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# Einstein–Maxwell fields with plane symmetry. I

By S. Patnaik
*Department of Mathematics, Université de Montréal, Montreal, Canada*†

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An exact static solution of the system of Einstein–Maxwell field equations in the absence of charge is obtained when both the metric tensor $g_{\alpha\beta}$ and the Maxwell field tensor $F_{\alpha\beta}$ exhibits plane symmetry in the sense that both remain invariant under the group of motions that characterize plane symmetry. Our solution generalizes the empty space-time solution with plane symmetry previously obtained by Taub to the situation when static charge-free electromagnetic fields are present. We hope to discuss non-static solutions of Einstein–Maxwell fields with plane symmetry as well as some other considerations in a subsequent paper.

1. *Introduction.* The system of Einstein–Maxwell field equations of general relativistic electrodynamics in the absence of charge is given by (1),

$$R^{\mu}_{\nu} - \tfrac{1}{2} g^{\mu}_{\nu} R = -kc^2 T^{\mu}_{\nu}, \quad k = \frac{8\pi G}{c^2}, \tag{1·1}$$

$$T^{\mu}_{\nu} = F^{\mu\sigma} F_{\nu\sigma} - \tfrac{1}{4} g^{\mu}_{\nu} F^{\rho\sigma} F_{\rho\sigma}, \tag{1·2}$$

$$F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0, \tag{1·3}$$

$$\frac{1}{\sqrt{-g}} (F^{\mu\nu} \sqrt{-g})_{,\mu} = 0. \tag{1·4}$$

(Here Greek letters run from 1 through 4, comma denotes differentiation with respect to the variable indicated and $x^1 = x, x^2 = y, x^3 = z, x^4 = t$.)

Equations (1·3) and (1·4) are the eight Maxwell equations and if a solution of these Maxwell equations for the electromagnetic field tensor $F_{\alpha\beta}$ could be obtained in terms of the coordinates $x^{\alpha}$ and the pre-supposed metric tensor $g_{\alpha\beta}$, then the equations (1·2) would determine the energy tensor $T^{\mu}_{\nu}$. The energy tensor $T^{\mu}_{\nu}$ thus determined will satisfy the divergence relations $T^{\mu}_{\mu;\nu} = 0$ and so the Einstein equations (1·1) would be integrable. In this manner a simultaneous solution for $g_{\alpha\beta}, F_{\alpha\beta}$ of the system of equations (1·1) through (1·4) could be obtained. We shall follow this procedure when both the metric tensor $g_{\alpha\beta}$ and the Maxwell field tensor $F_{\alpha\beta}$ exhibit plane symmetry (2). The metric tensor $g_{\alpha\beta}$ or the space-time determined by it is said to exhibit plane symmetry if it remains invariant under the group of transformations given by

$$\left.\begin{aligned} x^{2*} &= x^2 \cos\theta + x^3 \sin\theta + a, \\ x^{3*} &= -x^2 \sin\theta + x^3 \cos\theta + b, \\ x^{1*} &= x^1 \quad x^{4*} = x^4, \end{aligned}\right\} \tag{1·5}$$

† Present address: Institute of Mathematics, Università di Pisa, Pisa, Italy.

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where $a$, $b$, $\theta$ are arbitrary constants. It has been shown by Taub(2) that the line-element of such space-times whose metric tensor exhibits plane symmetry reduce to

$$ds^2 = -e^{2u}[(dx^1)^2-(dx^4)^2]-e^{2v}[(dx^1)^2+(dx^3)^2], \tag{1·6}$$

where $u$, $v$ are functions of $x^1$ and $x^4$ only. The infinitesimal vector $\xi^\alpha_a$ ($\alpha = 1, 2, 3, 4$; $a = 1, 2, 3$) corresponding to the transformations (1·5) is given by the matrix

$$\|\xi^\sigma_a\| = \begin{Vmatrix} 0 & 0 & 0 \\ 1 & 0 & -x^3 \\ 0 & 1 & x^2 \\ 0 & 0 & 0 \end{Vmatrix}. \tag{1·7}$$

The system of Killing equations(3)

$$g_{\alpha\beta,\sigma}\xi^\sigma + g_{\alpha\rho}\xi^\rho{}_{,\beta} + g_{\tau\beta}\xi^\tau{}_{,\alpha} = 0 \tag{1·8}$$

state the condition that the metric tensor $g_{\alpha\beta}$ remain invariant under the infinitesimal transformations given by

$$x^{*\sigma} = x^\sigma + \xi^\sigma(x^\alpha). \tag{1·9}$$

For the metric tensor $g_{\alpha\beta}$ given by (1·6), the Killing vector given in (1·7) satisfies the equations (1·8).

It can be seen that the invariance of the antisymmetric Maxwell tensor $F_{\alpha\beta}$ under the infinitesimal transformations given by (1·9) give rise to the equations.

$$\xi^\sigma F_{\alpha\beta,\sigma} + F_{\alpha\rho}\xi^\rho{}_{,\beta} + F_{\tau\beta}\xi^\tau{}_{,\alpha} = 0 \tag{1·10}$$

which are similar to the Killing equations (1·8).

In other words, equations (1·10) express the statement that the Lie derivative of $F_{\alpha\beta}$ with respect to the vector field $\xi^\alpha$ vanishes.

When the values for $\xi^\sigma_a$ as given by (1·7) are substituted in the equations (1·10), the following conditions are obtained

$$\left.\begin{aligned} F_{13} = 0, \quad F_{12} = 0, \quad F_{43} = 0, \quad F_{42} = 0; \\ F_{14,i} = 0, \quad F_{23,i} = 0 \quad (i = 2, 3). \end{aligned}\right\} \tag{1·11}$$

Hence, for a Maxwell tensor exhibiting plane symmetry in the sense that it remains invariant under the group of motions given by (1·5), the only non-vanishing components are $F_{14}$ and $F_{23}$, both depending only on the coordinates $x^1$ and $x^4$. (The component $F_{14}$ gives the electric field and $F_{23}$ gives the magnetic field.)

2. The metric tensor for plane-symmetric space-times, as has been noted before, is given by

$$\left.\begin{aligned} g_{11} = -g_{44} = -e^{2u}, \quad u = u(x^1, x^4), \\ g_{22} = g_{33} = -e^{2v}, \quad v = v(x^1, x^4), \\ g_{\alpha\beta} = 0 \quad (\alpha \neq \beta). \end{aligned}\right\} \tag{2·1}$$

Maxwell's field equations (1·3) and (1·4) give, in our case, using (1·11) and (2·1), for the only non-vanishing components of $F_{\alpha\beta}$,

$$F_{23} = c_1, \quad F_{14} = c_2 e^{2u-2v} \tag{2·2}$$

where $c_1, c_2$ are arbitrary constants.

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The energy tensor $T^{\mu}_{\nu}$ is given by the equations (1·2) and the only non-vanishing components $F_{14}$ and $F_{23}$, as given above. Thus,

$$\left.\begin{aligned} &T^1_1 = T^4_4 = -T^2_2 = -T^3_3 = -\tfrac{1}{2}(c_1^2+c_2^2)\,e^{-4v} \\ &T^{\alpha}_{\beta} = 0 \quad (\alpha \neq \beta); \quad T = T^{\alpha}_{\alpha} = 0. \end{aligned}\right\} \tag{2·3}$$

It can be easily seen that for the energy tensor $T^{\mu}_{\nu}$ as given above, the divergence relations $T^{\mu}_{\mu;\,\nu} = 0$ are identically satisfied as should be the case because $T^{\mu}_{\nu}$ is built out of the Maxwell field tensor that satisfies the Maxwell's field equations.

3. We restrict ourselves to the static case in this paper, thus all the functions involved depend on only one space coordinate $x^1$. We shall obtain a complete mathematical solution of the entire system of the Einstein–Maxwell equations (1·1)–(1·4) with the metric tensor $g_{\alpha\beta}$ and the Maxwell tensor $F_{\alpha\beta}$ both exhibiting plane symmetry. Using equations (2·1) and (2·2), as the functions $u$ and $v$ depend on the coordinate $x^1$ only, the Einstein–Maxwell equations reduce to the following ordinary differential equations

$$u_{,11} + 2v_{,1}^{\,2} + 2v_{,11} - 2v_{,1}u_{,1} = \frac{k}{2}(c_1^2+c_2^2)\,e^{-4v}e^{2u}, \tag{3·1}$$

$$\tfrac{1}{2}e^{-2u}(v_{,11} + 2v_{,1}^{\,2}) = -\frac{k}{2}(c_1^2+c_2^2)\,e^{-4v}e^{2u}, \tag{3·2}$$

$$u_{,11} + 2v_{,1}u_{,1} = \frac{k}{2}(c_1^2+c_2^2)\,e^{-4v}e^{2u}, \tag{3·3}$$

where $u$ and $v$ are functions of $x^1$ and $c_1, c_2$ are arbitrary constants. The fact that the above system of three ordinary differential equations is consistent for the solution of the two unknown functions $u$ and $v$ is a consequence of the Bianchi identities and the divergence relations for the energy tensor which in our case are identically satisfied by our construction. The system of equations (3·1)–(3·3) is equivalent to the following system,

$$A = 0, \quad B = 0, \quad C = 0$$

where

$$A \equiv v_{,1}^{\,2} + 2v_{,1}u_{,1} + \frac{k}{2}(c_1^2+c_2^2)\,e^{-4v}e^{2u},$$

$$B \equiv v_{,11} + v_{,1}^{\,2} - 2v_{,1}u_{,1},$$

$$C \equiv u_{,11} + 2u_{,1}v_{,1} - \frac{k}{2}(c_1^2+c_2^2)\,e^{-4v}e^{2u}.$$

The above defined expressions $A$, $B$, $C$ satisfy the differential equation

$$A_{,1} - 2(v_{,1}+u_{,1})(A+B) - 2v_{,1}C = 0. \tag{3·4}$$

We first solve the two ordinary differential equations $A = 0$, $B = 0$ and determine $u(x^1)$ and $v(x^1)$. The functions $u$ and $v$ thus obtained must satisfy the third equation $C = 0$ as a consequence of (3·4). The equation $B = 0$ is written as

$$\frac{v_{,11}}{v_{,1}} + v_{,1} - 2u_{,1} = 0$$

or

$$(\log v_{,1})_{,1} + (v - 2u)_{,1} = 0.$$



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From the above equation, we have on integration,

$$e^{2u} = c_3 e^v v_{,1} \tag{3·5}$$

where $c_3$ is an arbitrary constant. Instead of the equation $B = 0$, we choose for our second equation, $A + B = 0$, that is

$$v_{,11} + 2v_{,1}^2 + \frac{k}{2}(c_1^2 + c_2^2)\, e^{-4v} e^{2u} = 0$$

which may be rewritten as

$$(e^{2v})_{,11} - c_3 k (c_1^2 + c_2^2)\,(e^{-v})_{,1} = 0,$$

in which use is made of the equation (3·5). On integration, the above equation yields,

$$e^{2v}{}_{,1} - c_3 k (c_1^2 + c_2^2)\, e^{-v} + c_4 = 0,$$

where $c_4$ is an arbitrary constant. The above equation is transformed into the following with a change of the dependent variable defined by

$$e^v = w, \quad 2w^2 w_{,1} - c_3 k (c_1^2 + c_2^2) + c_4 w = 0.$$

This ordinary differential equation is integrated by separation of the variables and one quadrature. The solution for $w$ is implicitly given by the equation

$$\frac{w^2}{2} + \frac{c_3}{c_4} k (c_1^2 + c_2^2)\, w + \frac{c_3^2 k^2}{c_4^2} (c_1^2 + c_2^2) \log [c_3 k (c_1^2 + c_2^2) - c_4 w] + \tfrac{1}{2} c^4 (x^1 + c_5) = 0,$$

where $c_5$ is an arbitrary constant. Returning to the original variable $v$, we summarize our solution for the Einstein–Maxwell equations for charge-free static plane-symmetric fields

$$ds^2 = -e^{2u(x^1)}[(dx^1)^2 - (dx^4)^2] - e^{2v(x^1)}[(dx^2)^2 + (dx^3)^2],$$

where $u$ and $v$ are given by the equations

$$-c_4 x^1 - c_4 c_5 = e^{2v} + \frac{2c_3}{c_4}(c_1^2 + c_2^2)\, e^v + \frac{2c_3^2 k^2}{c_4^2}(c_1^2 + c_2^2) \log [c_3 k (c_1^2 + c_2^2) - c_4 e^v],$$

$$\left.\begin{aligned} e^{2u} &= c_3 e^v v_{,1}, \\ F_{14} &= -c_1 e^{2u} e^{-2v} = -F_{41}, \\ F_{23} &= c_2 = -F_{32}, \\ F_{12} &= 0 = F_{13}, \end{aligned}\right\} \tag{D}$$

where $c_j\,(j = 1, 2, 3, 4, 5)$ are arbitrary constants; $c_1$ and $c_2$ characterize the electric and the magnetic fields respectively. When one proceeds to the limiting case, $c_1 \to 0$ and $c_2 \to 0$, the electromagnetic field vanishes, and so, $T_\nu^\mu = 0$ (in this case, we have $R_{\mu\nu} = 0$ but this empty space-time is curved because $R^\alpha_{\beta\lambda\delta} \neq 0$) and the above solution reduces to the empty space-time solution given by Taub (2),

$$ds^2 = -(c_4 x^1 + c_5)^{-\frac{1}{2}}[(dx^1)^2 - (dx^2)^2],$$
$$-(c_4 x^1 + c_5)[(dx^2)^2 + (dx^3)^2],$$

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where $c_4$ and $c_5$ are arbitrary constants. Thus our solution (D) is a true generalization of the Taub's empty space-time to the case when static charge-free electromagnetic fields are present.

When the constant $c_4 \to 0$, but the constants $c_1$ and $c_2$ do not vanish at the same time, the solution (D) reduces to the following.:

$$ds^2 = -e^{2u}[(dx^1)^2 - (dx^4)^2] - e^{2v}[(dx^2)^2 + (dx^3)^2],$$

where

$$\begin{aligned}
e^{2u} &= \frac{c_3^2 k}{2}(c_1^2+c_2^2)\,[\tfrac{3}{2}c_3 k(c_1^2+c_2^2)\,x^1 + c_6]^{-\frac{2}{3}},\\
e^{2v} &= [\tfrac{3}{2}c_3 k(c_1^2+c_2^2)\,x^1 + c_6]^{-\frac{2}{3}},\\
F_{14} &= -F_{41} = -\tfrac{1}{2}c_1 c_3^2 k(c_1^2+c_2^2)\,[\tfrac{3}{2}c_3 k(c_1^2+c_2^2)\,x^1 + c_6]^{-\frac{4}{3}},\\
F_{23} &= -F_{32} = c_2,\\
F_{12} &= 0 = F_{23},
\end{aligned}$$

where $c_1, c_2, c_3, c_6$ are arbitrary constants. In a sense, our solution as given by (D) is the plane symmetric analogue of the Reissner–Nordstrom solution (4),

$$ds^2 = -\left(1 - \frac{2m}{\gamma} + \frac{\epsilon}{\gamma^2}\right)^{-1} dr^2 - \gamma^2(d\theta^2 + \sin^2\theta\, d\phi^2) + \left(1 - \frac{2m}{\gamma} + \frac{\epsilon}{\gamma^2}\right) dt^2$$

which generalizes the well-known Schwarzschild solution giving the exterior field of a spherically charged distribution of mass. We shall discuss the meaning of the arbitrary constants occurring in our solution (D), the problem of boundary conditions, the meaning of the energy content of the system, the meaning of singularities, the uniqueness of the solution at time $t = t_1 > 0$ when the solution (D) corresponding to $t = 0$ is given and the possible physical models which our solution represents in a subsequent paper.

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