \in \{0, \ldots, s-2\}$, such that*

$$\inf\left\{p_{k_0}^{(j)}, p_{k_0+1}^{(j)} : j \in \mathbb{N}\right\} > 0,$$

then μ-almost all numbers in $[0, 1]$ are normal to base r.

(ii) *If there exists $i_0 \in \{0, 1, \ldots, s-1\}$ such that $p_{i_0}^{(n)} \geq 1/s + \varepsilon$, $n \in \mathbb{N}$, for some $\varepsilon > 0$, then μ-almost all numbers in $[0, 1]$ are not simply normal to every base s^k, $k = 1, 2, \ldots$.*

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(iii) *If the sequence* $\max\{|p_k^{(j)} - 1/s| : k = 0, 1, \ldots, s-1\}$ *tends to* 0, *as* j *tends to* $+\infty$, *then* μ*-almost all numbers are normal to base* s.

This is a generalization of a result given by Feldman and Smorodinsky [**5**] and Pushkin [**8**]. In their work they are concerned with measures with $p_k^{(j)}$ independent of j; namely, under μ the s-digit process was an independent identically distributed process.

We also refer to the papers of Cassels [**3**] and Schmidt [**9**], where the authors make use of Cantor–Lebesque-type measures in order to prove independently that the cardinality of the reals which are normal to base r but not even simply normal to base s is equal to the cardinality of the reals. For another proof see [**7**].

Analogous results for Riesz products measures are given in [**2**], using the transcendence theory of Baker.

It is well known that the Fourier–Stieltjes coefficients $\hat{\mu}(w)$, $w \in \mathbb{Z}$, of the measure μ are given by the relation

$$\hat{\mu}(w) = \prod_{j=1}^{\infty} \left(\sum_{k=0}^{s-1} p_k^{(j)} e\left(\frac{kw}{s^j}\right)\right),$$

where $e(kw/s^j) = e^{2\pi(kw/s^j)i}$ and so

$$|\hat{\mu}(w)| = \Pi_p(s; w), \tag{1}$$

where

$$\Pi_p(s; w) = \prod_{j=1}^{\infty} \left| \sum_{k=0}^{s-1} p_k^{(j)} e\left(\frac{kw}{s^j}\right)\right|.$$

Using the ideas of [**9**] we give bounds for sums of the form

$$\sum_{N_1 < n,m \le N_2} \Pi_p(s; ur^n + vr^m), \quad u, v \in \mathbb{Z}, N_1, N_2 \in \mathbb{N}.$$

We mention that in [**9**, Lemma 7B, p. 668] the following fact is proved:

$$\sum_{N_1 < n,m \le N_2} \Pi(s, t; ur^n + vr^m) \le C(u, v)(N_2 - N_1)^{2-b}, \tag{2}$$

where

$$\Pi(s, t; ur^n + vr^m) = \prod_{j=1}^{\infty} \left| \left(\sum_{k=0}^{t-1} e\left(\frac{kw}{s^j}\right)\right) \Big/ t \right|, \quad 1 < t < s,$$

$C(u, v)$ is a constant which is dependent only on u, v and $b \in (0, 1)$.

2. *On a result of Schmidt*

In this section we give a modification of a result due to Schmidt [**9**]. More precisely, we prove an inequality analogous to (2). For the convenience of the reader we use the terminology of Schmidt.

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LEMMA 1. *Let $w = c_g \dots c_2 c_1$ in the scale of s. Assume there are at least z pairs of digits $c_{i+1}c_i$ with*

$$1 \le c_{i+1}c_i \le s^2 - 2$$

(here $c_{i+1}c_i = sc_{i+1} + c_i$). If there exists $k_0 \in \{0, \dots, s-2\}$ such that $\inf\{p_{k_0}^{(j)}, p_{k_0+1}^{(j)} : j \in \mathbb{N}\} > 0$, then

$$\Pi_p(s; w) \le a_1^z,$$

where $a_1 = a_1(s) \in (0, 1)$.

Proof. By assumption we have that there are at least z numbers i having

$$\frac{1}{s^2} \le \left\{\frac{w}{s^i}\right\} \le 1 - \frac{1}{s^2},$$

where $\{w/s^i\}$ is the fractional part of w/s^i. For such an i and since for some $k_0 \in \{0, \dots, s-2\}$ we have that $\inf\{p_{k_0}^{(j)}, p_{k_0+1}^{(j)} : j \in \mathbb{N}\} > 0$, we obtain that

$$\begin{aligned}
\left|\sum_{k=0}^{s-1} p_k^{(n)} e\left(\frac{kw}{s^i}\right)\right| &\le \left|p_{k_0}^{(n)} e\left(\frac{k_0 w}{s^i}\right) + p_{k_0+1}^{(n)} e\left(\frac{(k_0+1)w}{s^i}\right)\right| + \sum_{\substack{k=0 \\ k \ne k_0,\, k_0+1}}^{s-1} p_k^{(n)} \\
&= \left|p_{k_0}^{(n)} + p_{k_0+1}^{(n)} e\left(\frac{w}{s^i}\right)\right| + 1 - (p_{k_0}^{(n)} + p_{k_0+1}^{(n)}) \\
&\le \left|p_{k_0}^{(n)} + p_{k_0+1}^{(n)} e\left(\frac{1}{s^2}\right)\right| + 1 - (p_{k_0}^{(n)} + p_{k_0+1}^{(n)}) \\
&\le a_1(s) \in (0, 1),
\end{aligned}$$

and the lemma is proved (see also [**9**, Lemma 2]). □

LEMMA 2. *Assume $r \not\sim s$, $u \ne 0$, $v \ne 0$, $u, v \in \mathbb{Z}$. Then*

$$\sum_{N_1 < n, m \le N_2} \Pi_p(s, ur^n + vr^m) \le C(u, v)(N_2 - N_1)^{2-a},$$

where $C(u, v)$ is a constant which is dependent only on u, v and $a \in (0, 1)$.

Proof. The process for our proof is indeed *word-for-word* the same as the process for the proof of relation (2) [**9**, Lemma 7B, pp. 665–669], except that one should use our Lemma 1 (given above) instead of Schmidt's second lemma and our product $\Pi_p(s; ur^n + vr^m)$ instead of Schmidt's product $\Pi(s, t; ur^n + vr^m)$. □

3. *Normal numbers to base r*

In order to prove the normality to base r we make use of the following.

PROPOSITION 1. *Let μ be a positive measure on $[0, 1]$. If*

$$\sum_{N=1}^{\infty} \frac{1}{N^3} \sum_{n,m=1}^{N} \hat{\mu}((r^n - r^m)h) = \sum_{N=1}^{\infty} \frac{1}{N} \int \left|\frac{1}{N} \sum_{n=1}^{N} e^{i2\pi r^n hx}\right|^2 d\mu(x) < \infty,$$

for all $h \ne 0$, $h \in \mathbb{Z}$, then $\{r^n x\}$ is uniformly distributed μ-a.e.

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Proof. This can be obtained from a result of Davenport, Erdös and LeVeque [**4**]. See also [**2**, Proposition 1, p. 13], and its proof. □

Proof of Theorem 1. (i) Using relation (1), we have from Lemma 2 that

$$\sum_{N_1<n,m\le N_2} |\hat{\mu}(ur^n + vr^m)| = \sum_{N_1<n,m\le N_2} \Pi_p(s; ur^n + vr^m) \le C(u, v)(N_2 - N_1)^{2-a}.$$

So we have

$$\sum_{N=1}^{\infty} \frac{1}{N^3} \sum_{n,m=1}^{N} \hat{\mu}((r^n - r^m)h) \le \sum_{N=1}^{\infty} \frac{1}{N^3} C(h, -h)N^{2-a} < \infty.$$

By Proposition 1 we have that μ-almost all numbers are normal to base r.

(ii) Let $x = \sum_{j=1}^{\infty} \varepsilon_j^{(k)}(x)/s^{kj}$, $\varepsilon_j^{(k)}(x) \in \{0, \dots, s^k - 1\}$, $k \in \mathbb{N}$. It is easy to check that given $k \in \mathbb{N}$ and $m \in \{0, \dots, s^k - 1\}$, the sequence of functions $X_{E_{m,k,j}}(x)$, $j \in \mathbb{N}$, where $X_{E_{m,k,j}}$ is the characteristic function on $E_{m,k,j} = \{x \in [0, 1] : \varepsilon_j^{(k)}(x) = m\}$, is a sequence of independent random variables with respect to μ, $\mu(E_{m,k,j}) = \prod_{i=0}^{k-1} p_{c_{k-i}}^{(i+1+(j-1)k)}$ where $m = \sum_{i=1}^{k} c_i s^{i-1}$, $c_i \in \{0, 1, \dots, s - 1\}$. So by the law of large numbers we have that

$$\lim_{N\to+\infty} \frac{1}{N}\left(\sum_{j=1}^{N} X_{E_{m,k,j}}(x) - \prod_{i=0}^{k-1} p_{c_{k-i}}^{(i+1+(j-1)k)} \right) = 0, \quad \mu\text{-a.e.} \tag{3}$$

For $c_i = i_0$, $i = 1, \dots, k$ ($m = i_0(s^k - 1)/(s - 1)$), we take, since by hypothesis we have $p_{i_0}^{(n)} \ge 1/s + \varepsilon$,

$$\liminf_{N\to+\infty} \frac{1}{N} \sum_{j=1}^{N} X_{E_{m,k,j}}(x) \ge \left(\frac{1}{s} + \varepsilon\right)^k, \quad \mu\text{-a.e.},$$

which gives our result; see [**6**].

(iii) By the hypothesis we have that

$$\lim_{N\to+\infty} \frac{1}{N} \sum_{j=1}^{N} \prod_{i=0}^{k-1} p_{c_{k-i}}^{(i+1+(j-1)k)} = \frac{1}{s^k}$$

and so by (3) we have that

$$\lim_{N\to+\infty} \frac{1}{N} \sum_{j=1}^{N} X_{E_{m,k,j}}(x) = \frac{1}{s^k}, \quad \mu\text{-a.e.}$$

This gives that μ-almost all the numbers are normal to base s; see [**6**].

For the above measures we have the following question.

Question. Is it true that any measure of the above form which is continuous (see [**1**] for details about the continuity) has full measure on a set of normal numbers to the base r?

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